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GUIDE TO ESTIMATION UNDER CONDITIONS OF UNCERTAINTY AND CONSTRAINED NUMBER OF MEASUREMENTS

Part I
Estimation of a constant value and a set of parameters of a linear dependence

On behalf of the Initiative Group

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Abstract

This Guide was developed as a cooperative work of

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International Organization for Standardization (ISO),…, Comite International des Poids et Mesures (CIPM), … , National Insitute of Standards and Technology (USA), … , Deutsches Institut fur Normung, … , Federal Agency on Technical Regulation and Metrology of Russian Federation (GOSSTANDART), Ural Scientific-Research Institute of Metrology (UNIIM), Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences (IMM of UrB RAS), Center for Certification and Standardization (CERTIMET) of Presidium of UrB RAS, …

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…

It establishes general rules for estimation and procession of results of measuring under conditions of uncertainty of probabilistic characteristics in measurements and constrained number of measurements in the sample of values of dependences linear in the parameters to be estimated.

The Guide is elaborated on the basis of the Interval Analysis Theory and its applied branches. The Guide intends to be applicable to a broad spectrum of measurements and complements the ISO Standard "Guide to the Expression of Uncertainty in Measurements" (1993) onto the mentioned situation of processing data corrupted by errors and disturbances.

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SUPPLEMENTS

Foreword

A note of the Authors: *Here, under agreement of all participants on further stages of elaboration of this Guide, a short presentation has to be inserted about the approach on the basis of the Interval Analysis and its principal connection with estimation, identification, and procession of corrupted experimental results under conditions of uncertainty and constrained number of measurements ...*

Introduction

The overwhelming majority of Standards, Regulations, Guides and Rules for estimation and procession of corrupted signals, experimental data and information with errors are based on application of the probabilistic approaches. For instance, the methods of the mathematical statistics are used in the State Russian Standards (GOST) and in the ISO Standards of the European Community; the statistical Kalman Filter methods are used for estimation of the dynamical systems. Assumptions on the probabilistic properties of errors are made in the advanced ISO Standard "Guide to the Expression of Uncertainty in Measurements".

The main condition for application of these methods is direct knowledge of the probabilistic characteristics of disturbances and errors of measuring, or hypothesis on their distribution laws. As an output result in these statistical methods, a pointwise estimation of parameters of the physical quantity or the vector of parameters of a dependence are found, and the confidence intervals of the pointwise estimates are shown for the given value of the confidence probability.

The statistical methods work wrong or their application is impossible if the length of the sample is small, or the probabilistic characteristics of the disturbances and errors of measuring are incomplete or absolutely absent.

Nowadays, special methods of estimation and procession corrupted data on the basis of Interval Analysis Theory have been successfully elaborated in theoretical aspects. Methods of such a type are based on application of only a value of geometric constraint on disturbances and errors in measurements, some appropriate description of dependence (satisfying the researcher) to be estimated, and corresponding estimation procedures.

The typical formulation of such a problem corresponds to the case that is wide known and very important in practice. Here, the error in each measurement obeys only the geometric constraint. Namely, the constraint on the maximal value of its modulus is given. The internal structure of the error may be arbitrary – the absolute, relative or combined type. Inside of this constraint and in the sequence of measurements or along a dependence, the error may change arbitrarily, i.e., can take arbitrary value. On the basis of the given constraint, an ***uncertainty set*** is put in correspondence to each obtained measurement. This is a set of possible values of the dependence or the physical quantity that are consistent with this measurement and the structure of the error.

The problem of estimation is formulated in the following typical form.

Estimation of a physical quantity (value). The domain of its admissible values is estimated by intersection of all uncertainty sets of measurements of the sample. Such a domain is called the ***membership set*** or the ***informational set*** of all admissible states (values) consistent with the given sample of measurements and the structure of the error.

For the input sample, the minimal value of the error (i.e., actually presented in the sample) is found, for which the informational set is non-empty and consists of one point only.

If the input sample has outliers (for example, because of gross arbitrary errors), the informational set of the whole sample becomes empty. If so, the specialized procedures based on the graph theory are used. The collection of the uncertainty sets of measurements is described as a graph, which, further, is divided into a collection of complete subgraphs, and, respectively, the input sample is divided into a collection of consistent subsamples.

For obtained consistent subsamples, particularly, for the consistent subsample of the ***maximal length***, the necessary procession is carried out and output estimates calculated. The outliers are revealed by deviation of their uncertainty sets from the constructed informational sets of consistent subsamples.

Estimation of a set of admissible values of parameters for a given dependence (function). A dependence or a process to be estimated is described by some function of the given type (exact or approximate). This function depends on the argument and a vector of its parameters. A sample of corrupted measurements of the dependence is given. In a general case, the errors may be both in the process measurements and in the values of the argument.

To obtain fast numerical procedures and to have the opportunity to analyze consistency of the input sample, the special techniques are applied. The techniques for estimation are based on application of methods of the Interval Analysis Theory.

To each group of measurements of the input sample the ***partial informational set*** of parameters is put in correspondence. The length of each group is chosen to be sufficient for convenient description of corresponding partial informational set.

The ***informational set*** of parameter values of the dependence is sought for, i.e., the domain of values of the parameter vector, which are consistent with the given sample, is to be found. The informational set is found as intersection of the partial informational sets.

For the input sample, the ***minimal value of the error*** (i.e., actually presented in the sample) is found, for which the informational set is non-empty and consists of one point only.

Further, the found informational set is used to build the ***tube*** of the admissible values of the describing function, i.e., the domain of the dependence values consistent with the given sample.

If the input sample has outliers (for example, because of gross arbitrary errors), the informational set of the whole sample becomes empty. If so, the specialized procedures based on the graph theory are used. The collection of the partial informational sets of parameters is described as a graph, which, further, is divided into a collection of complete subgraphs, and, respectively, the input sample is divided into a collection of consistent subsamples.

For obtained consistent subsamples, particularly, for the consistent subsample of the ***maximal length***, the necessary procession is carried out and output estimates calculated.

The ***outliers*** are revealed by deviation of corresponding partial sets from the constructed informational sets of consistent subsamples.

In this Guide, the rules for two cases of estimation and procession of the corrupted sample under conditions of uncertainty of probabilistic characteristics of errors and disturbances and short length of the input measurement sample are considered: estimation of a constant physical quantity and estimation of a dependence linear in the parameters.

1 Scope

1.1 This document, hereafter referred to as the *Guide*, establishes general rules for estimation and procession of corrupted measurements under conditions of uncertainty of probabilistic characteristics of errors in physical measurements and short length of measurement sample. These rules should be followed at all levels of accuracy and in all fields – from shop floor to fundamental research. Therefore, the principles of this *Guide* are intended to be applicable to a broad spectrum of measurements including those required for:

- maintaining quality control and quality assurance in production;
- complying with and enforcing laws and regulations;
- conducting basic research, and applied research and development, in science and engineering;
- calibrating standards and instruments and performing tests throughout a national measurement system in order to achieve traceability to national standards;
- developing, maintaining and comparing international and national physical reference standards, including reference materials.

1.2 This *Guide* is primarily concerned with the estimation of uncertainty and procession of the measurement of a physical quantity for obtaining its unique, invariant value – the measurand – or a set of such quantities.

> NOTE – If the phenomenon of interest can be represented as a value or is dependent on one or more parameters, such as time, then the measurands required are the set of invariant quantities describing that dependence.

1.3 This *Guide* is also applicable to estimation and procession of the uncertainty associated with the conceptual design and theoretical analysis of experiment, and complex components and systems. Because a measuring result and its uncertainty may be conceptual and based entirely on hypothetical data, the term "result of a measurement" as used in this Guide should be interpreted in this broader context.

1.4 This *Guide* provides general rules for estimation and procession, and not detailed technology–specific instructions. Further, it does not discuss how the uncertainty of a particular measurement result, one estimated, may be used for different purposes, for example, to draw conclusions about the compatibility of that result with other similar results, to establish tolerance limits in a manufacturing processes, or to decide if a certain course of action may be safely undertaken. It may be necessary, therefore, to develop particular standards based on this *Guide* that deal with the problem peculiar to specific fields of measurement or with the various uses of quantitative measures of uncertainty. These standards should include the detail that is appropriate to the level of accuracy and complexity of the measurements and uses addressed.

2 Basic concepts and terms

2.1 Measurement

This notion coincides with the standard one. Namely. a ***measurement*** is a numerical result of measuring the state of an object, value, process, etc.

2.2 Description of the dependence measured

It is a ***theoretical or empirical function*** that to each true value of the argument (independent variable) puts in correspondence the true value of the dependence.

In the case of estimation of a constant value, the describing function has the form

$$y_{\mathrm{tr}i} = \mathrm{Const}, \quad \text{for all } i = 1, N, \tag{2.1}$$

where i is the number of the measurement; N is the length of the sample.

In the case of a linear dependence the describing function has the form

$$y_{\mathrm{tr}i} = y(x_{\mathrm{tr}i}) = a\, x_{\mathrm{tr}i} + b, \quad i = 1, N, \tag{2.2}$$

where i is the number of the measurement; N is the length of the sample; $x_{\mathrm{tr}i}$ is the ith true value of the argument; $y_{\mathrm{tr}i}$ is the true value of the dependence at the ith true value of the argument; a is the parameter (coefficient) of inclination; b is the constant (shift) parameter.

Each pair of parameters (a, b) is called the ***vector of parameters*** that defines some concrete describing function.

2.3 Errors and disturbances in measuring

The notion of the ***corrupted measurement*** is introduced in the following form:

$$y_i = y_{\mathrm{tr}i} + e_i, \tag{2.3}$$

where y_i is the corrupted measurement; $y_{\mathrm{tr}i}$ is the ture value to be measured; e_i is the resultant (summary) error of measuring.

In the general case the ***model of the resultant corrupting* error** (disturbance) is introduced in the following form:

$$e_i = \varepsilon_i + (1 + \delta_i)\, y_{\mathrm{tr}i} \tag{2.4}$$

where ε_i is the ***absolute component*** of the resultant error; δ_i is the relative error generating the ***relative component*** of the resultant error that depends of the true value $y_{\mathrm{tr}i}$ to be measured.

In one (important in practice) particular case the model of the resultant corrupting error can have only the absolute component

$$e_i = \varepsilon_i. \tag{2.5}$$

In another (also important in practice) particular case the model of the resultant corrupting error can have only relative component

$$e_i = (1 + \delta_i)\, y_{\mathrm{tr}i}\ . \tag{2.6}$$

The following quantitative property of errors is introduced.

It is said that the error is ***bounded geometrically*** if the following constraints on the maximal (in modulus) values of the errors are given:

$$|\ \varepsilon_i\ | \le \varepsilon_{max}\ , \tag{2.7}$$

$$|\ \delta_i\ | \le \delta_{max}\ , \tag{2.8}$$

where ε_{max} and δ_{max} are the geometrical constraints (bounds).

These constraints are to be given *a priori* by theoretical or empitical reasons.

Let call (2.4) – (2.8) the ***error*** **(*disturbance*)** ***model***.

2.4 Uncertainty set of a measurement

To each corrupted measurement y_i from the given input sample, the ***Uncertainty Set*** (***US***) is put in corresponcence. This set H_i is a totality of all values of the measured true value y_{tri} , which are consistent with this measurement and the given model of the error (2.4) – (2.8).

In the general case of the error model (2.4), the lower h_{lowi} and upper h_{uppi} boundaries of the *US* are calculated as follows:

$H_i= [h_{lowi}, h_{uppi}]$:

– if $y_i \le -\varepsilon_{max}$,

$$h_{lowi} = (y_i - \varepsilon_{max}) / (1 - \delta_{max}), \quad (2.9a)$$

$$h_{uppi} = (y_i + \varepsilon_{max}) / (1 + \delta_{max}), \quad (2.10a)$$

– if $-\varepsilon_{max} < y_i < \varepsilon_{max}$,

$$h_{lowi} = (y_i - \varepsilon_{max}) / (1 - \delta_{max}), \quad (2.9b)$$

$$h_{uppi} = (y_i + \varepsilon_{max}) / (1 - \delta_{max}), \quad (2.10b)$$

– if $\varepsilon_{max} < y_i$,

$$h_{lowi} = (y_i - \varepsilon_{max}) / (1 + \delta_{max}), \quad (2.9c)$$

$$h_{uppi} = (y_i + \varepsilon_{max}) / (1 - \delta_{max}), \quad (2.10c)$$

In the case of the error model (2.5) with only absolute component, the lower h_{lowi} and upper h_{uppi} boundaries of the *US* are calculated as follows:

$H_i = [h_{lowi} , h_{uppi}]$:

$$h_{lowi} = y_i - \varepsilon_{max} , \quad (2.11)$$

$$h_{uppi} = y_i + \varepsilon_{mai} . \quad (2.12)$$

In the case of the error model (2.6) with only relative component, the lower h_{lowi} and upper h_{uppi} boundaries of the *US* are calculated as follows:

$H_i = [h_{lowi} , h_{uppi}]$:

– if $y_i \le 0$,

$$h_{lowi} = y_i / (1 - \delta_{max}), \quad (2.13a)$$

$$h_{uppi} = y_i / (1 + \delta_{max}), \quad (2.14a)$$

– if $0 < y_i$,

$$h_{lowi} = y_i / (1 + \delta_{max}), \quad (2.13b)$$

$$h_{uppi} = y_i / (1 - \delta_{max}), \quad (2.14b)$$

2.5 Admissible parameters and functions

A value (a,b) of the parameter vector is ***admissible*** and corresponding function (2.2) is ***admissible***, if the following system of the double-side inequalities is satisfied:

$$(a,b): \ y(x_i,a,b) \in H_i, \text{ for all } i = \overline{1,N} . \quad (2.15a)$$

or

$$h_{lowi} \le a\,x_i + b \le h_{uppi} , \text{ for all } i= \overline{1, N}, \quad (2.15b)$$

2.6 Partial informational set

A totality G_{ij} of values (a, b) of the parameters vector defined by two corrupted measurements i and j is called the ***Partial Informational Set*** (***PIS***) if for them the following system of double-side inequalities holds

$$G_{ij}(a,b) = \{(a,b):\ ax_i + b \in H_i \text{ and } ax_j + b \in H_j. \quad (2.16a)$$

or

$$G_{ij}(a,b) = \{(a,b):\ h_{\text{lo}i} \le ax_i + b \le h_{\text{up}i} \text{ and } h_{\text{lo}j} \le ax_j + b \le h_{\text{up}j}. \quad (2.16b)$$

2.7 Informational set of parameters

A totality $I(a, b)$ of values (a, b) of the parameter vector defined by ***all*** measurements of the input sample is called the ***Informational Set*** (***IS***) if for them the following system of double-side inequalities holds

$$I(a, b) = \{ (a, b) : h_{\text{low}i} \le a x_i + b \le h_{\text{upp}i},\ \text{for all } i = 1, N \}. \quad (2.17)$$

For the linear describing function, the informational set is a convex polygon and is defined exactly by its vertices and linear boubndaries between them

$$I(a, b) = \{ (a_v, b_v),\quad v = 1, V \}, \quad (2.18)$$

where v is the number of the vertex; a_v, b_v are coordinates of the vertex; V is the number of vertices.

2.8 Consistent sample of measurements

A sample given is ***consistent*** if the set of admissible values of parameters is ***not empty***, i.e., the *IS* is not empty. Otherwise the input sample is called ***inconsistent***.

2.9 Tube of admissible values of a dependence

The ***tube*** T_i of admissible values of the describing function is a totality of its values generated by the admissible values of the parameters vector, i.e., by the informational set. For each value of the argument from the given interval $x_i \in [x_1, x_N]$, the lower $T_{\text{low}i}$ and upper $T_{\text{upp}i}$ frontiers of the tube are calculated. For a linear finction the calculation is implemented through all vertices (4.22) of the *IS*:

$$I(a, b) = \{ (a, b) : h_{\text{low}i} \le a x_i + b \le h_{\text{upp}i},\ \text{for all } i= 1, N \}. \quad (2.19)$$

$$T_{\text{lo}}(x_i) = \min\{a_m x_i + b_m\},\ \text{over } (a_m, b_m),\ m = 1, M, \quad (2.20)$$

$$T_{\text{up}}(x_i) = \max\{a_m x_i + b_m\},\ \text{over } (a_m, b_m),\ m = 1, M. \quad (2.21)$$

2.10 Practical considerations

In practice, model (2.3) – (2.6) of the error is chosen by some theoretical and empiric reasons. The level of constraints (2.7) and (2.8) on the maximal value of the errors componenets are given by *a priori* levels from the corresponding sertificate of the measuring unit or approximately on the basis of some previous or experimental research. For complete investigation of the input sample of corrupted measurements, the level of the constraints can be variated around the previously given *a priori* values.

3 Estimation of the admissible interval for an unknown constant value under conditions of uncertainty

3.1 Formulation of the problem

Some unknown constant value C is to be estimated. A corrupted measurement sample is given in the following form:

$$\{x_i, i = 1,...,N\}, \tag{3.1}$$

where x_i is a measurement; N is the length of the sample. Without losing generality, a homogeneous sample is considered.

The structure of corrupting is the following:

$$x_i = C + \varepsilon_i, \tag{3.2}$$

where ε_i is the error in the *i*th measurement.

Any probability characteristics of the error (such as probability distribution low, mean or square mean, etc.) are unknown. The errors in measurements obey the geometrical (in maximal modulus) constraint (consider only the case with the absolute component)

$$|\varepsilon_i| \le \varepsilon_{max}, \tag{3.3}$$

and the error "behavior" can be arbitrary in the limits of constraint (3.2). In more general case both the absolute and relative components can be present.

It is necessary to find the set I_C of admissiblee values of the constant C consistent with the given sample (3.1), the structure of corrupting (3.2), and constraint (3.3).

The set I_C is called the ***Informational Set*** (*IS*) and consists only of admissible values of the constant to be estimated.

3.2 Uncertainty sets of the measurements

The following ***Uncertainty Set*** (*UNS*) H_i is put in correspondence to each measurement of the sample (3.1):

$$H_i = [h_{loi}, h_{upi}], \; i = 1, N \tag{3.4a}$$

$$h_{loi} = x_i - \varepsilon_{max}, \tag{3.4b}$$

$$h_{upi} = x_i + \varepsilon_{max}, \tag{3.4c}$$

Here, h_{loi}, h_{upi} are the lower and upper bounds of the *UNS*. Each *UNS* of the constant to be estimated is a segment.

3.3 Analysis of the sample consistency

On the basis of the bounds (3.4b) and (3.4c) of the *UNS* H_i, the following auxiliary values are calculated (through all numbers $i = 1,...,N$):

$$h_{min} = \max_i h_{loi}, \tag{3.5}$$

$$h_{max} = \min_i h_{upi}, \tag{3.6}$$

These values are compared:

– if $h_{\min} > h_{\max}$, (3.7a)

it means that the sample is inconsistent, and further analysis is implemented by procedures of the Subection 3.5.

– if $h_{\min} \le h_{\max}$, (3.7b)

it means that the sample is consistent, and the values (3.5) and (3.6) are the lower and upper bounds of the informational set (IS) I_C

$$I_C = [h_{\min}, h_{\max}], \quad (3.8)$$

– if $h_{\min} = h_{\max}$,

the limit case of the sample consistency is present, and interval (3.8) degenerates into one point.

If some *a priori* interval

$$I^{\mathrm{ap}} = [x^{\mathrm{ap}}_{\min}, x^{\mathrm{ap}}_{\max}] \quad (3.9)$$

of possible values of the constant under estimation is known, then the bounds (3.8) can be enhanced by intersection of (3.8) and (3.9).

3.4 Output information

Besides interval (3.8), the following used in practice estimations are calculated.

The central estimation x_c is calculated as the center of the segment (3.8)

$$x_c = (h_{\max} + h_{\min})/2. \quad (3.10)$$

The maximal deviation Δx is calculated as a half of segment (3.8)

$$\Delta x = (h_{\max} - h_{\min})/2. \quad (3.11)$$

The deviation γ_i of each measurement from the central estimation is determined

$$\gamma_{\mathrm{i}} = x_{\mathrm{i}} - x_c, \quad i = 1,\dots,N. \quad (3.12)$$

3.5 Detection of single outliers

The procedures of this subsection is implemented in the case of detection by (3.7a) the inconsistency of the origin sample (3.1). The collection of all pairwise intersections of all *UNS* pairs H_i and H_j is calculated

$$P_{ij} = H_i \cap H_j, \quad i = 1,\dots,N-1, \quad j = i+1,\dots,N. \quad (3.13)$$

This procedure is realized by formulas

$$p_{\mathrm{lo}ij} = \max\{h_{\mathrm{lo}i}, h_{\mathrm{lo}j}\}, \quad (3.14)$$

$$p_{\mathrm{up}ij} = \min\{h_{\mathrm{up}i}, h_{\mathrm{up}j}\}. \quad (3.15)$$

Then

– if $p_{\mathrm{low}ij} > p_{\mathrm{upp}ij}$,

it means that the set P_{ij} is ***empty***, and this pair of measurements has ***zero*** value of the ***incidence flag*** $S_{ij} = 0$.

– if $p_{\mathrm{low}ij} \le p_{\mathrm{upp}ij}$, it means thet the set P_{ij} exists (is ***nonempty***), the values (3.14) and (3.15) are its bounds, and for this pair of measurements the unit value of the ***incidence flag*** is given $S_{ij} = 1$.

So the ***table of incidence*** has been composed

$$\{S_{ij},\ S_{ji} = S_{ij},\quad i = 1, N,\ j = 1, N,\ j \neq i\}, \tag{3.16}$$

which shows a pair-incidence of measurements in the origin sample.

The ***single outliers (gross errors)***. The measurement with a number i is a single outlier, if its row $\{S_{ij},\quad j = 1, N\quad j \neq i\}$ is entirely composed of zeros.

The outliers detected are eliminated out of the sample, i.e., the sample is truncated.

The truncated sample is processed again by the procedures of the Subsection 3.3. If the inconsistency stays, the further processing of the truncated sample is implemented by procedures of the Subsection 3.6.

3.6 Partition of the inconsistent sample into a collection of consistent subsamples

The sample is represented as an imaginary graph, where the measurements are the graph vertices. Consistency of each pair of measurements is given by the value of the flag S_{ij} in the table of incidence.

The partition of the original inconsistent sample into the collection of consistent subsamples is, in its essence, the standard procedure of detection the collection of all complete subgraphs from the original graph. Special fast version of such a procedure has been elaborated.

The most important for practice is the ***consistent subsample of the maximal length***.

3.7 Estimation of the minimal level of the error

For the given sample (3.1) special practically meaning index can be calculated, which could give a lower estimation of the error level in the sample. Namely, the ***minimal level of the error***. It is calculated by the following procedures:

– if the sample is consistent,

then in relations (4b) and (4c) the level $\varepsilon_{\max}$ of the constraint is decreased down to the value $\varepsilon^*_{\max}$, for which segment (3.8) degenerates to a point

$$\varepsilon^*_{\max}:\ I^* = x^* = h_{\min} = h_{\max}; \tag{3.18}$$

– if the sample is inconsistent,

then in relations (3.4b) and (3.4c) the level $\varepsilon_{\max}$ of the constraint is increased up to the value $\varepsilon^*_{\max}$, for which the pointwise set (3.9) appears.

In the case of estimation of an unknown constant, the limit values can be calculated directly (without some searching process)

$$\varepsilon^*_{\max} = (x_{\max} - x_{\min})/2, \tag{3.19}$$

$$x^* = (x_{\max} + x_{\min})/2, \tag{3.20}$$

where x_{max}, x_{min} are the maximal and minimal values of measurements in the input sample (3.1).

Application of the described approach to analysis of measurements sample of an unknown constant is illustrated by Examples in the **Supplement I**. The comparison of the results with ones obtained by the standard statistical procedures is also given there.

4 Estimation of the set of parameters for the linear function under conditions of uncertainty

4.1 Formulation of the problem

The dependence is described by a linear function

$$y(x) = ax + b, \qquad (4.1)$$

where x is the argument; y is a value of the function; a,b are unknown parameters, the coefficient of inclination and the constant, correspondingly. Without losing the discussion generality, the case is considered where the incline coefficient is definite $|a| < \infty$.

A corrupted measurement sample is given in the following form:

$$\{y_i, x_i, i = 1,..., N\}, \qquad (4.2)$$

where y_i is a measurement for the value x_i of the argument; N is the length of the sample. Without losing generality, a homogeneous sample is considered.

Consider the most known in practice case when in measuring the argument values are known exactly, and they are ordered by increasing $x_1 < x_2 < ... < x_{N-1} < x_N$.

The structure of corrupting is the following:

$$y_i = y_{\text{tr}i} + \varepsilon_i, \qquad (4.3)$$

where $y_{\text{tr}i}$ is the unknown true value; ε_i is the error in the ith measurement.

Any probability characteristics of the error (such as probability distribution low, mean or square mean, etc.) are unknown. The errors in measurements obey the geometrical (in maximal modulus) constraint (consider only the case with the absolute component)

$$|\varepsilon_i| \le \varepsilon_{max}, \qquad (4.4)$$

and the error “behavior” can be arbitrary in the limits of the constraint (4.4).

***It is necessary*:**

a) to find the ***two-dimensional*** set $I(a,b)$ of admissible values of the coefficient a and constant b consistent with the given sample (4.2), the structure of corruptimg (4.3) and constraint (4.4);
b) to build the set $T(x)$ of admissible values of the function (4.1).

The set $I(a,b)$ is called the ***Informational Set*** (*IS*) and consists only of admissible values of the parameters to be estimated. The set $T(x)$ is called the ***Tube*** (*TB*) of the function and consists of curves of the admissible functions only.

4.2 Uncertainty sets of the measurements

The following ***Uncertainty Set*** (*UNS*) H_i is put in correspondence to each measurement of sample (4.2):

$$H_i = [h_{\text{lo}i},\, h_{\text{up}i}],\ i = \overline{1,N} \qquad (4.5a)$$

$$h_{\text{lo}i} = y_i - \varepsilon_{\max}, \qquad (4.5b)$$

$$h_{\text{up}i} = y_i + \varepsilon_{\max}, \qquad (4.5c)$$

Here, $h_{\text{lo}i}$, $h_{\text{up}i}$ are the lower and upper bounds of the *UNS*. Each *UNS* is an interval.

For the case of linear function the following definitions are introduced:

a) A value (a,b) of the parameter vector is ***admissible*** and corresponding function (4.1) is ***admissible***, if the following system of the double-side inequalities is satisfied:

$$(a,b):\ y(x_i,a,b) \in H_i,\ \text{ for all } i = \overline{1,N},$$

or

$$h_{\text{lo}i} \le ax_i + b \le h_{\text{up}i},\ \text{ for all } i = \overline{1,N}. \qquad (4.6)$$

b) A sample given is ***consistent*** if the set of admissible values of parameters is ***not empty***, i.e., the *IS* is not empty.

c) Take a pair of measurements (x_i, y_i) and (x_j, y_j) for arbitrary $i < j$. A set $G_{ij}(a,b)$ of parameter values (a,b) consistent with the uncertainty sets H_i and H_j of these measurements is called the ***Partial Informational Set*** (*PIS*) if the following system of double-side inequalities holds

$$G_{ij}(a,b) = \{\ (a,b):\ ax_i + b \in H_i \text{ and } ax_j + b \in H_j\ \},$$

or

$$G_{ij}(a,b) = \{\ (a,b):\ h_{\text{lo}i} \le ax_i + b \le h_{\text{up}i} \text{ and } h_{\text{lo}j} \le ax_j + b \le h_{\text{up}j}\ \}. \qquad (4.7)$$

Remark 4.1. *It follows from* (4.7) *that the IS* $I(a,b)$ *to be found satisfies the similar system of inequalities*

$$I(a,b) = \{(a,b): h_{\text{lo}i} \le ax_i + b \le h_{\text{up}i},\ \text{ for all } i = \overline{1,N}\ \}. \qquad (4.8)$$

4.3 Possible intervals of parameters

In the plane of parameters $a \times b$, the standard right coordinate system $a0b$ is introduced. The axis $0a$ is the abscissa, the axis $0b$ is the ordinate.

The first step of the sample analysis is in calculating the ***possible intervals*** of values of the parameters. For each pair of measurements the interval of possible values of parameter a is

$$a_{\min ij} = (h_{\text{lo}j} - h_{\text{up}i})/(x_j - x_i), \qquad (4.9a)$$

$$a_{\max ij} = (h_{\text{up}j} - h_{\text{lo}i})/(x_j - x_i). \qquad (4.9b)$$

By intersection of these intervals through all pairs of the sample (4.2), the auxiliary values are obtained

$$a_{\min} = \max_{i=1,N-1;\,j=i+1,N} \{a_{\min ij}\}, \tag{4.10}$$

$$a_{\max} = \min_{i=1,N-1;\,j=i+1,N} \{a_{\max ij}\}. \tag{4.11}$$

Similar computations are fulfilled for parameter b:

$i = 1, N-1$, $j = i+1, N$,

$$b_{\max ij} = (h_{\mathrm{lo}_i} / x_i - h_{\mathrm{up}_j} / x_j) \cdot x_j \cdot x_i / (x_j - x_i), \tag{4.12a}$$

$$b_{\max ij} = (h_{\mathrm{up}_i} / x_i - h_{\mathrm{lo}_j} / x_j) \cdot x_j \cdot x_i / (x_j - x_i), \tag{4.12b}$$

$$b_{\min} = \max_{i=1,N-1;\,j=i+1,N} \{b_{\min ij}\}, \tag{4.13a}$$

$$b_{\max} = \min_{i=1,N-1;\,j=i+1,N} \{b_{\max ij}\}. \tag{4.13b}$$

4.4 Preliminary analysis of consistency of the input sample and unconditional intervals of parameters

If $a_{\min} \le a_{\max}$, (4.14)

then the ***unconditional interval*** of a exists, i.e., is ***non-empty***, and equals

$$I_a = [a_{\min}, a_{\max}]: \tag{4.15}$$

If $b_{\min} \le b_{\max}$, (4.16)

then the ***unconditional interval*** of b exists, i.e., is ***non-empty***, and equals

$$I_b = [b_{\min}, b_{\max}]: \tag{4.17}$$

Remark 4.2. *The simultaneous* ***non-emptiness*** *of both intervals* I_a *and* I_b *is the* ***necessary condition*** *of the sample* (4.2) *consistency. In this case, the precise analysis has to be carried out.*

Remark 4.3. *If non-empty, both intervals* (4.12) *and* (4.17) *compose the rectangle* $I_a \times I_b$ *that gives the minimal upper box-approximation of the exact IS for the consistent original sample for the case of the lunear function estimated. If the intervals are non-empty, but, nevertheless, the original sample is not consistent, then the rectangle is the minimal upper box-approximation of a collection of the exact informational sets of all consistent subsamples of the original sample.*

Remark 4.4 *If both intervals* (4.12) *and* (4.17) *are* ***empty****, then the precise analysis of inconsistency is necessary.*

Remark 4.5 *If only one of the intervals* (4.12) *or* (4.17) *is* ***empty****, then the precise analysis of inconsistency is necessary.*

4.5 Calculation of partial informational sets

Recall that the non-emptiness of the unconditional box-estimationgives only necessary conditions for the sample consistency. So, having constructed these intervals, the more sophisticated information for the precise analysis is to be prepared.

The collection of ***Partial Informational Sets*** (*PIS*'es) which are originated by each pair of measurements is considered.

$$\{G_{ij},\ j=\overline{1,N-1},\ j=\overline{i+1,N}\}. \qquad (4.18)$$

Each *PIS* in (4.18) is a closed bounded convex polygon with four vertices, namely, the parallelogram. Let us to count the vertices clockwise in the mentioned coordinate system in the plane of parameters. The vertices coordinates are calculated by the formulas:

$$\begin{aligned}
&\{G_{ij},\ i=\overline{1,N-1},\ j=\overline{i+1,N}\}:\\
&a_1(i,j)=(h_{\text{lo}j}-h_{\text{up}i})/(x_j-x_i),\\
&b_1(i,j)=h_{\text{up}i}-a_1(i,j)x_i,\\
&a_2(i,j)=(h_{\text{up}j}-h_{\text{lo}i})/(x_j-x_i),\\
&b_2(i,j)=h_{\text{lo}i}-a_2(i,j)x_i,\\
&a_3(i,j)=(h_{\text{lo}j}-h_{\text{lo}i})/(x_j-x_i),\\
&b_3(i,j)=h_{\text{lo}i}-a_3(i,j)x_i\\
&a_4(i,j)=(h_{\text{lo}j}-h_{\text{up}i})/(x_j-x_i),\\
&b_4(i,j)=h_{\text{up}i}-a_4(i,j)x_i.
\end{aligned} \qquad (4.19)$$

4.6. Exact analysis of the sample consistency and computation of the informational set

By the special fast program of intersection of convex polygons, the sequential intersection of sets (4.19) is fulfilled

$$\{\bigcap_{j=\overline{2,N}} G_{1j}\}\cap\{\bigcap_{j=\overline{3,N}} G_{2j}\}\cap\ldots\cap G_{N-1,N}. \qquad (4.20)$$

If on some step of the procedure (4.20) the ***emptiness*** of intersection appears, it means that the original sample (4.2) is inconsistent; then its further analysis is to be implemented (see Subsection 4.9 and 4.10).

If the emptiness does not appear, the operation (4.20) is fulfilled to the end. Its result is the desirable informational set, i.e., the totality of all admissible values of the parameter vector

$$IS=\bigcap_{i=\overline{1,N-1},\ j=\overline{i+1,N}} G_{ij}, \qquad (4.21)$$

consistent with the given function (4.1), sample (4.2), structure of corrupting (4.3), and constraint (4.4).

In the plane of parameters, the *IS* is a convex polygon with linear boundaries and a collection of vertices:

$$IS:(a_m,b_m),\ m=\overline{1,M}, \qquad (4.22)$$

where (a_m,b_m) are coordinates of the vertices; M is the number of vertices.

If some *a priori* domain of possible parameter vector values is known

$$IS_0 = [a^{ap}_{min}, a^{ap}_{max}] \times [b^{ap}_{min}, b^{ap}_{max}], \qquad (4.23)$$

then the *IS* (4.21) and (4.22) can be enhanced by intersection with the domain (4.23). When having made it, the number of vertices in (4.22) can be varied.

The *IS* found allows to obtain necessary output estimations.

Remark 4.6 *Note that if the IS has a form of a narrow inclined band* (*from one corner of the box to another its corner*), *the box-approximation by* (4.23) *can be very inexact. But in this case, the IS gives exact description of boundaries for the set of admissible parameter values.*

4.7 Conditional intervals of parameters

More sophisticated information about structure of the *IS* (4.22) is given by constructing ***conditional intervals*** of admissible parameter values. Such intervals are sections of the *IS* by the lines, parallel to the abscissa or ordinate axis of the mentioned coordinate system in the plane of coordinates.

For each value of the parameter b from the unconditional interval (4.17), the conditional interval for the parameter a is calculated

$$I(a/b) = [a_{\min}(b),\ a_{\max}(b)],\ \ b \in I_b\,. \qquad (4.24)$$

Similarly, for each value of the parameter a from the unconditional interval (4.15), the conditional interval for the parameter b is calculated

$$I(b/a) = [b_{\min}(a),\ b_{\max}(a)],\ \ a \in I_a\,. \qquad (4.25)$$

4.8 Calculation of the admissible tube of the dependence

On the basis of the *IS* (4.22), the ***tube*** of admissible values of the function is constructed. The lower $T_{lo}(x_i)$ and upper $T_{up}(x_i)$ doundaries of the tube are calculated for each value of the argument from the given interval $x_i \in [x_1,\ x_N]$. In the case of r\eatimation of the linear function, it is sufficient to implement the calculation only through all vertices (4.22) of the *IS*:

$$T_{lo}(x_i) = \min\{a_m x_i + b_m\},\ \text{over}\ (a_m, b_m),\ m = \overline{1, M}\,, \qquad (4.26)$$

$$T_{up}(x_i) = \max\{a_m x_i + b_m\},\ \text{over}\ (a_m, b_m),\ m = \overline{1, M}\,. \qquad (4.27)$$

4.9 Partition of the inconsistent sample into a collection of consistent subsamples

The procedures of this subsection is implemented in the case of detecting the inconsistency of the origin sample (4.2) during calculation of the unconditional intervals (4.12) and (4.17) in Subsection 4.3 or during calculation (4.20) of the informational set in Subsection 4.5.

The *PIS'*es are enumerated as line sequence

$$\{G_{ij},\ i = \overline{1, N-1},\ j = \overline{i+1, N}\} \Rightarrow \{\Gamma_k,\ k = \overline{1,\ N(N-1)/2}\}. \qquad (4.28)$$

By the special fast program of intersection of convex polygons, the pair-intersections of the partial informational sets (4.28) are calculated, and the ***table of incidence flags*** is composed

$$S_{kl} = \begin{cases} 1, & \text{if } \Gamma_k \cap \Gamma_l \text{ is non-empty,} \\ 0, & \text{if } \Gamma_k \cap \Gamma_l \text{ is empty,} \end{cases} \qquad (4.29)$$
$$k = 1,\ N(N-1)/2-1,\ \ l = k+1,\ N(N-1)/2.$$

The collection (4.28) is regarded as some graph, where each set Γ_k is the graph vertex. Consistency of each pair of the vertices is given by the value of the flag S_{kl} in the table of incidence.

The partition of the original inconsistent sample into the ***collection of consistent subsamples*** is, in its essence, the standard procedure of detection the ***collection of all complete subgraphs*** from the original graph.

Связь каждой пары парциальных информационных множеств $\{G_{ij},\ i=1,N-1,\ j=i+1,N\}$ в их последовательности $\{\Gamma_k\}$ является ребром графа

Incidences of each pair of the partial informational sets $\{G_{ij},\ i=1,N-1,\ j=i+1,N\}$ in their sequence $\{\Gamma_k\}$ (4.28) can be regarded as the edges of the graph. So, the operation of intersection $\Gamma_k \cap \Gamma_l$ in (4.29) is, in its essence, a check of consistency of a face of the original graph, i.e., the check of consistency of the triple of measurements of the original sample of measurements. So, each complete subgraph of $\{\Gamma_k, k \in K\}$, where K is a collection of this subgraph vertex numbers, determines a corresponding sequence $\{i \in N_K\}$ of measurements numbers, composing some ***consistent subsample*** of measurements from the original sample.

The most for practice is the ***consistent subsample of the maximal length***. Having this subsample been selected, all necessary calculations are fulfilled: the unconditional intervals, informational set, tube and the output estimations.

In practice, construction of informational sets for other consistent subsamples of representative length can be useful.

Special fast version of the procedure for partition of the inconsistent sample (4.2) into a collection of consistent subsamples has been elaborated.

4.10 Estimation of the minimal level of the error

For the given sample (4.2) special practically meaning index can be calculated, which could give a lower estimation of the error level that has realized in the sample. Namely, the ***minimal level of the error***. It is calculated by the following procedures.

If the sample is consistent, then in relations (4.5b) and (4.5c) the level $\varepsilon_{\max}$ of the constraint is decreased down to the value $\varepsilon^*_{\max}$, for which the informational set (4.22) degenerates to a point

$$\varepsilon^*_{\max}:\ I^*(a,b) = (a^*, b^*)\,. \qquad (4.30)$$

If the sample is inconsistent, then in relations (4.5b) and (4.5c), the level $\varepsilon_{\max}$ of the constraint is increased up to the value $\varepsilon^*_{\max}$, for which the pointwise set (4.22) appears.

Application of the described approach to analysis of measurements sample of an unknown linear function (4.1) is illustrated by Example in the **Supplement II**. The comparison of the results with ones obtained by the standard statistical procedures is also given there.