UDC 62-50

Numerical Solution of a Third-Order Directed Game*

M. A. ZARKH AND V. S. PATSKO
(Sverdlovsk)

An examination is made of a linear antagonistic differential game between two persons with a fixed instant of termination. The controls of the players are assumed to be scalar. The purpose of the first player is to bring three specified coordinates of the phase vector to a closed convex bounded set, and the interests of the second player are opposite to this. An algorithm for numerical construction of the set of all initial positions from which the problem can be solved by the first player is described. Examples of computer calculations are included.

* * *

INTRODUCTION

Many problems of control in the presence of noise can be formalized as antagonistic differential games [1 - 5]. An important class of antagonistic differential games is the class of games with geometrical constraints on the controls, and among these are differential games which lead toward a target set at a fixed instant of time.

The purpose of the first player in such a game is to bring the system being controlled into a target set M at a given instant ϑ, and the second player hinders this. One of the central concepts in a directed game is the solvability set W for the problem, which is the set of all positions (t_*, x_*) from which the first player, using a control formed according to the feedback principle, guarantees that the system will be brought into M at the instant ϑ no matter what actions the second player takes.

At the present time, numerical methods have been developed for constructing the solvability set using the constructions introduced in the works of Isaacs, Bellman, Pontryagin, and Pshenichnyy. Starting with a grid of instants t_i and going back in time from the instant ϑ, one constructs successively the sections $W(t_i)$. The rule for transition from $W(t_i)$ to $W(t_{i+1})$, $t_{i+1} < t_i$, is monotonic with i and constitutes the essence of the reverse construction. Questions of numerical construction of the set W for nonlinear systems have been examined in [6, 7] and for linear systems in [8 - 11].

Knowledge of how to construct the solvability set in directed problems is not important just in its own right: using it, one can obtain solutions of other types of differential games. Thus, in a game with fixed instant of termination ϑ and terminal payment γ, the level set $\{(t, x): \Gamma(t, x)\leq c\}$ of the cost function Γ coincides with the solvability set W_c of the directed game, in which the target set has the form $M_c = \{x: \gamma(x)\leq c\}$. If we take, together with the grid of instants t_i, a finite set of values of the parameter c and construct for each c in that set the sections $W_c(t_i)$, we can determine numerically the optimal strategies of the two players in the game with payment γ on the basis of them.

A directed game with linear dynamics has a number of distinctive features which enable us considerably to simplify the computational procedures for finding the sections $W(t)$ in comparison with the general nonlinear case. The two most important of these features are the

*Originally published in Tekhnicheskaya Kibernetika, No. 6, 1987, pp. 162-169.

ISSN0882-4002/88/0004-0092$7.50/0

following: If the target set M is convex, so are the sections $W(t)$. If M is cylindrical with respect to all the coordinates except for certain m of them, then the dimension of the problem can be lowered to m by moving to an equivalent directed game [2, 4] without the phase variable on the right-hand side. We note that the section of the solvability set for the instant t, written in the coordinates of the equivalent problem coincides with the alternation integral [3], constructed in the interval $[t, \vartheta]$, from the set M.

The methods of constructing sections of the solvability set (in the coordinates of the equivalent problem) have been made into standard computational programs for $m = 2$ [11]. In the present article, we consider the case $m = 3$. The target set is assumed to be convex and the controls of the two players are assumed to be scalar. A standard program for constructing the sections of the solvability set has been worked out on the basis of the method to be described.

The transition from the original differential game (which can be of high dimension) to a third-order equivalent game does not present any great difficulty. The complexity consists entirely in solving the equivalent game. This is the reason for "third-order" in the title of the article.

At the end of the article, we shall give two examples, calculated on a computer. The first of these relates to the problem of control of the lateral motion of a landing airplane when there is a perturbing wind. The linearized system of differential equations is of seventh order, and the target set depends on the three coordinates of the phase vector. The second example demonstrates the possibility of using a program of constructing the sections of the solvability set of a three-dimensional differential game in order to find the cost function in a two-dimensional game with terminal payment. This possibility depends upon the results of [12].

1. Statement of the Problem

Consider a linear differential game

$$\dot{x}=Ax+Bu+Cv,\quad x\in R^n,\quad u\in P\subset R,\quad v\in Q\subset R \tag{1.1}$$

between two persons with a fixed instant of termination ϑ and a closed convex target set $M\subset R^n$. Let us assume that the set M is cylindrical with respect to all except three specified coordinates of the vector x. We assume without loss of generality that these are the first three coordinates. Thus, the set M is of the form $M = \{x\in R^n : (x_1, x_2, x_3)'\in M^*\}$, where M^* is a convex compact subset of R^3. The first player strives to get the system (1.1) in M at the instant ϑ and the second player hinders this. Suppose that W is the set of all initial positions (t_*, x_*), $t_*\leqslant\vartheta$, from which the problem of getting the system to M at the instant ϑ can be solved by the first player in the class of controls formed according to the feedback principle [2].

We denote by $X^*(\vartheta, t)$ the matrix consisting of the first three rows of the fundamental Cauchy matrix $\exp A(\vartheta-t)$. Making the change of variable $y(t)=X^*(\vartheta, t)x(t)$, we move from (1.1) to the third-order equivalent directed game

$$\dot{y}=D^*(t)u+E^*(t)v,\quad D^*(t)=X^*(\vartheta, t)B,\quad E^*(t)=X^*(\vartheta, t)C,\quad y\in R^3,\quad u\in P,\quad v\in Q \tag{1.2}$$

with target set M^*. We denote by W^* the solvability set for the game (1.2). The sections $W(t)$ and $W^*(t)$ are related by $W(t)=\{x\in R^n : X^*(\vartheta, t)x\in W^*(t)\}$. We are required to construct a numerical procedure for finding the sections $W^*(t)$.

2. The Approximating Game

Partitioning the t-axis to the left of ϑ with step Δ by means of the points t_i, $i = 0, 1, 2, \ldots$, $t_0 = \vartheta$, we consider at the piecewise-constant functions $D(t)=D^*(t_i)$, $E(t)=E^*(t_i)$, $t\in(t_{i-1}, t_i]$, $i = 0, 1, 2, \ldots$. We replace the convex compact set M^* by the convex polyhedron M and the differential equation (1.2) by the equation

$$\dot{y}=D(t)u+E(t)v,$$

$$y \in R^3, \quad u \in P, \quad v \in Q. \tag{2.1}$$

Let W denote the solvability set for the approximating game (2.1). For arbitrary $t \leqslant \vartheta$, the section $W(t)$ is a convex polyhedron [13] (if $W(t) \neq \varnothing$). Estimates of the closeness of $W(t)$ and $W^*(t)$ can be found in [8, 10].

We shall be interested in the sections $W(t_i)$, $i = 0, 1, 2, \ldots$. We set

$$P_i = D(t_i)P, \quad Q_i = E(t_i)Q, \quad W_i = W(t_i).$$

We have $W_0 = M$. Suppose that $F_i = W_i - \Delta P_i$. We denote by q_i^* and q_i^0 the extreme points of the segment Q_i. For any $i = 0, 1, 2, \ldots$, the polyhedron W_{i+1} is the intersection of the polyhedrons $F_i - \Delta q_i^*$ and $F_i - \Delta q_i^0$ [13]. The intersecting polyhedrons differ from each other only by a displacement, which is small for small Δ. This detail enables us to find the result of the intersection more economically than in the case of two arbitrary polyhedrons [14].

3. The Support Function for the Polyhedron W_{i+1}

Let us describe the polyhedrons with the aid of the support function ρ. We set $\eta_i(l) = \rho(l, F_i) - \rho(l, \Delta Q_i)$, $l \in R^3$. The support function $\rho(\cdot, W_{i+1})$ of the polyhedron W_{i+1} coincides with the convex hull of the function η_i [13]. This last function is convex in each of the half-spaces defined by the plane through the origin orthogonal to the segment Q_i. Consequently, the local convexity of the function η_i can be violated only on the plane indicated. The difference between η_i and its convex hull depends upon the size of the step Δ. These circumstances, represented in the language of a dual description reformulating the above-mentioned features of formating the polyhedron W_{i+1}, make it possible, using a finite iteration process, to construct the support function $\rho(\cdot, W_{i+1})$ rather rapidly.

We shall describe the method of representing an arbitrary continuous positively continuous piecewise-linear function with convex cones of linearity. In particular, the support function for a polyhedron is such a function. We shall treat the algorithm for constructing the support function $\rho(\cdot, W_{i+1})$ in terms of this representation.

The support function of any convex polyhedron is a continuous positively homogeneous piecewise-linear function. Specifically, corresponding to each vertex $\bar{y}$ of the polyhedron is a convex cone in which the support function is linear. If the polyhedron is nondegenerate, the cone of linearity is polyhedral and the generators of the cone are the external normals to the faces of the polyhedron that contain the vertex $\bar{y}$. In what follows, by normals we shall mean unit normal vectors.

Let ξ denote a continuous positively homogeneous (though not necessarily convex) piecewise-linear function in R^3 with convex cones of linearity. On the unit sphere S, we construct a net $G(\xi)$ defined by the intersection of the sphere with the cones of linearity of the function ξ. For example, if I is a segment in R^3 and $\xi = \rho(\cdot, I)$, then $G(\xi)$ is a circle on S lying in the plane passing through the origin and orthogonal to the segment I. If I is a nondegenerate convex polyhedron, then, corresponding to each cone of linearity of the function $\xi = \rho(\cdot, I)$ in the net $G(\xi)$, there is a polygonal sector on the sphere.

We denote by l^k the nodes of the net $G(\xi)$ (when the function ξ has polyhedral cones of linearity). We shall refer to an arc of the net that connects adjacent nodes as a connection: The function ξ with polyhedral cones of linearity is completely determined by its net $G(\xi)$ and the values of $\xi(l^k)$ at the nodes. In a computer, we can store the net in the form of the set $\{l^k\}$ and the sets J^k which assign to each node numbered k the numbers of all the nodes adjacent to it. The convex hull conv ξ of the function ξ is the support function of the polyhedron obtained by intersection of the half-space $l^{k\prime} y \leqslant \xi(l^k)$ in sorting all the nodes l^k of the net $G(\xi)$.

We shall assume that the reverse procedure of constructing the sections of the solvability set is terminated not only when the polyhedron W_{e+1} becomes empty (that is, conv η_e does not exist) at the $(e+1)$-st step but also when the polyhedron W_{e+1} is degenerate. This is natural since, in the case of degeneracy, the process of reverse construction is unstable. Thus, in formation of the polyhedron W_{i+1}, we assume the polyhedron W_i to be nondegenerate. Then, the polyhedron F_i will also be nondegenerate.

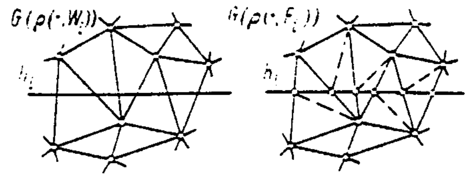

Fig. 1

Let us agree that all cones of linearity of the function $\rho(\cdot, W_i)$ are trihedral (and hence, the sectors of the net $G(\rho(\cdot, W_i))$ are triangular). For this, we need to partition the sectors of the original net $G(\rho(\cdot, W_0))=G(\rho(\cdot, M))$ into triangular sectors and then introduce an additional partition when forming new nets if necessary. In speaking of such a partition, we assume that it is done without increasing the number of nodes in the net. Only connections are added.

The net $G(\rho(\cdot, F_i))$ is the result of superposing the nets $G(\rho(\cdot, W_i))$ and $G(\rho(\cdot, P_i))$. Since P_i is a segment, $G(\rho(\cdot, P_i))$ is a circle h_i on the sphere S (the intersection of the sphere with the plane through the origin orthogonal to P_i). The net $G(\rho(\cdot, F_i))$ contains all the nodes of the net $G(\rho(\cdot, W_i))$. The new nodes are the intersections of h_i with $G(\rho(\cdot, W_i))$ (Fig. 1). Beginning with an arbitrary connection of the net $G(\rho(\cdot, W_i))$ that intersects the circle h_i, let us move along the circle, successively fixing new nodes and new connections. For all sectors of the net $G(\rho(\cdot, F_i))$ to be triangular, let us establish additional connections (indicated in Fig. 1b by dashes). The net $G(\rho(\cdot, F_i))$ is completely formed when we move all the way around the circle h_i and return to the connection from which we started.

Suppose that p_i^* and p_i^0 are the vertices of the segment P_i. The values of the support function of the polyhedron F_i at the nodes of the net $G(\rho(\cdot, F_i))$ are calculated according to the following rule: If l is a node of the net $G(\rho(\cdot, W_i))$, then

$$\rho(l, F_i)=\rho(l, W_i)+\Delta \max\{-l'p_i^*, -l'p_i^0\}.$$

If the node l is new, it is the intersection of the circle h_i with some connection l_1l_2 of the net $G(\rho(\cdot, W_i))$. Therefore, $l'p_i^*=l'p_i^0$ and $l=\alpha l_1+\beta l_2$ for certain $\alpha, \beta > 0$. We have

$$\rho(l, F_i)=\alpha\rho(l_1, W_i)+\beta\rho(l_2, W_i)-\Delta l'p_i^*.$$

We denote by φ_i a continuous positively homogeneous piecewise-linear function such that $G(\varphi_i)=G(\rho(\cdot, F_i))$ and $\varphi_i(l^k)=\eta_i(l^k)$ at each node l^k. We have $\operatorname{conv}\varphi_i=\operatorname{conv}\eta_i=\rho(\cdot, W_{i+1})$. It is convenient to start the iteration process of constructing $\rho(\cdot, W_{i+1})$ not with the function η_i but with the function φ_i since it is simpler to construct: its net coincides with $G(\rho(\cdot, F_i))$.

4. Convexification Algorithms

The algorithm is based on testing for local convexity. To give the appropriate definition, we consider the function ξ with the properties listed above. All the cones of linearity of the function ξ are assumed to be trihedral. Suppose that $l_1l_2l_3$ and $l_1l_2l_4$ are two sectors of the net $G(\xi)$ with common connection l_1l_2. We shall say that a function ξ is locally convex on the connection l_1l_2 if the splice of sections of it on cone$\{l_1, l_2, l_3\}$ and cone$\{l_1, l_2, l_4\}$ is convex for any small segment that intersects cone$\{l_1, l_2\}$. Here, "cone" means the convex conical envelope. An equivalent definition, appropriate for calculations, is the following: Suppose that x_0 is a solution of the system $l_s'y=\xi(l_s)$, $s=1, 2, 3$, and that x^0 is a solution of the system $l_s'y=\xi(l_s)$, $s=1, 2, 4$. Then, each of the inequalities

$$l_4'x_0\leqslant\xi(l_4),\ l_3'x^0\leqslant\xi(l_3)$$

is equivalent to local convexity of the function ξ on the connection $l_1 l_2$. Below, we shall use the first inequality as the criterion of local convexity. We mention that, if the function ξ is locally convex on any connection of its net, it is globally convex in R^3.

We give an iterative algorithm for constructing the convex hull of the function φ_i. We denote by Z_i the plane passing through the origin that is orthogonal to the segment Q_i. Suppose that Z_i^+ and Z_i^- are the half-spaces defined by Z_i. We assume that one of these is closed, the other open. As was mentioned, the function η_i is convex in each of the half-spaces Z_i^+, Z_i^-. Therefore, remembering the relationship between the functions η_i and φ_i, we conclude that local convexity of the function φ_i can be violated only on connections intersecting Z_i or on connections for which both nodes belong to the same half-space with respect to Z_i and are connected with a node in the other half-space. We put all the connections of the net $G(\varphi_i)$ of the type indicated in a special set Π.

Let us set $\varphi^{(1)}=\varphi_i$ and let us make a local analysis of the connections in Π. In the case of local convexity, a connection in the set Π is derived and the following connection is tested. If this discloses nonconvexity, we shift to the function $\varphi^{(2)}$, then $\varphi^{(3)}$, etc.

The shift from the function $\varphi^{(j)}$ to the function $\varphi^{(j+1)}$ consists in a local change of the net $G(\varphi^{(j)})$. Here, the set Π is also corrected. As a result, the set Π will contain all the connections of the net $G(\varphi^{(j+1)})$ on which local convexity of the function $\varphi^{(j+1)}$ may be absent.

Let us describe the shift from the function $\varphi^{(j)}$ to the function $\varphi^{(j+1)}$ and the correction of the set Π when local nonconvexity is disclosed. Suppose that the function $\varphi^{(j)}$ is nonconvex on the connection $q_* = l_1 l_2$ in Π, that is, that $l_4'x>\varphi^{(j)}(l_4)$, where x is a solution of the system $l_s'y=\varphi^{(j)}(l_s)$, $s=1, 2, 3$, and l_3 and l_4 are nodes, distinct from l_1 and l_2, of the triangular sectors of the net $G(\varphi^{(j)})$ that are adjacent along the connection $l_1 l_2$. We express the vector l_4 in terms l_1, l_2, l_3:

$$l_4=\lambda_1 l_1+\lambda_2 l_2+\lambda_3 l_3.$$

Since the sectors $l_1 l_2 l_3$ and $l_1 l_2 l_4$ are adjacent, it follows that $\lambda_3 < 0$. Further actions depend on the signs of the coefficients λ_1 and λ_2.

1) If $\lambda_1 > 0$ and $\lambda_2 \leq 0$ (resp. ($\lambda_1 \leq 0$, and $\lambda_2 > 0$), we remove the node l_1 (resp. l_2) from the net $G(\varphi^{(j)})$. We remove from the net and also from the set Π all the connections containing l_1 (resp. l_2). On the sphere S, we obtain a sector K_* bounded by the connections of the net $G(\varphi^{(j)})$, both nodes of which had been connected with l_1 (resp. l_2) by means of the connections removed. We partition K_* with new connections into triangular sectors without adding new nodes in so doing. Suppose that G^* is the new net. There is one node fewer in it than in $G(\varphi^{(j)})$. We define $\varphi^{(j+1)}$ as the function for which $G(\varphi^{(j+1)})=G^*$ and $\varphi^{(j+1)}(l^k)=\varphi^{(j)}(l^k)$ on each node of the net G^*. After filling the sector K_* with new connections, we enter them in the set Π. We also put in the set Π those boundary connections of the sector K_* that are not in it.

2) In the case $\lambda_1 > 0$, $\lambda_2 > 0$, we replace the connections $q_* = l_1 l_2$ with a new connection $q^* = l_3 l_4$. We obtain the net G^* which differs from $G(\varphi^{(j)})$ only in this replacement. We define the function $\varphi^{(j+1)}$ (as in case 1). We derive the connection $l_1 l_2$ from the set Π and put in the set Π those of the connections that were not in it before.

3) If $\lambda_1 \leq 0$ and $\lambda_2 \leq 0$, the intersection of the half-spaces $l_s'y\leq\varphi^{(j)}(l_s)$, $s = \overline{1, 4}$, is empty. Consequently, $W_{i+1}=\varnothing$. The constructions are terminated.

Thus, in cases 1) and 2), the shift to a new function is simply a change of net. We have $\operatorname{conv}\varphi^{(j+1)}=\operatorname{conv}\varphi^{(j)}$.

If case 3) is not realized in the iteration process, the process stops at some number r when the set Π is exhausted. Here, the function $\varphi^{(r)}$ is locally convex on all connections of its net. It coincides with $\operatorname{conv}\varphi_i$.

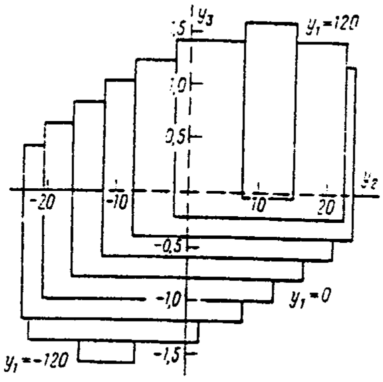

Fig. 2

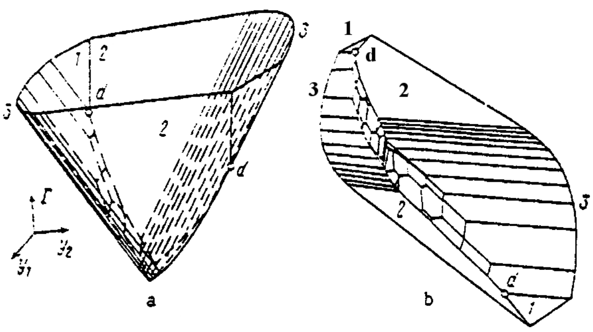

Fig. 3

With numerical implementation of procedures associated with testing the inequality for local convexity and with analysis of the signs of the coefficients λ_1 and λ_2, some small additions are made. These additions ensure termination of the calculations when the set W_{i+1} is degenerate or nearly so.

Another algorithm for convexification of the function φ_i is described in [15]. It is based on the complex logic of encirclement of connections on which there can be no local convexity.

Having made the function φ_i convex, we obtain the support function $\rho(\cdot, W_{i+1})$ of the polyhedron W_{i+1}. Information about the support function is stored in the form of the net $G(\rho(\cdot, W_{i+1}))$ and the values $\rho(l^k, W_{i+1})$ at the nodes of the net. Each node l^k is the outer normal vector to one of the faces of the polyhedron W_{i+1}. Since a polyhedron is completely defined by the set of its normals and the values of the support function for them, information about the net $G(\rho(\cdot, W_{i+1}))$ is superfluous. However, apart from the fact that this information is necessary when using the algorithm for constructing the polyhedron W_{i+2} at the next step of the reverse procedure, its presence makes it possible rapidly to obtain a graphical representation of the polyhedron W_{i+1}.

5. Examples

We consider the problem, in linearized form, of controlling the lateral motion of an airplane that is landing under conditions of a wind perturbation [5, 16]. The dynamics of the object are described by a vector differential equation $\dot{x}=Ax+Bu+Cv$, $x\in R^7$, $u\in P$, $v\in Q$, where

$$A=\begin{Vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0762 & -5.34 & 0 & 9.81 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -0.0056 & -0.392 & -0.0889 & -0.0378 & -0.17 & 0.0378 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -0.0129 & -0.9016 & -0.2045 & -0.0869 & -0.89 & 0.0869 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{Vmatrix}$$

$$B=(0,0,0,0,0,0,1)',\ C=(0,0.0762,0,0.0056,0,0.0129,0)'.$$

$$P=\{u\in R: |u|\leqslant 0.14\},\ Q=\{v\in R: |v|\leqslant 10\}.$$

The coordinate x_1 denotes the lateral deviation of the center of mass of the airplane from the center line of the landing strip, x_2 is the velocity of lateral deviation, x_3 is the angle of yaw, x_4 is the rate of change of the angle of yaw, x_5 is the angle of bank, and x_6 and x_7

are auxiliary variables. The lateral deviation is measured in meters, the angles in radians, and time in seconds. The controlling parameter u is the "given" angle of bank (rad), the parameter v is the lateral component of the wind velocity (m/sec). Let M^* denote the paralellepiped in $x_1x_2x_3$-space with vertices at (20, 0, 0), (20, -10, 0), (20, -10, 0.26), (20, 0, 0.26), (-20, 0, 0), (-20, 10, 0), (-20, 10, -0.26), (-20, 0, -0.26). We need, by choosing a control u, to get the first three coordinates of the vector x into the set M^* at the fixed instant ϑ = 15 sec. We regard the instant ϑ as the instant at which the airplane flies over the end of the landing strip. In essence, getting the three coordinates into the set M^* means getting the controlled system into "admissibility." We set the initial instant of time equal to zero.

Suppose that y_1, y_2, and y_3 are the coordinates of an equivalent differential game of the form (1.2). Figure 2 shows the results of constructing the set $W^*(0)$ on a computer. The sections of the set $W^*(0)$ by planes orthogonal to the y_1-axis with step 30 on that axis are shown. In the calculations, the step Δ for the reverse procedure was taken equal to 0.1.

It follows from the meaning of the set $W^*(0)$ that if the point $y(0) = X^*(15, 0) \times (0)$ belongs to $W^*(0)$, we can, by choosing u properly, get the system in M^* at the instant $\vartheta = 15$. If $y(0) \notin W^*(0)$, we do not have this guarantee.

As a second example, let us look at a model game with fixed instant of termination ϑ and terminal payment function γ [17]:

$$\dot{x}_1 = x_2 + v,\ \dot{x}_2 = u,\ |u| \leqslant 1,\ |v| \leqslant 1,\ \gamma(x) = \max\{|x_1|, |x_2|\}.$$

An equivalent game without phase variable in the right-hand member has the form

$$\begin{gathered}\dot{y}_1 = (\vartheta - t)u + v,\\ \dot{y}_2 = u,\ |u| \leqslant 1,\ |v| \leqslant 1,\\ \gamma^*(y) = \max\{|y_1|,\ |y_2|\}.\end{gathered} \tag{5.1}$$

Let us add to equations (5.1) the equation $\dot{c} = 0$. We obtain the system

$$\begin{gathered}\dot{y}_1 = (\vartheta - t)u + v,\\ \dot{y}_2 = u,\\ \dot{c} = 0,\ |u| \leqslant 1,\ |v| \leqslant 1.\end{gathered} \tag{5.2}$$

Let us fix the number $\bar{c} > 0$. Suppose that $M^* = \{(y_1, y_2, c)': \gamma^*(y) \leqslant c \leqslant \bar{c}\}$. The set M^* is that part of the subgraph of the function γ^* lying below the level $\bar{c}$. Taking the instant $t \leqslant \vartheta$ and constructing for it the section of the solvability set of the three-dimensional game (5.2) with target set M^*, we obtain a cut of the subgraph of the cost function in the game (5.1) at level $\bar{c}$ [12].

Figure 3 shows the results of computer computation for $t = \vartheta - 2$, $\bar{c} = 2.5$, $\Delta = 0.1$. By using a central projection, we obtain the side view (Fig. 3a) in the graph of the cost function Γ^*, and the view from below (Fig. 3b). The numbers 1 and 2 indicate plane sections of the surface of the cost function; 3 indicates ruled sections. In the limit, as the accuracy of the constructions is increased, the line polyhedral sections become smooth. In the region of the coordinates y_1 and y_2 corresponding to the plane and linear sections, the cost $\Gamma^*(t, y)$ coincides with the maximin. In the remaining portion of the variable space y_1 and y_2, the cost is different from the maximin. The point d corresponds to the value $\Gamma^* = 2$. The smallest value of the numerically constructed cost function Γ^* is equal to 0.6415 and is attained at the point $y = 0$. The theoretically calculated value [17] of the cost at the point $y = 0$ is 0.6518. In Fig. 3b, the center of symmetry corresponds to the point $y = 0$.

REFERENCES

1. Isaacs, R. P. Differential Games: A Mathematical Theory with Applications to Welfare and Pursuit, Control and Optimization. Wiley, New York, 1965 (Russian translation, Mir, Moscow, 1967).
2. Krasovskiy, N. N. and A. I. Subbotin. Positsionnyye differentsial'nyye igry (Positional Differential Games). Nauka, Moscow, 1974.
3. Pontryagin, L. S. Linear differential games of pursuit. Matematicheskiy sbornik, Vol. 112 (154), No. 3(7), 1980.

4. Subbotin, A. I. and A. G. Chentsov. Optimizatsiya garantii v zadachakh upravleniya (Optimization of a Guarantee in Control Problems). Nauka, Moscow, 1981.
5. Keyn, V. M. Optimizatsiya sistem upravleniya po minimaksnomy kriteriyu (Optimization of Control Systems According to a Minimax Criterion). Nauka, Moscow, 1985.
6. Ushakov, V. N. The problem of constructing stable bridges in a differential approach-avoidance game. Izv. AN SSSR. Tekhnicheskaya Kibernetika, 18, No. 4, 1980 (Engineering Cybernetics, 18, No. 4, 1980).
7. Ostapenko, V. V. Methods of solving a certain class of approach-avoidance problems. Avtomatika i telemekhanika, No. 6, 1984.
8. Ponomarev, A. P. Estimation of the error in the numerical method of constructing the alternating Pontryagin integral. Vestnik Moskovskogo gosudarstvennogo universiteta, ser. 15, Vychislitel'naya Matematika i Kibernetika, No. 4, 1978.
9. Nikol'skiy, M. S. Pontryagin's alternating integral. Matematicheskiy sbornik, Vol. 116(158), No. 1(9), 1981.
10. Botkin, N. D. Estimation of error in numerical constructions in a differential game with a fixed instant of termination. Problemy upravleniya i teorii informatsii, 11, No. 4, 1982.
11. Algoritmy i programmy resheniya lineynykh differentsial'nykh igr (Algorithms and Programs for Solving Linear Differential Games). AN SSSR, UNTs, IMM, 1984 (Information on computer software).
12. Subbotin, A. I. and A. M. Taras'yev. Conjugate derivatives of the cost function of a differential game. Doklady AN SSSR, Vol. 283, No. 3, 1985.
13. Pshenichnyy, B. N. and M. I. Sagaydak. Differential games with fixed time. Kibernetika, No. 2, 1970.
14. Maller, D. and F. Preparata. Finding the intersection of two convex polyhedrons. In: Kiberneticheskiy sbornik. Novaya seriya (Cybernetic Handbook. New Series), No. 20. Mir, Moscow, 1983.
15. Zarkh, M. A. and V. S. Patsko. Numerical solution of three-dimensional linear differential games with fixed instant of termination. Manuscript deposited in VINITI Archives under No. 7509-B-85, Moscow, 1985.
16. Botkin, N. D., V. M. Keyn, and V. S. Patsko. A model problem on the control of the lateral motion of an airplane on landing. Prikladnaya matematika i mekhanika, Vol. 48, No. 4, 1984.
17. Taras'yev, A. M. A certain nonregular differential game. Prikladnaya matematika i mekhanika, Vol. 49, No. 4, 1985.

□◆◆◆□