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ON LEVEL SETS WITH "NARROW THROATS" IN LINEAR DIFFERENTIAL GAMES

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Examples with zero-sum linear differential games of fixed terminal time and a convex terminal payoff function depending on two components of the phase vector are considered. Such games can have an indifferent zone with constant value function. The level set of the value function associated with the indifferent zone is called the "critical" tube. In the selected examples, the critical tube and the neighboring level sets exhibit "narrow throats". Presence of such throats requires extremely precise computations for constructing the level sets. The paper presents different forms of critical tubes with narrow throats and indicates the combinations of problem parameters that can produce them.

Keywords: Differential games; value function; level sets; numerical constructions.

1. Introduction

The solution of a two-person zero-sum differential game is given by the triplet of the players' optimal strategies and the value function (Isaacs, 1965). The value function, which can be represented by its level sets, is the solution of the corresponding Hamilton-Jacobi-Isaacs partial differential equation and its knowledge allows deriving the respective optimal strategies. Solving this equation for a general zero-sum differential game is very complicated. Fortunately, linear zero-sum differential games yield in many cases simpler numerical and, sometimes, analytical solutions. Games with terminal payoff and fixed terminal time belong to this category. Their solution can be well illustrated by the ensemble of the level sets of the value function, which are also the loci of optimal trajectories.

The paper deals with linear differential games with fixed terminal time and a convex terminal payoff function depending on two components of the phase vector with constrained controls. Such a game can be transformed (Krasovskii and Subbotin (1988), pp. 89–91) to a two-dimensional "equivalent" game, for which the solution can be visualized by the level sets of the value function and their time-sections.

The examples considered in the paper belong to a class of problems called in Russian mathematical literature on differential games the "generalized L.S. Pontryagin's test example" (Pontryagin (1972)). This class of games is characterized by the eventual existence of an indifferent zone of the game space, where the value function is constant and the optimal strategies are arbitrary. The level set of the value function associated with the indifferent zone is called the "critical" tube. The critical tube and the neighboring level sets exhibit "narrow throats". The narrow throat of the critical tube is located where the tube has, at least in one direction, zero width, as indicated by its name. The form of the critical level set and the neighboring ones near the throat can be rather complex. Presence of such throats requires extremely precise computations for constructing the level sets.

A numerical method for computing level sets of the value function in linear differential games with fixed terminal time (Isakova *et al.* (1984), Patsko (1996)) was developed at the Institute of Mathematics and Mechanics (Russian Academy of Sciences, Ekaterinburg, Russia). Simultaneously, specialized visualizational programs were created (Averbukh *et al.* (1999)). The existing software allows illustrating the different forms of critical tubes with narrow throats and analyzing their dependence on the problem parameters.

There are several publications describing other numerical methods for constructing level sets of the value function in linear differential games with fixed terminal time (Ushakov (1981), Subbotin and Patsko (1984), Taras'ev *et al.* (1988), Grigorenko *et al.* (1993), Bardi and Dolcetta (1997), Kurzhanski and Valyi (1997), Cardaliaguet *et al.* (1999), Polovinkin *et al.* (2001)). The examples presented in this paper can be used for testing the accuracy and efficiency of such methods.

The structure of the paper is the following. In the next section, the formulation of zero-sum linear differential games with fixed terminal time is presented, including the form of the generalized L.S.Pontryagin's test example and the description of the transformation of the original game to a reduced order equivalent game. In Sec. 3, the analytical solution of the equivalent game is presented and the eventual existence of a narrow throat is explained by a simple example. In Sec. 4, the numerical method used for constructing the level set of the value function is outlined. In Subsec. 5.1, an air-to-air interception problem is formulated as a linear pursuit-evasion game with fixed terminal time and bounded controls. Subsections 5.2 and 5.3 present results of a numerical investigation of the value function level sets for two variants of the game parameters. Subsection 5.3 also shows an example of the game, for which the numerical computations failed. The lessons learned from the investigations are discussed in Sec. 6, followed by some concluding remarks.

2. Linear Differential Games with Fixed Terminal Time

The standard form of linear differential games with bounded controls, fixed terminal time T is

$$\dot{z} = A(t)z + B(t)u + C(t)v, \quad t \in [t_0, T], \ \ z \in R^n, \ \ u \in P, \ \ v \in Q. \tag{2.1}$$

One of the simplest cases is a zero-sum game with convex terminal payoff function depending on two components z_i, z_j of the state vector z:

$$J = \varphi\big(z_i(T), z_j(T)\big).$$

This function is minimized by the first player (with control u) and maximized by the second (with control v).

Let $\Phi(T,t)$ be the fundamental Cauchy (transition) matrix of the original homogeneous system of (2.1). Let the symbol K denote a constant rectangular $2 \times n$ matrix extracting the i-th and j-th rows from an $n \times n$ matrix.

By using the transformation (Krasovskii and Subbotin (1988), pp. 89–91)

$$\xi(t) = K\Phi(T,t)z(t), \tag{2.2}$$

the standard form (2.1) of the original game can be reduced to a two-dimensional "equivalent" differential game. The term "equivalent" means that if the states ξ and z satisfy (2.2), the value $\mathcal{V}(t,\xi)$ of the value function $\mathcal{V}$ of the new game is equal to the value $V(t,z)$ of the value function V of the original game. Moreover, the optimal strategies of the new game are also identical to the optimal strategies of the original game.

The dynamics of the equivalent two-dimensional game is independent of the state vector ξ:

$$\begin{aligned} \dot{\xi} &= D(t)u + E(t)v, \quad t \in [t_0, T], \quad \xi \in R^2, \quad u \in P, \quad v \in Q, \\ D(t) &= K\Phi(T,t)B(t), \quad E(t) = K\Phi(T,t)C(t) \end{aligned} \tag{2.3}$$

with the payoff function

$$J = \varphi\big(\xi_1(T), \xi_2(T)\big).$$

Let us call the sets $\mathcal{P}(t) = D(t)P$ and $\mathcal{Q}(t) = E(t)Q$ the vectograms of the first and second players at the instant t. The vectogram $\mathcal{P}(t)$ ($\mathcal{Q}(t)$) describes the collection of all possible contributions of the first (second) player to the speed $\dot{\xi}$ of the equivalent system. Comparison of the sets $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ indicates the advantage of one player with respect to the other at different directions. Very often, it is useful to see the tubes of the set $-\mathcal{P}(t)$ and $-\mathcal{Q}(t)$ enrolled in the backward time. Let us call these tubes the tubes of vectograms of the first and second player and denote them by $\mathcal{P}$ and $\mathcal{Q}$ respectively.

There is a subclass of linear pursuit-evasion games, which is called the "L.S.Pontryagin's generalized test example" in Russian literature on differential games. Games from this class are of the following type

$$\begin{aligned} x^{(k)} + a_{k-1}x^{(k-1)} + \cdots + a_1\dot{x} + a_0 x = u, \quad u \in P, \\ y^{(s)} + b_{s-1}y^{(s-1)} + \cdots + b_1\dot{y} + b_0 y = v, \quad v \in Q. \end{aligned} \tag{2.4}$$

Here, x and y are the position coordinates of two objects in Euclidian space, and u and v are their controls, constrained to respective compact sets. The coefficients in the equations are assumed constant. The game terminates at a given prescribed time T.

The game has a terminal payoff function depending on the distance between the two players at T:

$$J = \varphi(x(T), y(T)) = |x(T) - y(T)|. \tag{2.5}$$

If the vectors x and y are only two dimensional, the change of variables

$$\begin{aligned}
z_1 &= x_1 - y_1, & z_{2k+1} &= y_1,\\
z_2 &= x_2 - y_2, & z_{2k+2} &= y_2,\\
z_3 &= \dot{x}_1, & z_{2k+3} &= \dot{y}_1,\\
z_4 &= \dot{x}_2, & z_{2k+4} &= \dot{y}_2,\\
z_5 &= \ddot{x}_1, & z_{2k+5} &= \ddot{y}_1,\\
z_6 &= \ddot{x}_2, & z_{2k+6} &= \ddot{y}_2,\\
&\vdots & &\vdots\\
z_{2k-1} &= x_1^{(k-1)}, & z_{2(k+s)-1} &= y_1^{(s-1)},\\
z_{2k} &= x_2^{(k-1)}, & z_{2(k+s)} &= y_2^{(s-1)}
\end{aligned}$$

transforms the system (2.4) to the standard form (2.1) with constant matrices A, B, C and the payoff function

$$J = \varphi(z_1(T), z_2(T)) = \sqrt{z_1^2(T) + z_2^2(T)}. \tag{2.6}$$

For some particular cases, some more convenient variable exchange can exist.

In the case when the original game dynamics is of the (2.4) type, the matrices $D(t)$ and $E(t)$ of the equivalent game have the form $D(t) = \zeta(t) \cdot I_2$, $E(t) = \eta(t) \cdot I_2$, where I_2 is the 2×2 unit matrix, and $\zeta(t)$ and $\eta(t)$ are scalar functions of time. So, for this class of games, the players' vectograms are computed as $\mathcal{P}(t) = \zeta(t)P$ and $\mathcal{Q}(t) = \eta(t)P$.

3. Solving the Equivalent Game

The first step in the solution of the equivalent game according to the Isaacs method is by using the necessary conditions of game optimality. The Hamiltonian function of the game is

$$H(\xi, \lambda, t, u, v) = \lambda'[D(t)u + E(t)v],$$

where $\lambda(t)$ represents (along optimal candidate trajectories) the gradient of the payoff function and has to satisfy the adjoint equation

$$\dot{\lambda} = -\frac{\partial H(\xi, \lambda, t, u, v)}{\partial \xi} = 0 \tag{3.7}$$

and the corresponding transversality condition

$$\lambda(T) = \operatorname{grad} J. \tag{3.8}$$

Relations (3.7) and (3.8) yield

$$\lambda(t) = \lambda(T) = \alpha^*, \tag{3.9}$$

where α^* is a constant vector on each optimal trajectory.

The candidate optimal strategy pair u^* and v^* can now be determined by

$$\begin{aligned} u^*(t) &= \arg\min_{u \in P} \max_{v \in Q} H(\xi, \lambda(t), t, u, v) = \arg\min_{u \in P} \lambda'(t) D(t) u = \arg\min_{u \in P} \alpha^{*\prime} D(t) u, \\ v^*(t) &= \arg\max_{v \in Q} \min_{u \in P} H(\xi, \lambda(t), t, u, v) = \arg\max_{v \in Q} \lambda'(t) E(t) v = \arg\max_{v \in Q} \alpha^{*\prime} E(t) v, \end{aligned} \tag{3.10}$$

where the explicit expressions depend on the control constraints of the game. Since α^* is a constant vector, the equations of the candidate optimal trajectories can be directly integrated backwards in time from any point at the terminal time T.

Each level set of the value function is generated by the family of such candidate optimal trajectories, all starting (in reverse time) at different points on the boundary of a level set of the payoff function at the terminal time T. If the ensemble of candidate optimal trajectories, generated by backwards integration in time, succeed to fill the entire equivalent game space without intersecting each other, the sufficiency conditions of game optimality (Isaacs (1965)) are satisfied by (3.10). In this case, we can obtain level sets of the value function and the value function itself.

Unfortunately, singularities in differential games are rather a rule, not an exception. In most cases, completely "regular" solutions do not exist for two reasons. The candidate optimal trajectories frequently intersect and create singular surfaces of the game space, where at least one component of $\lambda(t)$ is discontinuous. This means that the solution (3.9) is not valid any more. If the game space cannot be filled by regular candidate optimal trajectories, the empty regions also involve some singularities. Some of the empty regions become "indifferent" zones, where the value function is constant and the optimal strategies are arbitrary.

Let us illustrate this situation with a simple example. Assume that the payoff function is of the (2.6) type. Then its level sets are circles and the solution of the adjoint system is defined by formula

$$\lambda(t) = \lambda(T) = \alpha^* = \frac{\xi(T)}{|\xi(T)|}.$$

Also suppose that the control constraints are circular (with radii $u_{\max}$ and $v_{\max}$ respectively). In this case, the game solution will be identical in every plane of arbitrary direction (for example, in the plane $\xi_2 = 0$). Therefore, in the sequel a planar case is considered, as shown in Figs. 1, 2.

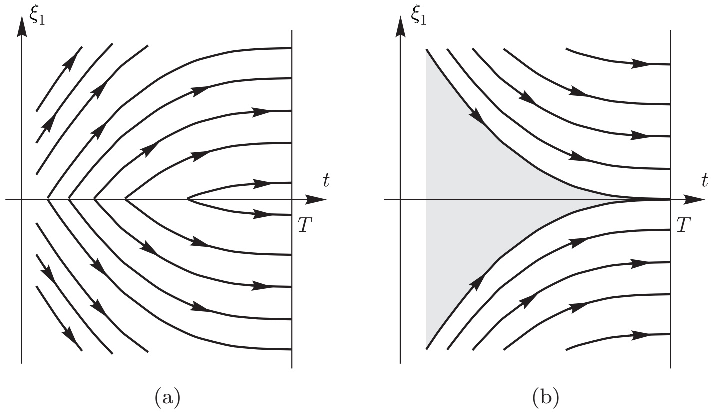

Fig. 1. Behavior of the optimal trajectories: (a) $\Gamma(t) > 0$, (b) $\Gamma(t) < 0$. Arrows show the direction of motions (in real time).

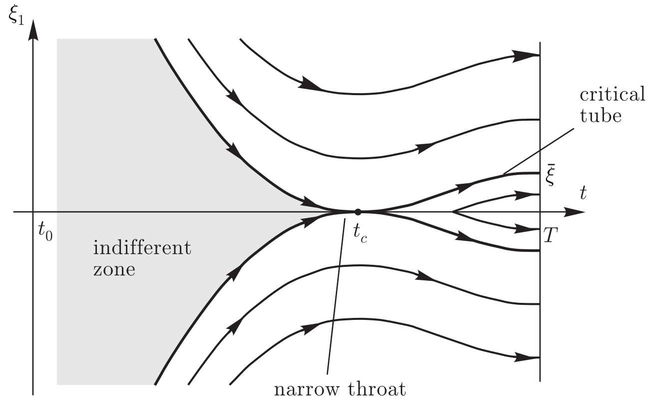

Fig. 2. Presence of a throat and an indifferent zone in the problem with circular vectograms.

The candidate optimal strategies (3.10) in this case become

$$u_1^* = -u_{\max} \operatorname{sign}\{\zeta(t)\} \operatorname{sign}\{\xi_1(T)\}, \quad v_1^* = v_{\max} \operatorname{sign}\{\eta(t)\} \operatorname{sign}\{\xi_1(T)\}.$$

The trajectory dynamics in the plane $\xi_2 = 0$ can be written as

$$\begin{aligned} \dot{\xi}_1^* &= \zeta(t)\, u_1^* + \eta(t)\, v_1^* = -u_{\max} \,|\zeta(t)| \operatorname{sign}\{\xi_1(T)\} + v_{\max}\,|\eta(t)| \operatorname{sign}\{\xi_1(T)\} \\ &= (-u_{\max}\,|\zeta(t)| + v_{\max}\,|\eta(t)|)\operatorname{sign}\{\xi_1(T)\} = \Gamma(t) \operatorname{sign}\{\xi_1(T)\}. \end{aligned}$$

Thus, candidate optimal trajectories in each half plane differ from each other by a shift along the axis ξ_1 depending on the initial point $\xi_1(T)$.

If $\Gamma(t)$ is positive for all $t < T$, then the two families (upper and lower) of candidate optimal trajectories, generated by backwards integration, will intersect. A candidate optimal trajectory that reaches (backwards) the time axis ceases to be optimal. Thus, its backwards integration cannot continue. The time axis is a "dispersal" line of the game dominated by the maximizing player. In this case, both halves of the equivalent game space are filled (see Fig. 1a).

If $\Gamma(t)$ is negative for all $t < T$, then the pair of candidate optimal trajectories, generated by backwards integration from the point $\xi_1(T) = 0$, serves as the boundaries of an "indifferent" zone, expanding backwards to any initial time. The value of the game in this region is zero, so, this indifferent zone is also the "capture zone" of the game (see Fig. 1b).

Let $\Gamma(t)$ be positive at the neighborhood of T and change its sign once in the interval $t_0 < t < T$ from negative to positive (in direct time). Then a pair of optimal trajectories corresponding to a value $\bar{J} = |\bar{\xi}|$ of the payoff function will reach the time axis tangentially at $t = t_c$ (see Fig. 2) where $\Gamma(t_c) = 0$. The pair of two such symmetric trajectories (from the upper and lower semiplanes) generates a level set of the value function of the game, which we shall call the "critical level set". The region in the neighborhood of the point of tangency is called the "throat" and the instant associated with it can be named as "instant of degeneration of the critical level set".

On the left from the tangency point, an empty zone appears. All optimal trajectories starting in this zone between the critical trajectories associated with $t < t_c$, can leave it (in direct time) only at the point $\xi_1(t_c) = 0$. From this point, further motion with $t_c \leq t \leq T$ must continue along one of the critical trajectories yielding the same cost $\bar{J}$. Therefore, the optimal strategies in the entire region are arbitrary and the value function of the game is constant equal to $\bar{J}$. So, this particular singular region of the game is an "indifferent zone". The point associated with "instant of degeneration of the level set" is a singular dispersal point and the section $t_c < t < T$ of the t-axis is a dispersal line of the game, both dominated by the maximizing player.

In the general case, $\Gamma(t)$ can change sign more than once and this can lead to the existence of several throats of one level set or several indifferent zones generated by different level sets.

The first non-trivial example of a linear game with an indifferent zone was solved in (Pashkov (1971)). In this work, the problem "a boy and a crocodile" (Pontryagin and Mischenko (1969)), which played an important role in the differential game theory, was studied as a fixed terminal time game. Later, examples with indifferent zones were observed in missile guidance problems formulated as zero-sum pursuit-evasion games using different linearized models of the engagement (Gutman and Leitmann (1976), Gutman (1979), Shinar and Gutman (1980), Shinar (1981), Shinar *et al.* (1984), Melikyan and Shinar (2000)).

In the case of circular control constraints, all time sections of the level sets of the value function are circular and the narrow throat is also circular around a point in the time axis. However, if the circular symmetry is disturbed, for example by elliptical control constraints, the time sections of the level sets near the throat become elongated in some direction, which can change during the time. In such case, the time sections of the critical tube can be of a very complicated structure and the construction of the level sets of the value function becomes a very complex task that requires extremely precise numerical methods. The complexity of the form of the critical level set and its neighbors is illustrated in several examples in the sequel.

4. Backward Procedure for Constructing Level Sets

Now, let us describe the algorithm (Isakova *et al.* (1984)) for constructing level sets of the value function (maximal stable bridges) of games of the type (2.1), (2.3). All constructions are made in the equivalent coordinates with the transformation to the equivalent game being done numerically in an automatic way. The procedures created are of the backward type and can be treated as the dynamic programming principle applied to differential games.

To do the numerical construction, let us take a sequence of instants $t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T$ in the time interval $[t_0, T]$ of the game. Uniformity of the grid is unessential. For a given constant c, the result of the procedure is a collection of sets, each corresponding to a chosen fixed time instant t_i and approximating the time section $\mathcal{W}_c(t_i)$ of the level set $\mathcal{W}_c = \big\{(t,\xi) : \mathcal{V}(t,\xi) \le c\big\}$ of the value function $\mathcal{V}$ of the game (2.3) at these instants. The symbol $\boldsymbol{W}_c(t_i)$ will denote the set approximating the original time section $\mathcal{W}_c(t_i)$.

Change the dynamics of the game (2.3) by a piecewise-constant dynamics

$$\dot{\xi} = \boldsymbol{D}(t)u + \boldsymbol{E}(t)v, \quad \boldsymbol{D}(t) = D(t_i), \quad \boldsymbol{E}(t) = E(t_i), \qquad t \in [t_i, t_{i+1}). \tag{4.11}$$

Instead of the original constraints P and Q for the controls of the players, let us consider their polyhedral approximations $\boldsymbol{P}$ and $\boldsymbol{Q}$. Let $\hat{\varphi}$ be the approximating payoff function. It is defined so that for any number c, its level set $\boldsymbol{M}_c = \big\{\xi : \hat{\varphi}(\xi) \le c\big\}$ is a convex polygon close in Hausdorff metrics to the level set $\mathcal{M}_c$ of the original payoff function.

The approximating game (4.11) is chosen such that in each step $[t_i, t_{i+1}]$ of the backward procedure we deal with a game with simple motions (Isaacs (1965)) and polyhedral convex constraints for the players' controls. At the first step $[t_{N-1}, t_N] = [t_{N-1}, T]$, a solvability set $\boldsymbol{W}_c(t_{N-1})$ for a game of homing with target set $\boldsymbol{W}_c(t_N) = \boldsymbol{M}_c$ is constructed. Here, the first player tries to guide the system to the set $\boldsymbol{W}_c(t_N)$ at the instant t_N, and the second one opposes this. Continuing in the same way, a set $\boldsymbol{W}_c(t_{N-2})$ is constructed on the base of $\boldsymbol{W}_c(t_{N-1})$, and so on. As a result, we obtain a collection of convex polygons $\boldsymbol{W}_c(t_i)$ approximating (Ponomarev and Rozov (1978), Botkin (1982), Polovinkin *et al.* (2001)) the sections $\mathcal{W}_c(t_i)$ of the original level set W_c of the value function of the game (2.3) in Hausdorff metrics. An algorithm for a posteriori estimating the numerical construction error is developed (Botkin and Zarkh (1984)).

The procedure of moving from the section $\boldsymbol{W}_c(t_{i+1})$ to the next one $\boldsymbol{W}_c(t_i)$ is described in terms of support functions of the sets under consideration. Recall that the value $\rho(l, A)$ of the support function of a convex bounded closed set A on the vector l is calculated by formula

$$\rho(l, A) = \max_{a \in A} l'a,$$

where the prime denotes transposition.

The support function $l \mapsto \rho\big(l, \boldsymbol{W}_c(t_i)\big)$ of the polygon $\boldsymbol{W}_c(t_i)$ coincides (Pschenichnyi and Sagaidak (1970)) with convex hull of the function

$$\begin{aligned}\gamma(l, t_i) &= \max_{\xi \in \boldsymbol{W}_c(t_{i+1})} l'y + \Delta \max_{u \in \boldsymbol{P}} l'(-D(t_i)u) + \Delta \min_{v \in \boldsymbol{Q}} l'(-E(t_i)v) \\ &= \rho(l, \boldsymbol{W}_c(t_{i+1})) + \Delta\rho(l, -D(t_i)\boldsymbol{P}) - \Delta\rho(l, E(t_i)\boldsymbol{Q}). \qquad (4.12)\end{aligned}$$

The function $\gamma(\cdot, t_i)$ is positively homogeneous and piecewise-linear (because the support functions of the polygons $\boldsymbol{W}_c(t_{i+1})$, $-D(t_i)\boldsymbol{P}$ and $E(t_i)\boldsymbol{Q}$ are of this type). The property of local convexity can be violated only on the boundary of the linearity cones of the function $\rho(\cdot, E(t_i)\boldsymbol{Q})$, that is, on the boundary of the cones generated by normals to the neighboring edges of the polygon $E(t_i)\boldsymbol{Q}$. This fact is useful for developing fast and effective convexification algorithms (Isakova *et al.* (1984), Patsko (1996)).

As a result of the backward procedure in the interval $[t_0, T]$, one obtains a collection of the sets $\boldsymbol{W}_c(t_i)$ for a value of the parameter c. The collection is used by a visualizational software to construct a solid tube to be drawn. Let us denote this tube by $\boldsymbol{W}_c$. This object is not an exact level set of the value function for the approximating game (4.11), but is very close to it.

5. Pursuit-Evasion Games with Elliptical Constraints

5.1. *Air-to-air interception and the corresponding differential game*

In the works Shinar *et al.* (1984), Shinar and Zarkh (1996), and Melikyan and Shinar (2000), three-dimensional air-to-air interception problems were formulated as pursuit-evasion games. The pursuer is the interceptor missile, while the evader is a maneuverable aerial target (an aircraft or another missile). The natural payoff function of the game is the distance of closest approach, the miss distance, to be minimized by the pursuer and maximized by the evader. For sake of simplicity, point mass models with velocities of constant magnitudes V_P, V_E were selected. The lateral accelerations of both players, normal to the respective velocity vectors, have constant bounds a_P and a_E. The evader controls its maneuver with ideal dynamics, while the pursuer's maneuver dynamics is represented by a first order transfer function with the time constant τ_P.

The origin of the Cartesian coordinate system is collocated with the pursuer. The direction of the X-axis is along the initial line of sight. The XY plane is the nominal "collision plane" determined by the initial velocity vector of the evader $(V_E)_0$ and

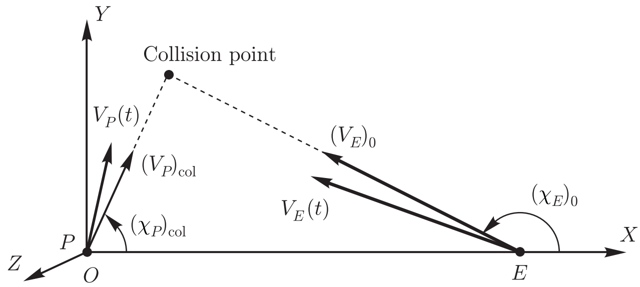

Fig. 3. The system of coordinates in the problem of three-dimensional pursuit. The actual vectors of $V_P(t)$ and $V_E(t)$ differ only slightly during the engagement from the nominal vectors $(V_P)_{\rm col}$ and $(V_E)_0$, respectively.

the initial line of sight. The Z-axis completes a right handed coordinate system, as illustrated in Fig. 3.

It is assumed the initial conditions are near to a "collision course", defined by

$$V_P \sin(\chi_P)_{\rm col} = V_E \sin(\chi_E)_0, \tag{5.13}$$

and the actual velocity vector $V_P(t)$ of the pursuer remains close to the collision requirement $(V_P)_{\rm col}$, satisfying

$$\sin(\chi_P(t) - (\chi_P)_{\rm col}) \approx \chi_P(t) - (\chi_P)_{\rm col}, \quad \cos(\chi_P(t) - (\chi_P)_{\rm col}) \approx 1. \tag{5.14}$$

It is also assumed that the actual velocity vector $V_E(t)$ of the evader will remain close enough to its initial direction satisfying

$$\sin(\chi_E(t) - (\chi_E)_0) \approx \chi_E(t) - (\chi_E)_0, \quad \cos(\chi_E(t) - (\chi_E)_0) \approx 1. \tag{5.15}$$

Based on the small angle assumptions (5.14) and (5.15), the relative trajectories can be linearized with respect to the initial line of sight. Moreover, the relative motion in the X direction can be considered as uniform. Thus, this coordinate can be replaced by the time, transforming the original three-dimensional motion to a two-dimensional motion in the YZ plane. For a given initial range, the uniform closing velocity determines the final time T of the engagement. Therefore, the problem of minimizing (maximizing) the three-dimensional miss distance at a free terminal time can be changed by the minimization (maximization) of the distance in the YZ plane at the fixed terminal time of the nominal collision.

Since in general the velocity vectors $(V_P)_{\rm col}$ and $(V_E)_0$ of the players are not aligned with the initial line of sight, the projections of the originally circular control constraints, normal to the respective velocity vectors, become elliptical.

The equations of motion of the linearized pursuit-evasion game are

$$\begin{array}{ll} \ddot{x} = F, & t \in [0, T], \quad x,\ y \in R^2,\ \ u \in P,\ \ v \in Q, \\ \dot{F} = -(F - u)/\tau_P, & \\ \ddot{y} = v, & \varphi(x(T), y(T)) = |x(T) - y(T)|, \end{array} \tag{5.16}$$

where x and y are the positions of the players in the plane normal to the initial line of sight, and u and v are their respective acceleration commands. Eliminating F in the dynamics (5.16) and denoting $\alpha = 1/\tau_P$ yield

$$\dddot{x} + \alpha\,\ddot{x} = \alpha\,u, \quad \ddot{y} = v, \tag{5.17}$$

which is in the form of a typical “generalized L.S.Pontryagin’s test example”.

To reduce the dynamics (5.17) to the form (2.1), a variable change

$$\begin{aligned} z_1 &= x_1 - y_1, \quad & z_2 &= \dot{x}_1 - \dot{y}_1, \quad & z_3 &= \ddot{x}_1, \\ z_4 &= x_2 - y_2, \quad & z_5 &= \dot{x}_2 - \dot{y}_2, \quad & z_6 &= \ddot{x}_2 \end{aligned}$$

can be applied, leading to the following standard form of the game

$$\begin{aligned} &\dot{z} = Az + Bu + Cv, \\ &A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_1 \end{array}\right], \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/\tau_P \end{bmatrix}, \\ &B' = (1/\tau_P)\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \\ &C' = \left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right], \end{aligned}$$

with constraints for the players’ controls taken as ellipses

$$\begin{aligned} u \in P &= \left\{ u \,:\, u' \begin{bmatrix} 1/\cos^2(\chi_P)_{\mathrm{col}} & 0 \\ 0 & 1 \end{bmatrix} u \le a_P^2 \right\}, \\ v \in Q &= \left\{ v \,:\, v' \begin{bmatrix} 1/\cos^2(\chi_E)_0 & 0 \\ 0 & 1 \end{bmatrix} v \le a_E^2 \right\}, \end{aligned}$$

and the payoff function

$$\varphi\big(z_1(T), z_4(T)\big) = \sqrt{z_1^2(T) + z_4^2(T)}.$$

The transformation (2.2) to the equivalent game yields

$$\dot{\xi} = D(t)u + E(t)v, \quad t \in [0,T],\ \ \xi \in R^2,\ \ u \in P,\ \ v \in Q,\ \ \varphi(\xi(T)) = |\xi(T)|,$$

where

$$D(t) = \zeta(t) \cdot I_2, \quad \zeta(t) = (T-t) + \tau_P e^{-(T-t)/\tau_P} - \tau_P, \tag{5.18a}$$

$$E(t) = \eta(t) \cdot I_2, \quad \eta(t) = -(T-t) \tag{5.18b}$$

and I_2 is the 2×2 unit matrix.

In order that achieving a small miss distance be feasible, in all realistic pursuit-evasion examples the pursuer must have some advantage in maximum lateral acceleration in every direction. This means that the control constraint set P of the pursuer has to cover completely the control constraint set Q of the evader. In other words, the inequalities

$$a_P/a_E > 1, \quad a_P|\cos(\chi_P)_{\text{col}}| > a_E|\cos(\chi_E)_0|, \tag{5.19}$$

describing the relations of the semiaxes of the ellipses P and Q, have to be valid. Such an advantage allows reducing an initial launching error and overcoming long duration constant evader maneuvers. However, due to the first order dynamics of the pursuer's control function and the ideal dynamics of the evader, zero miss distance cannot be achieved against an optimally maneuvering evader.

In Shinar *et al.* (1984), as well as in Melikyan and Shinar (2000), the parameters of the problem were of an interception of a manned aircraft, assuming that $V_P > V_E$. Thus using (5.13) one obtains

$$|\cos(\chi_P)_{\text{col}}| > |\cos(\chi_E)_0|.$$

In Shinar and Zarkh (1996), an interception of a tactical ballistic missile was considered with $V_P < V_E$, leading to

$$|\cos(\chi_P)_{\text{col}}| < |\cos(\chi_E)_0|.$$

This difference influences considerably the form of the critical tube in the equivalent game and the associated singular surfaces.

5.2. *Numerical constructions in the case of a faster pursuer*

Let us start with the results of numerical investigations for the case when the pursuer's velocity V_P is greater than the velocity V_E of the evader. The nominal collision geometry for this case is shown in Fig. 4.

Relation (5.13) of the nominal collision and the relation $V_P > V_E$ of the players' velocities yield that the eccentricity of the ellipse P is smaller than the eccentricity of the ellipse Q (see Fig. 5).

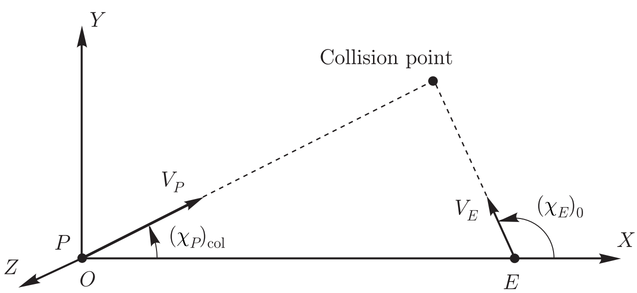

Fig. 4. The nominal collision geometry for the case of a faster pursuer.

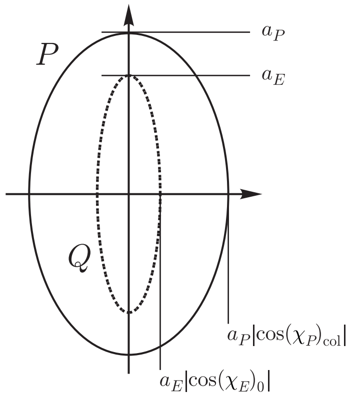

Fig. 5. Elliptical constraints of the players' controls in the case of a faster pursuer. The ellipse P is drawn by a solid line, Q is drawn by a dashed one. The eccentricity of P is smaller than the eccentricity of Q.

In the numerical investigation the following data were chosen:

$$\frac{V_E}{V_P} = 0.666, \quad \frac{a_P}{a_E} = 5.0, \quad |\cos(\chi_P)_{\mathrm{col}}| = 0.87, \quad |\cos(\chi_E)_0| = 0.66, \quad \tau_P = 1\,\mathrm{sec}.$$

Consequently, the elliptical control constraints are:

$$P = \left\{ u \in R^2 : \frac{u_1^2}{0.87^2} + \frac{u_2^2}{1.00^2} \leq 5.0^2 \right\}, \quad Q = \left\{ v \in R^2 : \frac{v_1^2}{0.66^2} + \frac{v_2^2}{1.00^2} \leq 1 \right\}.$$

From here, the symbol τ denotes the backward time: $\tau = T - t$.

This example has been computed in the interval $\tau \in [0, 2]$ sec of backward time. The time step Δ was taken equal to 0.025 sec. The circles of the level sets of the payoff function and the ellipses P and Q of constraints for the players' controls were approximated by 100-gons (polygons with 100 vertices).

In Fig. 6, the vectogram tubes for this example are shown. The first player's tube is light gray, the second one's is dark gray.

An enlarged part of the previous picture from another point of view can be seen in Fig. 7. On the vectogram tube of the second player, the contours of some time sections are shown.

Since $\mathcal{Q}(t) = E(t)Q = \eta(t)I_2Q = \eta(t)Q$, where $\eta(\cdot)$ is described by (5.18b), the dark gray tube grows linearly with τ. For the tube $\mathcal{P}$, we have $\mathcal{P}(t) = \zeta(t)P$, where $\zeta(\cdot)$ is taken from (5.18a). So, initially (for small values of τ), the light gray tube grows slower than linearly, but for large values of τ it becomes almost linear and starts to grow faster than the tube $\mathcal{Q}$ does. This faster growth is provided by inequalities (5.19).

So, for $\tau < \tau_*$, the second player (the maximizer) has a complete advantage, that is, the vectogram $\mathcal{Q}(\tau)$ of the second player completely covers the vectogram $\mathcal{P}(\tau)$ of the first player (Fig. 8a). The instant τ_* is characterized by the fact that the horizontal size of the ellipses $\mathcal{P}(\tau_*)$ and $\mathcal{Q}(\tau_*)$ are equal (Fig. 8b). In the interval

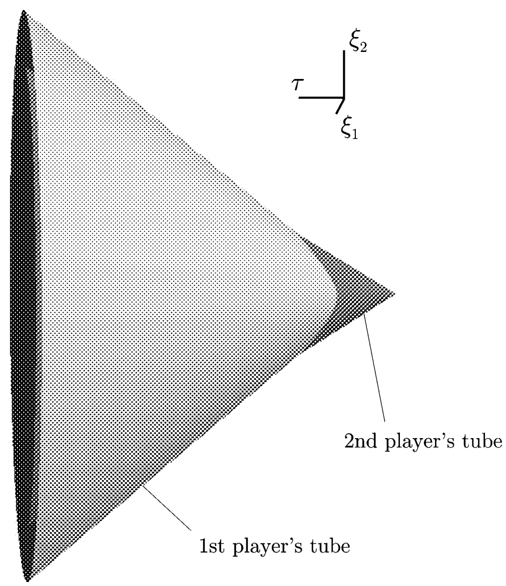

Fig. 6. Vectogram tubes for the case of a faster pursuer.

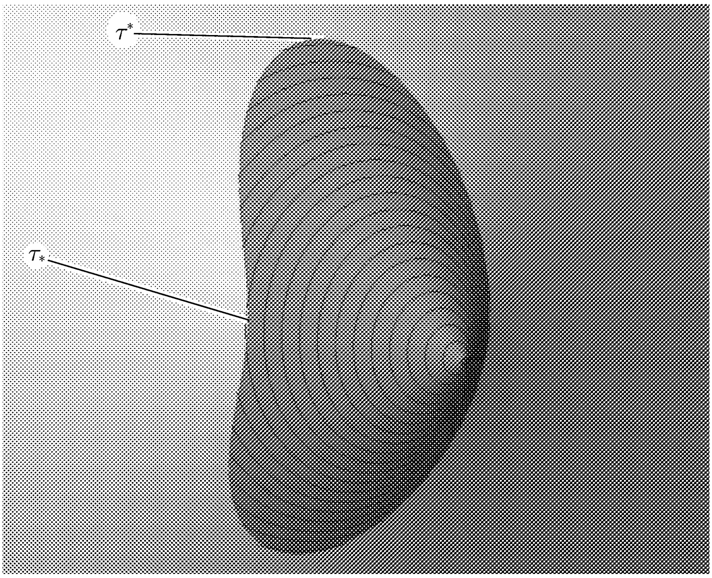

Fig. 7. An enlarged fragment of the vectogram tubes. The first player gains advantage in horizontal direction at the instant τ_* and a complete advantage at the instant τ^*.

$\tau_* < \tau < \tau^*$, none of the players has a complete advantage: the first player is stronger in horizontal direction, the second player is stronger in directions near to the vertical (Fig. 8c). When $\tau = \tau^*$, the vertical sizes of the ellipses become equal

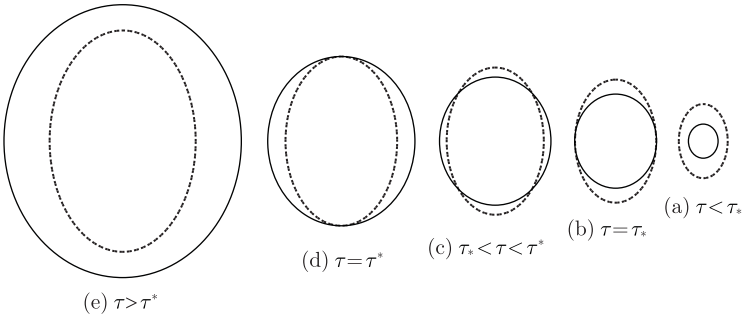

Fig. 8. Sections of the vectogram tubes at some time instants. The vectograms of the first player are shown by the solid lines, the dashed lines denote the vectograms of the second player.

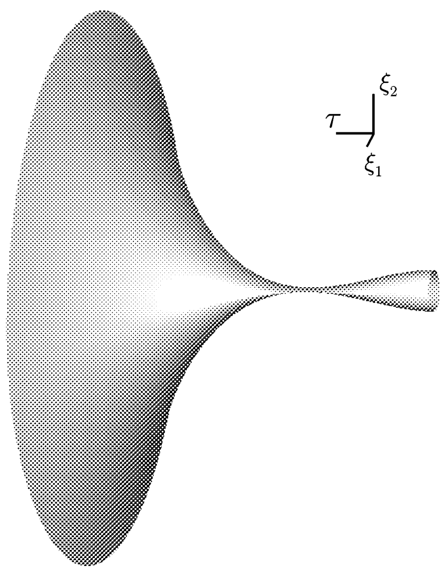

Fig. 9. A level set close to the critical one, $c = 0.141\,\mathrm{m}$. The instant of the most narrow place $\tau^* = 0.925\,\mathrm{sec}$.

(Fig. 8d) and for $\tau > \tau^*$ the first player has complete advantage (Fig. 8e). Such a change in the relationship of the vectograms $\mathcal{P}(\tau)$ and $\mathcal{Q}(\tau)$ can be explained by the difference between the eccentricities and the sizes of the ellipses P and Q and the form of the functions $\zeta(t)$ and $\eta(t)$.

Such a shift of the advantage from the maximizing player to the minimizer leads to creating a narrow throat. Figure 9 shows a level set close to the critical one. This level set is computed for $c = 0.141\,\mathrm{m}$. The narrow throat is located at $\tau^* = 0.925\,\mathrm{sec}$.

A close view of the narrow throat is shown in Fig. 10. Contours of some time sections of the level set are shown. One can see that the t-sections $\boldsymbol{W}_c(t)$ of the

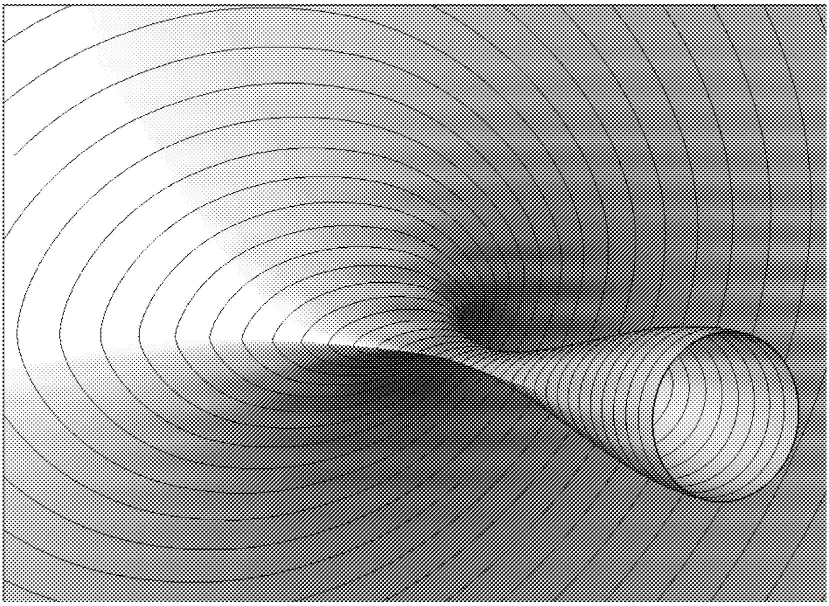

Fig. 10. An enlarged view of the level set close to the terminal instant with the contours of some time sections.

level set near the narrow throat are elongated horizontally. This is due to the relation of the players' capabilities. For $\tau < \tau_*$, the second player is stronger in the vertical direction than horizontally. Accordingly to this, the sections $\boldsymbol{W}_c(t)$ are compressed more in the vertical direction. When τ is slightly greater than τ^*, the first player's advantage is stronger in the horizontal direction (Fig. 8e), which leads to a horizontal expansion of the sections.

The following Figure 11 shows again a view of the vectogram tubes. The second player's vectogram is transparent. The τ-axis goes from right to left, and the axis ξ_2 is directed upwards. The axis ξ_1 is orthogonal to the sheet.

Using the point of view of the previous figure, a scene is given (Fig. 12) containing also the level set close to the critical one. Both vectogram tubes are transparent now. Such an overlap demonstrates clearly the influence of the players' vectograms on the geometry of the level set surface. For example, one can easily see that, when the first player gains complete advantage (at $\tau^* = 0.925\,\text{sec}$) the narrow throat ends (the tube of the level set starts to enlarge). In addition, it is seen that before that instant the tube contracts due to the advantage of the second player.

The results shown here agree qualitatively with the ones obtained in an analytical investigation of the problem with a faster pursuer made in Shinar *et al.* (1984) and Melikyan and Shinar (2000). In these papers, it shown that in the case of a faster pursuer the geometry of the critical level set is the same for any combination of parameters of the problem.

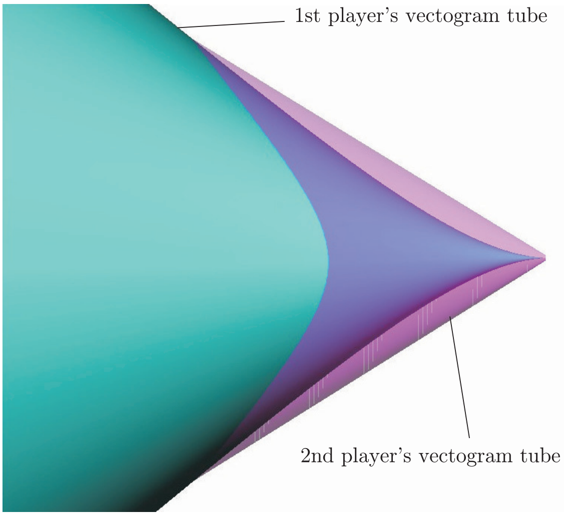

Fig. 11. Superposition of the vectogram tubes. The second player's tube is transparent.

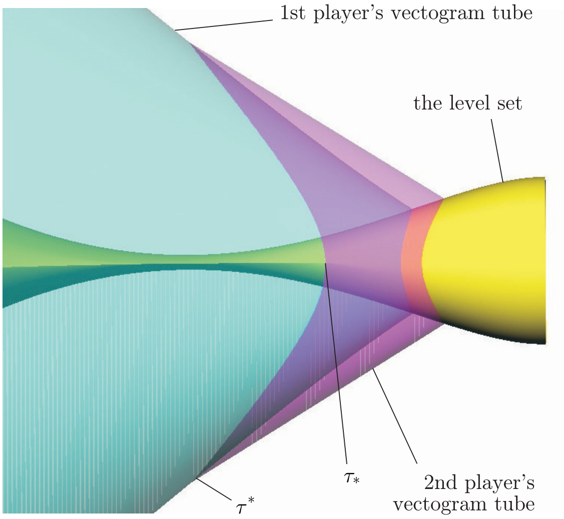

Fig. 12. Superposition of the vectogram tubes and the level set close to the critical one. The players' vectogram tubes are transparent.

5.3. *Numerical constructions in the case of a slower pursuer*

In this subsection, the results with a slower pursuer $V_P < V_E$ are presented. The nominal collision geometry for this case is shown in Fig. 3. The eccentricity of the ellipse P is greater than the eccentricity of the ellipse Q (see Fig. 13).

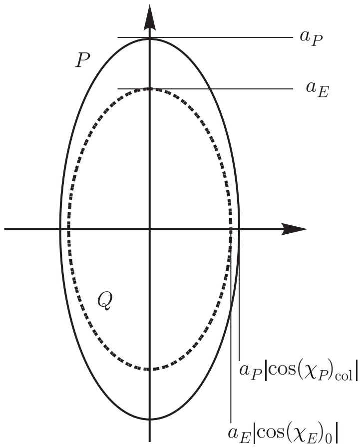

Fig. 13. Elliptical control constraints of the players in the case of a slower pursuer. The ellipse P is drawn by a solid line, Q is drawn by a dashed one. The eccentricity of P is greater than the eccentricity of Q.

Based on the data of the original problem

$$\frac{V_E}{V_P} = 1.054, \quad \frac{a_P}{a_E} = 1.3, \quad |\cos(\chi_P)_{\rm col}| = 0.67, \quad |\cos(\chi_E)_0| = 0.71, \quad \tau_P = 1\,\text{sec},$$

in the construction, the following data were used: $\alpha = 1/\tau_P = 1$,

$$P = \left\{ u \in R^2 : \frac{u_1^2}{0.67^2} + \frac{u_2^2}{1.00^2} \leq 1.30^2 \right\}, \quad Q = \left\{ v \in R^2 : \frac{v_1^2}{0.71^2} + \frac{v_2^2}{1.00^2} \leq 1 \right\}.$$

This example has been computed in the interval $\tau \in [0, 7]$ sec with the time step $\Delta = 0.01$ sec. The circles of the level sets of the payoff function and the ellipses of the players' control constraints, P and Q, were approximated by 100-gons.

Like in the example of the previous subsection, there is a narrow throat also here. Figure 14 shows a general view of the level set $\boldsymbol{W}_c$ computed for the parameter $c = 1.546$ m, which is slightly greater than the critical one. But unlike the example described in Subsec. 5.2, here the narrow throat has a much more complex structure: the orientation of the t-sections' elongation changes very tricky near the throat. An enlarged view of the throat is shown in Fig. 15.

Let us use the players' vectogram tubes for this problem to explain the shape of the level set. The vectogram tubes are shown in Fig. 16. The tube of vectograms of the first player ($\mathcal{P}$) is drawn in red, and the tube of the second player ($\mathcal{Q}$) is in green. Here also, the tube $\mathcal{Q}$ grows linearly with τ, whereas the tube $\mathcal{P}$ grows slower than linearly at small values of τ and becomes almost linear later. Eventually, for large values of τ, it will grow faster than the tube $\mathcal{Q}$, because (5.19).

Since the ellipses P and Q have different eccentricities, the first player's ellipse $\mathcal{P}(\tau)$ starts to cover the ellipse $\mathcal{Q}(\tau)$ of the second player in different directions at

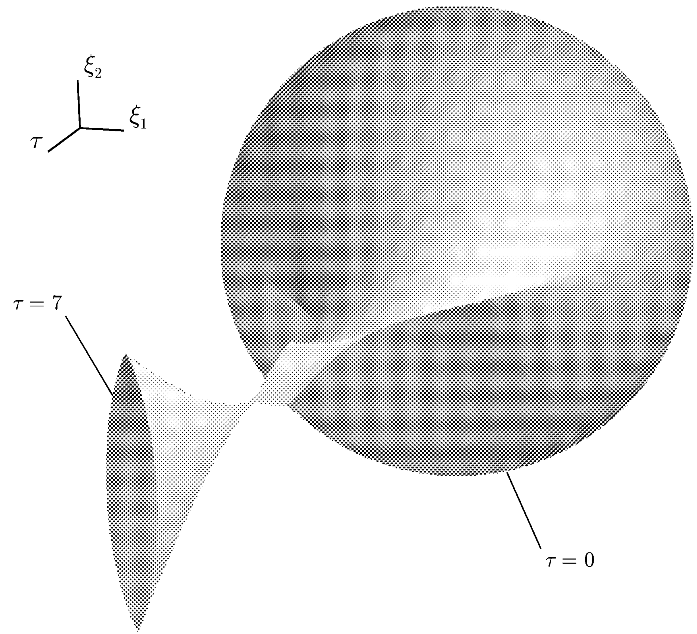

Fig. 14. General view of the level set of the value function with a narrow throat.

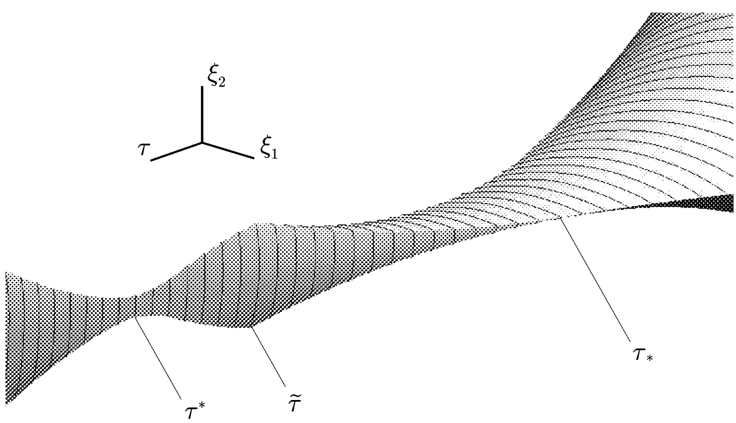

Fig. 15. Enlarged view of the narrow throat.

different instants. So, for $\tau < \tau_* = 4.18\,\text{sec}$, the ellipse $\mathcal{Q}(\tau)$ includes the ellipse $\mathcal{P}(\tau)$ completely (see Fig. 17a). At $\tau = \tau_*$, the first player's ellipse reaches the ellipse of the second player in the vertical direction (see Fig. 17b). In the interval $\tau_* < \tau < \tau^*$, the ellipse $\mathcal{P}(\tau)$ covers more and more the ellipse $\mathcal{Q}(\tau)$. Finally, at $\tau = \tau^* = 5.3\,\text{sec}$ the set $\mathcal{P}(\tau)$ covers the set $\mathcal{Q}(\tau)$ even in the horizontal direction (see Fig. 17c). And for $\tau > \tau_*$, $\mathcal{P}(\tau)$ covers $\mathcal{Q}(\tau)$ completely (see Fig. 17d).

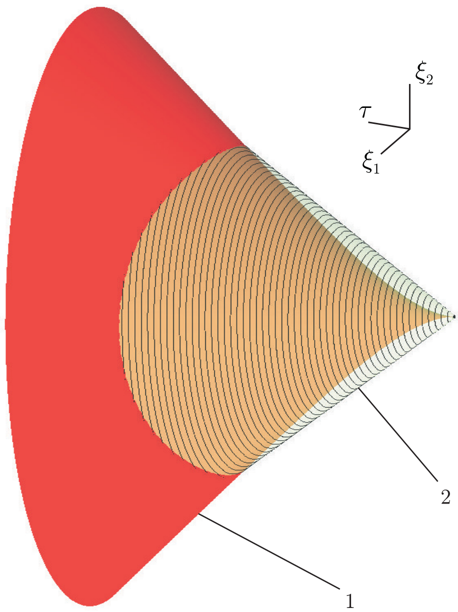

Fig. 16. General view of the vectogram tubes of the first (1) and second (2) players. The vectogram tube of the second player is transparent, showing contours of some sections.

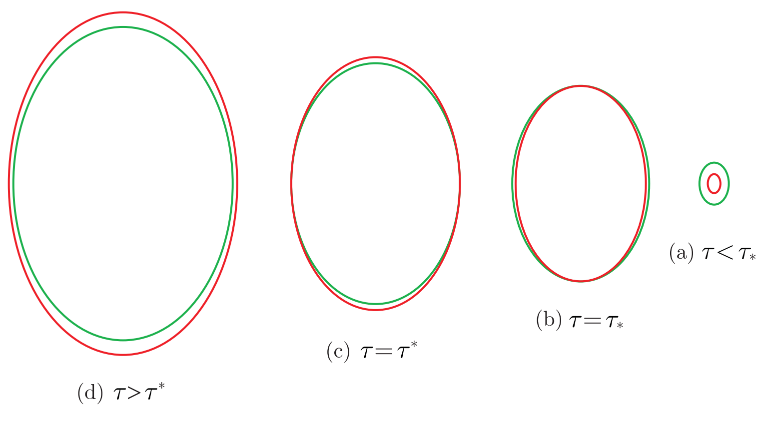

Fig. 17. Sections of the vectogram tubes at some time instants. The vectograms of the first player are shown by the red lines, the green lines denote the vectograms of the second player.

The relationship between the players' vectograms leads to an intricate changing of the level set's t-sections near the narrow throat, as it can be seen in Figs. 14, 15, and 18. The latter shows groups of sections in different intervals of τ to demonstrate the different phases of the sections' changing.

For $\tau < \tau_*$, the second player has complete advantage over the first one. Since in backward time the second player tries to contract the sections of the level sets

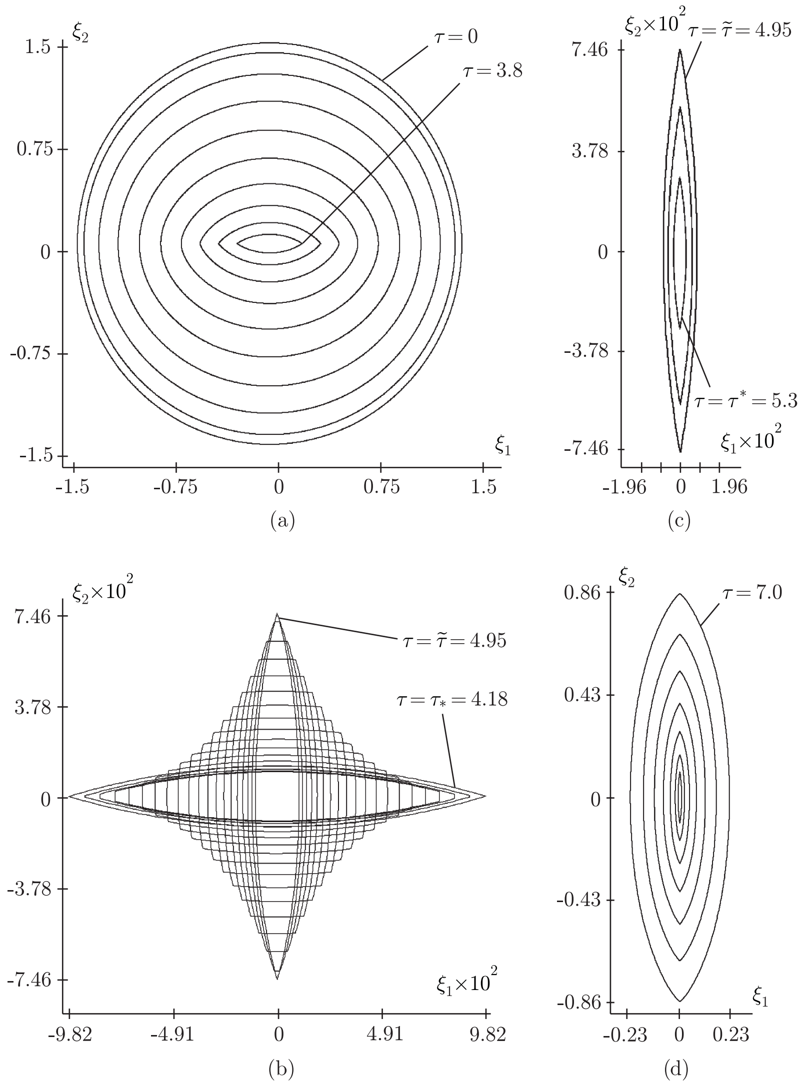

Fig. 18. Groups of time sections of a level set close to the critical tube in some intervals of the backward time: (a) $\tau \in [0, 3.8]$ sec; (b) $\tau \in [\tau_*, \tilde{\tau}]$; (c) $\tau \in [\tilde{\tau}, \tau^*]$; (d) $\tau \in [5.41, 7.0]$ sec.

as much as possible, the t-sections of the tube $\boldsymbol{W}_c$ are reduced in the interval $0 < \tau < \tau_*$. In Fig. 18a, the sections are shown in the interval $\tau \in [0, 3.8]$ sec. Since the second player's advantage is greater in the vertical direction, the tube starts to contract more in the vertical direction than in the horizontal one. Therefore, at $\tau = \tau_*$ the t-section of the level set is elongated horizontally.

In the interval $\tau_* < \tau < \tau^*$, the first player gains advantage gradually, starting in the vertical direction, while the second player keeps its horizontal advantage. For this reason the t-sections of the level set start expanding vertically, while they

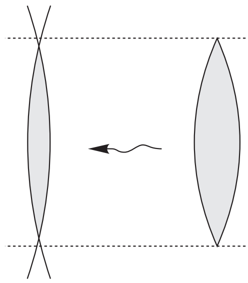

Fig. 19. Decreasing the time sections in vertical direction due to horizontal contraction.

continuing being reduced in the horizontal direction. This interval can be subdivided into two parts.

Between τ_* and $\tilde{\tau} = 4.95\,\text{sec}$ the time sections have the shape of "curvilinear rectangles" as it can be seen in Fig. 18b. Their form is gradually changing from a horizontal elongation to a vertical one.

At $\tilde{\tau}$ the horizontal arcs disappear, and the t-sections start having a vertical lens shape. Simultaneously, the vertical expansion becomes a contraction despite of the vertical advantage of the first player (Fig. 18c), because the horizontal contraction enforces a contraction due to the lens shape as explained in Fig. 19. The side arcs of the previous section $\boldsymbol{W}_c(\tau_i)$ enlarge vertically under action of the first player and become closer to each other horizontally under action of the second one. The next section $\boldsymbol{W}_c(\tau_{i+1})$ is bounded by the intersection of the arcs.

Finally, at $\tau = \tau^*$, when the first player gains a complete advantage, one obtains the narrowest section of the throat (Figs. 15 and 18c) with vertical elongation. For $\tau > \tau^*$ the first player keeps the complete advantage and the t-sections start to expand in all directions monotonically. The rate of expansion is, however, non uniform, but the direction of elongation remains vertical (see Fig. 18d).

The following two figures show the critical level set in comparison with level sets close to it. Figure 20 shows the critical tube (drawn in transparent yellow) and the tube computed for the value of $c = 1.48\,\text{m}$ of the payoff function, which is less than the critical one. This tube is finite in time and drawn in red. In Fig. 21, the critical level set (in red) and the one computed for $c = 1.67\,\text{m}$ (in transparent yellow) are presented. One can see that the latter has smooth boundary. These figures demonstrate that the majority of peculiarities of the value function are found near the narrow throat, emphasizing the necessity of extremely accurate computations near the throat.

Note that the unique features of the narrow throat, described in this subsection, appear also for other linearized pursuit-evasion game examples (of the same mathematical model) with a slower pursuer. Of course, by varying the parameters of the game, the duration of the narrow throat and the size of the t-sections near the throat will change.

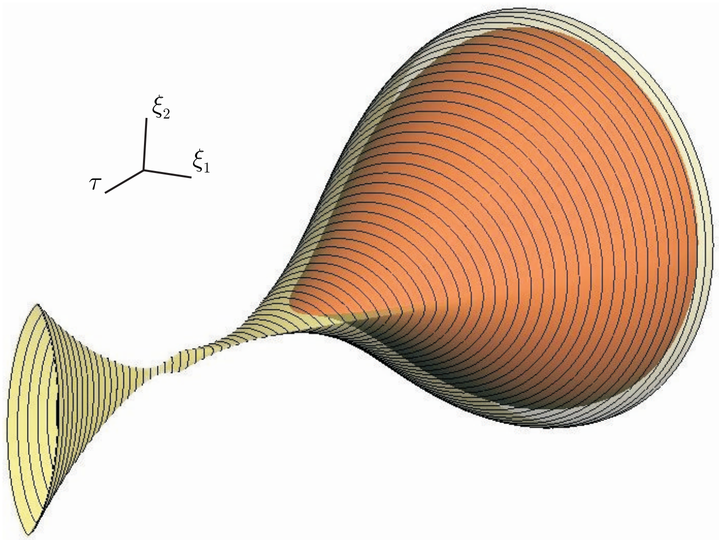

Fig. 20. The level set with narrow throat for the parameter $c = 1.546\,\mathrm{m}$ (yellow transparent) and the level set for a less value of the parameter $c = 1.48\,\mathrm{m}$ (red).

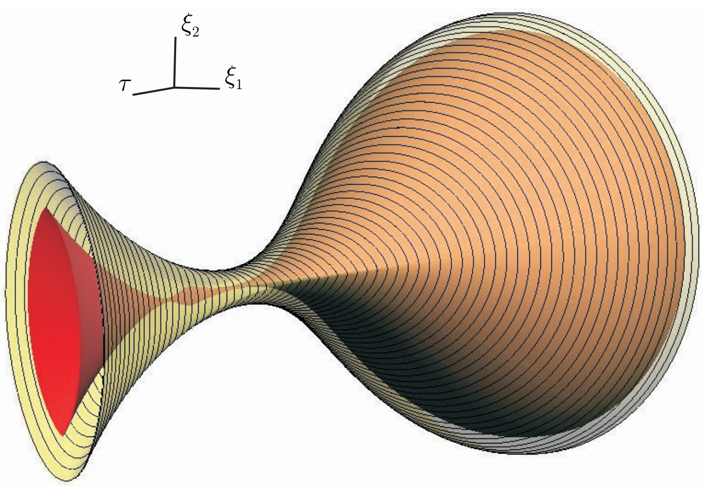

Fig. 21. The level set with narrow throat for the parameter $c = 1.546\,\mathrm{m}$ (red) and the level set for a greater value of the parameter $c = 1.67\,\mathrm{m}$ (yellow transparent).

In the paper Shinar and Zarkh (1996), in particular, an example with the following parameters

$$\frac{V_E}{V_P} = 1.03, \quad \frac{a_P}{a_E} = 1.3,$$

$$|\cos(\chi_P)_{\mathrm{col}}| = 0.98, \quad |\cos(\chi_E)_0| = 0.9812, \quad \tau_P = 1\,\mathrm{sec}$$

was analyzed, leading to $\alpha = 1/\tau_P = 1$,

$$P = \left\{ u \in R^2 : \frac{u_1^2}{0.98^2} + \frac{u_2^2}{1.0^2} \le 1.3^2 \right\}, \quad Q = \left\{ v \in R^2 : \frac{v_1^2}{0.9812^2} + \frac{v_2^2}{1.0^2} \le 1 \right\}.$$

The tube of the critical level set in this example was associated with the value of $c = 1.534\,\text{m}$.

The attempt to reconstruct numerically this level set by the algorithm of Sec. 4 was unsuccessful. Although in general, the tube was reproduced, the shape of t-sections near the narrow throat was incorrect.

One of the reasons of the failure was that in this example the time duration of the narrow throat turned out to be extremely short. In order to reconstruct the shape adequately, computations should have been carried out (at least, in the interval of the throat) with a time step $\Delta = 5 \cdot 10^{-4}$ sec. Moreover, the maximal length of the approximating polygons' edges should have been also adjusted to this small time step, resulting in polygons of thousands of edges. All these reasons would have lead to an unacceptable time of computation.

The reasons for the very short duration of the narrow throat are the almost circular control constraints and the negligible difference in their eccentricities. The physical origin of this feature is the small difference in velocity magnitudes, as well as the near frontal initial geometry ($(\chi_P)_{\text{col}} = 11.5$ degree, $(\chi_E)_0 = 168.9$ degree).

The analytical results of the paper Shinar and Zarkh (1996) shows that in the case of a slower pursuer the geometry of the critical tube differs qualitatively for different combinations of the parameters of the problem. The dependence of the critical tube geometry on the parameters of the problem (and how it affects the singular surfaces) is investigated in that paper. The example computed numerically in this subsection corresponds to the case of the most complicated structure of the narrow throat.

6. Summary and Concluding Remarks

The paper presents examples of a linear zero-sum differential game with bounded controls, fixed terminal time and terminal payoff. The payoff function is convex and depends only on two components of the state vector. This game is a mathematical model of a pursuit-evasion problem between a guided missile and its aerial target, where the kinematics can be linearized. The original game can be transformed to a two-dimensional "equivalent" game, which is much more convenient for analysis.

The main contribution of the paper is the demonstration of the "narrow throat" phenomenon the value function's level sets. This phenomenon is closely connected to the existence of "indifferent" zones in the equivalent game space, where the value function is constant and the optimal controls are arbitrary. Such indifferent zones can exist only if the players' vectograms in the equivalent game are such that none of them has a complete advantage during the entire game. If the control constraints (and consequently, the vectograms) have circular symmetry, the indifferent zone (bounded by the "critical" level set of the value function) starts in reverse time at a single point on the time axis and the neighborhood of the point is called the "throat". Depending on the number of advantage changes, there can be more than one throats. If the circular symmetry of the control constraints is perturbed, for example (as in the examples considered in the paper) by elliptical vectograms, the time sections of the critical level set near the throat become elongated in some direction. Contraction of the time sections of a level set in direct time indicates the advantage of the minimizing player, while expansion indicates the advantage of the maximizer. The change in the orientation of the level sets' time sections near the narrow throat indicates the shift of advantage in the components of their vectograms. This last phenomenon may have a great importance in the development of optimal evasive maneuvers from guided missiles.

The discovery and the analysis of such complex behavior could be achieved only by an intricate and very accurate computational algorithm for construction of the level sets of the value function, developed by the first two authors in a multi-year effort. The analytical investigations made in the papers of the third author allowed to check the correctness of the numerical constructions and to find an example with some variant of input data, which could not been reproduced yet by the currently available numerical algorithm.

Acknowledgment

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