Analytical and numerical study of the dolichobrachistochrone problem

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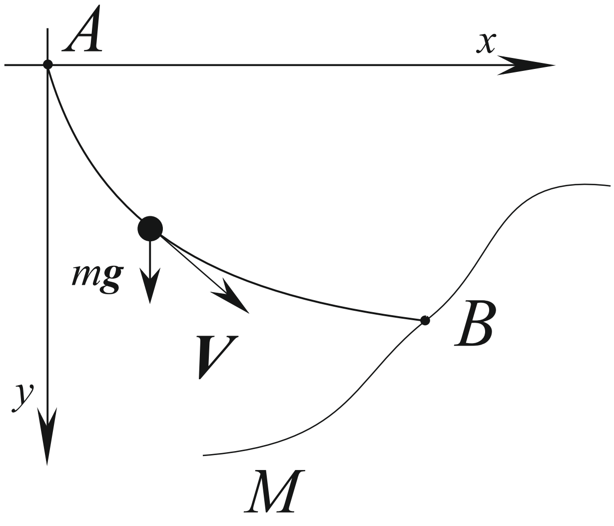
The presentation is devoted to analytical and numerical study of a time-optimal game problem in the plane. This problem is a game extension of the model brachistochrone problem.

The classical brachistochrone problem



Velocity of the mass point:

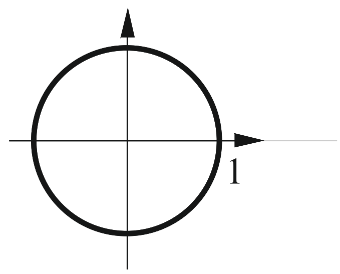
$$V(x,y)=\sqrt{2gy}$$

Controlled system:

$$\dot{x}=\sqrt{2gy}\,u_1$$
$$\dot{y}=\sqrt{2gy}\,u_2$$

Admissible controls:

$$u=(u_1,u_2)\in P=$$



Space of states:

$$N=\{(x,y)\in \mathrm{R}^2 : y\geq 0\}$$

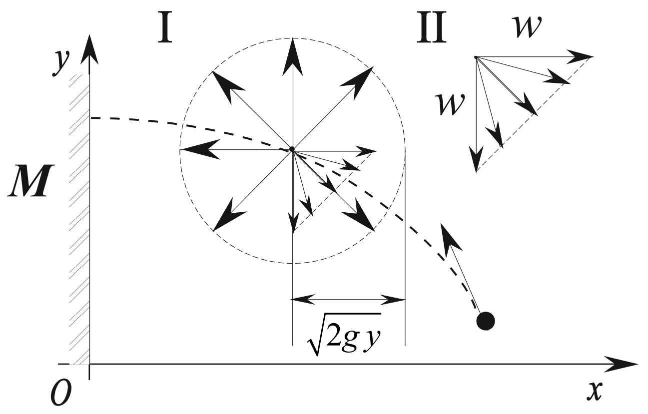
t_f is the time of attaining the terminal set M

The aim of control: $t_f \to \min$

Let A and B be arbitrary points in the vertical plane. One looks for a curve connecting A and B such that a massive point being started from A with zero initial velocity attains B along this curve for the minimal time. In the general case, the terminal point B is replaced by a given terminal curve M. For any trajectory, the velocity of the mass point at (x, y) is $(2gy)^{1/2}$, where g is the gravitation constant.

The brachistochrone problem can be reformulated as the following control problem: instead of choice of the trajectory, we assume that the magnitude of the velocity is $(2gy)^{1/2}$ and the direction is defined by the unit vector u. Thus, u is chosen from the unit circle P. The problem is considered for $y \geq 0$. The control objective is to minimize the time of attaining the terminal set M. This control problem is equivalent to the brachistochrone problem.

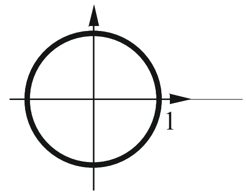
Isaacs' game statement of the problem



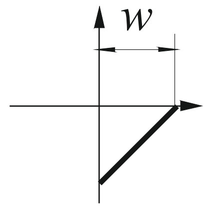
$$\dot{x}=\sqrt{2gy}\,u_1+v_1$$
$$\dot{y}=\sqrt{2gy}\,u_2+v_2$$

Admissible controls :

$u=(u_1,u_2)\in P=$



$v=(v_1,v_2)\in Q=$

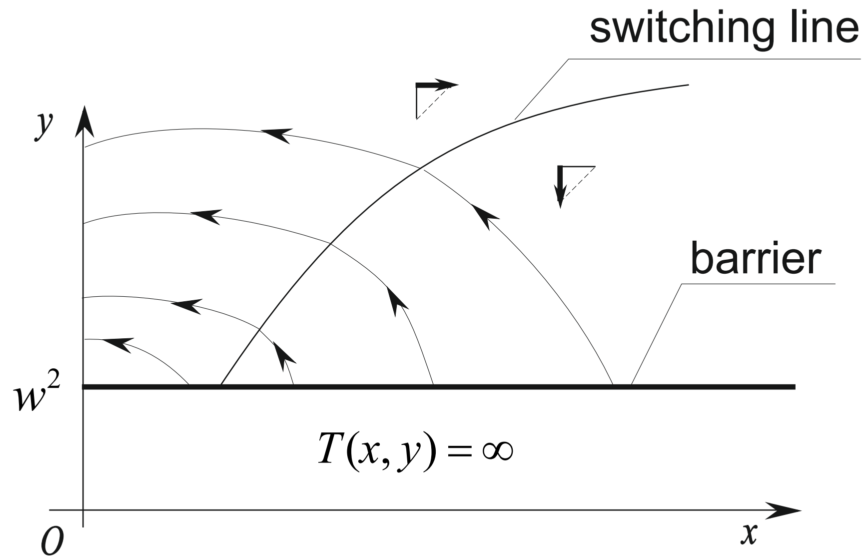


Game space:

$$N=\{(x,y)\in \mathrm{R}^2 : x\geq 0, y\geq 0\}$$

$\mathrm{I}:\ t_f \to \min$

$\mathrm{II}:\ t_f \to \max$



When considering the brachistochrone problem in the book “Differential games”', R. Isaacs introduced a disturbance influencing the dynamics of the system. The disturbance can be interpreted as a second player, whose objective is to increase the time of attaining the terminal set.

R. Isaacs considered the first quadrant as the state space. The terminal set was the positive semiaxis y. So, it was unbounded. The vectograms of the players were: the circle of radius $(2gy)^{1/2}$ for the first player, and the diagonal of a square with the side w for the second player. The first player minimizes the time of attaining the terminal set M, the second player has the opposite objective.

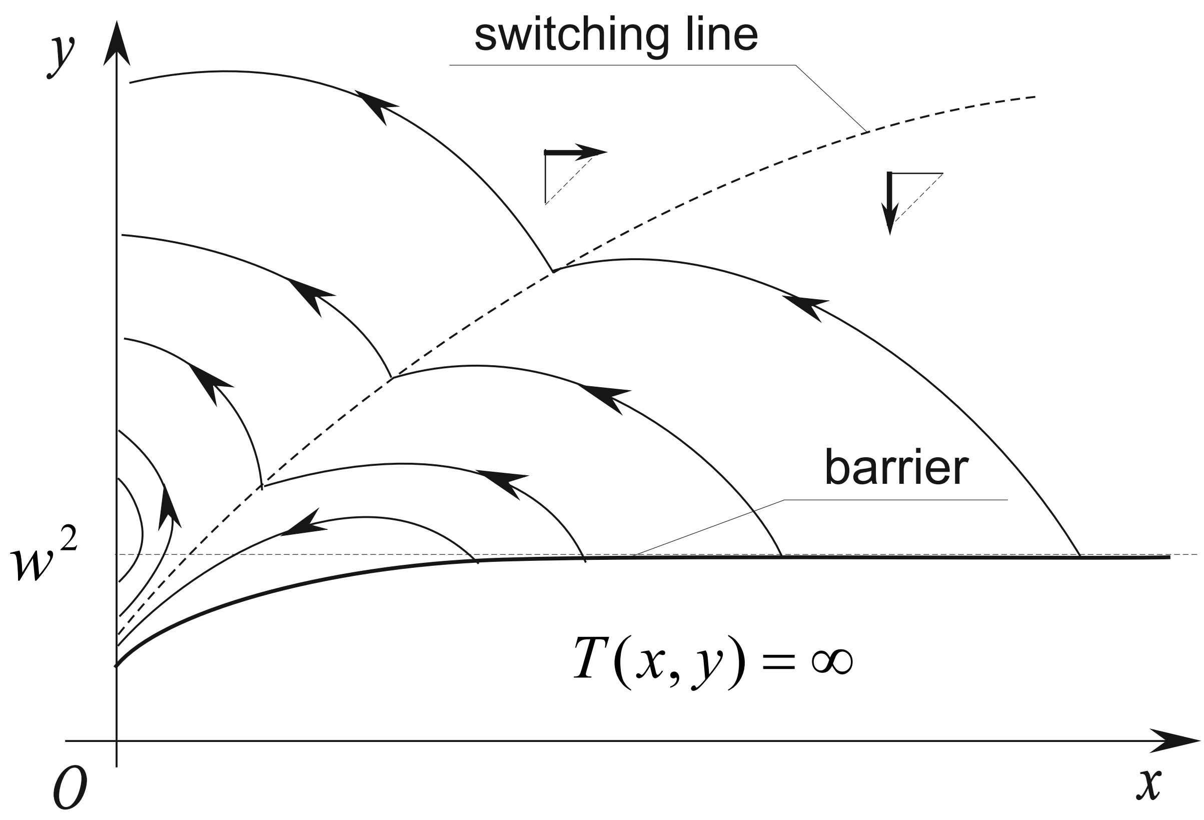
It is clear that the coefficient $(2g)^{1/2}$ does not play any role, hence, it can be replaced by 1.

A solution to this differential game was given by R. Isaacs in his book. However, this solution is not absolutely correct. Namely, Isaacs supposed that the value function is infinite below the horizontal line $y = w^2$. Above this horizontal line, the solution is defined by a switching line of the second player. On one side of the switching line, the second player utilizes one of two extremal controls; on the other side of the switching line, he uses the other extremal control of the opposite sign.

Improved solution to the Isaacs' problem

M.L. Lidov "On a differential game problem"
Avtomat. i Telemekhan., no. 4, 1971, pp.173–175 (in Russian)

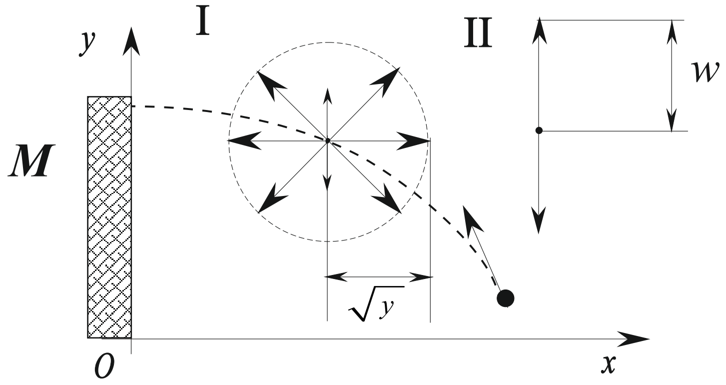
S.A. Chigir' "The game problem on the dolichobrachistochrone"
Journal of Appl. Math. Mech., Vol. 40, no. 6, 1976, pp 950–960



M.L. Lidov pointed out to an error in the Isaacs' solution. Later on, S.A. Chigir has improved the R. Isaacs' solution. The horizontal line $y = w^2$ is not a barrier, as it was erroneously claimed in the Isaacs' book. Namely, there are points below this line, for which the game is solvable. The correct form of the barrier is shown on the slide. The optimal trajectories are broken on the switching line of the second player. The value function is differentiable in the whole solvability domain.

It is interesting to modify the game statement of the brachistochrone problem in such a way that the value function would become non-differentiable. Besides, we wanted to find out whether the singular equivocal lines can arise in the game brachistochrone problem. According to the book of R. Isaacs, the equivocal singular lines are only inherent to differential games.

Our statement of the problem



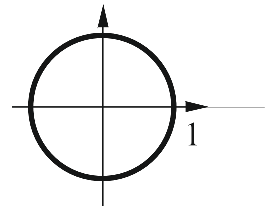
Dynamical system:

$$\dot{x} = \sqrt{y}\, u_1$$

$$\dot{y} = \sqrt{y}\, u_2 + v$$

Game space:

$$N = \{(x, y) \in \mathrm{R}^2 : x \ge 0, y \ge 0\}$$

Admissible controls: $u = (u_1, u_2) \in P =$ 

$$v \in Q = \{v \in \mathrm{R}^1 : |v| \le w\}$$

Terminal set: $M = [-d, 0] \times [0, h]$

$\mathrm{I} :\ t_f \to \min$ $\qquad$ $\mathrm{II} :\ t_f \to \max$

Isaacs-Bellman equation: $\min\limits_{u \in P} \max\limits_{v \in Q} \{T_x \dot{x} + T_y \dot{y}\} = -1$

For our case: $-\sqrt{y}\,\rho + w|T_y| + 1 = 0 \qquad \rho = \sqrt{T_x^{\,2} + T_y^{\,2}}$

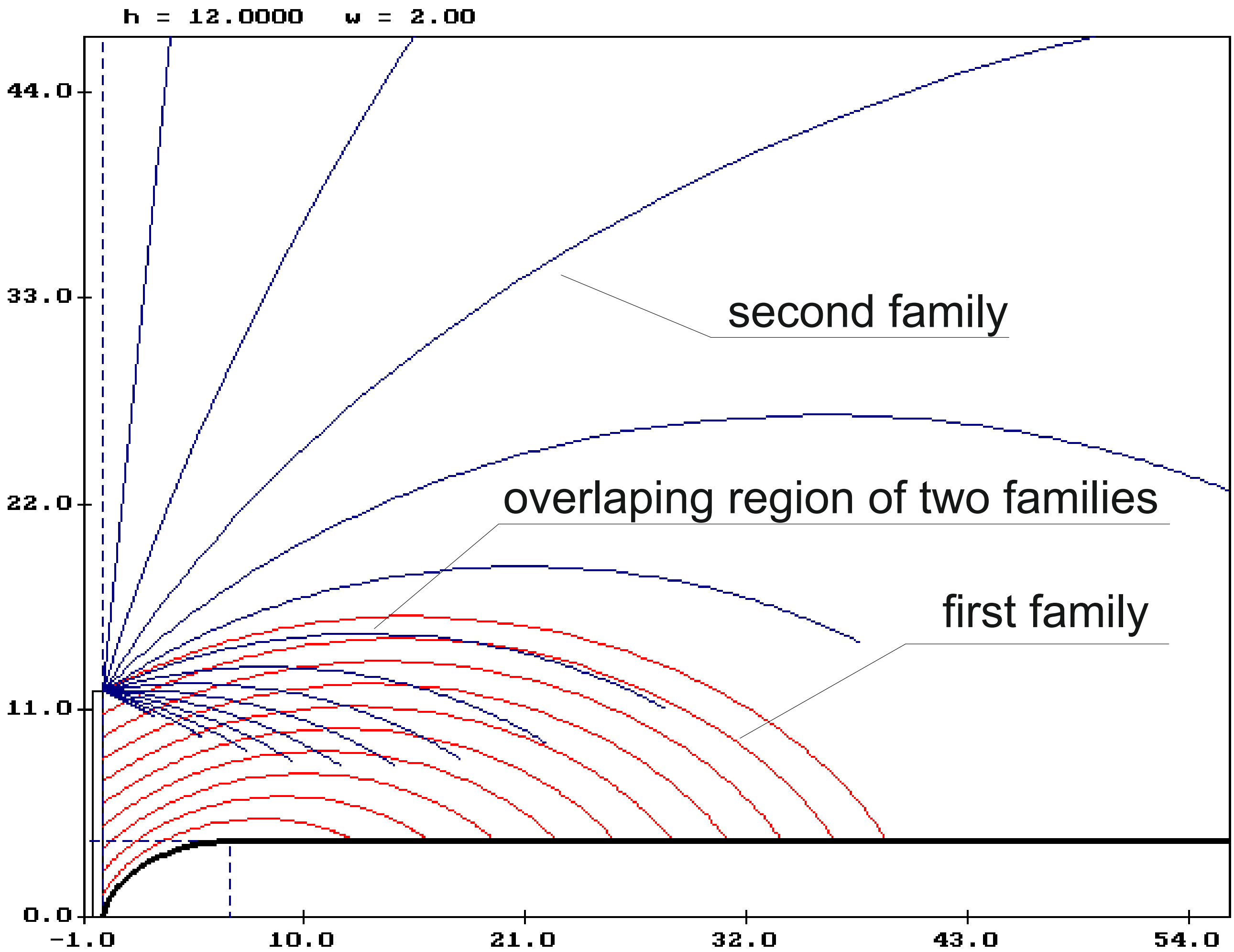
Let the vectograms of the players are of the following form: a circle of radius $y^{1/2}$ for the first player and a vertical segment of the length $2w$ for the second player. The symbols u_1 and u_2 denote control parameters of the first player in the description of the system dynamics. The variable v is a scalar control of the second player.

The terminal set M is chosen to be a rectangle of the height h with the base edge on the x-axis. Since the right-hand side of the dynamic equations does not depend on x, the location of the set M on the x-axis and its width do not play any essential role. This implies that the solution has to be symmetric with respect to the vertical line passing through the center of M.

The Bellman-Isaacs equation for this problem is written in the lower part of the slide.

Thus, we will consider only the right half of the solution. We begin with the case $h > w^2$.

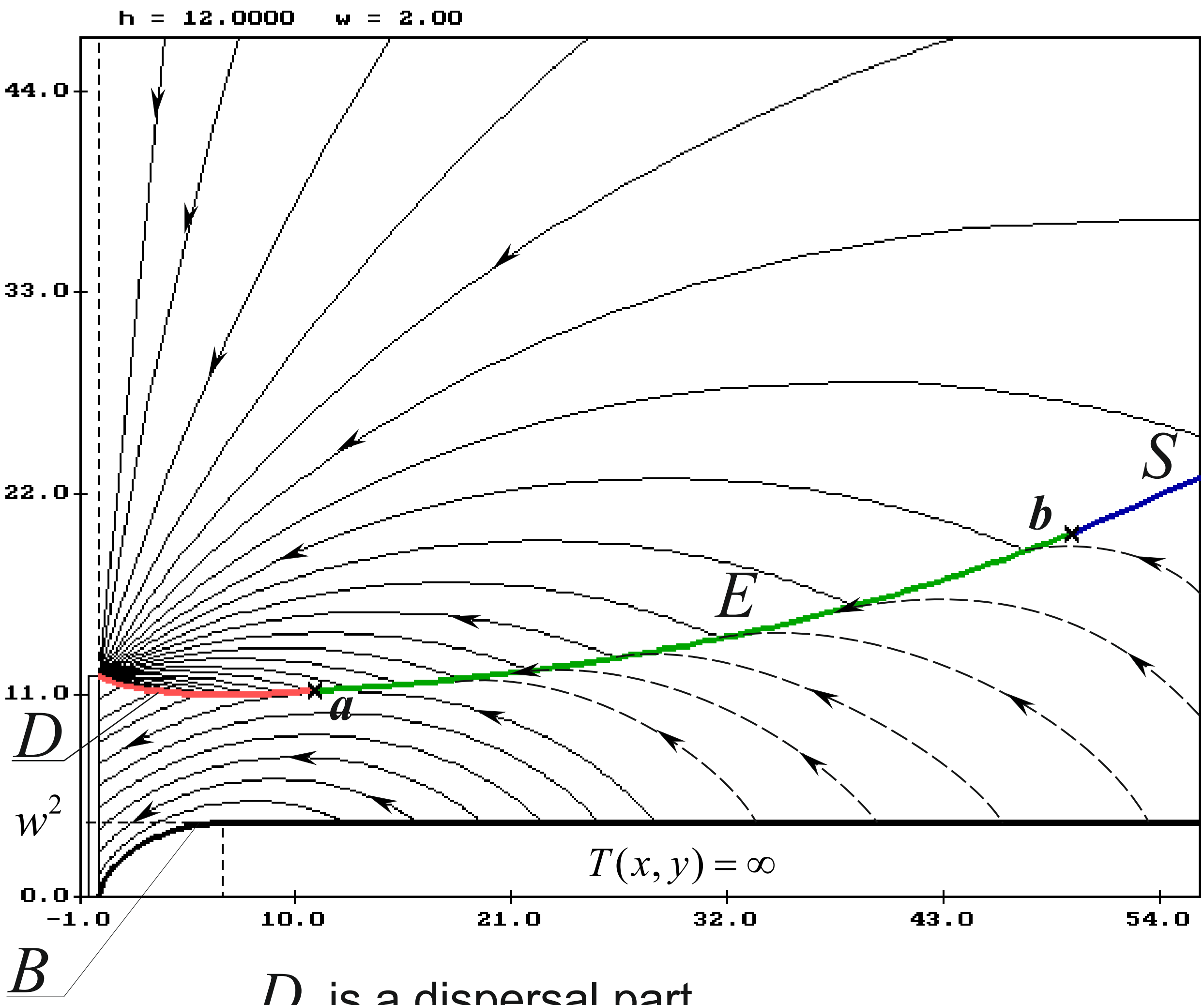
Fields of characteristics
(case $h > w^2$)



Using the Isaacs' method, we obtain three families of characteristics that are constructed taking into account the boundary condition on the terminal set. The characteristics of the first, second, and third family emanate from the vertical part of the boundary, from the right upper vertex of M, and from the horizontal part of the boundary, respectively. The third family consists of vertical lines. For all families, the characteristics belonging to the same family do not intersect each other. The first and second families overlap partially. The second and third families adjoint each other smoothly.

Relations on the moving time along the characteristics of all families are obtained. Unfortunately, they do not define the guaranteed attainment time explicitly. To obtain the optimal trajectories, the characteristics of the first and second families were processed additionally.

Solvability set and optimal trajectories



D is a dispersal part

E is an equivocal part

S is a switching (with respect to the second player control) part

The singular dispersal line D was computed from the condition that the attainment instants along characteristics of the first and second families are equal. Parts of characteristics computed backwards are removed beyond intersection points with D. Each point of D is a source for two optimal trajectories. The construction of the dispersal line is stopped if it becomes tangent to one of the trajectories of the first family. A point corresponding to this situation is denoted by $\boldsymbol{a}$ on the slide.

The point $\boldsymbol{a}$ gives rise to a singular equivocal line E. The property of equivocal lines is that two optimal trajectories emanate from their points. One of the two optimal trajectories goes into the upper region and arrives at the right upper vertex of M along a characteristic of the second family. The second optimal trajectory goes along the equivocal line up to the point $\boldsymbol{a}$.

The equivocal line is described by a first-order ordinary differential equation.

Construction of the equivocal line is continued until it meets a curve that bounds the second family from below. The equivocal line approaches this curve tangentially. Let us denote the common point by $\boldsymbol{b}$. The point F divides the lower bounding curve of the second family into two parts: the right part is denoted by S on the slide. The curve S is a switching line of the second player.

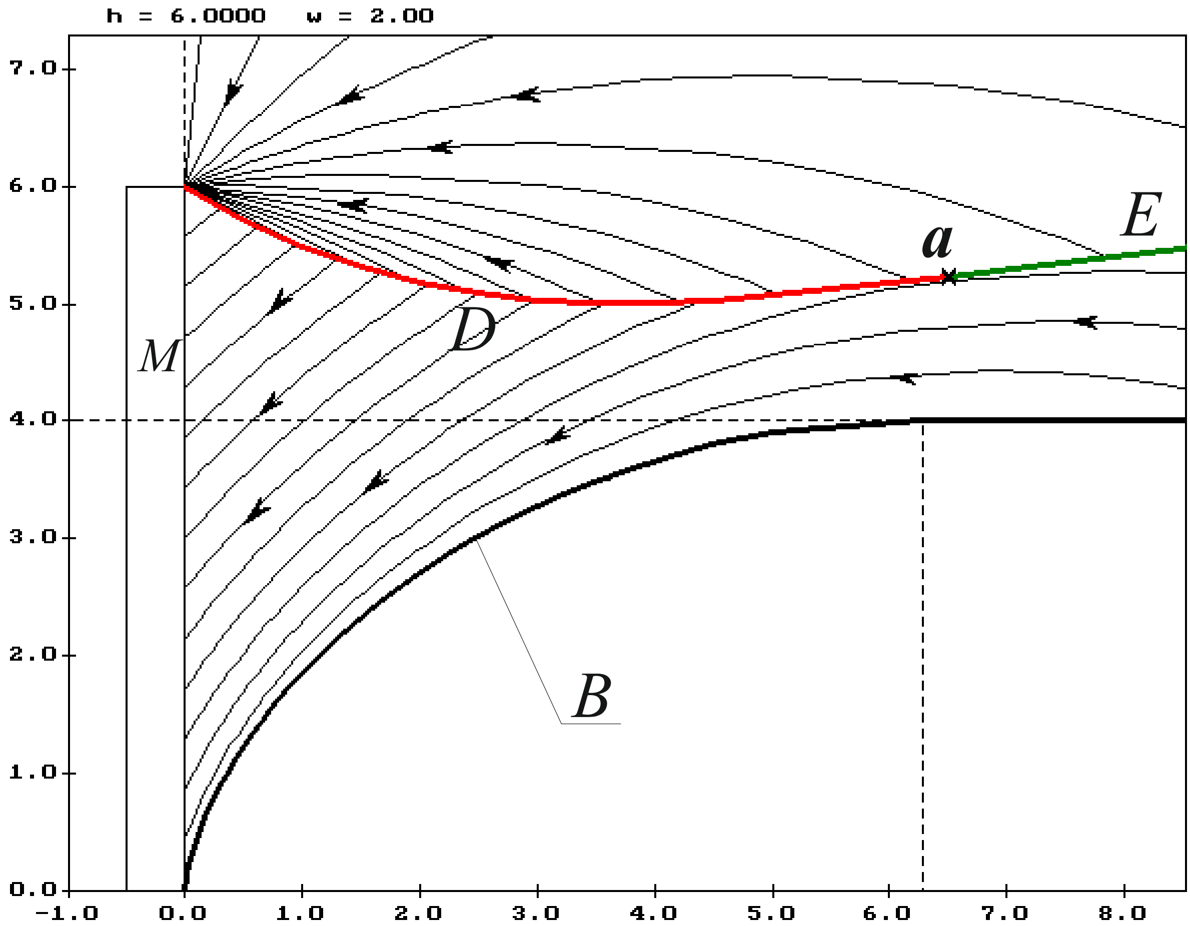
On the second stage of solving process, characteristics of the Bellman-Isaacs equation are issued backwards from the parts E and S of the singular line. But till now, complete description of these characteristics is absent.

Thus, the first arc of the singular line that defines the optimal solution posseses the dispersal property. This arc is computed numerically. The endpoint $\boldsymbol{a}$ of the dispersal arc can not be expressed analytically. The next arc of the singular line posseses the equivocal property. It is a numerical solution to a differential equation with $\boldsymbol{a}$ as the initial condition. The endpoint of the equivocal arc is determined numerically too. The third arc S is a switching line of the second player.

The value function is not differentiable on the arcs D and E, and it is differentiable on the line S. The solvability region of the problem is defined by the barrier line B that consists of a curviline segment and a horisontal line whose y-coordinate is equal to w^2.

The barrier line is smooth. The value function is infinite on the curviline part of the barrier. The second player can prevent the achievement of the terminal set whenever the trajectory starts from any point that lies below the barrier line.

Solution for $h = 6.0$, $w = 2$

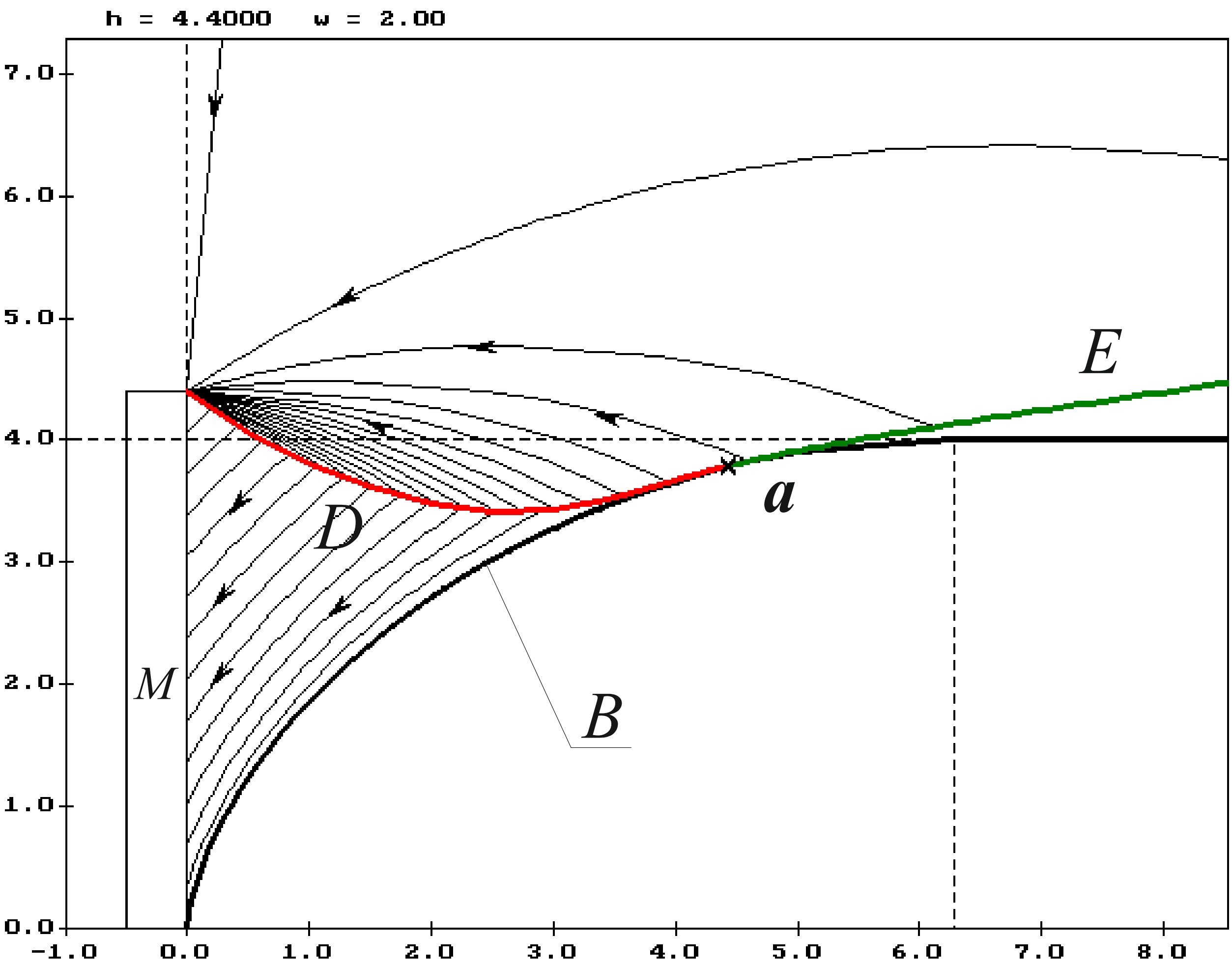


All three parts of the singular line are presented

It was the description of the solution structure for $h > w^2$. Now, we show how the solution changes if the value of the parameter h decreases. The value of w is fixed.

For this slide, $h = 6$. For the previous slide, $h = 12$. Principally, the qualitative structure of the solution remains the same. As before, the singular line consists of three arcs, and it is located over the barrier line B.

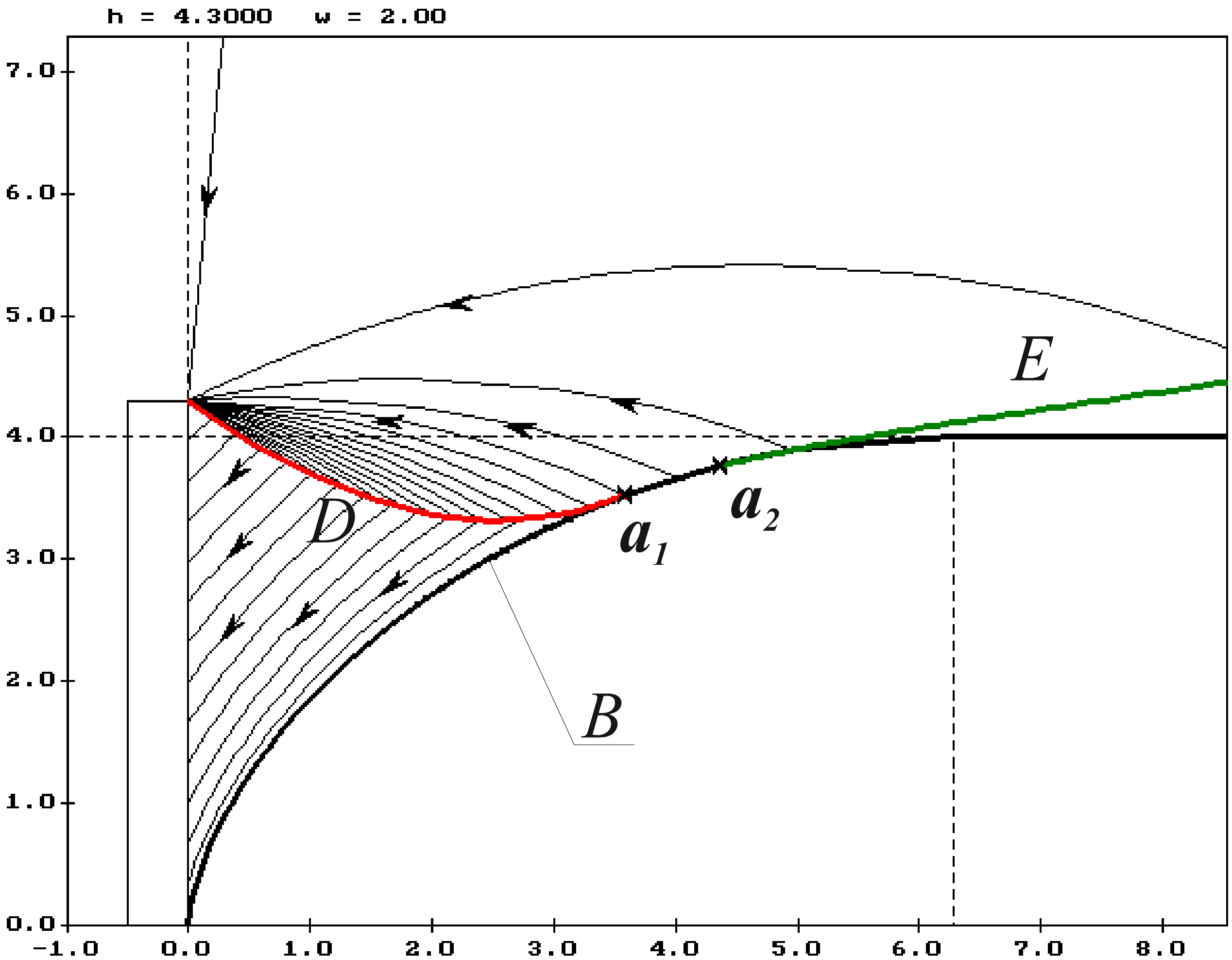
Solution for $h = 4.4, w = 2$



The singular line and the barrier have common tangent at the point $\boldsymbol{a}$

If h continues to decrease, the singular line approaches the barrier line near and near. The limiting case where the singular line touches the barrier is shown.

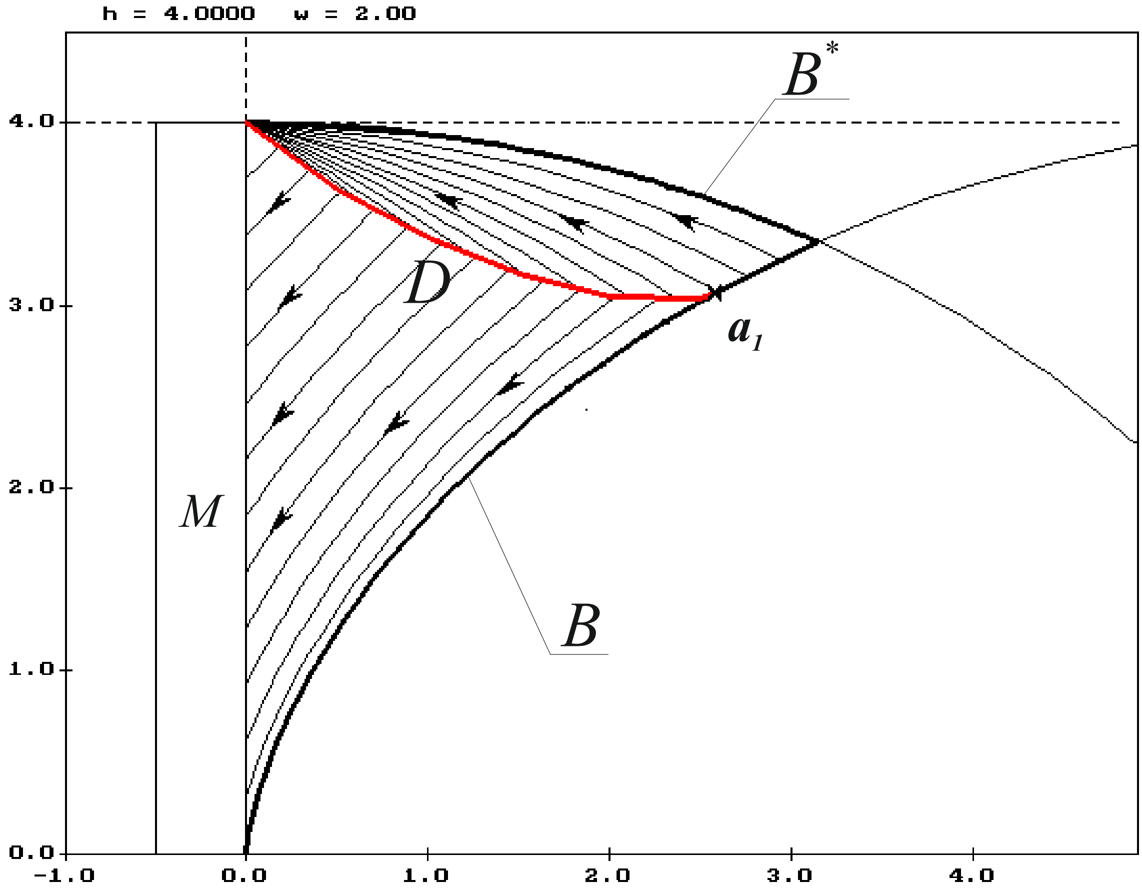
Solution for $h = 4.3, w = 2$



The equivocal part is not any continuation of the dispersal part

If h decreases further, the singular line breaks up: the dispersal arc approaches the barrier B tangentially at the point $\boldsymbol{a}_1$, but the equivocal arc emanates tangentially from another point $\boldsymbol{a}_2$ lying on the barrier on the right from the point $\boldsymbol{a}_1$. Basically, the singular line consists of three parts as before.

Solution for $h = 4.0, \quad w = 2$



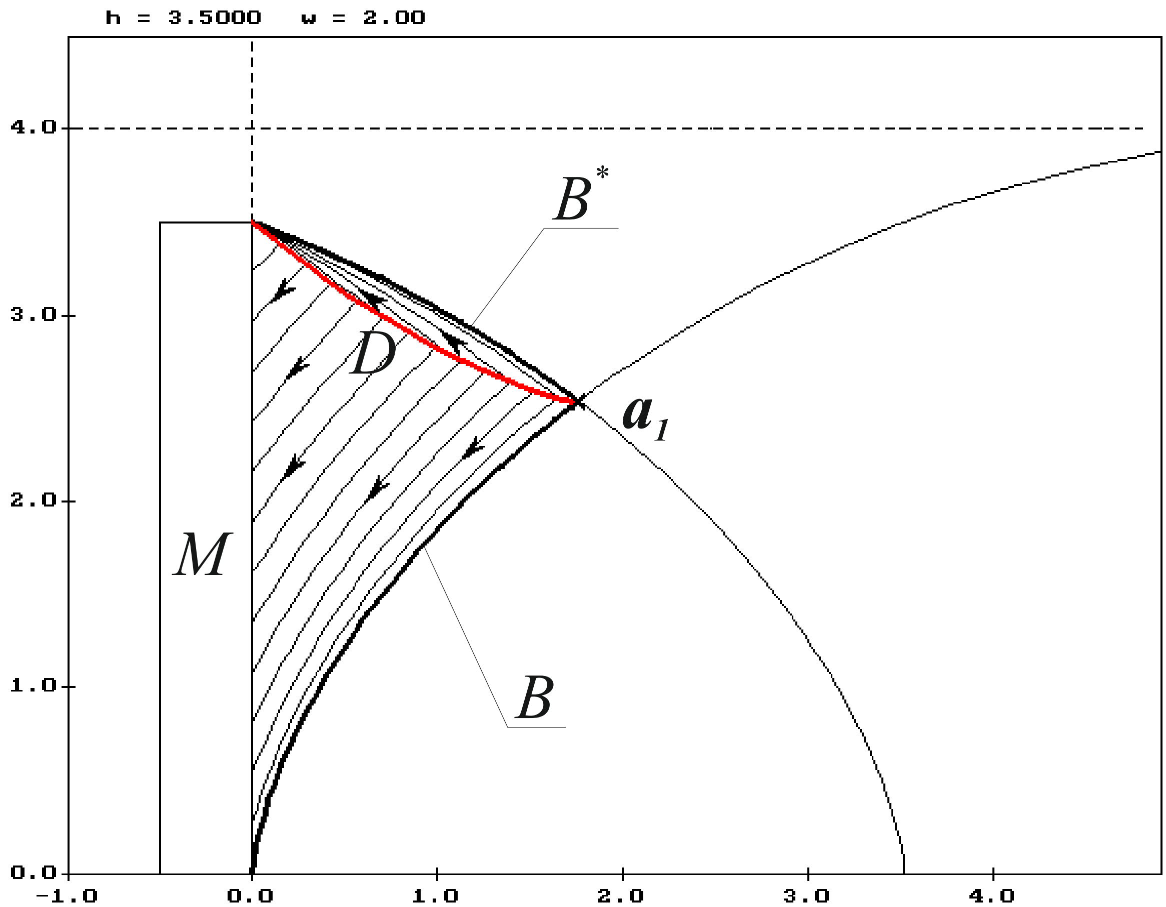
The solvability set changes jump-like.
Only dispersal line remains.

$T(x, y) < \infty$ for the barrier B

$T(x, y) = \infty$ for the barrier B^*

In the case $h = w^2$, the solvability region changes jump-like. The upper barrier line B^* bounding the solvability region from above appears. The equivocal arc and switching arc of the second player disappear. It is interesting to remark that the value function is infinite on the upper barrier line B^*. The value function is finite on the lower barrier B.

Solution for $h = 3.5, \quad w = 2$



The endpoint of the dispersal line goes on to the upper barrier

$T(x, y) < \infty$ for the both barriers

If h decreases, the solution structure remains the same: two barriers define the solvability region. The endpoint of the dispersal arc goes on from the lower barrier to the upper barrier. The value function is finite on both barriers.

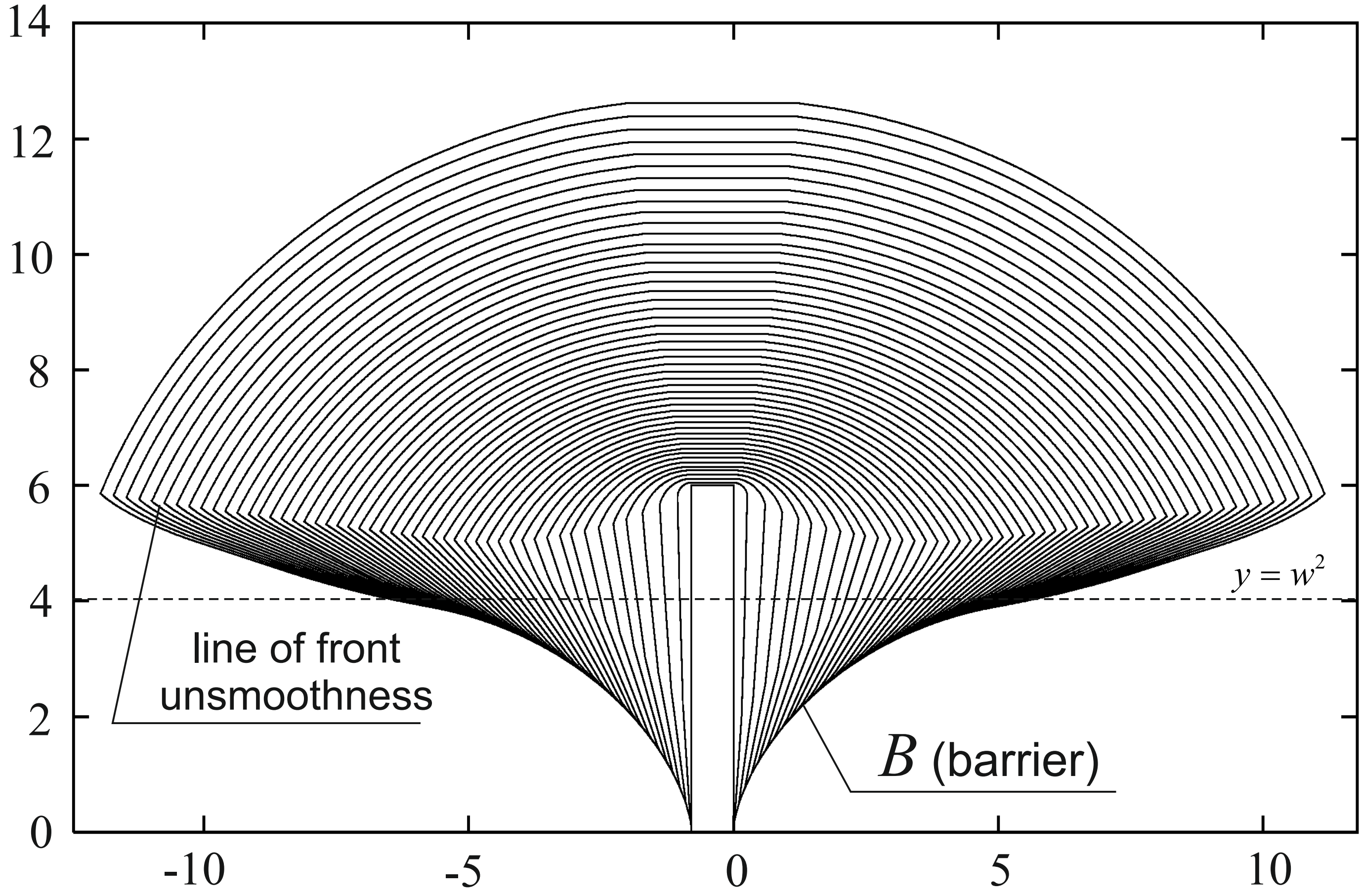
The results set forth above are of our study based on construction of the singular line. The solution is not trivial. We have a program for computation of the singular line.

The complete exact proof for the described solution structure is not implemented yet. Nevertheless, we are sure that our results are correct because they are in agreement with the independent computation of level sets of the value function. This independent computation utilizes an algorithm based on the backward propagation of fronts.

General view of the level sets of the value function

Height of the terminal set: $h = 6$

Parameter for the second player: $w = 2$



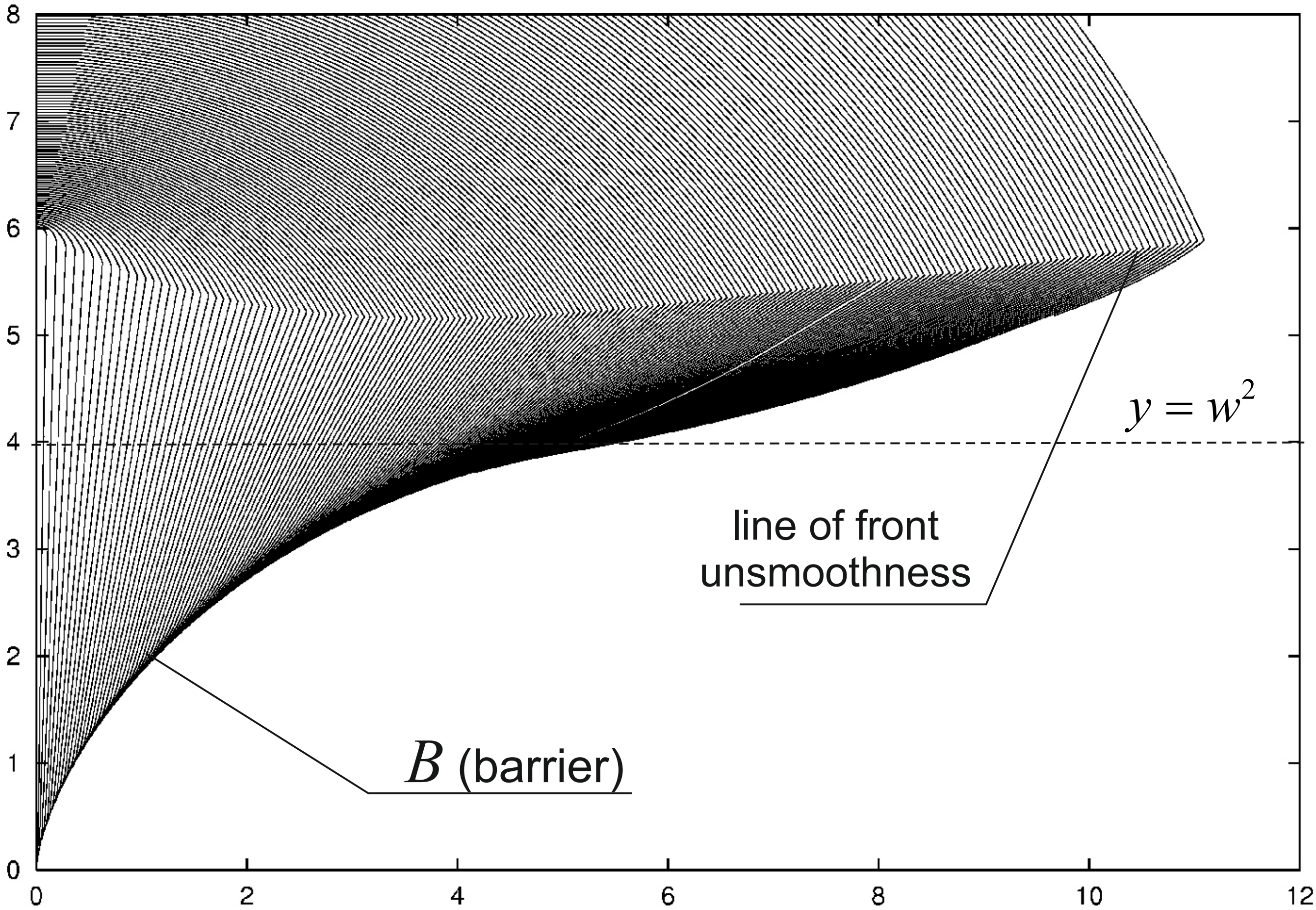
The step of calculation is 0.05, each third front is shown, computation is performed up to the instant 7.32

Here, level sets of the value function are presented for $h = 6$, $w = 2$. The curve consisting of the corner points of the fronts is clearly seen. This curve coincides with our singular line. The region that is filled out with the fronts coincides with the solvability region. The accumulation of the fronts near the horizontal line $y = w^2$ means that the value function goes to infinity when approaching $y = w^2$ from above.

Enlarged fragment of the previous view

Height of the terminal set: $h = 6$

Parameter for the second player: $w = 2$



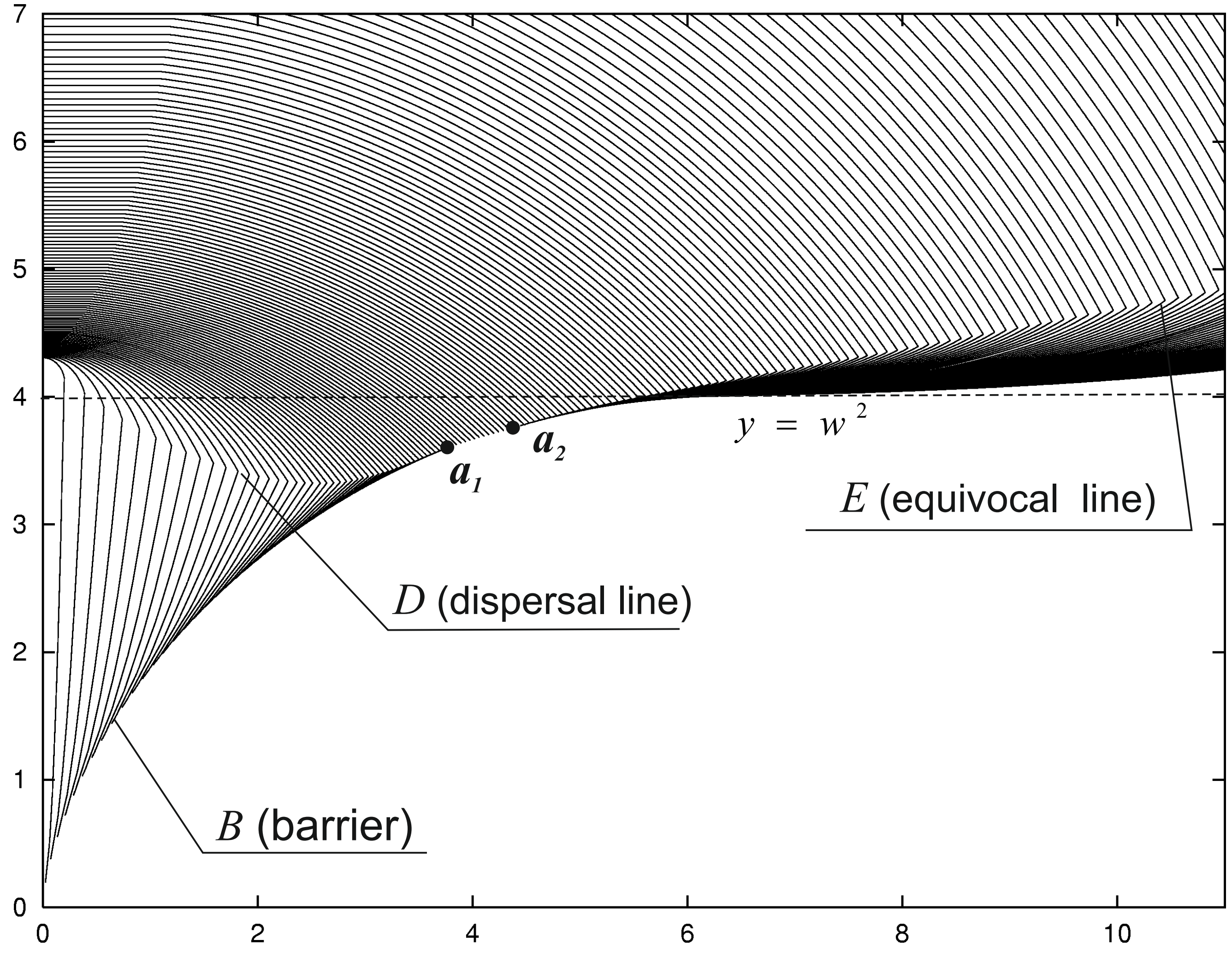
Computation is performed up to the instant 7.32

Here, a fragment of the previous picture is shown. More fronts for the same time interval are presented.

Separation of the dispersal and equivocal lines

Height of the target set: $h = 4.3$

Parameter for the second player: $w = 2$

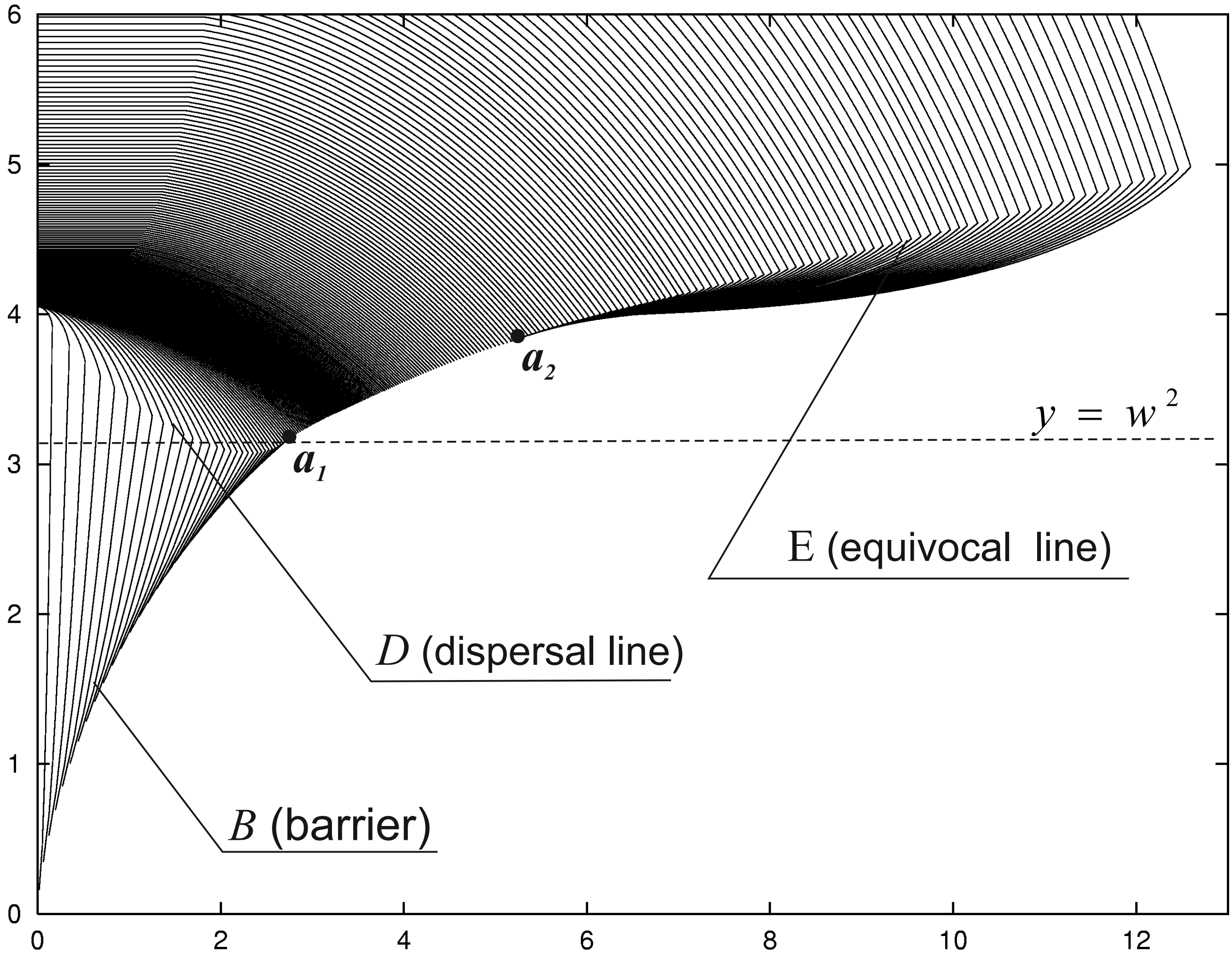


On this slide, a fragment of the collection of the level sets is shown for $h = 4.3$. One can see the curve composed of the corner points of the fronts is divided onto two parts. For h cosidered, the division point $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are close to earch other.

Regions of accumulation of the fronts

Height of the terminal set: $h = 4.05$

Parameter for the second player: $w = 2$

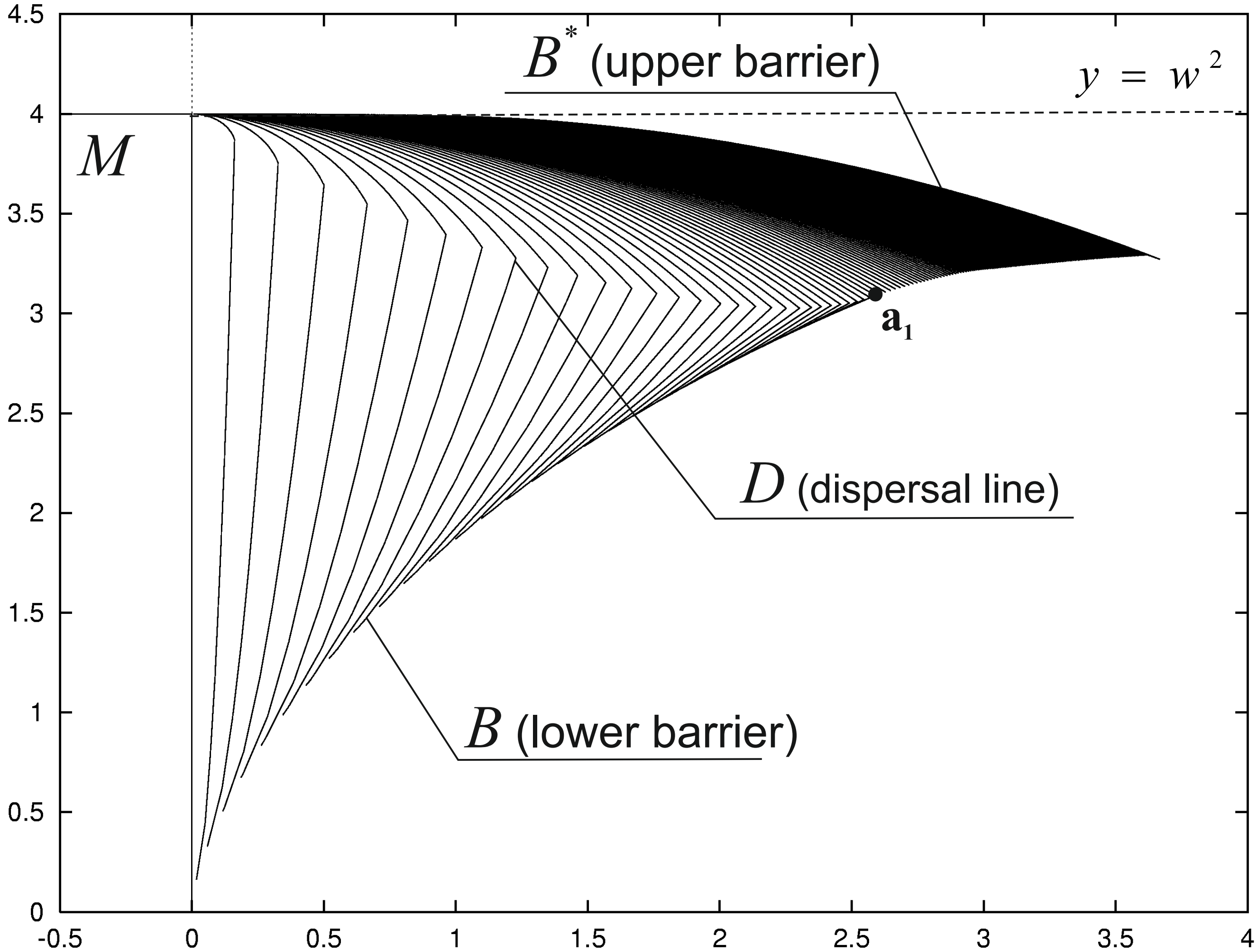


If h decreases, the points $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ go away each from other. An accumulation of the fronts arises over the terminal set in the left part of the picture.

Appearance of the upper barrier

Height of the terminal set: $h = 4.0$

Parameter for the second player: $w = 2$

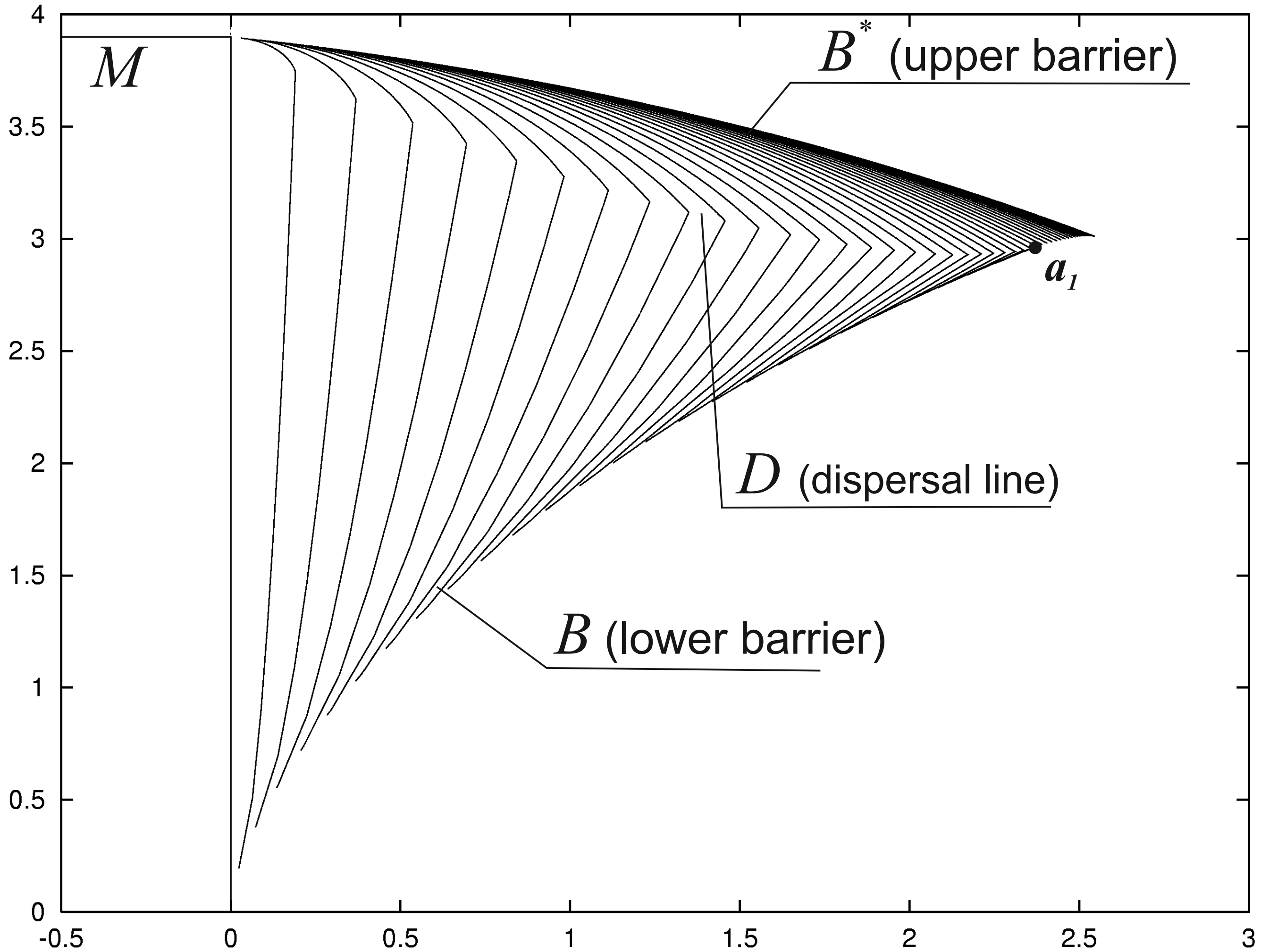


This picture is done for $h = w^2 = 4$. The accumulation of the fronts in the upper part of the picture gives the upper barrier B^*. The solvability region changes jump-like.

Decreasing of the regions of front concentration

Height of the terminal set: $h = 3.9$

Parameter for the second player: $w = 2$

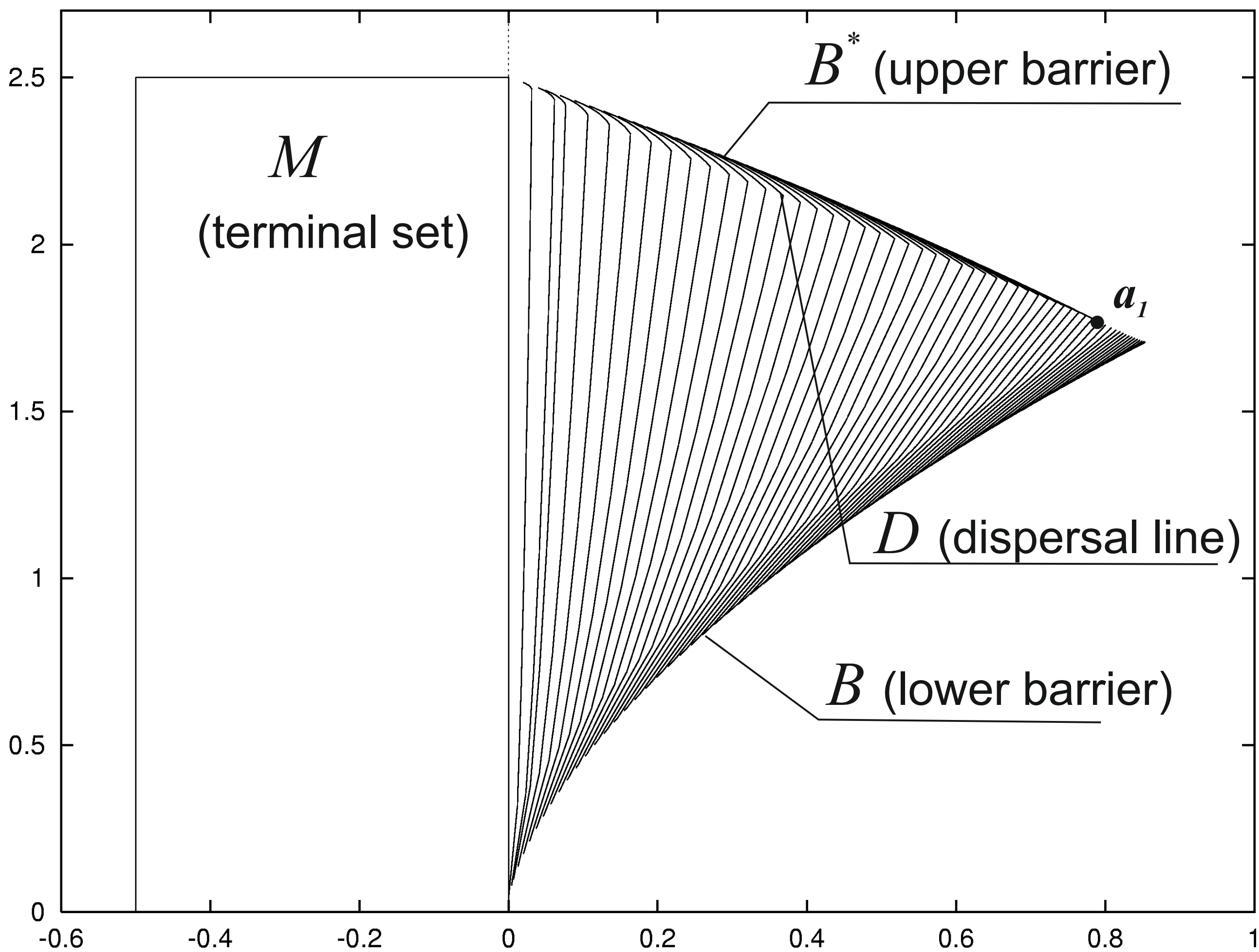


If h decreases further, the upper accumulation region of the fronts reduces. The curve composed of the corner points of the fronts meets the lower barrier as before.

Motion of the endpoint of the dispersal line

Height of the terminal set: $h = 2.5$

Parameter for the second player: $w = 2$



The computation are done for $h = 2.5$. The curve composed of the corner points of the fronts meets the upper barrier.