MINIMUM-TIME PROBLEM FOR LINEAR SECOND-ORDER CONFLICT-CONTROLLED SYSTEMS

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Abstract. Time-optimal problem for linear differential games in the plane is considered. An algorithm for the construction of level sets of the value function is proposed. Numerical examples are presented.

Keywords: *Differential games; time-optimal control; value function.*

INTRODUCTION

Linear stationary time-optimal control problems in the plane are canonical problems in the theory of optimal control (see, for example, Tsien (1) and Pontryagin et al (2)). Many textbooks contain pictures related to the construction of switch lines which determine the optimal feedback control in such problems. Also, the level sets $W(\theta, M)$ of the optimal result function (value function) can be constructed very easily in many cases. The set $W(\theta, M)$ is the set of all initial points for each of which the optimal time needed to bring the system to the terminal set M does not exceed time θ. A totality of such sets gives some geometrical image of the value function. The construction of the level sets can be considered as the basis for finding optimal feedback controls.

The theory of differential games (Isaacs (3), Krasovskii and Subbotin (4), Krasovskii and Subbotin (5)) is a natural extension of optimal control theory. In this paper, we consider a conflict-controlled system with linear dynamics and geometrical bounds on controls

$$\dot{x} = Ax + u + v, \tag{1}$$

$$x \in R^2, \ u \in P, \ v \in Q.$$

Here P and Q are convex closed polygons in the plane. The terminal set M (a convex polygon in the plane) is given. The first player who governs the control parameter u seeks to minimize the time of attaining M from some initial point x_0, the aim of the second player governing the control vector v is opposite. It is known (see (4),(5)) that optimal guaranteeing results of the players coincide in feedback (positional) controls for such problem.

We are interested in finding the sets $W(\theta, M)$, $\theta > 0$. Each of them is the set of all initial states x_0 such that the first player guarantees the transition of the state vector to M by the time θ. The set $W(\theta, M)$ is the level set (the Lebesque set) of the value function of the minimum-time game problem.

In terms of the works (4), (5), the set $W(\theta, M)$ is also called t-section of the maximal u-stable bridge corresponding to $t = \theta$.

The paper is devoted to the numerical construction of $W(\theta, M)$.

If there exists a polygon D such that $P = -Q + D$, the game (1) can be reduced (see (4), (5)) to the control problem

$$\dot{x} = Ax + w, \quad w \in D.$$

The most interesting cases are that ones where such reduction can not be done. In these cases the sets $W(\theta, M)$ can only be found numerically even for very simple examples (for instance, $\dot{x}_1 = x_2 + v$, $\dot{x}_2 = u$, $|u| \leq 1$, $|v| \leq 1$).

To find the sets $W(\theta, M)$, we use backward procedures. The application of backward procedures is the typical way for solving control and differential game problems. Essential role in the theory of differential games belongs to the backward procedures considered in the papers by L.S.Pontryagin (6) and B.N.Pshenichnii (7).

The most developed results related to the algorithmic implementations of backward constructions to differential games were obtained for linear differential games with fixed terminal time and convex target set (Subbotin and Patsko (8), Zarkh and Patsko (9), Taras'yev et al (10), Botkin and Ryazantseva (11), Zarkh and Ivanov (12)). In this case, the application of the backward procedure gives t-sections of the maximal u-stable bridge. The algorithms use the property: the convexity of the target set implies the convexity of the t-sections of the maximal stable bridge. This makes the problem easier and enables to apply numerical methods to some important practical problems (see, for example, Patsko et al (13), Sokolov and Turova (14)).

The above mentioned feature is not inherent to differential games with nonfixed time of termination: as a rule, t-sections of maximal stable bridges are not convex. Numerical methods for solving problems with nonfixed time of termination and nonconvex problems with fixed time are studied in papers by V.N.Ushakov and his collaborators (see Ushakov (15), Ivanov et al (16)). Recently, numerical methods for constructing value functions and their level sets based on the notion of viscosity solutions of Hamilton-Jacobi (Bellmann-Isaacs) equations were developed (Subbotin (17), Bardi and Falcone (18)).

The principal notion of our algorithm is the notion of a "front". Let Δ be the time step of constructions and let $W(i\Delta, M)$ be the level set corresponding to the time $i\Delta$. The front F_i is the set of all points on the boundary of the set $W(i\Delta, M)$ such that the minimum guaranteeing time of the attainment of the previous set $W((i-1)\Delta, M))$ is exactly equal to Δ. When operating with fronts, we use the ideas of the algorithms for constructing t-sections of maximal stable bridges proposed in (8) for linear games with fixed time of termination.

STATEMENT OF THE PROBLEM

We now define the set $W(\theta, M)$ more precisely (see (4), (5)). Let $\mathcal{U}$ be the set of all positional strategies U of the first player. Namely, this is the set of all functions defined on $[0, \theta] \times R^2$ and taking the values in P. Let σ be an arbitrary partition of the segment $[0, \theta]$ formed by the points $0 = t_1 < t_2 < ... < t_n = \theta$, let $d(\sigma)$ be its diameter, and let $v(\cdot)$ be a measurable function of time with values in Q. Let $y(\cdot\,; \sigma, x_0, U, v(\cdot))$ denote the solution of system (1) emanating from the point x_0 with the control $v(\cdot)$ of the second player and with the control u of the first player which is constant on each interval $[t_i, t_{i+1})$ of the partition σ and is chosen as $u = U(t_i, x(t_i))$. We denote by $W(\theta, M)$ the set of all points $x_0 \in R^2$ for each of which there exist a strategy $U \in \mathcal{U}$ and a mapping $\varepsilon \to \delta(\varepsilon)$ from R_+ to R_+ such that for any $\varepsilon > 0$, any σ with the diameter $d(\sigma) \leq \delta(\varepsilon)$, and any function $v(\cdot)$ with values in Q there exists a time $t \in [0, \theta]$ at which $y(t; \sigma, x_0, U, v(\cdot))$ belongs to the ε-neighborhood of the set M.

Such a definition is equivalent to other well-known definitions (see, for example, (5), (17)) of the solvability set $W(\theta, M)$ of the time-optimal game problem. We give this definition because it shows the properties of the optimal guaranteeing strategy of the first player in terms of the bundle of motions generated by different controls of the second player.

THE IDEA OF THE ALGORITHM

The set $W(\theta, M)$ is formed via step-by-step backward procedure giving a sequence of embedded sets

$$W(\Delta, M) \subset W(2\Delta, M) \subset W(3\Delta, M) \qquad (2)$$

$$\subset ... \subset W(i\Delta, M) \subset ... \subset W(\theta, M).$$

Each set $W(i\Delta, M)$ consists of all initial points such that the first player brings system (1) into the set $W((i-1)\Delta, M)$ within the time duration Δ. We put $W(0, M) = M$.

Before doing the first step of the backward procedure, we find a usable part Γ_0 of the boundary of M. In accordance with R.Isaacs (3), the usable part is a curve or several curves on the boundary of M attainable for trajectories of system (1) from points lying in the exterior of M close to the boundary of M. The usable part is defined by the following formula

$$\Gamma_0 = \mathrm{cl}\{x \in \partial M : \min_{u\in P} \max_{v\in Q} \langle \ell, Ax+u+v\rangle < 0, \forall \ell \in K_x\}.$$

Here K_x is the cone of outward normals to the set M at x. Since M is convex, each curve of the usable part is locally convex in the following sense: the normal to the curve at a point x keeps its rotation in only direction when x moves along the curve.

Suppose the usable part of M consists of one curve only. Let us introduce the term “front”. We put $F_0 = \Gamma_0$. The front F_i is the set of all points on the boundary of the set $W(i\Delta, M)$ for which the minimum guaranteeing time of the attainment of $W((i-1)\Delta, M)$ is equal to Δ. For other points of the boundary of $W(i\Delta, M)$ the optimal time is less than Δ. The line $\partial W(i\Delta, M) \setminus F_i$ possesses the properties of barriers defined in (3). The front F_i is designed using the previous front F_{i-1}.

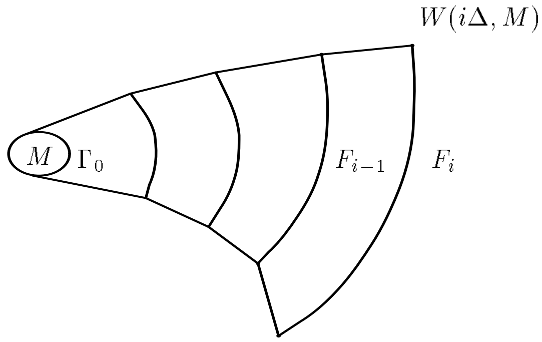
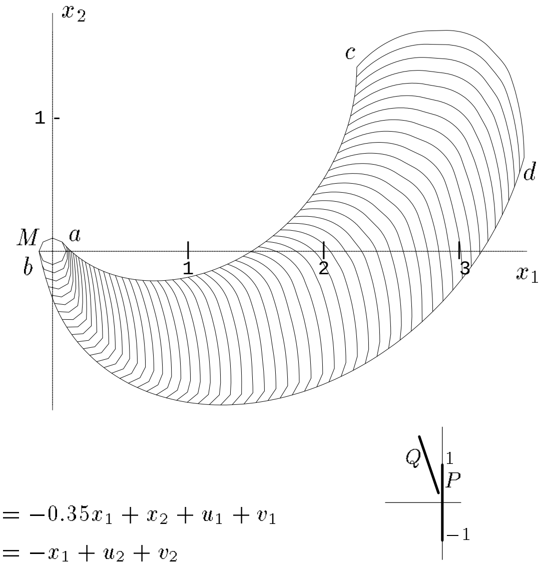


Fig. 1. Construction of the sets $W(i\Delta, M)$. “Convex” case.

Due to the linearity of system (1), the fronts F_1, F_2, ..., F_i, ... inherit (Fig. 1) the property of the local convexity of Γ_0 at the initial stage of constructions, and this property is kept until the current front F_i does not meet the set $W((i-1)\Delta, M)$. Straight lines connecting endpoints of F_i with the corresponding endpoints of F_{i-1} give the extension of the barrier lines. The boundary of the set $W(i\Delta, M)$ is formed by the front F_i, the above mentioned extensions of the barrier lines, and the line $\partial W((i-1)\Delta, M) \setminus F_{i-1}$. The property of the local convexity of fronts enable us to employ, with small modifications, procedures for the construction of t-sections of maximal stable bridges which were developed in (8) for linear differential games with a convex target set and fixed time of termination. An example of computing the sequence (2) for oscillating system in the case where the fronts do not meet the already constructed set is shown in Fig. 2. Here, the lines ac, bd are barriers.

If the next front F_i meets the already constructed set $W((i-1)\Delta, M)$, we say that the front collides with the set $W((i-1)\Delta, M)$. The situation of “collision” means that the current front meets the barrier part of the boundary of $W((i-1)\Delta, M)$ or the part $\partial M \setminus \Gamma_0$

of the boundary of M. To construct the next front F_{i+1}, we should take into account that F_i and the boundary of $W((i-1)\Delta, M)$ have the nonconvex conjunction (Fig. 3). Due to the properties of the plane, the complement of $W(i\Delta, M)$ is locally convex near the conjunction point. So, assuming that the second player seeks to bring the system to the complement and the first player has the opposite objective, we can use the ideas of the "convex" algorithms. After combining the curve which is the result of constructions from the convex part of the front with that one from the nonconvex conjunction, we obtain a new front F_{i+1} that may be not locally convex.



$$\dot{x}_1 = -0.35x_1 + x_2 + u_1 + v_1$$
$$\dot{x}_2 = -x_1 + u_2 + v_2$$

Fig. 2. Level sets of value function for oscillating system. Initial stage of constructions.

The continuation of computations depicted in Fig. 2 is given in Fig. 4. The first nonvexity appeared in the construction process due to the collision is shown within the enlarged square. The computations are carried out up to the reverse time $\tau = 7.2$ when the current front meets the barrier line marked as bd in Fig. 2.

The barrier lines of the set $W(i\Delta, M)$ are stored in the corresponding computer program as ordered collections of points. Updating these collections is easily done if not too many collisions happen. The program is not applicable to very complicated cases of collisions whose processing require the exhaustion of significantly large number of variants.

If the usable part of M consists of several fragments on the boundary of M, then our constructions can be carried out independently for each fragment until intersections of the sets sprouting from these segments do not occur.

So, the algorithm consists of the following operations:
1) Finding the usable part on the boundary of M.
2) Constructing the next front using the previous one.
3) Testing the intersections of the current front with the barrier part of the already constructed set and the boundary of M. If the intersection is detected, further computations are being carried out with taking into account the arising nonconvex conjunction and possible splitting of the front into several parts.

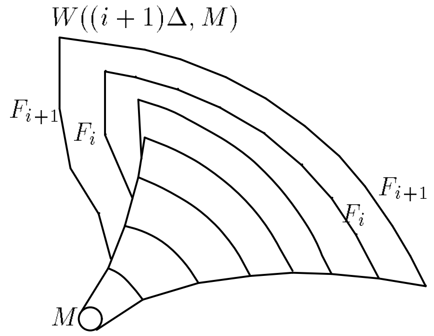


Fig. 3. Construction of the sets $W(i\Delta, M)$. "Nonconvex case".

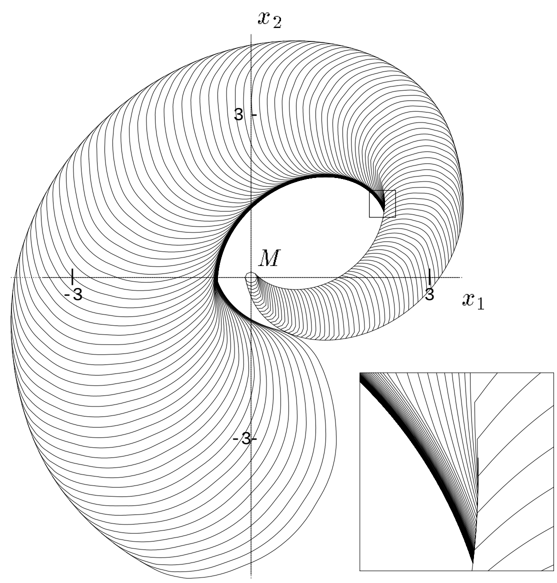


Fig. 4. Level sets of value function for oscillating system. Solution at $\tau = 7.2$.

EXAMPLES

1. The canonical example of the minimum-time problem in the theory of optimal control has the following form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \qquad | u | \leq 1. \end{aligned} \tag{3}$$

We add the disturbance v to the first equation and consider the following differential game:

$$\begin{aligned} \dot{x}_1 = &\quad x_2 + v \\ \dot{x}_2 = &\quad u, \qquad |\, u \,|\leq 1, \quad |\, v \,|\leq 1. \end{aligned} \tag{4}$$

The first player minimizes the time of the attainment of M, the aim of the second player is opposite.

Let M be the regular octagon inscribed into the circle of the radius 0.1 and with the center at the point $(0, 2)$. Here and below, we put $\Delta = 0.05$. The sets $W(\tau, M)$ ($W(\tau)$ briefly) for the time instants $\tau = k \cdot 4\Delta,\ k = \overline{1, 55}$, are shown in Fig. 5.

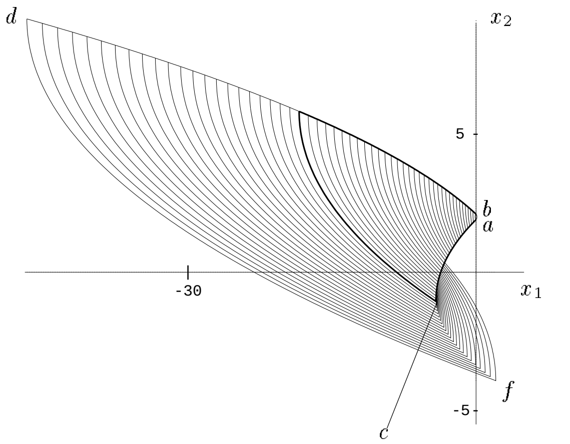


Fig. 5. Differential game (4). Solution at $\tau = 11$.

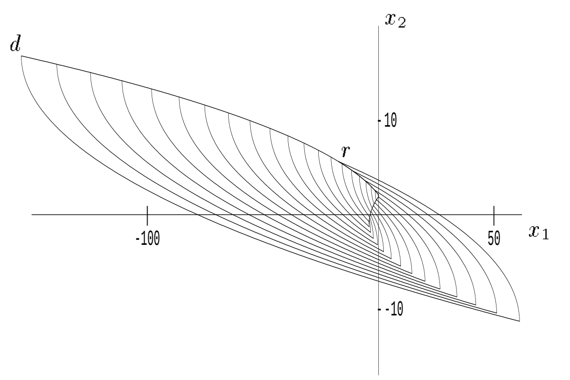


Fig. 6. Differential game (4). Solution at $\tau = 20$.

We denote by a, b the endpoints of the usable part Γ_0 of M. The curves ac and bd formed by the endpoints of fronts are barriers. The value function is discontinuous on these curves and also on the line $\partial M \setminus \Gamma_0$. The line cf formed by corners of fronts is the equivocal (see Ref. (3)) line. The value function is not differentiable on cf. The first situation of collision happens at $\tau = 6.6$, the set $W(6.6)$ is contoured. The fronts can be constructed analytically up to $\tau = 6.6$. After this time, it can not be done.

The sets $W(\tau)$ for $\tau = k \cdot 20\Delta,\ k = \overline{1, 20}$, are given in Fig. 6. For $\tau > 20$, the front's endpoint r which moves along the upper barrier overtakes another endpoint d. The upper barrier ceases to grow when r coincides with d, and this barrier (as well as the lower barrier ac) is extended by an equivocal line. Even if the level sets in the problem (4) look similar to the sets in the problem (3), they can not be calculated "by hand" because of the presence of equivocal lines.

2. Consider the oscillating system

$$\begin{aligned} \dot{x}_1 = &\quad 0.35x_1 + x_2 + v \\ \dot{x}_2 = &\quad -0.8x_1 + u, \\ -2 \leq u \leq 1.5, &\quad -6.1 \leq v \leq -4. \end{aligned} \tag{5}$$

The terminal set M is a regular octagon with the center at the origin. The level sets $W(\tau)$ for $\tau = k \cdot \Delta,\ k = \overline{1, 189}$, are given in Fig. 7. Up to $\tau = 5.7$, the front moves between the left and the right barrier lines emanating from the set M. The left barrier terminates at $\tau = 5.7$. For $\tau > 5.7$, the front begins to go around this barrier so that one of its endpoints slides along the outward side of the barrier. At $\tau = 8.15$, the front collides with the initial part of the left barrier from outside. For $\tau > 8.15$, the left and the right endpoints of the front move towards each other along the left barrier. The constructions are finished at $\tau = 9.45$.

In this example, the set filled up with the fronts by the time $\tau = 9.45$ is the set where optimal guaranteeing time is less than infinity. The first player cannot guarantee the transfer to M within any finite time from the initial points lying outside this set.

The singular lines for the game (5) are depicted in Fig. 8. The barrier line $acdef$ terminates at the point f. After that it is continued by the equivocal line fg which splits into the switch line gc of the first player and the switch line gr of the second player at the point g. The curve $bhkprs$ is the barrier, the curve dk is the equivocal line, and the curve ec is the switch line of the second player. The singular lines listed above divide the set where the problem has a solution into subsets so that the optimal controls of the players take constant values in the interior parts of the subsets. These constant values are equal to the minimal and maximal values of controls: $u_* = -2$, $u^* = 1.5$, $v_* = -6.1$, $v^* = -4$. On the boundaries and near the boundaries of the subsets, the optimal controls are defined in a special manner.

For the example in Fig. 9, the bounds on controls are the same as in the previous one, but the dynamics is of the form

$$\begin{aligned} \dot{x}_1 = &\quad 0.6x_1 + x_2 + v \\ \dot{x}_2 = &\quad -x_1 + u. \end{aligned}$$

The sets $W(\tau)$, $\tau = k \cdot 2\Delta,\ k = \overline{1, 130}$, are shown. The peculiarity here is that the barrier lines are practically tangent at $\tau = 2.4$. So, the very narrow channel connects the set where the optimal time is less than $\tau = 2.4$ with the set where the optimal time is more than $\tau = 2.4$. If we do small changes of the

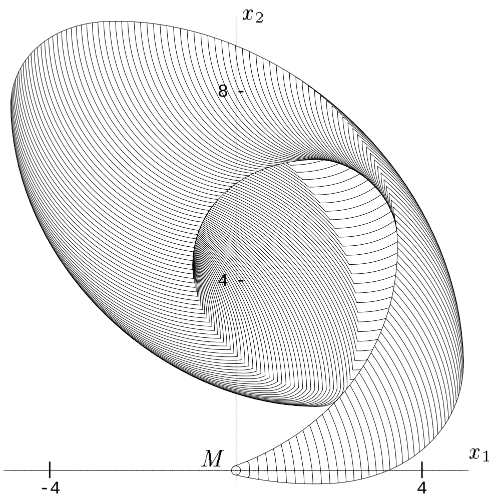


Fig. 7.

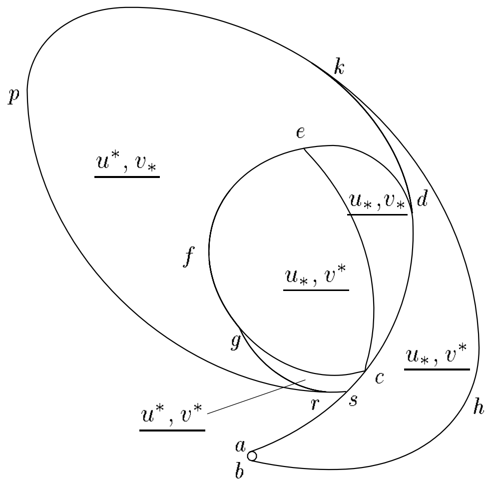


Fig. 8.

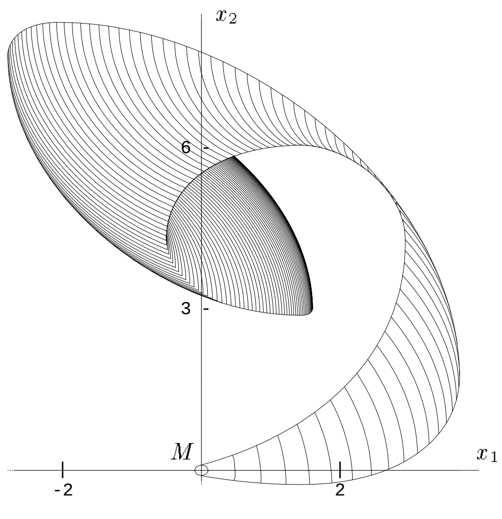


Fig. 9.

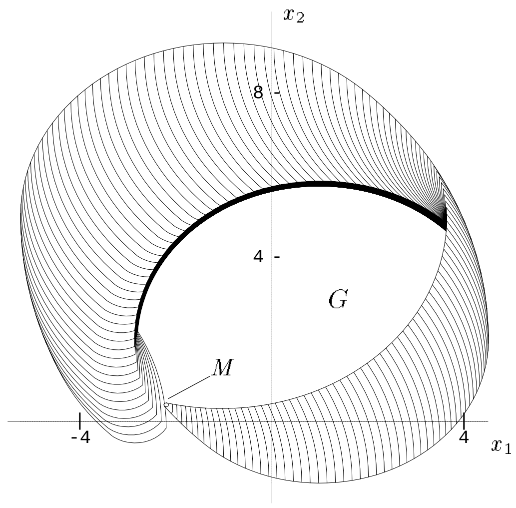


Fig. 10.

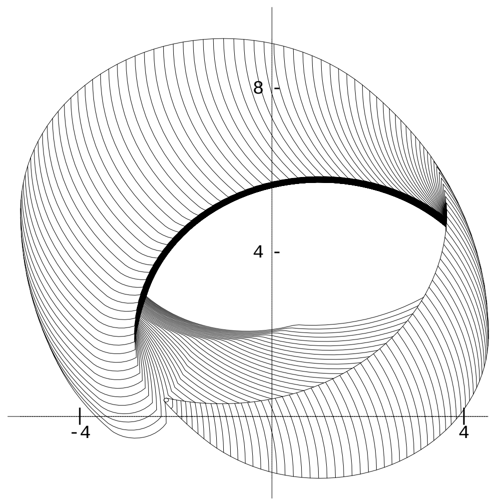


Fig. 11.

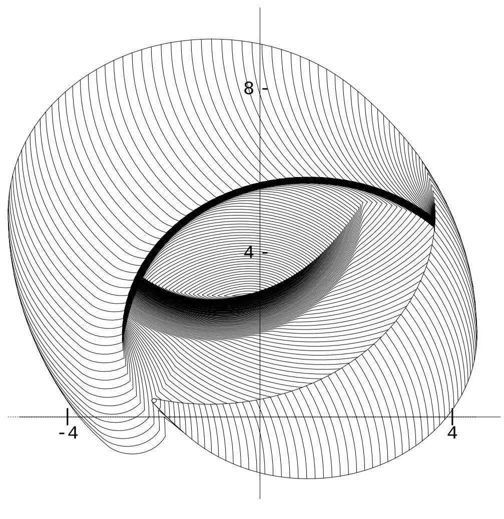


Fig. 12.

parameters of the problem (for example, if we reduce the set M), then the channel vanishes. As a result, the set where the optimal time is finite changes in a discontinuous manner.

Figures 10–12 correspond to the system with the dynamics

$$\begin{aligned} \dot{x}_1 &= \quad x_2 + u_1 + v_1 \\ \dot{x}_2 &= \ -x_1 + u_2 + v_2, \end{aligned}$$

$$u = (u_1, u_2)' \in P, \quad v = (v_1, v_2)' \in Q.$$

The set P is the vertical segment with the endpoints $(0, -2.5)$, $(0, 2.5)$, and Q is the segment with the apexes $(-5, 1.5)$, $(-1, -1.5)$. In Fig. 10, the calculations are carried out up to $\tau = 6.6$. At $\tau = 6.6$, the front collides with the terminal set M and is divided into two parts. Further constructions are made independently from these two parts. In Fig. 11, the constructions from the upper part are carried out until $\tau = 8$. The constructions are continued up to $\tau = 11.6$ in Fig. 12, and we are filling up the gap G. The front which corresponds to the maximal $\tau = 11.6$ is about the middle of G. In Fig. 12, only two fronts constructed from the lower part are shown. The accumulation of the fronts generates the dark regions in Figs. 10–12; that means very fast changing of the value function (though it is continuous).

CONCLUSION

The main difficulty in solving the minimax time-optimal problems is the necessity of doing (one way or another) the operations which are equivalent to the operations of summation and intersection of nonconvex sets. In the paper presented, the arising difficulties are overcome due to specific properties of the plane and dynamics' linearity. The level sets of the value function are constructed using backward procedures. On each step of the recurrent procedure, only some part of the boundary of the current level set is employed. The algorithm is fast enough. The computation time for the examples presented is small.

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