Model Problems of Aircraft Guidance in Education Program on Optimal Control and Differential Games

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The 9th IFAC Symposium
Advances in Control Education

Nizhny Novgorod, Russia, June 19 - 21, 2012

The authors of this talk work at the Institute of Mathematics and Mechanics in Ekaterinburg, Russia. The basic subject of their scientific investigations is differential game theory. Sergey Kumkov has a one-semester course on optimal control at the Physics Department of Urals University. Valerii Patsko reads a one-semester course on optimal control and differential games at the Mathematical Department.

Structure of the Course on Optimal Control and Differential Games

(14 lectures and home work of students)

1. Reachable sets of control system. Pontryagin's maximum principle for controls guiding the system to the boundary of a reachable set (linear and non-linear systems).

2. Open-loop control problems for different optimality criteria. Pontryagin's maximum principle.

3. Positional formalization by N.N.Krasovskii for differential games.

4. Maximal stable bridges and extremal strategies.

5. Model practical problems.

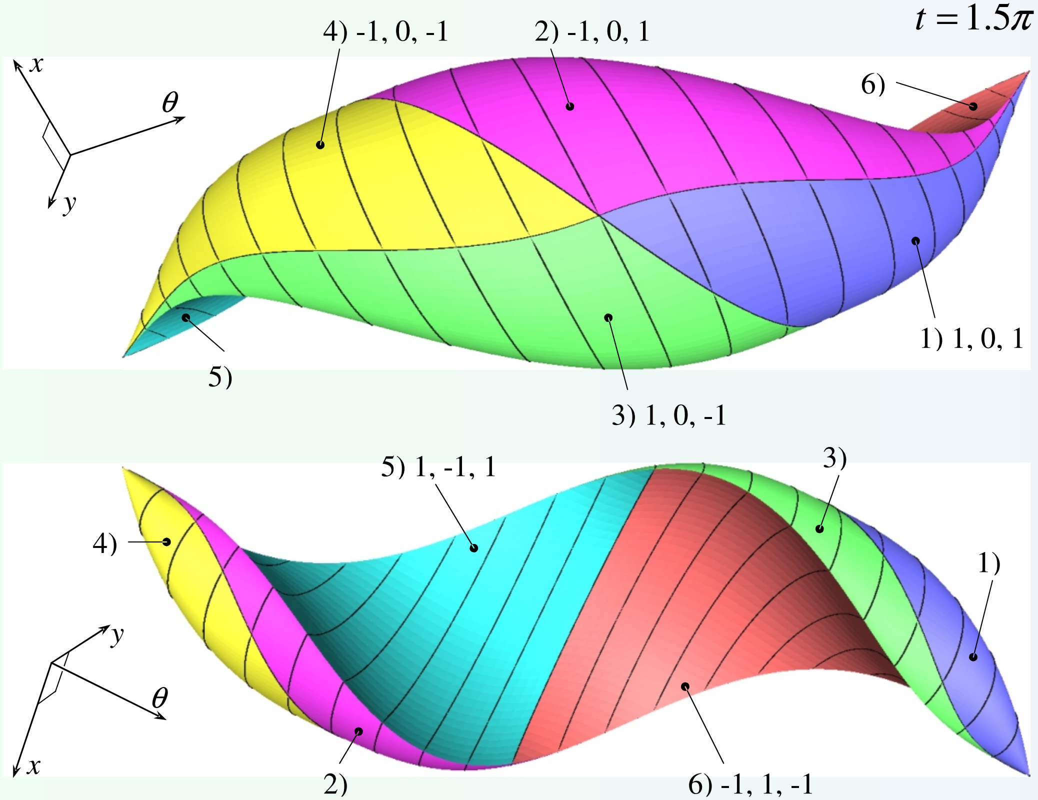
These are the main topics of the one-semester course for master students of the Mathematical Department. Concerning the mathematical control theory, the principal concept is the notion of reachable set. In differential game theory, the main object is maximal stable bridge. This is practically the same as solvability set and level set of the value function if we have a differential game with payoff function.

In our talk, we shall show four model airspace problems, which we use teaching students. Each of these problems has been formulated and investigated by the authors as an independent and very interesting control or differential game task.

Three-Dimensional Reachable Set for a Model of Aircraft Motion in Plane

$$\dot{x} = V\sin\theta, \quad \dot{y} = V\cos\theta, \quad \dot{\theta} = \frac{g}{V}\tan\gamma; \quad |\gamma| \le 30^\circ$$

$$\dot{x} = \sin\theta, \quad \dot{y} = \cos\theta, \quad \dot{\theta} = u; \quad |u| \le 1$$



For navigation computations, the model of aircraft motion in the horizontal plane written above is used. Here, x and y are the Cartesian coordinates of the aircraft, θ is the angle of the velocity vector, V is the magniude of the velocity, γ is the bank angle, and g is the gravity acceleration.

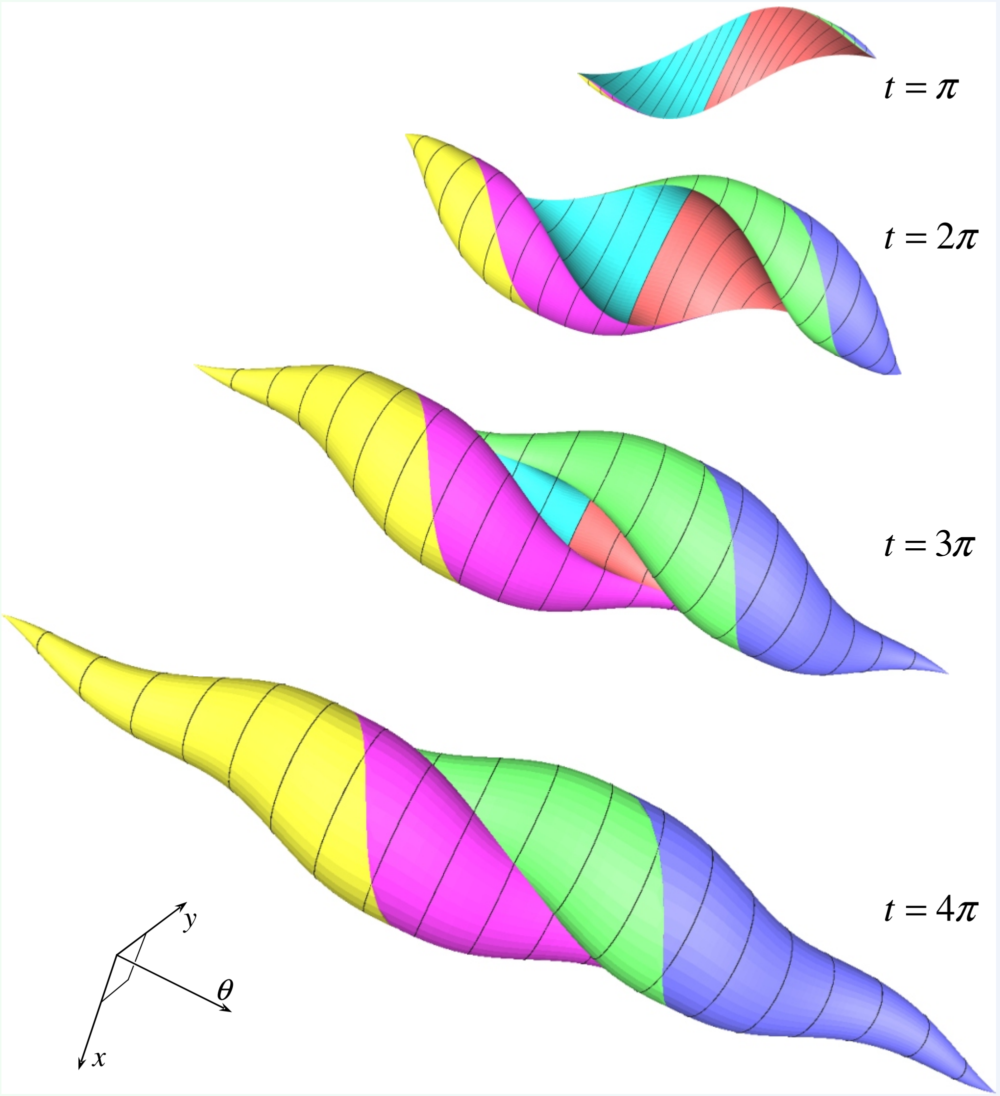
Assume that the value V is constant. Then after a normalization, we pass to the system in the second row. This model is also used in theoretical robotics. It is called the "Dubins' car".

The reachable set at a given instant t is the set of all points in the three-dimensional space, which can be reached by the system from the origin at the given instant t using admissible piecewise-continuous control.

It was established by means of the Pontryagin maximum principle (and students study this proof) that for arbitrary point on the boundary of the reachable set, the control guiding the system to the point has at most two switches. In addition, there are only six variants of changing the control. For example, the magenta part of the boundary is reachable by means of the control of the form -1, 0, +1 with two switches.

On this slide, we see the boundary of the reachable set at the instant $t = 1.5\pi$ from two points of view. Several lines of θ-sections are shown.

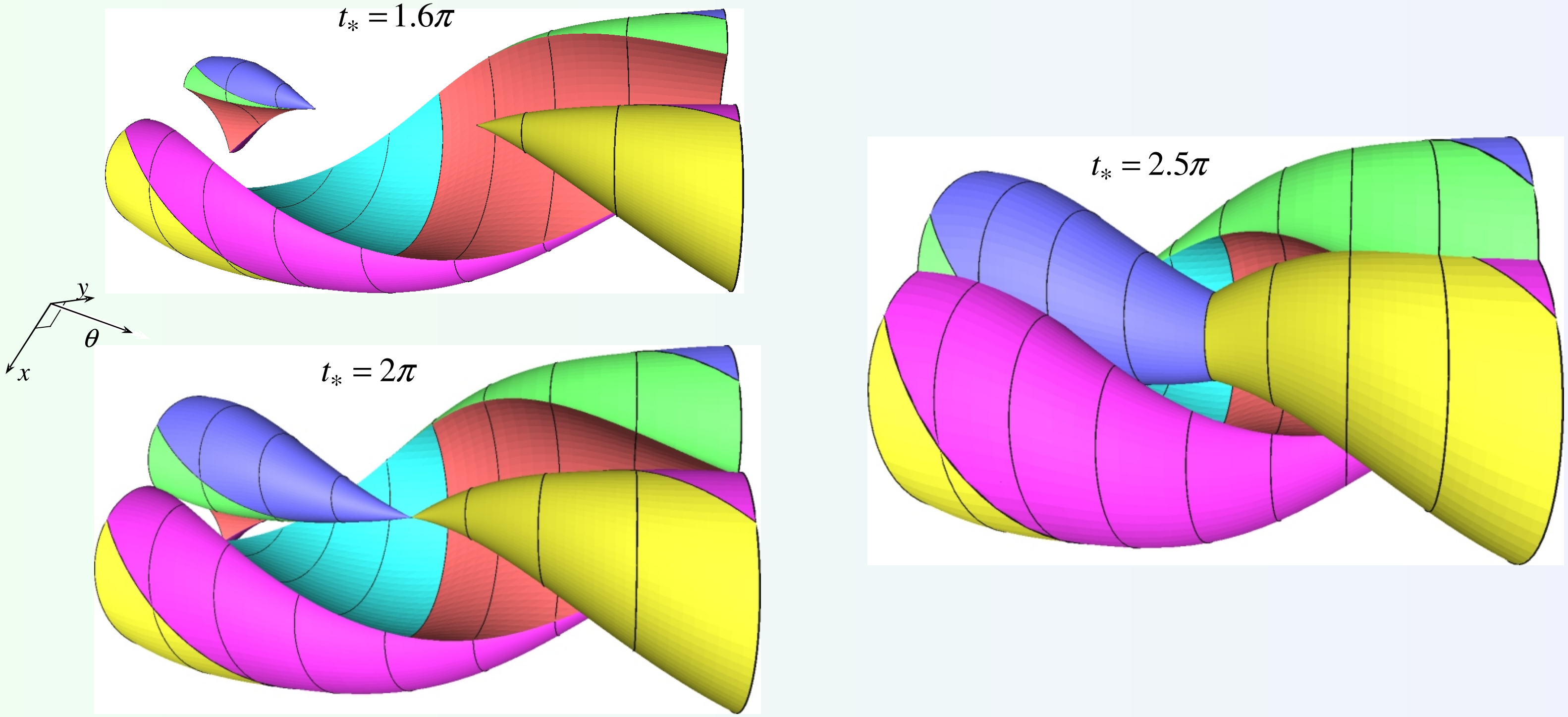
Development of the reachable set $G(t)$



These pictures show the development of the reachable sets with growing t.

Here, we assume that the angle θ can change in the range $(-\infty, +\infty)$. It allows us to recognize the laws of evolution of the reachable sets in time.

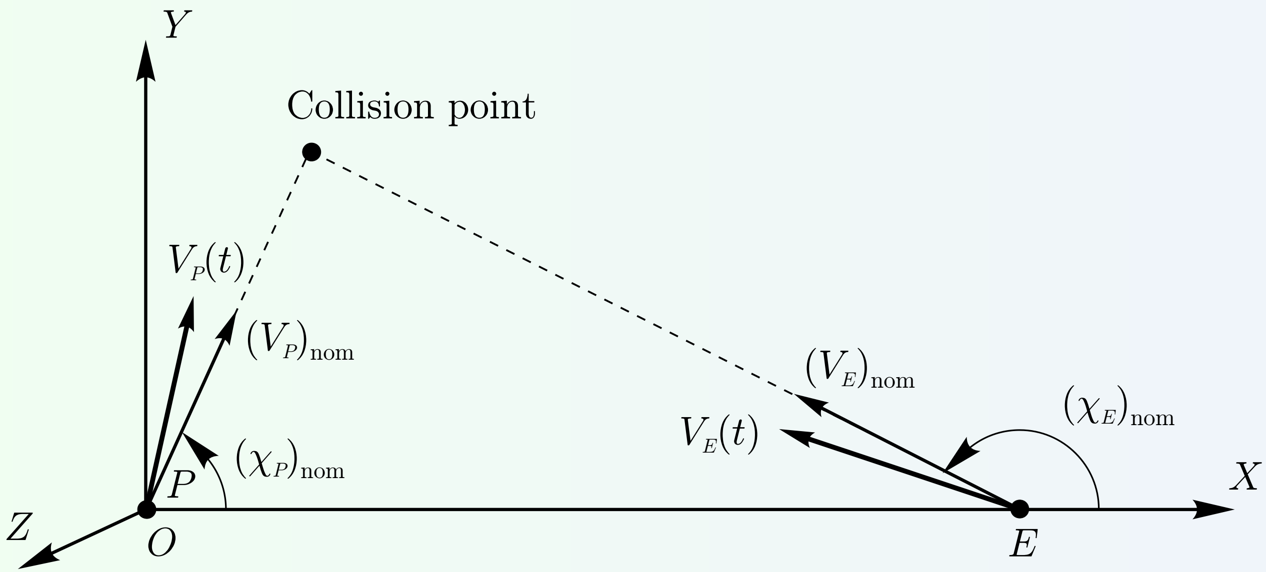
Reachable Sets with θ Computed Modulo 2π



It is possible to pass to the sets, for which the angle θ is calculated modulo 2π. This slide shows such sets for three instants.

When demonstrating such three-dimensional reachable sets to students, we emphasize that the Pontryagin maximum principle is only a necessary condition for the control leading a non-linear system to the boundary of the reachable set.

Linear Interception Problem



$$\ddot{x}_P = a_P, \qquad t \in [0, T], \quad x_P, x_E \in \mathbb{R}^2,$$
$$\dot{a}_P = (u - a_P)/\tau_P, \qquad u \in P, \quad v \in Q,$$
$$\ddot{x}_E = v, \qquad \varphi\big(x_P(T), x_E(T)\big) = \big|x_E(T) - x_P(T)\big|.$$

$y = x_E - x_P, \quad \xi_1(t),\ \xi_2(t) -$ values of components of y forecasted to the instant T (ZEM, *zero effort miss* coordinates)

J.Shinar

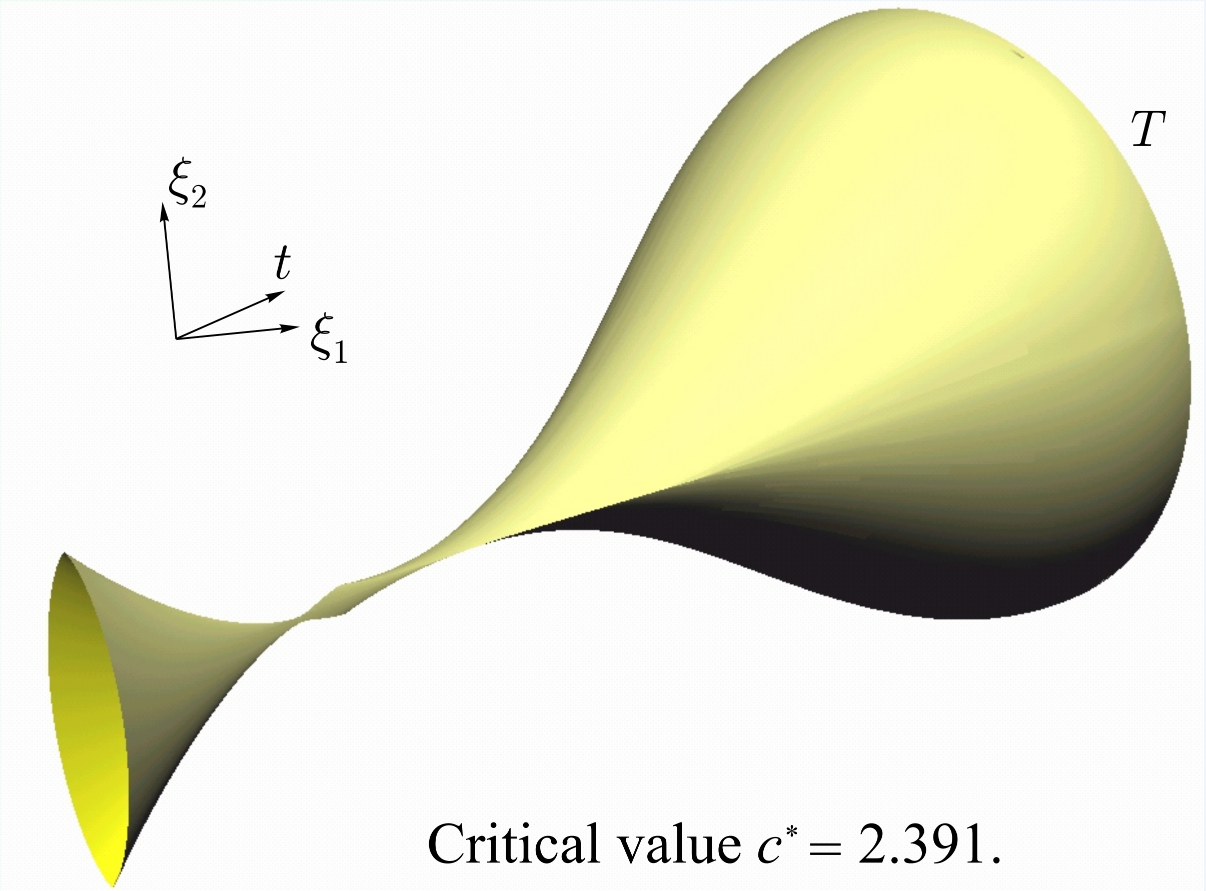
In the next model problem, we have the pursuer P (it is an anti-missile) and the evader E (it is a maneuvering target). Nominal initial points of P and E are connected by the initial nominal line-of-sight. The dashed lines denote nominal rectilinear trajectories. We consider the case when capabilities of the objects to change their velocities during the motion are small. The reference horizontal velocity is very large. Therefore, instead of the real miss (which is the closest distance between objects during the motion), we compute lateral miss in the plane YZ at the instant T of the nominal collision.

Dynamics of the objects after the linearization is shown in the center of this slide. Here, x_P is the two-dimensional position vector of the pursuer, u is his control vector, τ_P is the time constant characterizing the inertia of the pursuer's control action; x_E is the position of the evader, v is his control vector. The constraints for the controls u and v are ellipses. A payoff in the game is the geometric distance between the players at the termination instant. The fist player minimizes the payoff, the second one maximizes it.

By introducing two-dimensional reference vector y, the system can be rewritten, and we have linear differential game of the sixth order where payoff depends on Euclidean norm of the vector y at the instant T.

Using a standard way, which is well-known in the differential game theory, we pass to an equivalent differential game of the second order. The new phase variables are $\xi_1(t)$ and $\xi_2(t)$. They are values of two components of the vector y forecasted to the instant T under zero controls of the players (zero-effort miss coordinates).

General View of Critical Tube



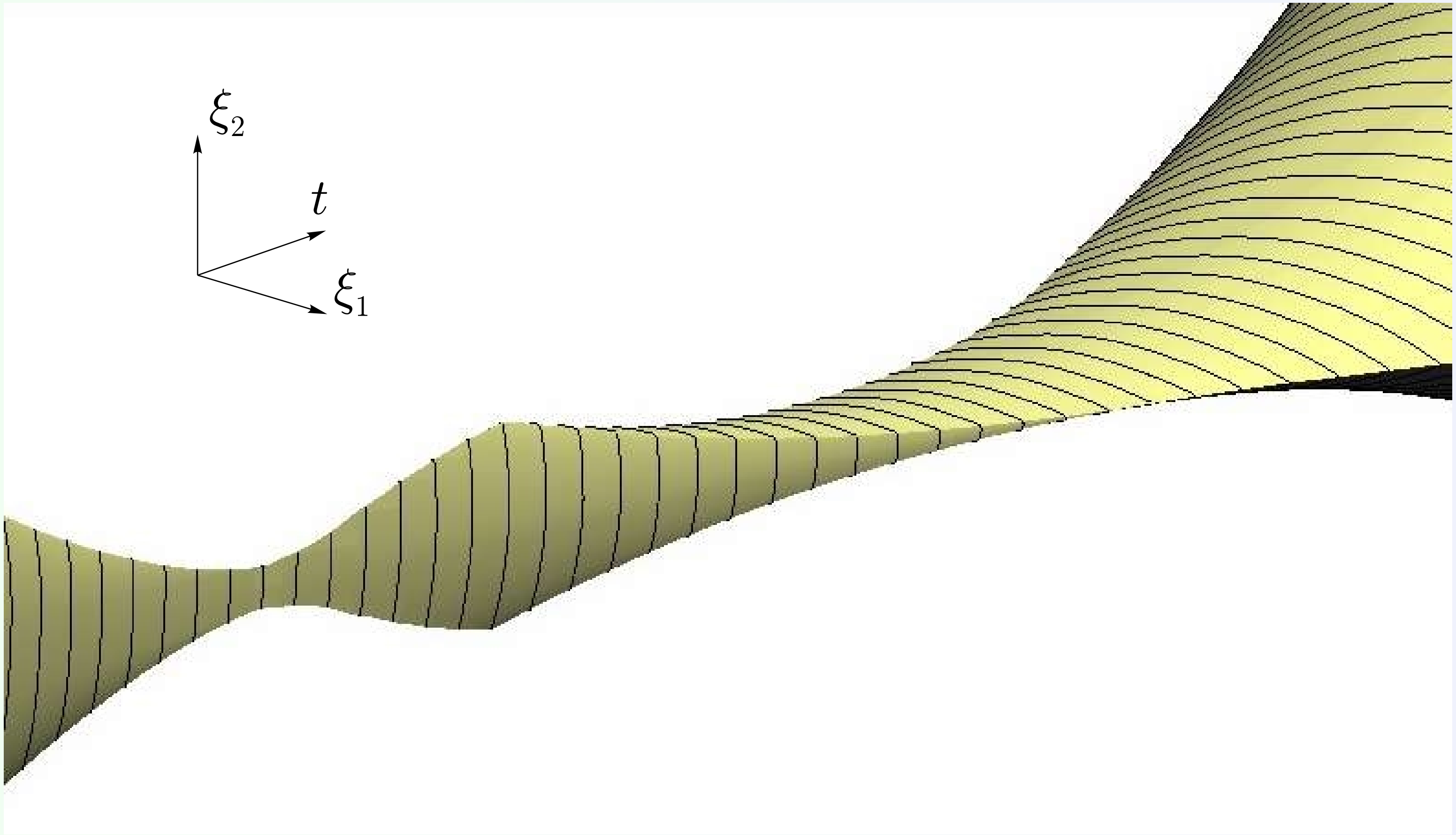
Critical value $c^* = 2.391$.

Backward instant
of the narrow throat $T - t^* = 5.13$.

The main aim to study this material by student is to know how exotic the level sets of the value function can be. On this slide, we see the level set of the value function corresponding to the value of the payoff $c^* = 2.391$. The tube has a narrow throat.

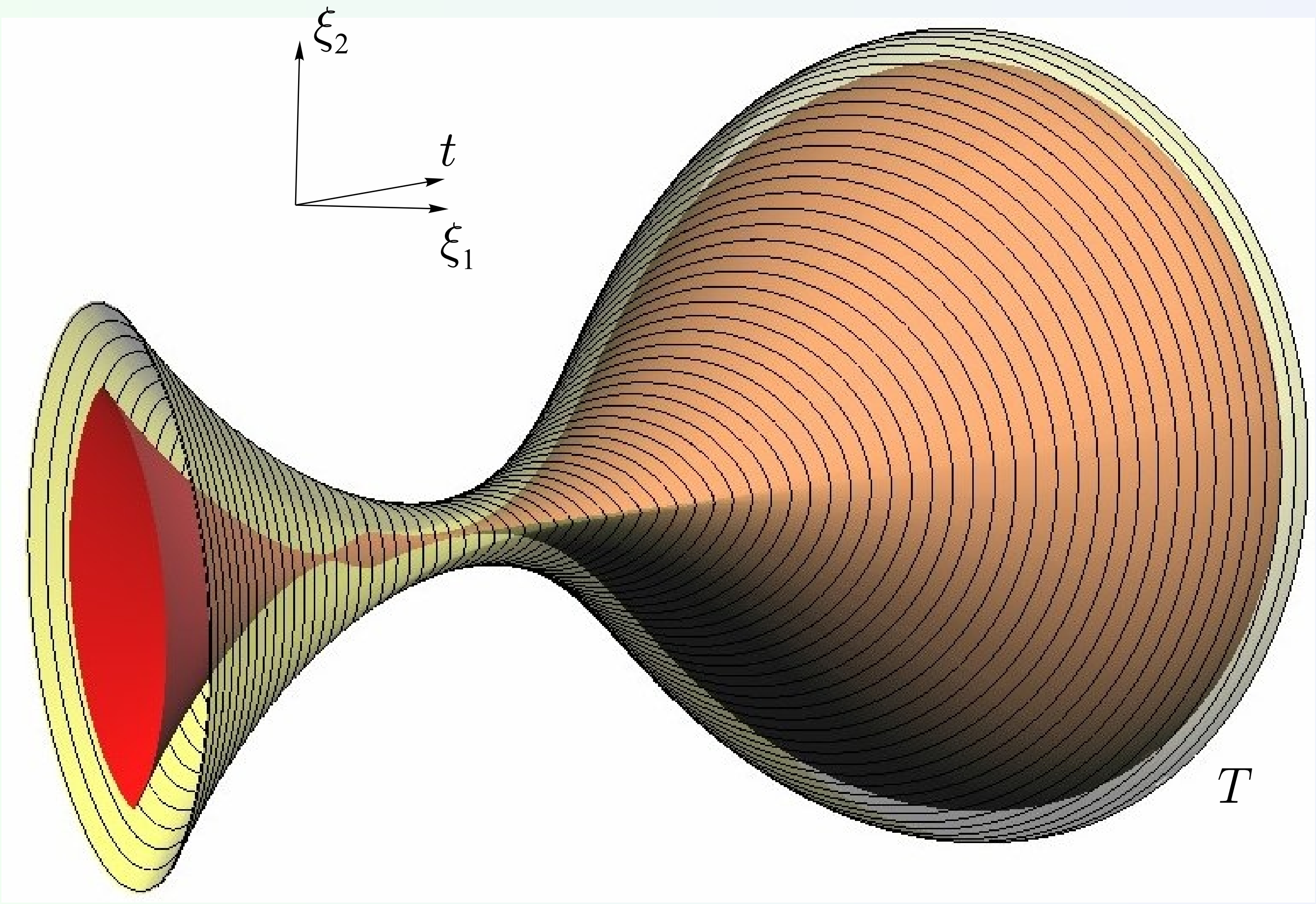
If the initial forecasted coordinates ξ_1, ξ_2 are inside this tube, then the first player guarantees the final miss not greater than c^*. If the initial vector (ξ_1, ξ_2) is outside the tube, then there is no such a guarantee.

Large View of the Narrow Throat



This is a large view of the throat. The time sections of the tube are convex, but their geometry is changing in time non-trivially. This is the result of numeric computations. It is very difficult to give analytical description of the value function in this region.

Level Set with the Narrow Throat and a Larger One



Here, we see one tube more. The value of the payoff for this additional tube (which is drawn semi-transparent) is a bit larger than c^*. But this new tube has smooth boundary and has good analytical description.

Interception Problem with Two Pursuers and One Evader

$$\ddot{x}_{P_1} = a_{P_1}, \qquad \ddot{x}_{P_2} = a_{P_2}, \qquad \ddot{x}_E = a_E,$$

$$\dot{a}_{P_1} = (u_1 - a_{P_1})/\tau_{P_1}, \quad \dot{a}_{P_2} = (u_2 - a_{P_2})/\tau_{P_2}, \quad \dot{a}_E = (v - a_E)/\tau_E,$$

$$|u_1| \le \mu_1, \qquad |u_2| \le \mu_2, \qquad |v| \le \nu,$$

$$a_{P_1}(t_0) = 0, \qquad a_{P_2}(t_0) = 0, \qquad a_E(t_0) = 0,$$

$$r_{P_1,E}(T_1) = x_E(T_1) - x_{P_1}(T_1), \qquad r_{P_2,E}(T_2) = x_E(T_2) - x_{P_2}(T_2),$$

$$\varphi = \min\Big\{ \big|r_{P_1,E}(T_1)\big|,\ \big|r_{P_2,E}(T_2)\big| \Big\}.$$

$$y_1 = x_E - x_{P_1}, \quad y_2 = x_E - x_{P_2}.$$

$\xi_1(t),\ \xi_2(t)$ − values of components of y forecasted to instants T_1 and T_2 (ZEM, *zero effort miss* coordinates)

J.Shinar, S. Le Menec

The third problem is similar to the previous one, but now we have two pursuers. If we would like to keep the second dimension of the geometric miss between each of the pursuers and the evader, then we pass finally to the equivalent differential game of the fourth order on the phase vector. It is very difficult both for analytical and numerical investigation.

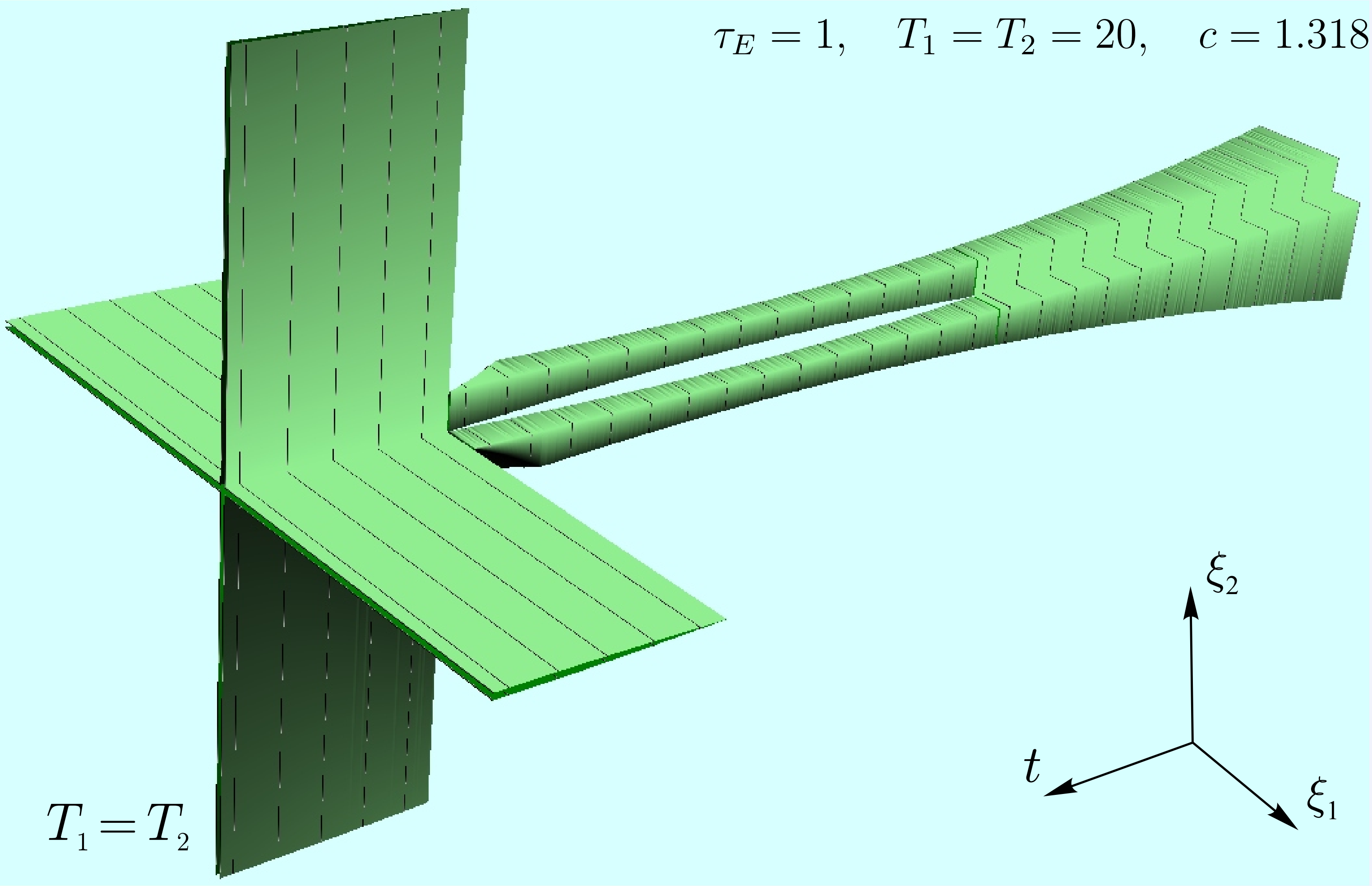
Therefore, we formulate the pursuit problem in such a way that the miss between each of pursuers and the evader is one-dimensional. Thus, we have a linear dynamics for the first and second pursuers. Here, x_{P1} and x_{P2} are scalar variables. The dynamics of the evader is similar.

Let T_1 and T_2 be the instants of the nominal collisions of the evader and the first and second pursuers, respectively. The crucial property of this problem is that the payoff is not convex.

Denote by ξ_1 and ξ_2 the zero-effort miss coordinates.

A Level Set of the Value Function

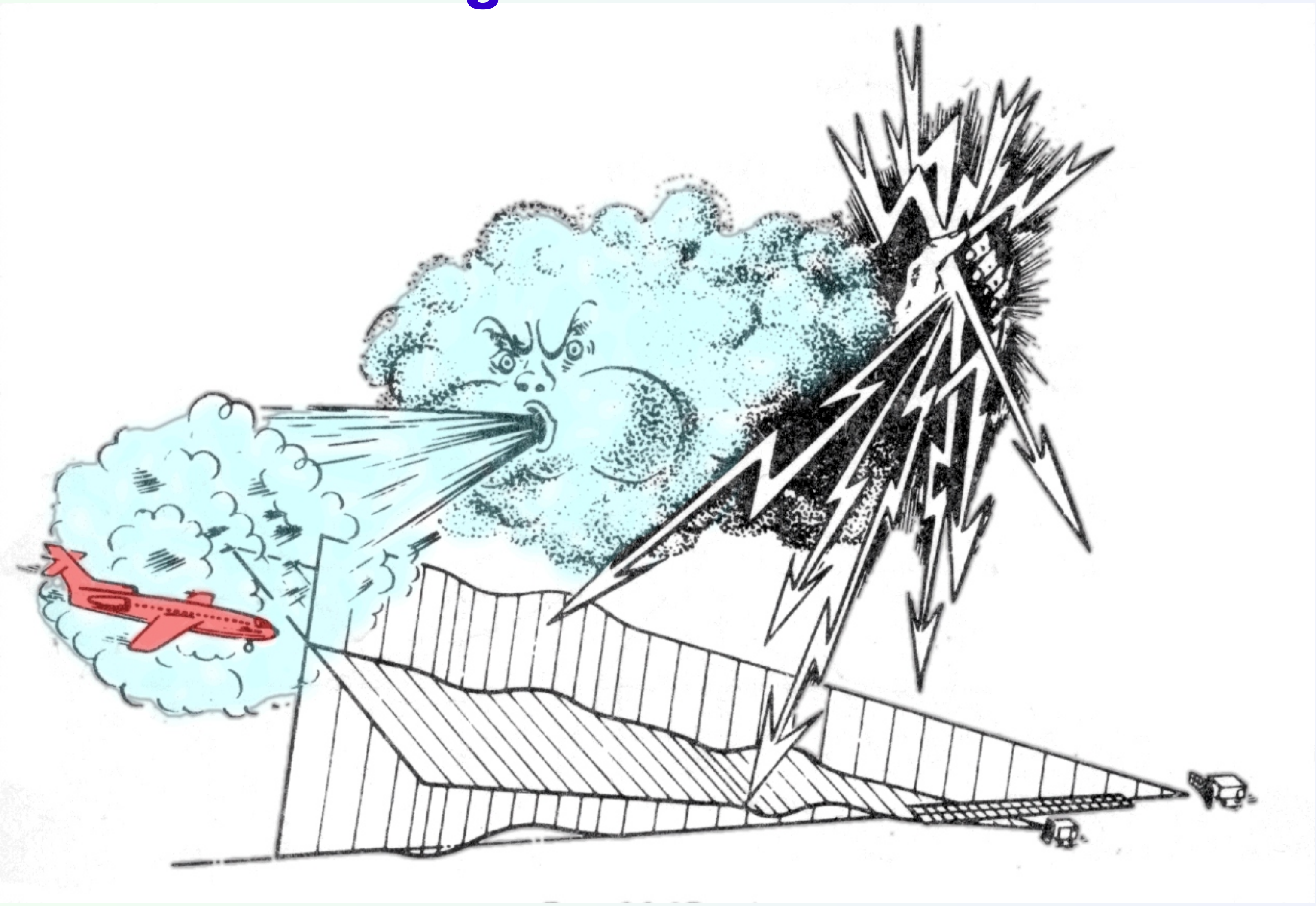
$$\mu_1 = \mu_2 = 1.1, \quad \nu = 1, \quad \tau_{P_1} = \tau_{P_2} = 1/0.6,$$
$$\tau_E = 1, \quad T_1 = T_2 = 20, \quad c = 1.318.$$



The solution of the problem depends essentially on the parameters of the game. Let us have the following values of the parameters μ_1, μ_2, ν, τ_{P1}, τ_{P2}, τ_E, T_1, and T_2, which are shown on the slide. For these parameters, we see here a level set of the value function obtained numerically. With growing of the backward time, the t-sections lose connectedness and disjoin into two parts, which join back further.

In the lectures, we show to the students the level sets of the value function for some variants of the parameters. We emphasize that non-convexity of the level sets of the value function and losing connectedness by them are stipulated by certain type of the payoff, which is specific for the considered problem.

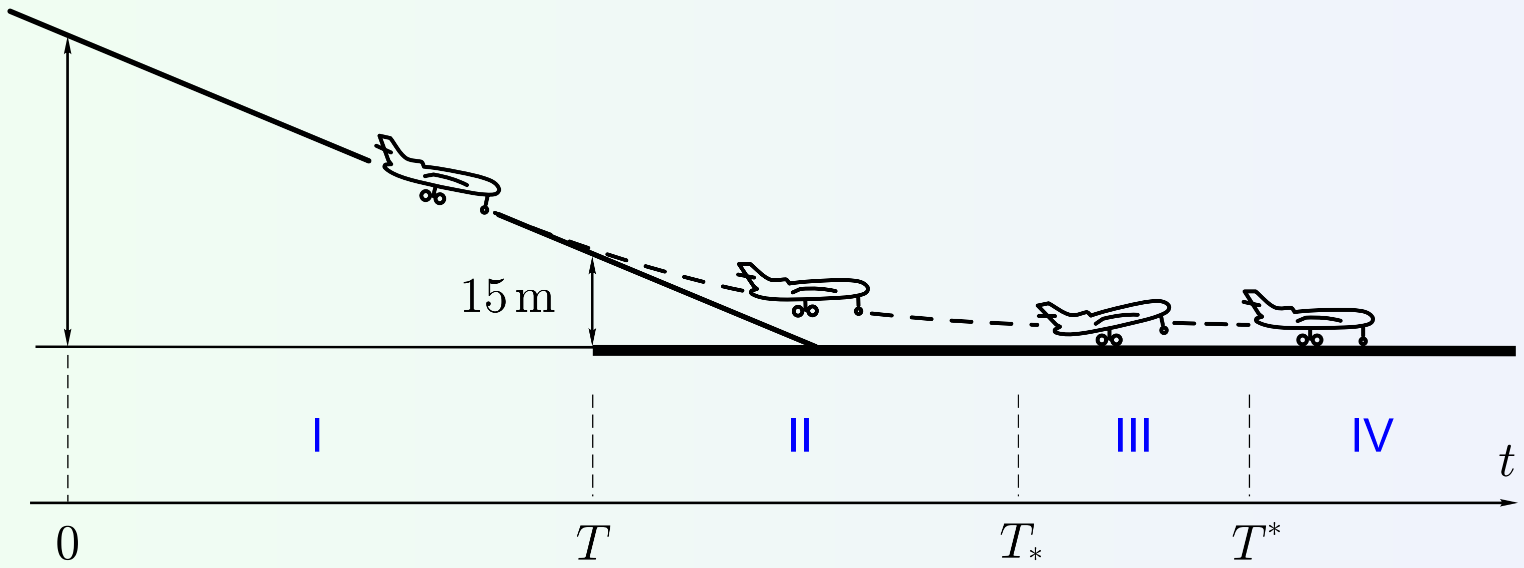
Aircraft Landing under Wind Disturbances



A.Miele, V.M.Kein

The problem of aircraft landing under wind disturbances is the natural example for mathematical control theory and differential games. But there are some difficulties for formalization of such a problem as a differential game. Namely, the aircraft has four main controls: thrust, elevator, rudder, and ailerons. Bounds for their ranges are known. Therefore, we can strictly describe the constraints for the useful control. In contrast, we do not know the real constraints for the wind disturbances.

Aircraft Landing Stages



I. Descent till passing the runway threshold – the problem under investigation

II. Levelling till contact with runway (the stage of flare)

III. Running on main wheels

IV. Running on all wheels

In our investigations, we consider the part of the landing process from the height about 400 meters till passing the runway threshold at the height about 15 meters, that is, the part when the nominal trajectory of the aircraft is a rectilinear glide path.

Landing Problem as Differential Game

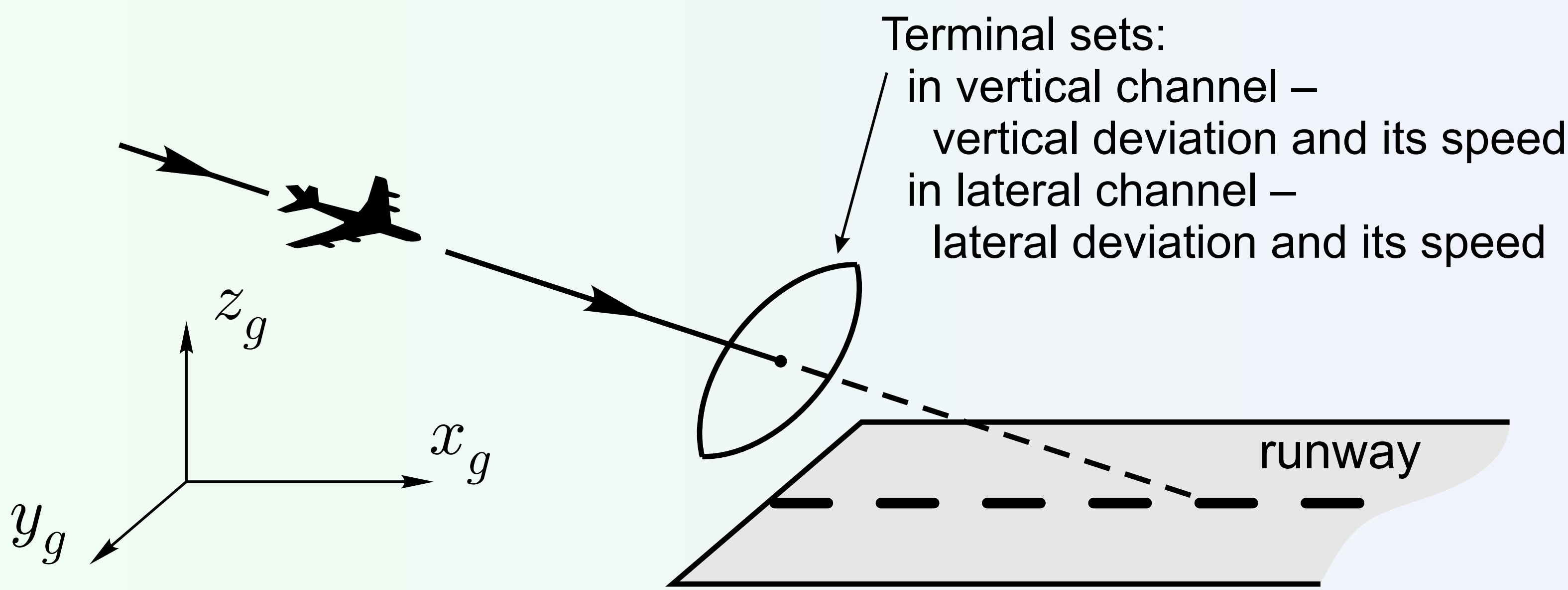
$$\dot{\eta} = f(\eta, u, v),$$

$$t \in [0, T], \quad \eta \in \mathbb{R}^{16},$$

$$u = (u_p, u_e, u_r, u_a),$$

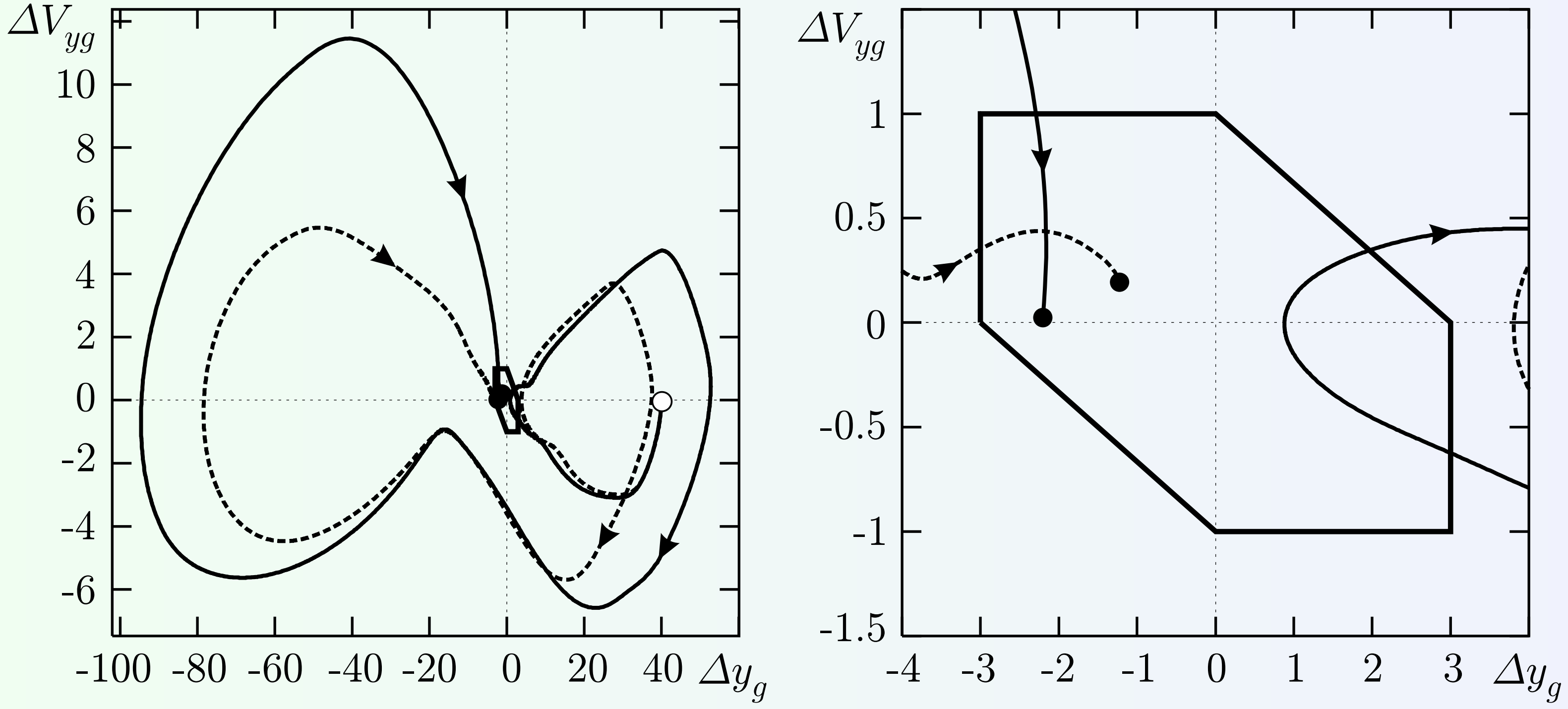
$$v = (v_x, v_y, v_z)$$

After linearization, the original non-linear system actually disjoins into two independent subsystems:
of lateral and vertical motions



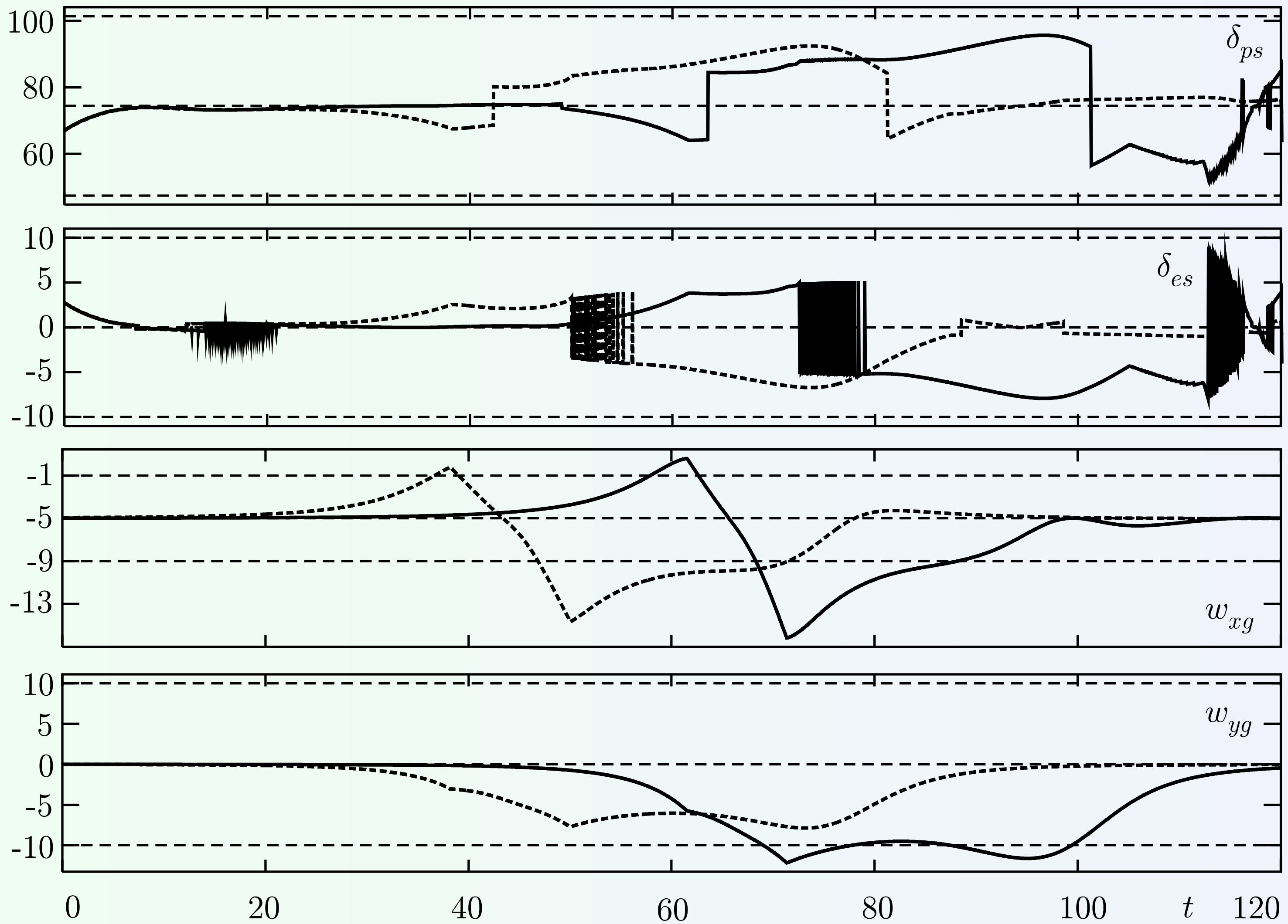
After linearization of the non-linear dynamics with respect to the nominal motion, the aircraft dynamics disjoins into two subsystems, which are almost independent. One of them is for the vertical channel, and another is for the lateral channel. We consider two corresponding auxiliary linear differential games where the payoff is computed at the terminal instant of passing the runway threshold. In the framework of these two games, we obtain a feedback adaptive control and further use it in the original non-linear dynamics. The method for generating the adaptive control does not demand a priori knowledge of the constraints for the wind disturbances.

Trajectories in the Space
Vertical Deviation × Velocity of Vertical Deviation



Here, the results of simulations of the non-linear system for two variants of the wind disturbances are given in the variables of the vertical channel. We see phase trajectories in the plane *vertical deviation from the nominal motion*, *the velocity of the vertical deviation*. At the right, we see a large view of the area near the terminal target set. The polygonal target set is the tolerance for the vertical channel.

Realization of Adaptive Controls and Disturbance



On this slide, we see the realizations of the thrust and elevator controls. These are command control signals. Below, the realizations of longitudinal and lateral wind disturbances are shown. They were created by means of a wind microburst model. We used two variants of the location of the microburst zone.

Studying the aircraft landing problem, students learn the description of the non-linear aircraft dynamics, linearized systems of the vertical and lateral motions, adaptive control method, and different model of the wind disturbances.

References

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On the slide of references, we give the book by N.N.Krasovskii and A.I.Subbotin and papers (by the authors) concerning the four problems of the presentation.

The last reference is devoted to the adaptive control method which have been elaborated by the authors on the basis of differential game theory.