CONSTRUCTION OF SINGULAR SURFACES IN LINEAR DIFFERENTIAL GAMES

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The main topic of the report concerns constructing singular surfaces in linear differential games with fixed terminal time. The considered algorithm is embedded into the backward procedures for constructing value function level sets. These procedures were elaborated in Sverdlovsk (now Ekaterinburg) at the beginning of 80's.

REFERENCES

Theoretical basis for numerical methods in DG:

Bellman R., Isaacs R., Fleming W.H., Breakwell J.V.,

Pontryagin L.S., Krasovskii N.N., Pshenichnii B.N., Subbotin A.I.

Singular surfaces investigations:

Bernhard P., Melikyan A.A.,

Merz A.W., Shinar J., Lewin J., Olsder G.J., Hovakimyan N.V.

Numerical constructions, algorithms and programs for DG:

Nikol'skii M.S., Ponomarev A.P., Rosov N.H., Ostapenko V.V.,

Ushakov V.N., Taras'ev A.M., Uspenskii A.A., Khripunov A.P.,

Shagalova L.G., Grigorieva S.V.,

Reshetova T.N., Lukoyanov N.Yu., Krasovskii A.N.,

Grigorenko N.L., Silin D.B., Polovinkin E.S., Ivanov G.E.,

Bardi M., Falcone M., Soravia P.,

Cardaliaguet P., Quincampoix M., Saint-Pierre P.,

Patsko V.S., Botkin N.D., Turova V.L., Zarkh M.A., Ivanov A.G.

While level sets of value function are constructed, the backward procedures based on papers of classics of DG are used.

Singular surfaces were studied theoretically (necessary conditions) and in the frames of concrete problems.

Nowadays, there are many papers devoted to numerical algorithms for DG solving. However algorithms for global construction and classification of singular surfaces are absent.

LINEAR DIFFERENTIAL GAME

$$\dot{x} = A(t)x + B(t)u + C(t)v,$$

$$x \in R^n, \quad u \in P, \quad v \in Q, \quad T$$

$$\varphi\big(x(T)\big) \to \min_u \max_v$$

φ depends on 2 coordinates of x

Level sets of value function, singular surfaces (for the case of scalar controls u, v)

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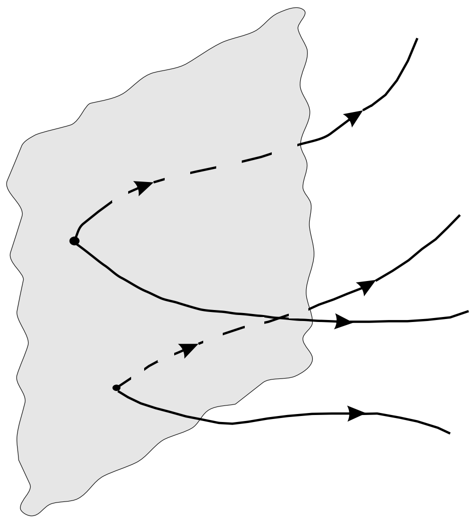
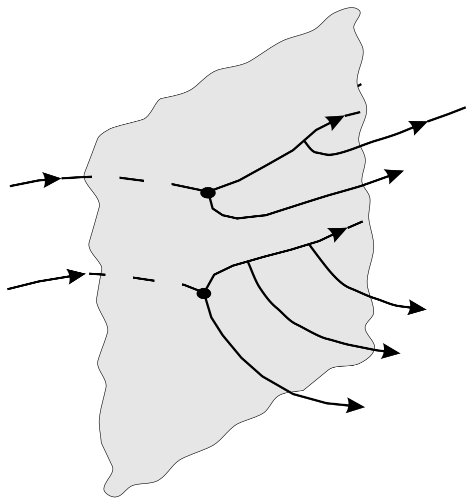
In this paper, linear differential games with fixed terminal time and convex payoff function depending on two components of the phase vector are considered. Computer programs for building level sets of the value function (stable bridges or Krasovskii bridges) were worked-out for arbitrary convex compact constraints *P*, *Q*. The algorithm of constructing singular surfaces is elaborated now only for the case of scalar players' controls.

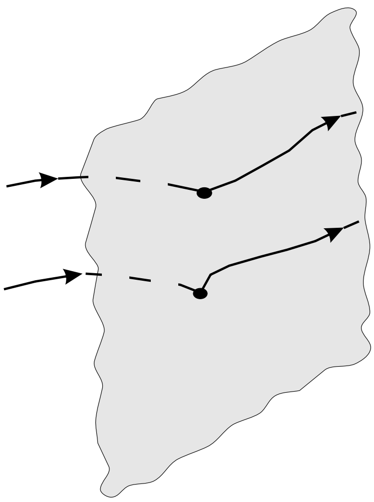
TYPES OF SINGULAR SURFACES



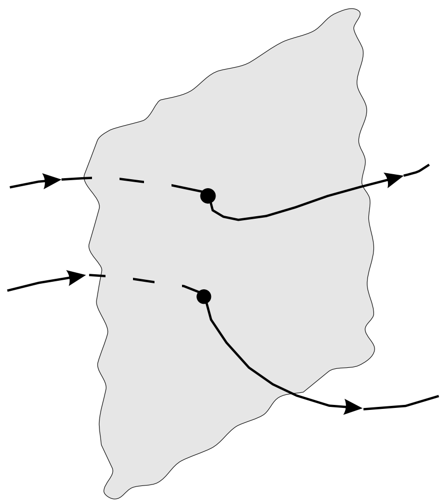
Dispersal surface



Equivocal surface



Switch surface without leaving



Switch surface with leaving

When we speak about singular surfaces, we mean Isaacs' classification. The type of a singularity is determined by behavior of optimal motions near the surface. In the class of games considered (with scalar players' controls), only the following types of singularity can appear: dispersal, equivocal, and switching.

In the base of our method, there is the algorithm for constructing level sets of the value function. Therefore, let us describe it shortly.

EQUIVALENT GAME

$$\dot{y} = D(t)u + E(t)v,$$

$$y \in R^2,\ u \in P,\ v \in Q$$

$$D(t) = X_{j,k}(T,t)B(t), \quad E(t) = X_{j,k}(T,t)C(t),$$

$$T, \quad \varphi\big(y(T)\big), \quad y(t) = X_{j,k}(T,t)x(t)$$

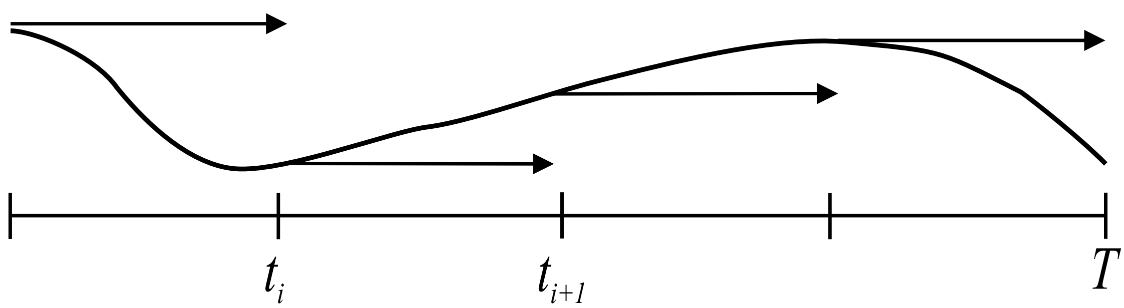
APPROXIMATING GAME

$$\dot{y} = D^*(t)u + E^*(t)v,$$

$$y \in R^2, \quad u \in P^*, \quad v \in Q^*$$

$$D^*(t) = D(t_i),$$

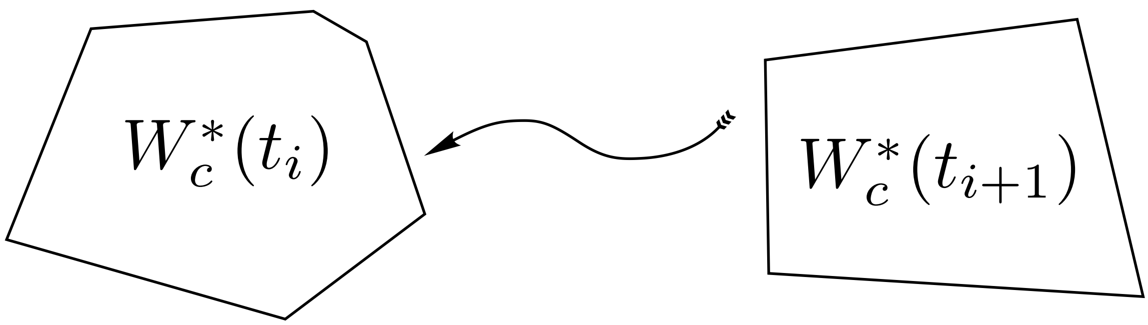
$$E^*(t) = E(t_i), \quad t \in [t_i, t_{i+1})$$



After transform to the equivalent coordinates, we have a DG of the second order on the phase variable. Changing the dynamics of the equivalent game by piecewise-constant one, we get the game with simple motion in every interval of time division. Also, original convex compacta *P* and *Q* are changed by convex polyhedra *P** and *Q**.

BACKWARD CONSTRUCTING

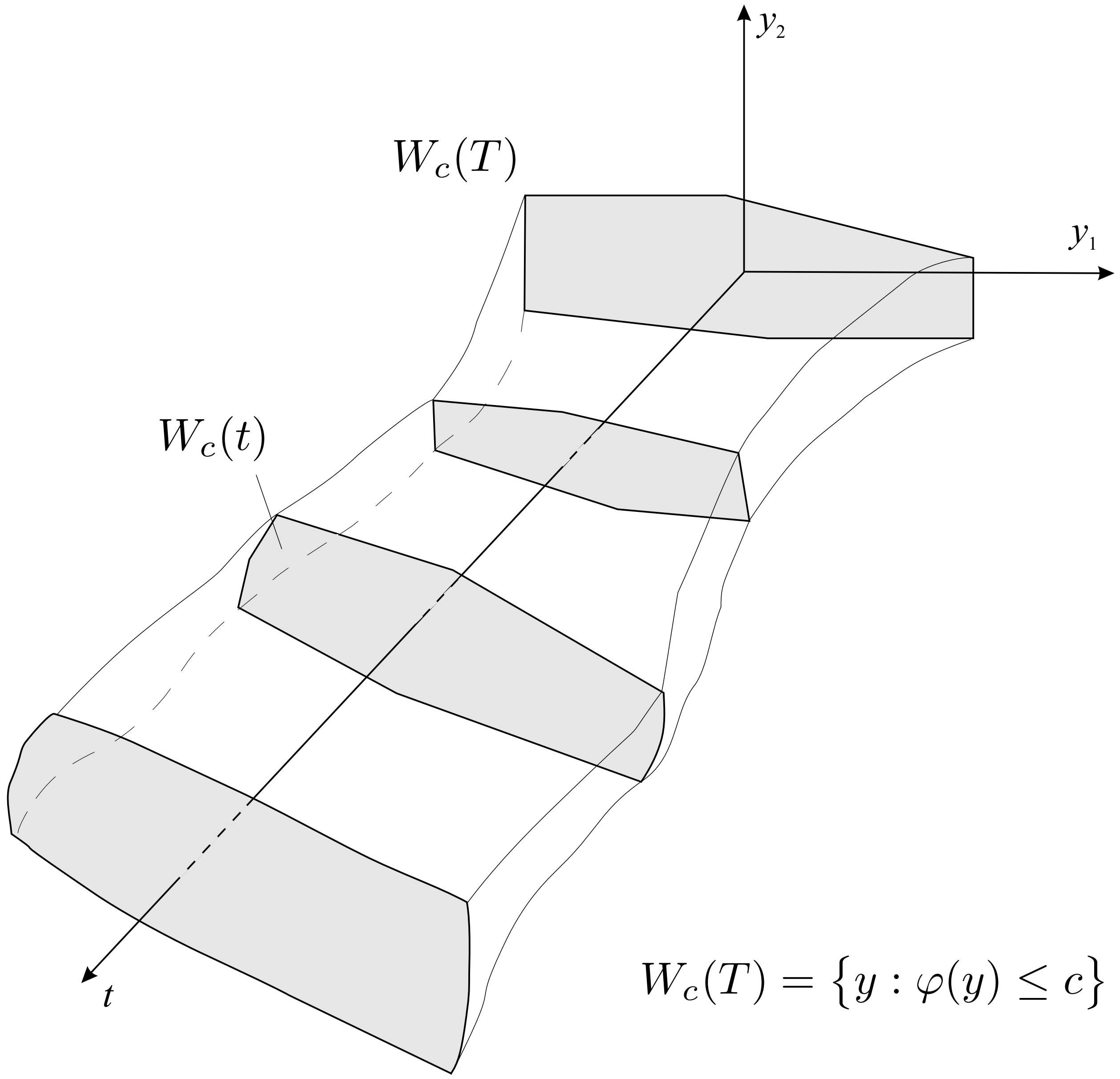
$$W_c^*(t_i) = \{y : \Gamma(t_i, y) \le c\}$$
$$W_c^*(T) = \{y : \varphi(y) \le c\}$$



$$\ldots < t_i < t_{i+1} < \ldots < t_{N-1} < t_N = T$$

Taking a value c of the payoff function φ, we build the corresponding level set of the payoff function. Then, making constructions in the reverse time, we find consecutively sections $W_c(t_i)$ of the level set W_c of the value function Γ.

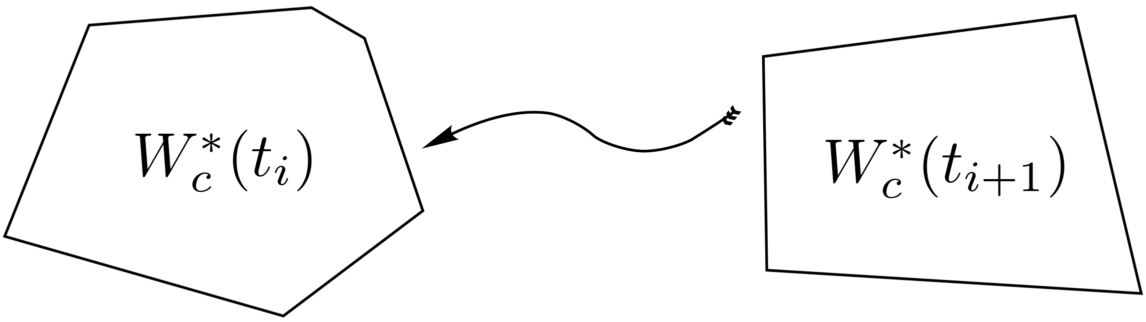
LEVEL SET OF VALUE FUNCTION



$$W_c(T) = \{y : \varphi(y) \leq c\}$$

So, the level set W_c in general is built as a collection of sections $W_c(t_i)$ on the time grid. It can be imagined as a "tube" in the three-dimensional space t, y_1, y_2.

CONVEX HULL CONSTRUCTING



$\rho\big(\cdot, W_c^*(t_i)\big)$ $\qquad\qquad$ $\rho\big(\cdot, W_c^*(t_{i+1})\big)$

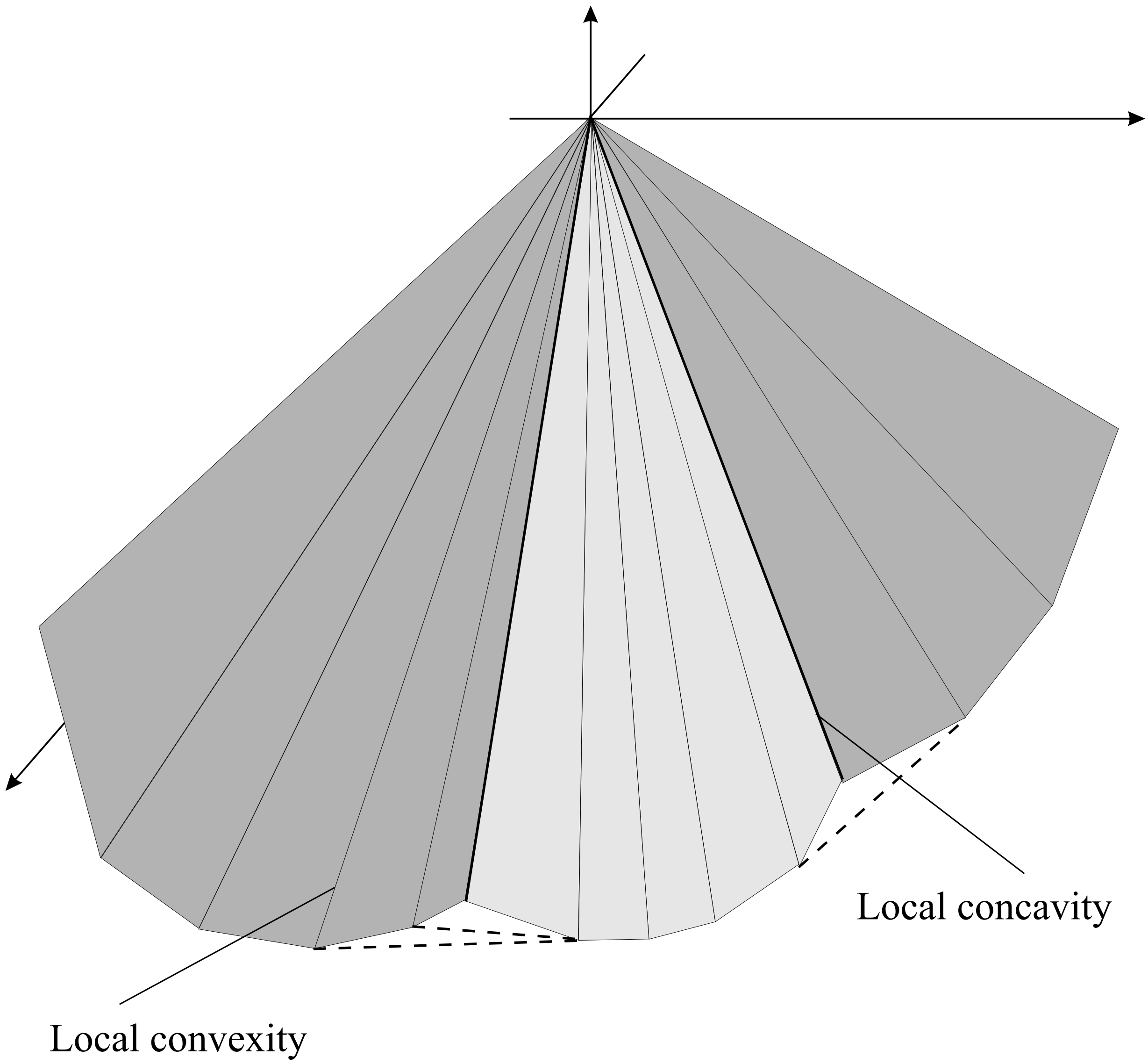
$$\rho\big(\cdot, W_c^*(t_i)\big) = \operatorname{conv}\gamma(\cdot, t_i), \quad \rho \text{ — support function}$$

$$\gamma(l, t_i) = \rho\big(l, W_c^*(t_{i+1})\big) + \Delta\rho\big(l, -D^*(t_i)P^*\big) - \Delta\rho\big(l, E^*(t_i)Q^*\big)$$

$$\Delta = t_{i+1} - t_i$$

Under this approximation, each time section $W_c(t_i)$ is a convex polygon described by its support function . The passage from one section to the next one is based on the operation of constructing convex hull of a piecewise-linear positively-homogeneous function . This function is calculated using support functions of the previous section and polygons, defined by players' controls.

LOCAL CONVEXITY AND CONCAVITY



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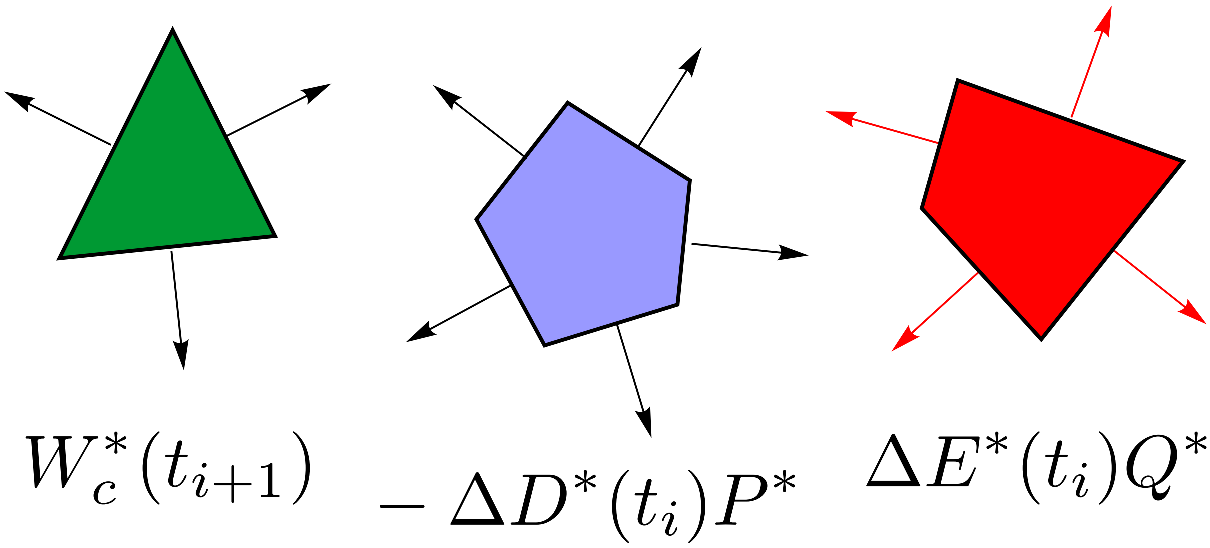
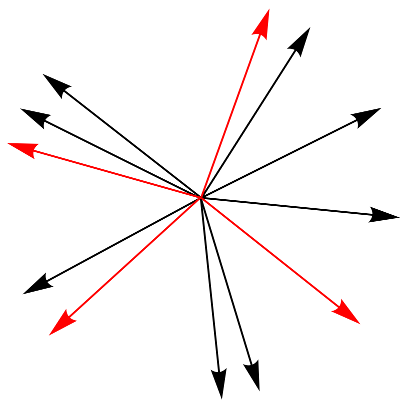
The procedure of convex hull construction is very fast because we have information about possible places of violation of local convexity. In this slide, the structure of the function graph is shown schematically. Dash lines point out “corrections” of the function during convexing process. Herewith, this process stops after a few steps.

PROCEDURE OF NORMALS GATHERING

Input sets



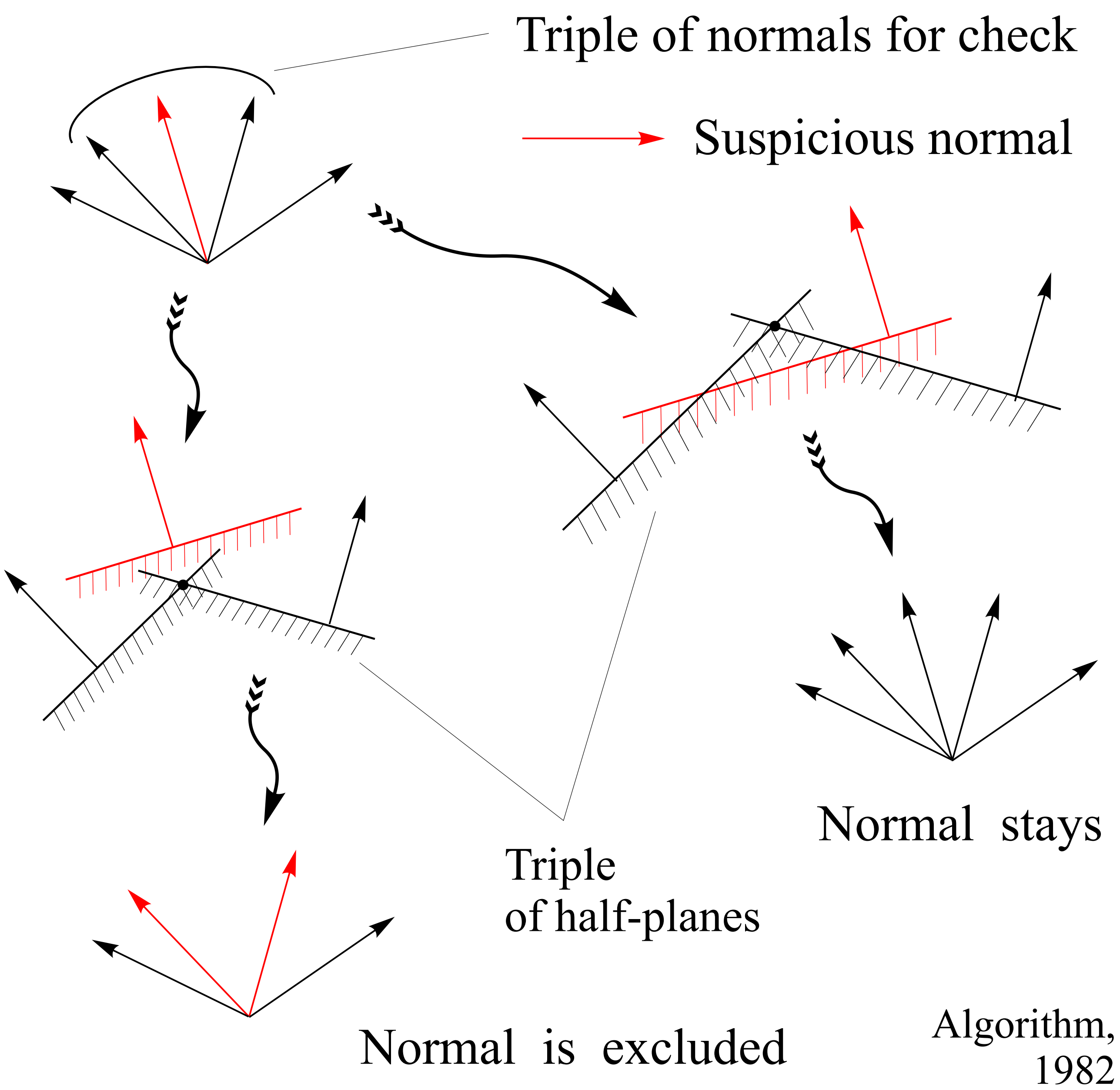
Resultant set of normals



Suspicious normals

The piecewise-linearity is defined by the bundle of normals taken from the following polygons: $W_c(t_{i+1})$, which is the previous section, – $P(t_i)$ and $Q(t_i)$. The polygons $P(t_i) = D^*(t_i)P^*$, $Q(t_i) = E^*(t_i)Q^*$ describe dynamic capabilities of players in the time interval $[t_i, t_{i+1})$. Normals taken from the polygon $Q(t_i)$ are named “suspicious” because the local violation of convexity can be only near them.

ELEMENTARY PROCEDURE OF CONVEX HULL CONSTRUCTING



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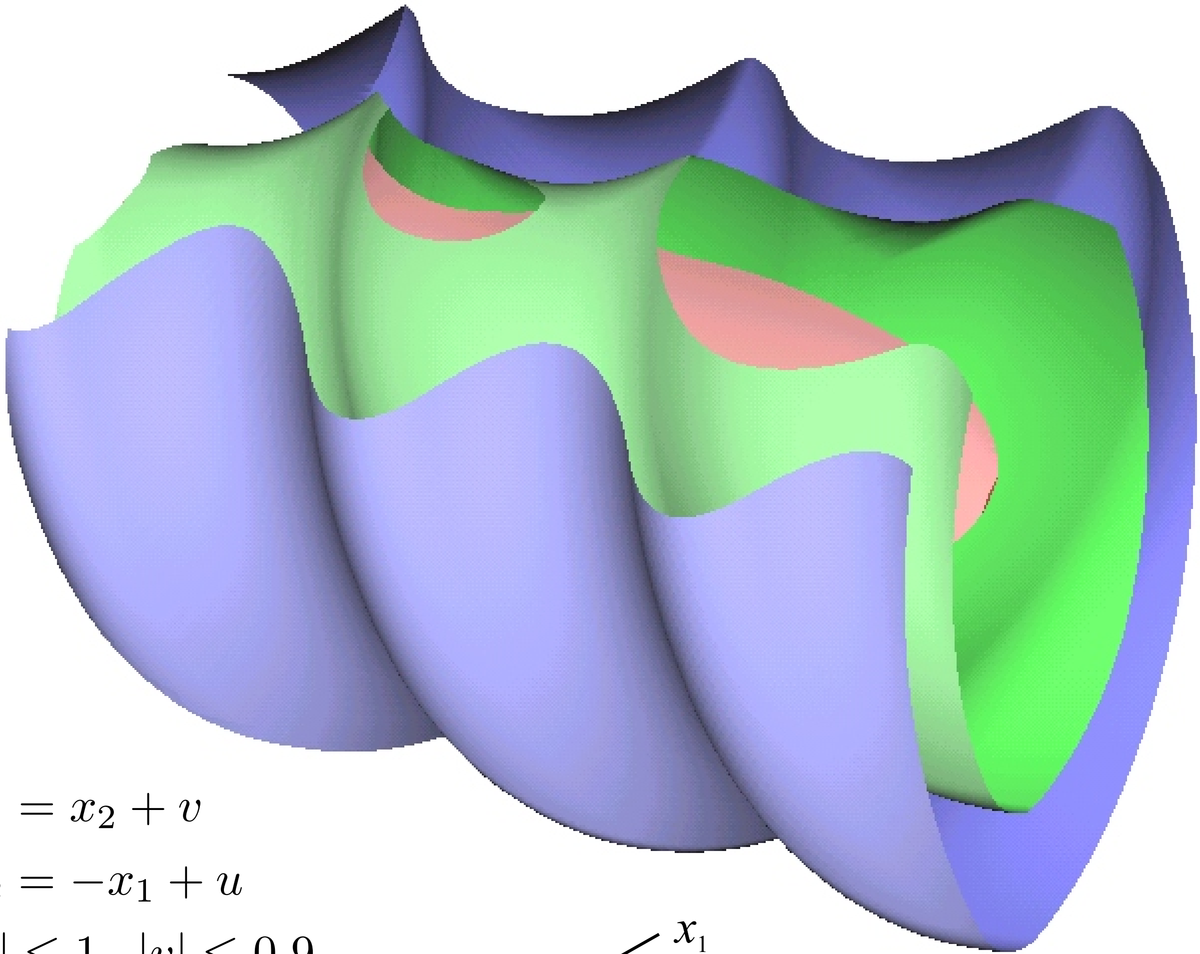
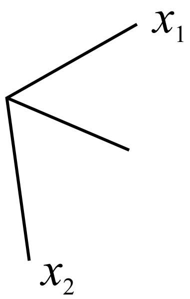
An elementary step of the convexing procedure consists of checking three linear inequalities defined by three neighbor vectors from current bundle and values of the function on these vectors. The middle vector is taken from the collection of suspicious ones. After the check, the middle vector either is removed from the bundle (and its neighbors become suspicious) or stays and looses the “suspicious” mark.

The algorithm of convex hull construction stops when the collection of suspicious vectors is empty after some elementary step. There is also a control for the case when the convex hull does not exist, that is, the tube W_c breaks at some time instant.

THREE LEVEL SETS FOR OSCILLATOR SYSTEM



$$\dot{x}_1 = x_2 + v$$
$$\dot{x}_2 = -x_1 + u$$
$$|u| \le 1,\ |v| \le 0.9$$
$$t \in [0, 8]$$
$$\varphi(x_1, x_2) = x_1^2 + x_2^2$$



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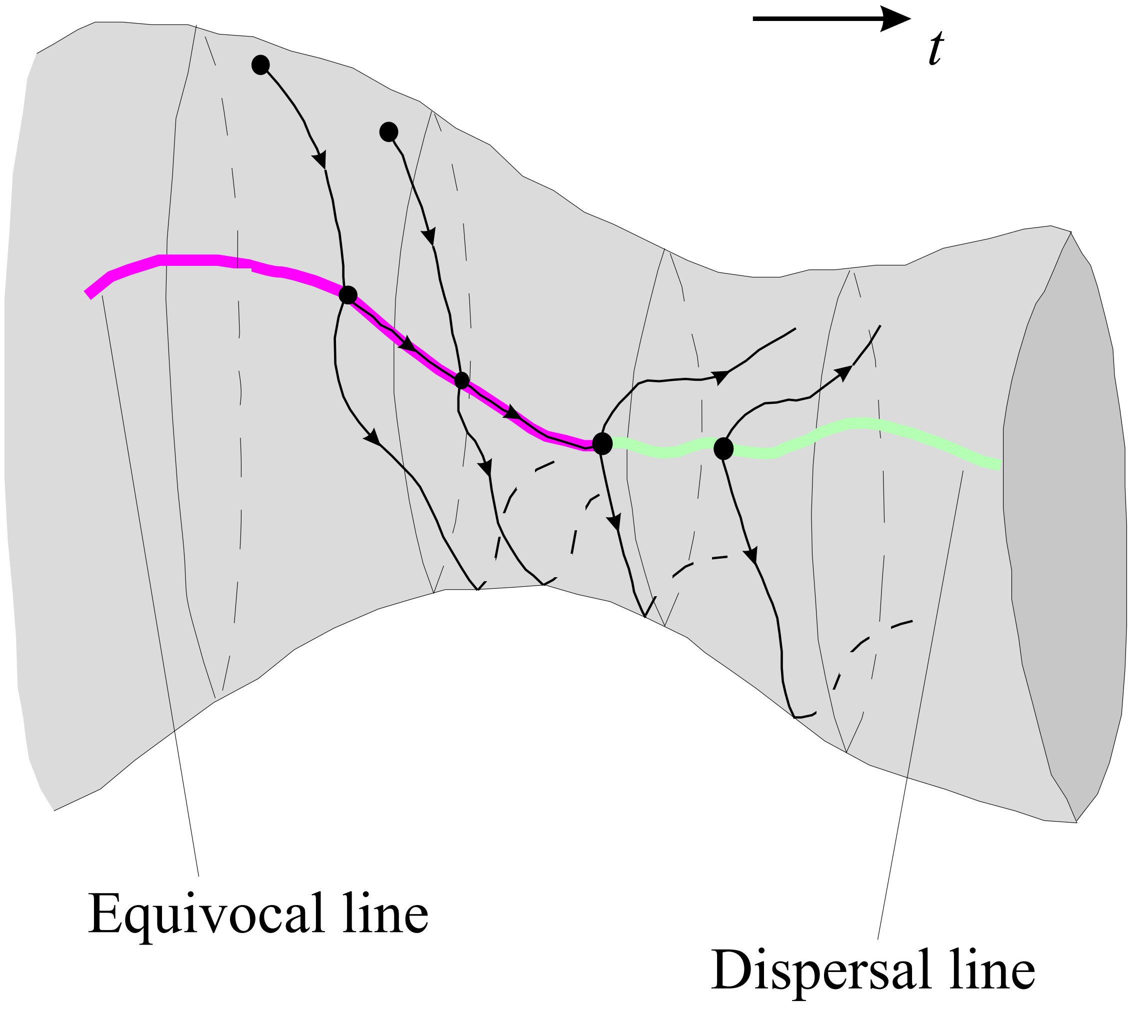
In this slide, three tubes for the DG with the oscillator dynamics are shown. Two outer tubes are cut by a plane. The internal one breaks. The tube visualization program is elaborated by Averbukh V.L., Yurtaev D.A., Zenkov A.I. and Shilov E.A. from the System Support Department of the Institute of Mathematics and Mechanics (Ekaterinburg). This program gives very convenient interface for manipulation and investigation of three-dimensional tubes, defined by their sections.

LEVEL SET AND SINGULAR LINES
(scheme sketch)



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In the base of singular surface construction, there is the algorithm for building singular lines going along concrete level set. Such lines in turn are combined from separate singular points, detected on the level set sections during their computing. When a singular point is found, it is classified.

TABBLE OF SINGULARITIES CLASSIFICATION

Number of normals	Flags	Additional condition	Object for marking	Secondary condition	Type of singularity
2	*FS* , *FS*	$\mathcal{Q}$-normal is strictly between	Vertex	$\mathcal{P}$-normal is not between	Dispersal for 2 nd
				$\mathcal{P}$-normal is between	Dispersal
2	*FS* , $N\mathcal{P}$+*FS*	$\mathcal{Q}$-normal is strictly between	$\mathcal{P}$-normal edge		Equivocal
1	$N\mathcal{P}$		Edge		Switching for 1 st

$\mathcal{P}$-normal – normal from the set $-P$

$\mathcal{Q}$-normal – normal from the set Q

FS – “former suspicious” normal

$N\mathcal{P}$ – flag of the normal from the set $-P$

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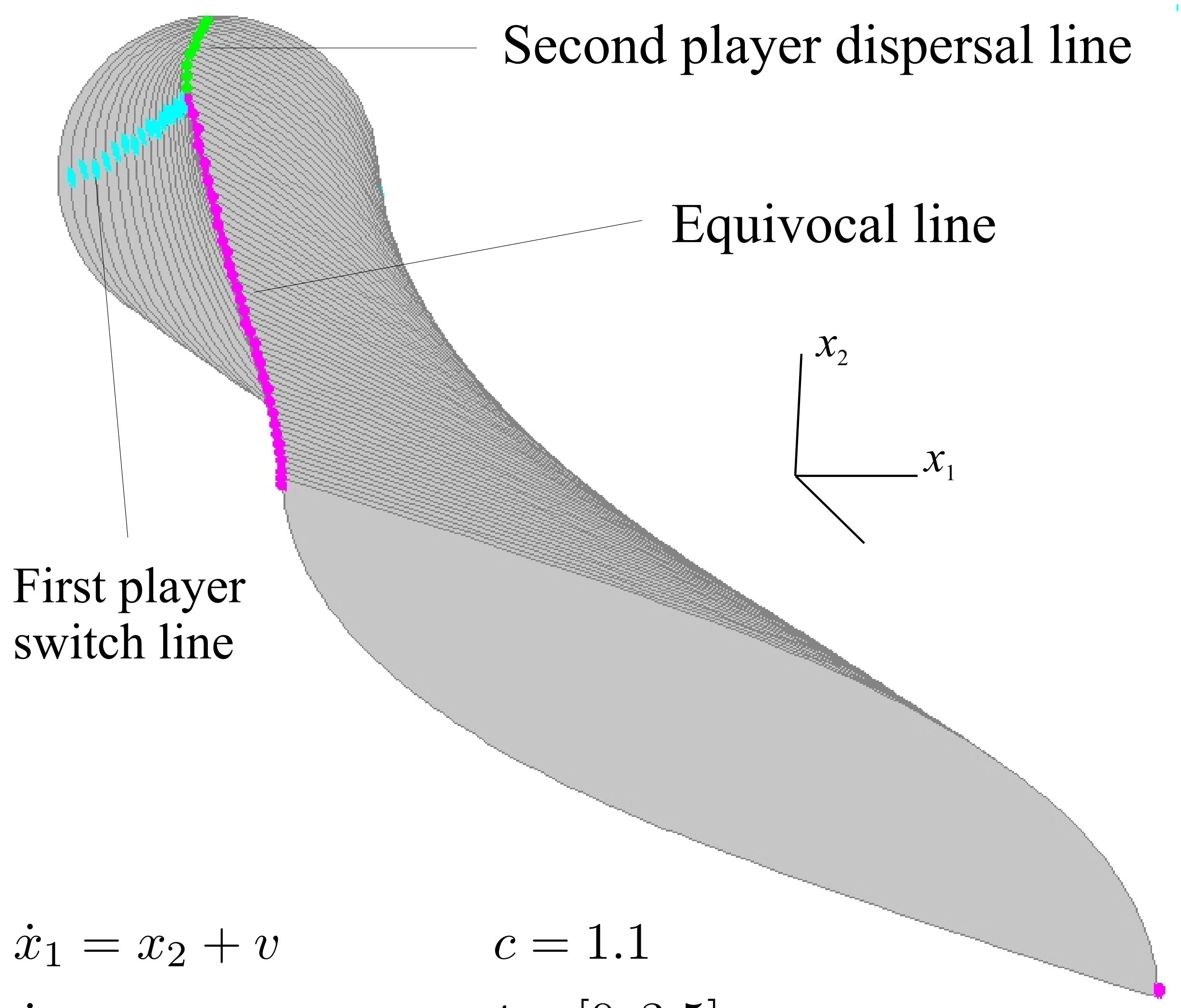
The algorithm for detection and classification of singular points on the next time section uses information obtained when the convex hull of the function is built. This information consists of some marks for vectors participated in the convexing process. Possible combinations of these marks and corresponding singularity types are shown in the table.

LEVEL SET AND SINGULAR LINES



$\dot{x}_1 = x_2 + v$ $\qquad c = 1.1$

$\dot{x}_2 = u$ $\qquad t \in [0, 3.5]$

$|u| \leq 1, \ |v| \leq 0.9$ $\qquad \varphi(x) = x_1^2 + x_2^2$

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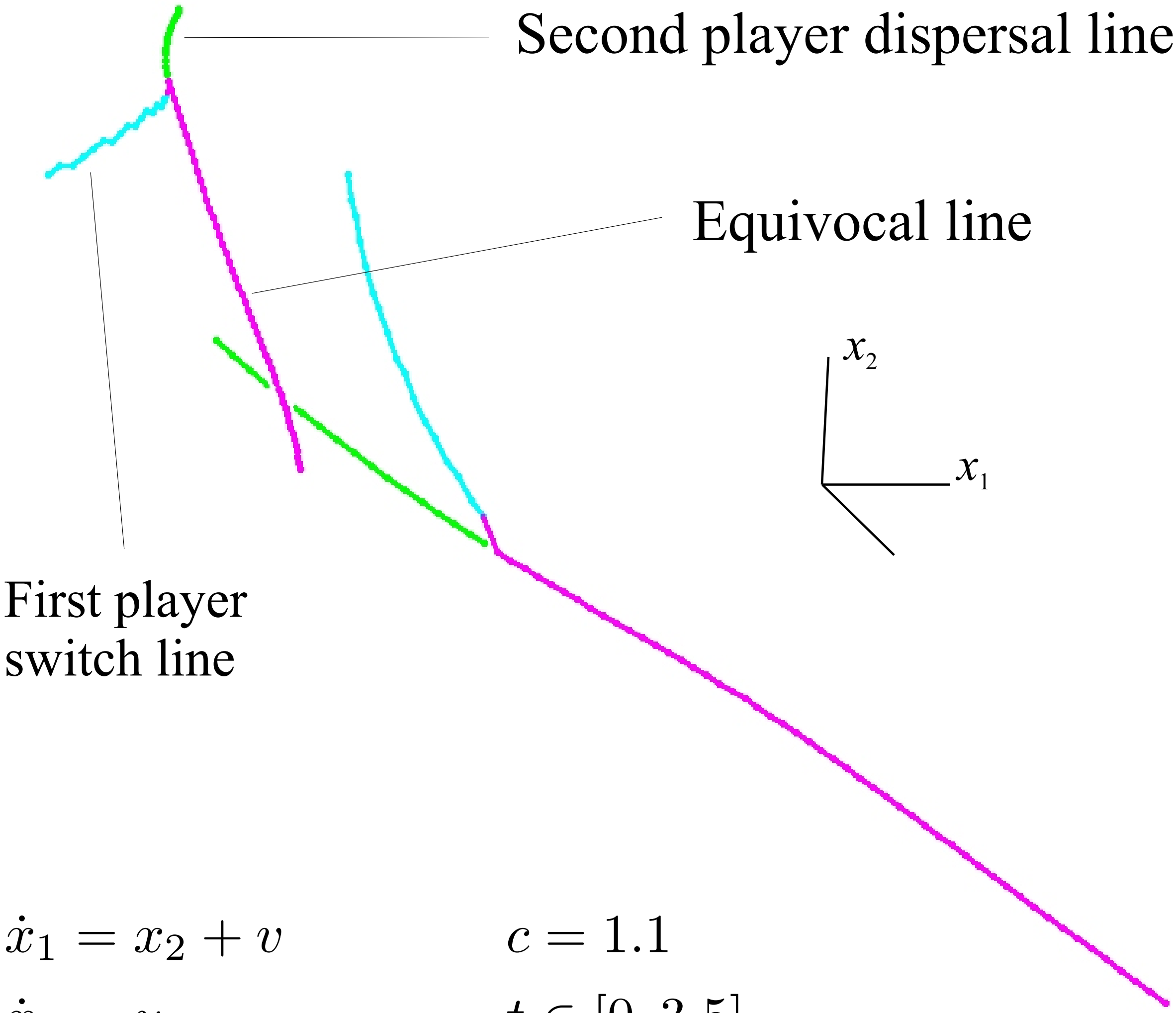
In this picture, a numerically calculated level set and singular lines on it are drawn. Here, the DG "material point" is considered. The dispersal line for the second player (green) and the switching line for the first one (blue) come (in reverse time) into the equivocal line (magenta). Due to symmetry of the problem with respect to zero, there is another system of singular lines, located on the invisible side of the level set surface.

SINGULAR LINES FROM LEVEL SET SURFACE



$$\dot{x}_1 = x_2 + v \qquad c = 1.1$$
$$\dot{x}_2 = u \qquad t \in [0, 3.5]$$
$$|u| \le 1,\ |v| \le 0.9 \qquad \varphi(x) = x_1^2 + x_2^2$$

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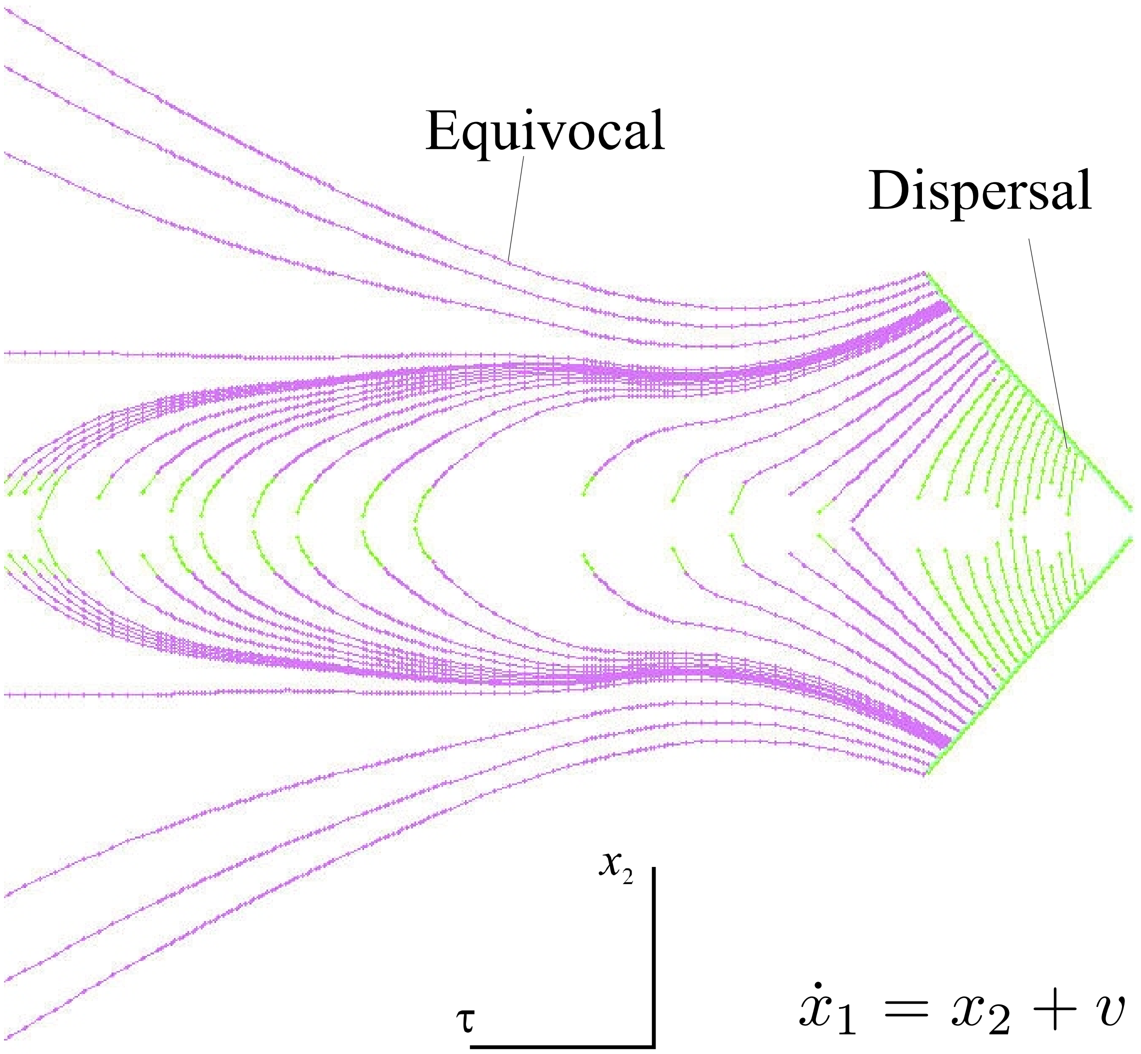
The singular lines taken from the tube shown in the previous slide are drawn.

SINGULAR SURFACES

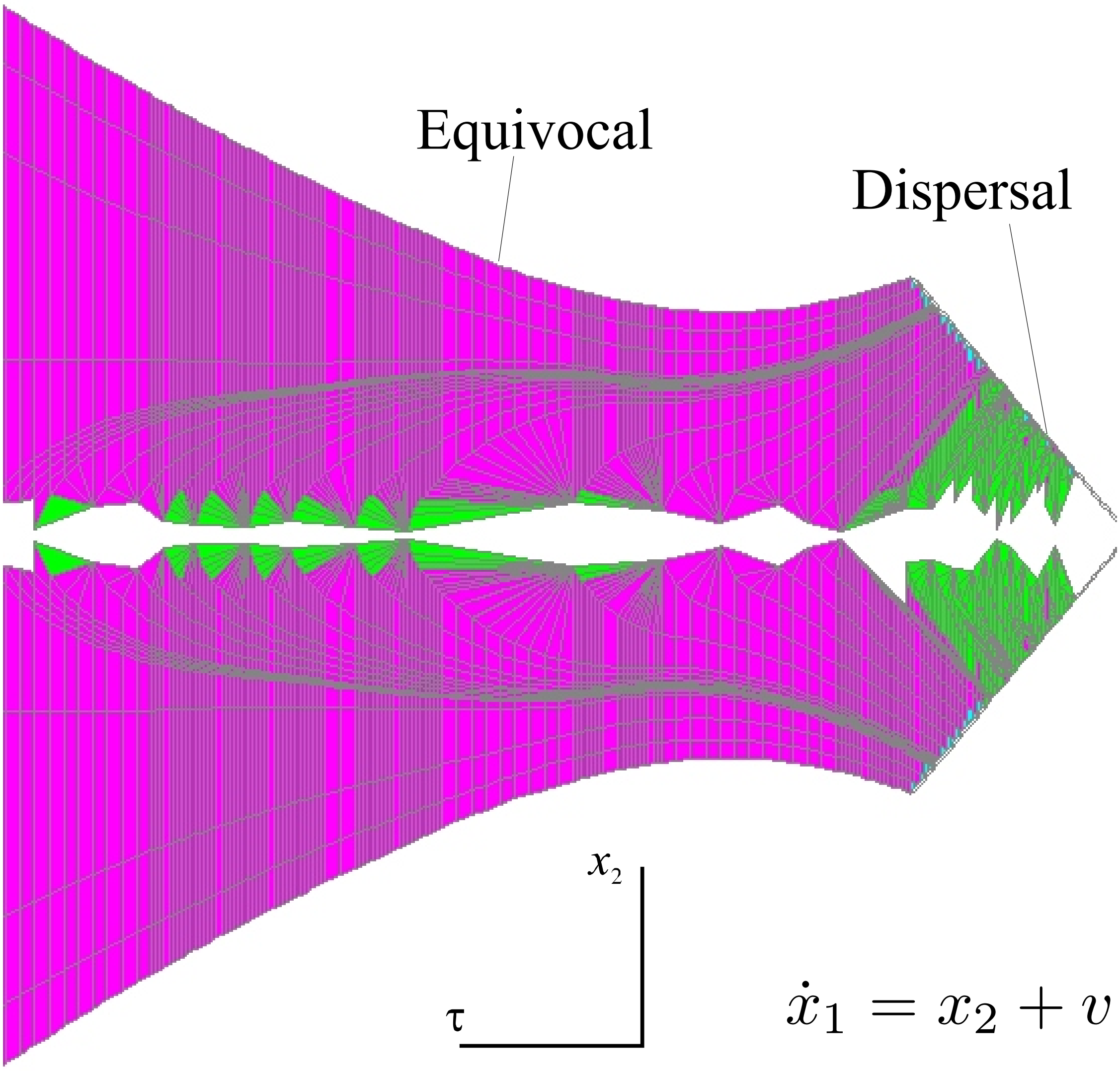


$$\dot{x}_1 = x_2 + v$$
$$\dot{x}_2 = u$$

$$\varphi(x) = |x_1|, \quad x_2(T) = 0 \qquad |u| \leq 1, \ |v| \leq 1$$

In this picture, the collection of singular lines from different tubes is given. Again, the "material point" dynamics is used. The payoff function describes the following interest of the first player: to lead the motion to the x_1-axis as close to zero as possible.

SINGULAR SURFACES



$$\dot{x}_1 = x_2 + v$$

$$\dot{x}_2 = u$$

$$\varphi(x) = |x_1|, \quad x_2(T) = 0 \qquad |u| \le 1, \ |v| \le 1$$

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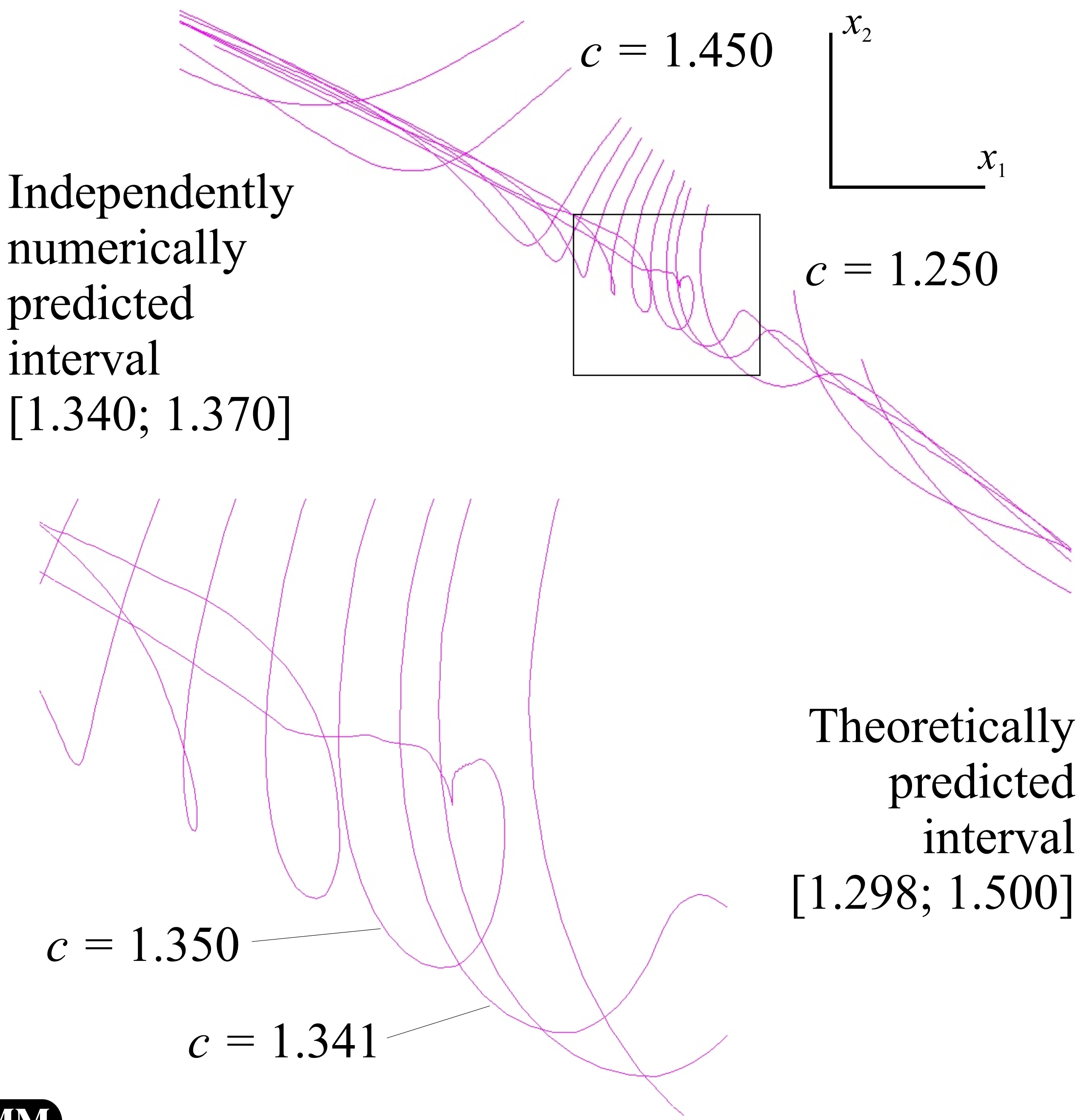
This slide demonstrates how separate singular lines join into singular surfaces. The equivocal surface is marked by the magenta color, the dispersal one by green. The gaps near the reverse time -axis can be filled by means of more delicate grids in time and value of the parameter *c*.

The singular surfaces appearing in this problem were investigated analytically in 1983. The numerical results coincide well with the analytical ones.

EQUIVOCAL LINES



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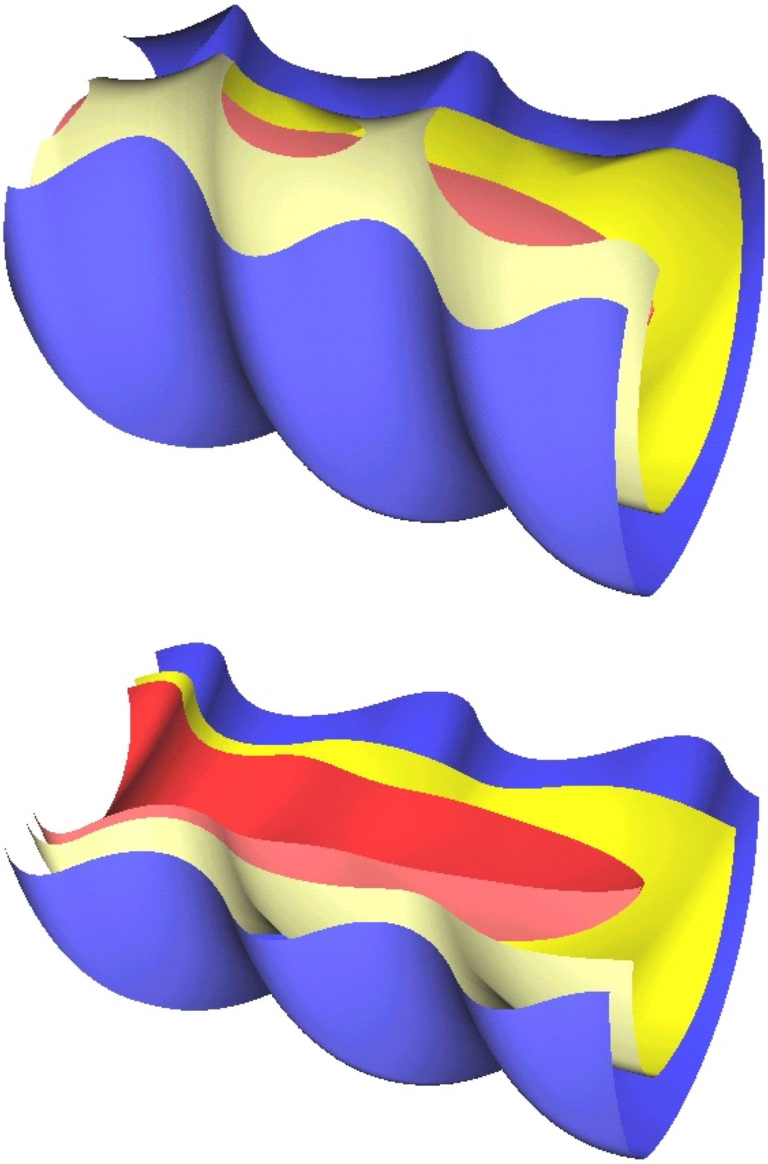
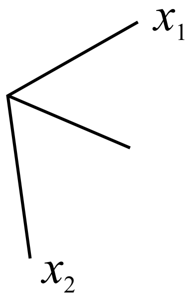
In particular, the behavior of equivocal lines from different tubes is very interesting. There is a critical value c^* of the parameter c, near which projections of equivocal lines onto x_1, x_2 plane look like spirals. The character of these spirals was investigated analytically. Computed equivocal lines are very similar.

THREE LEVEL SETS FOR OSCILLATOR SYSTEM





$$\dot{x}_1 = x_2 + v$$
$$\dot{x}_2 = -x_1 + u$$
$$|u| \le 1, \ \ |v| \le 0.9$$
$$t \in [0, 8]$$
$$\varphi(x_1, x_2) = x_1^2 + x_2^2$$

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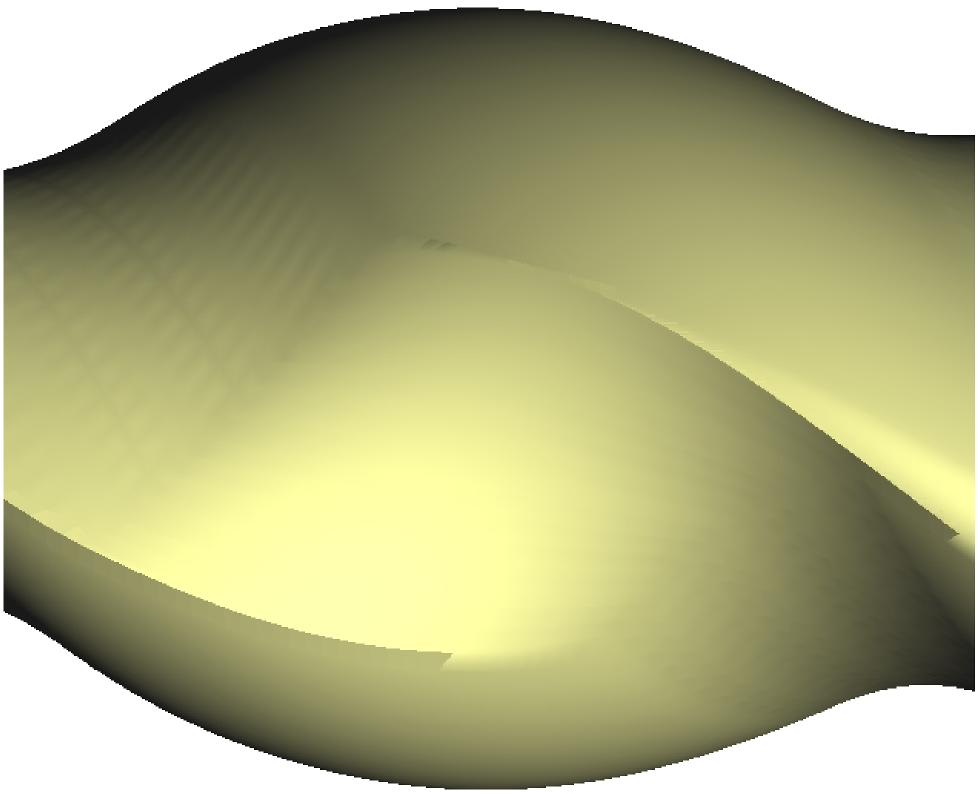
We calculated a lot of examples. The singular surfaces are the most complicated in oscillating systems. Quite naturally, this difficulty is generated by very tricky level sets in these systems. In this picture, the level sets for the "oscillator" DG are shown. Two variants of the sets cut are presented.

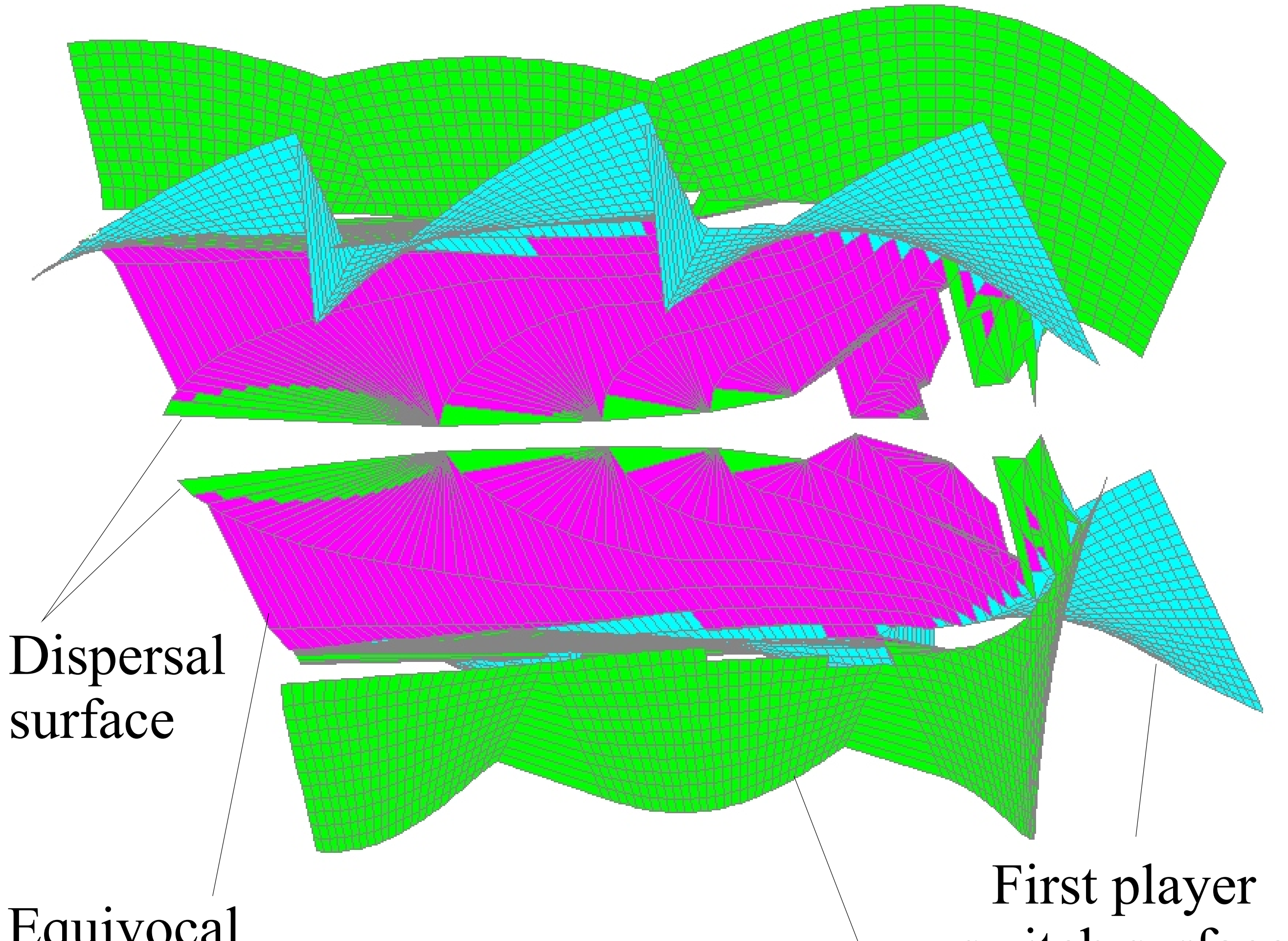
DISCONTINUOUS SINGULAR LINE

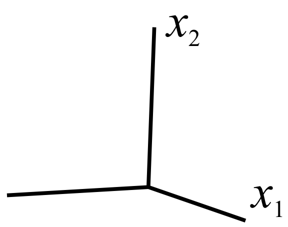


Oscillator system

Here, the fragment of the middle tube is drawn. One can see that the corner line has a jump. This line is singular. Presence of such peculiarities causes complexity of singular surfaces in general.

SINGULAR SURFACES





$$\dot{x}_1 = x_2 + v$$
$$\dot{x}_2 = -x_1 + u$$
$$|u| \le 1,\ |v| \le 0.9$$
$$\varphi(x) = \max\big(|x_1|, |x_2|\big)$$

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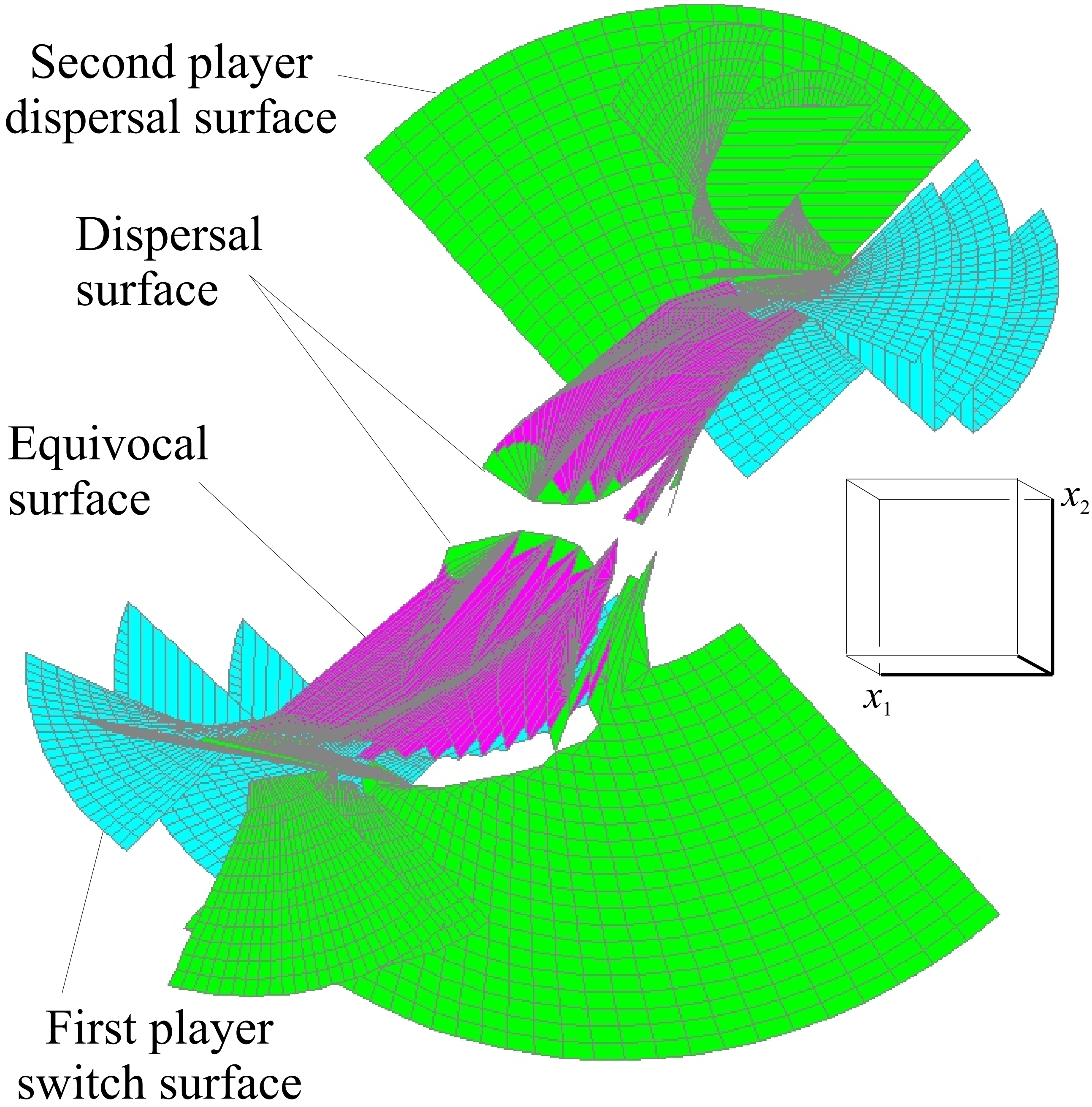
In this slide, the singular surfaces for the "oscillator" DG are viewed from the x_1-axis.

SINGULAR SURFACES



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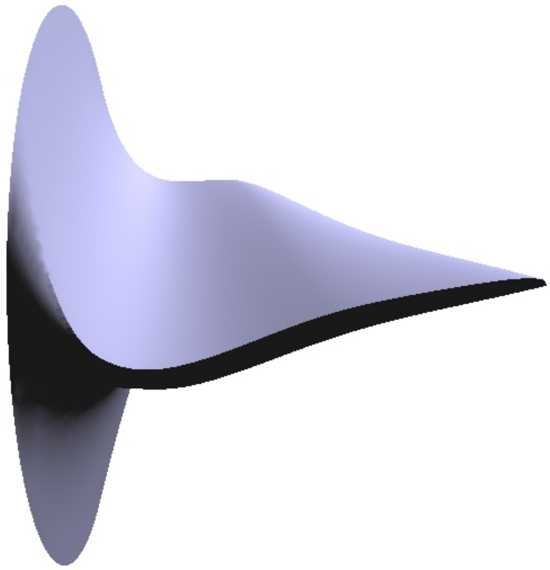
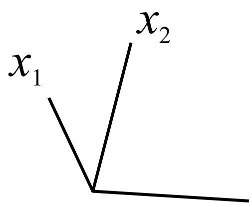
Here, the same surfaces are observed from the -axis.

DIFFERENT VIEWS OF LEVEL SETS FOR "DRAWING-PIN" SYSTEM



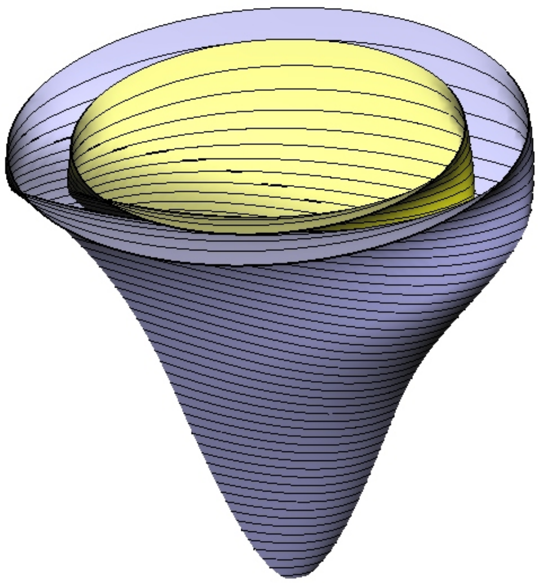


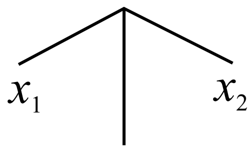
$$\dot{x}_1 = x_1 + 2x_2 + v$$

$$\dot{x}_2 = x_2 + u$$

$$|u| \le 1, \ |v| \le 0.9$$

$$\varphi(x) = x_1^2 + x_2^2$$





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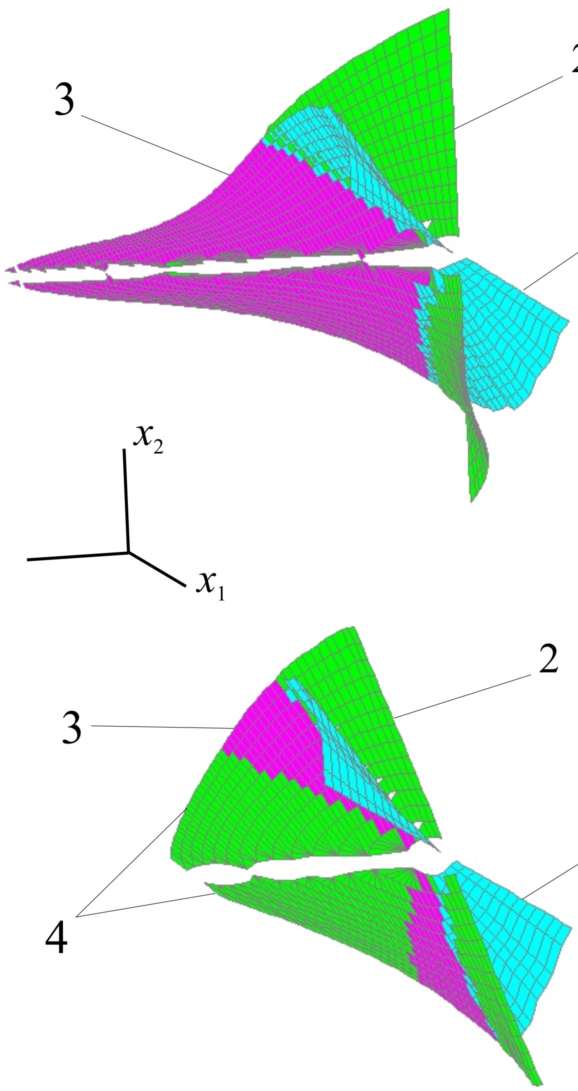
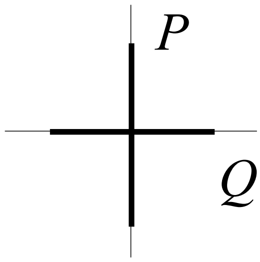
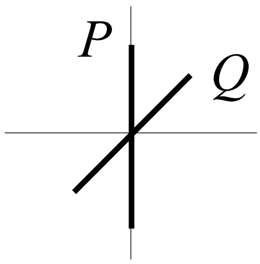
Next example has no any mechanical interpretation. But the level sets are very beautiful: they seem as a drawing-pin.

DEPENDENCE OF SINGULAR SURFACES ON THE 2ND PLAYER CONSTRAINT (case of the strong 2ND player)





$$\dot{x}_1 = x_1 + 2x_2 + v_1$$
$$\dot{x}_2 = x_2 + u + v_2$$
$$|u| \leq 1,\ (v_1, v_2)' \in Q$$
$$\varphi(x) = x_1^2 + x_2^2$$
$$\operatorname{diam} Q = 1.1 \cdot \operatorname{diam} P$$



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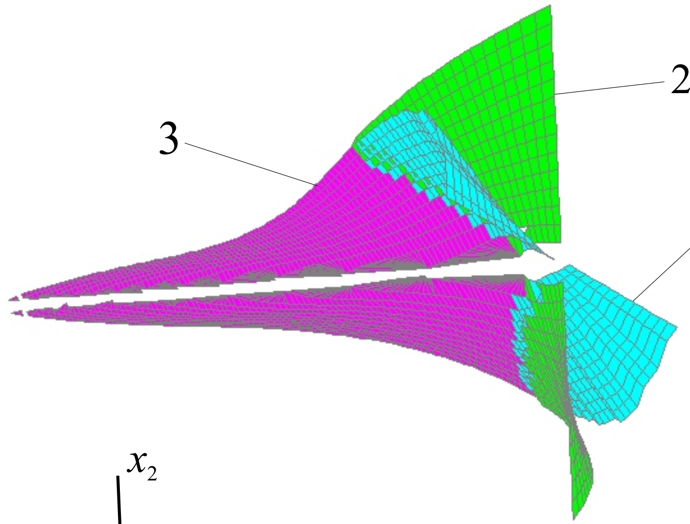
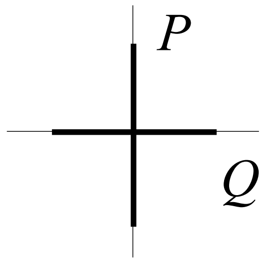
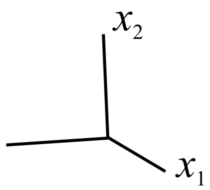
The dependence of the singular surfaces on parameters of the problem is shown. When the segment Q (the second player control constraint) goes to P (the first player control constraint), the dispersal surface near the -axis increases. It happens when the second player is "stronger" than the first one: the length of Q is greater than the length of P.

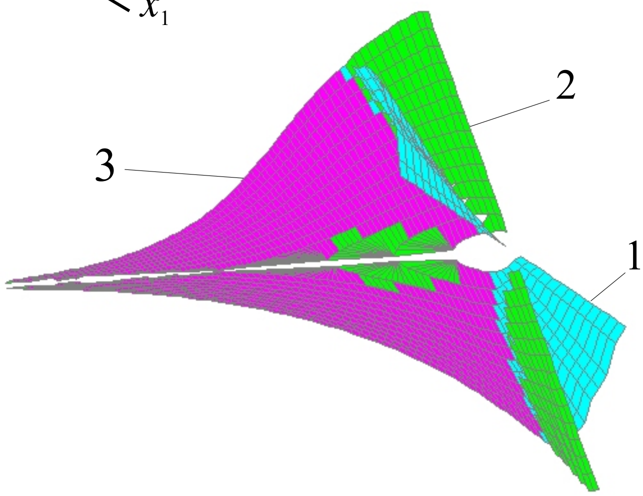
DEPENDENCE OF SINGULAR SURFACES ON THE 2ND PLAYER CONSTRAINT (case of the weak 2ND player)

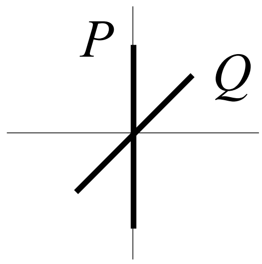






$$\dot{x}_1 = x_1 + 2x_2 + v_1$$
$$\dot{x}_2 = x_2 + u + v_2$$
$$|u| \le 1,\ (v_1, v_2)' \in Q$$
$$\varphi(x) = x_1^2 + x_2^2$$
$$\operatorname{diam} Q = 0.9 \cdot \operatorname{diam} P$$





IMM

If the situation is vice versa (that is, the first player is stronger), then the closing of P and Q does not change singular surfaces essentially.