Semigroup property of the programmed absorption operator in DGs with simple motions in the plane

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1. Problem statement

M-approaching DG with simple motions:

$$\dot{x} = p + q, \quad x \in \mathbb{R}^n, \quad t \in [0, \vartheta] \quad \text{(fixed time)},$$

$$p \in P \quad \text{(1st player)}, \quad q \in Q \quad \text{(2nd player)},$$

P, Q are closed convex sets in $\mathbb{R}^n$.

M is a closed terminal set:

the 1st player aims $x(\vartheta) \in M$, the 2nd player aims $x(\vartheta) \notin M$.

Consider the set

$$W := \{(t_0, x_0) \in [0, \vartheta] \times \mathbb{R}^n : \text{ the 1st player guarantees that } x(\vartheta) \in M\}.$$

W is widely known as the maximal stable set (Krasovskii bridge), or viability kernel, or Pontryagin's alternating integrals.

How to describe exactly and constructively the t-sections $W(t)$, $t \in [0, \vartheta]$, of the set W?

2.1 Some known results: Cauchy problem for HJBI equation

$$
\begin{aligned}
&w_t(t,x) + H(w_x(t,x)) = 0, \quad t \in (0,\vartheta), \qquad x \in \mathbb{R}^n,\\
&w(\vartheta,x) = \sigma(x), \qquad\qquad\qquad\qquad\quad x \in \mathbb{R}^n,
\end{aligned}
$$

$$
H(s) = \max_{q\in Q}\langle s,q\rangle + \min_{p\in P}\langle s,p\rangle = \rho(s;Q) - \rho(s;-P), \qquad \sigma(x) := \begin{cases} 0, & x \in M,\\ +\infty, & x \notin M, \end{cases}
$$

$\rho(s;A) := \sup\{\langle s,a\rangle : a \in A\}$ (support function).

There exists a unique lower semicontinuous generalized solution $w(t,x)$ (minimax or viscosity) [Subbotin: 1995], [Bardi, Capuzzo-Dolcetta: 1997] and

$$
W = \{(t,x) \in [0,\vartheta] \times \mathbb{R}^n : w(t,x) = 0\}.
$$

2.2 Some known results: Hopf formula for convex M

M is convex $\Rightarrow \sigma(x)$ is convex.

The Hopf formula:

$$w(t,x) = \sup_{s\in\mathbb{R}^n} \inf_{y\in\mathbb{R}^n} [\sigma(y) + \langle s, x-y\rangle + (\vartheta - t)H(s)] =$$

$$= \sup_{s\in\mathbb{R}^n} [\langle s, x\rangle + (\vartheta - t)H(s) - \sigma^*(s)] =$$

$$= \Big[\sigma^*(\cdot) - (\vartheta - t)H(\cdot)\Big]^*(x),$$

$$\sigma^*(s) = \sup_{s\in\mathbb{R}^n} [\langle s, x\rangle - \sigma(x)] \quad \text{(the Legendre conjugate)}$$

[Hopf: 1965], [Bardi, Evans: 1984], [Ishii, Barron, Alvarez: 1999]

2.3 Some known results: Pshenichnyi – Sagaidak formula

[Pontryagin: 1967], [Pshenichnyi, Sagaidak: 1970]

$$M \text{ is convex} \quad \Rightarrow \quad W(t) = (M - (\vartheta - t)P) \overset{*}{-} (\vartheta - t)Q.$$

Here,

$A + B := \{d : \; d = a + b, \; a \in A, \; b \in B\}$ is an algebraic (Minkowski) sum,

$A \overset{*}{-} B := \{d : \; d + B \subseteq A\}$ is a geometrical (Minkowski) difference.

Define the operator

$$M \to T_\tau(M) := (M - \tau P) \overset{*}{-} \tau Q, \quad \tau = \vartheta - t$$

(T_τ is known as the "programmed absorption operator".)

We have

$$W(t) = T_{\vartheta - t}(M) = \{(t, x) : \sup_{s \in \mathbb{R}^n} [\, \langle s, x \rangle - \rho(s; T_{\vartheta - t}(M)) \,] \leqslant 0\} = \ldots$$

$$= \{(t, x) : \sup_{s \in \mathbb{R}^n} [\, \langle s, x \rangle - \rho(s; M) + (\vartheta - t) H(s) \,] \leqslant 0\} = \ldots = \{(t, x) : w(t, x) \leqslant 0\}.$$

2.4 Some known results: semigroup property of T_τ

[Pshenichnyi, Sagaidak: 1970] $\quad M$ is closed (nonconvex) $\Rightarrow$

$$W(t) = \bigcap_{\tau_1+\tau_2+\cdots+\tau_m=\vartheta-t} T_{\tau_1}(T_{\tau_2}(\ldots T_{\tau_m}(M)\ldots)) =: \widetilde{T}_{\vartheta-t}(M).$$

(Operator $\widetilde{T}_\tau$ is called an operator with multiple recomputation or "positional absorption operator".)

Operators T_τ and $\widetilde{T}_\tau$ are equal if we have

$$T_{\tau_1}(T_{\tau_2}(M)) = T_{\tau_1+\tau_2}(M) \tag{1}$$

$\forall\ \tau_2, \tau_1 + \tau_2 \in [0,\tau]$ (semigroup property).

We have

$$M \text{ is convex} \Rightarrow (1).$$

Our problem is reduced to the following one: how to formulate conditions for M, P, Q, and τ_1, τ_2, which provide equality (1).

3.1 Our results: Theorem ($\mathbb{R}^2$; nonconvex M)

Suppose

(T1) $M \subset \mathbb{R}^2$ is closed & bounded & simply connected (without holes);
$P \subset \mathbb{R}^2$ is a convex k-polygon, $k \geq 2$ (a segment if $k = 2$);
$\mathcal{V}$ is a set of external normal vectors to P (for $k = 2$: $\mathcal{V} = \{\nu, -\nu\}$, $\nu \perp P$);
for any $x \in M$ and $\nu \in \mathcal{V}$, the set

$$\Pi_M(x, \nu) := M \cap \{z \in \mathbb{R}^2 : \langle z, \nu\rangle \leqslant \langle x, \nu\rangle\}$$

is connected;

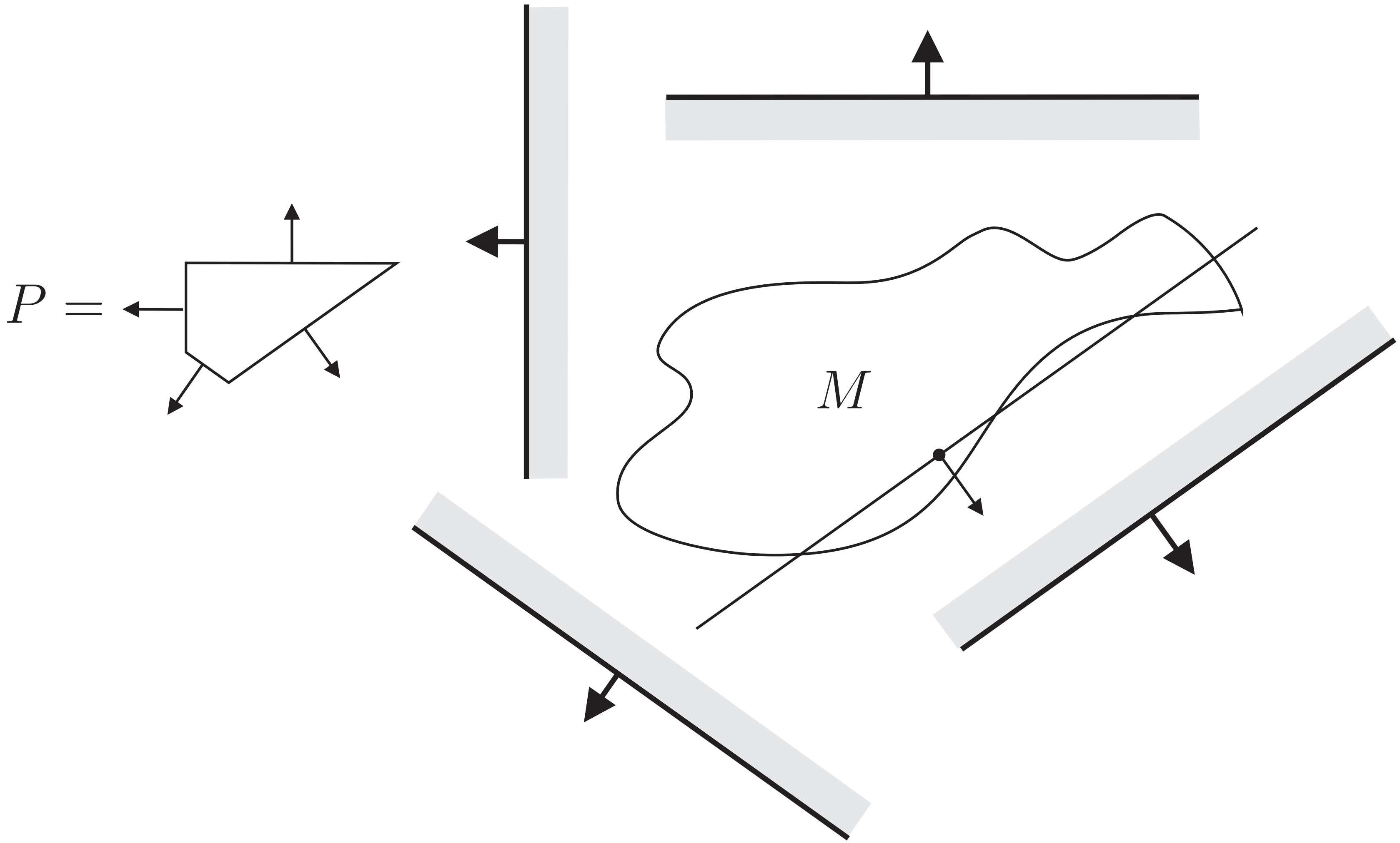
(T2) for any $\tau \in [0, \vartheta]$, the set $T_\tau(M) \neq \varnothing$ is connected; and for any $\nu \in \mathcal{V}$, the function

$$\tau \mapsto \delta_\nu(\tau) := \rho(-\nu; M) - \tau H(-\nu) - \rho(-\nu; T_\tau(M))$$

is non-decreasing in $[0, \vartheta]$.

Then the operator T_τ has a semigroup property over the segment $[0, \vartheta]$.

(And, consequently, $W(t) = T_{\vartheta - t}(M)$, $t \in [0, \vartheta]$.)



$P =$
M

3.2 Our results: Proposition (if M is a polygon)

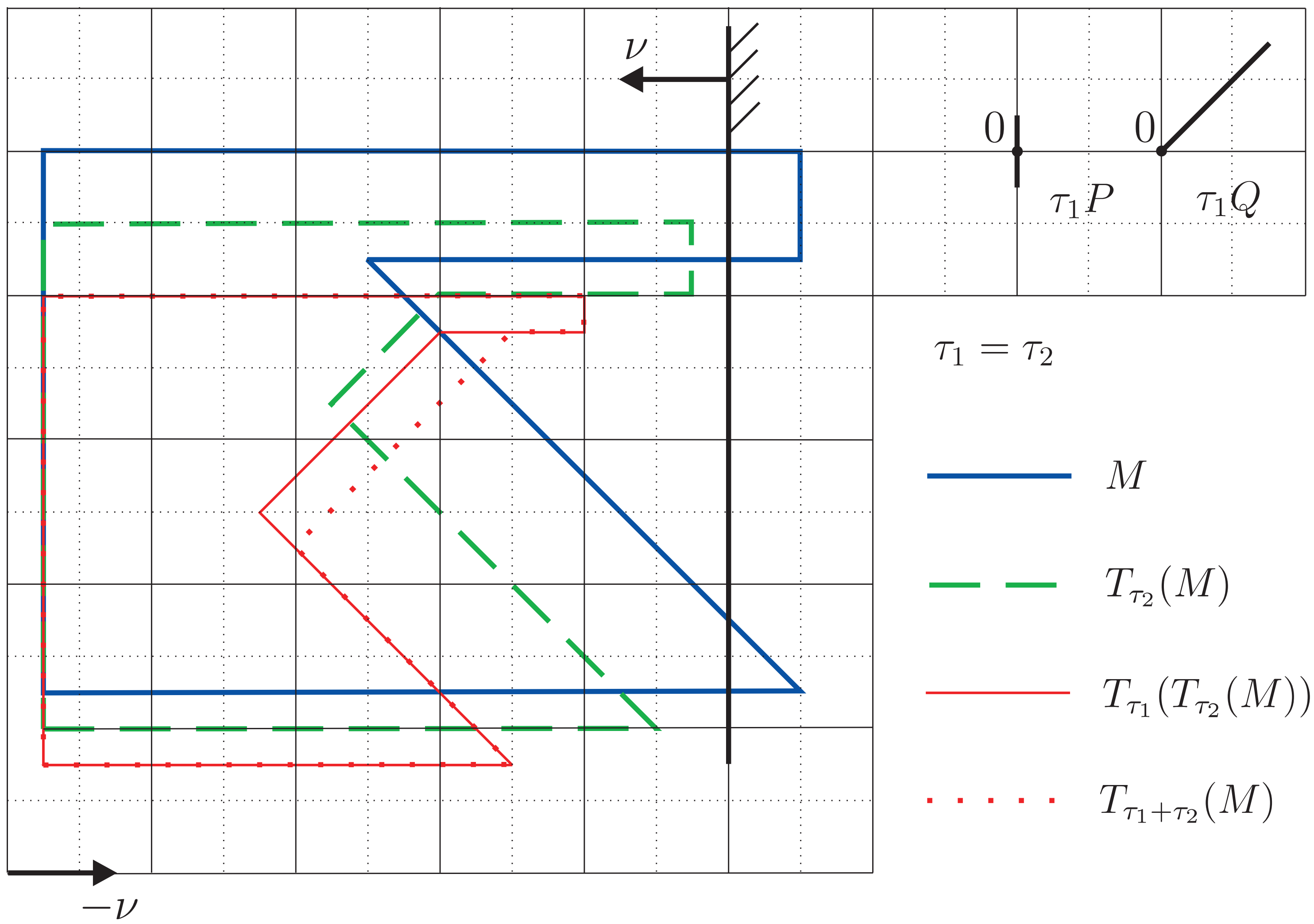
Suppose M is a non-degenerate polygon and condition (T1) of the theorem is satisfied.

Then $\exists\ \bar{\vartheta} > 0$ such that

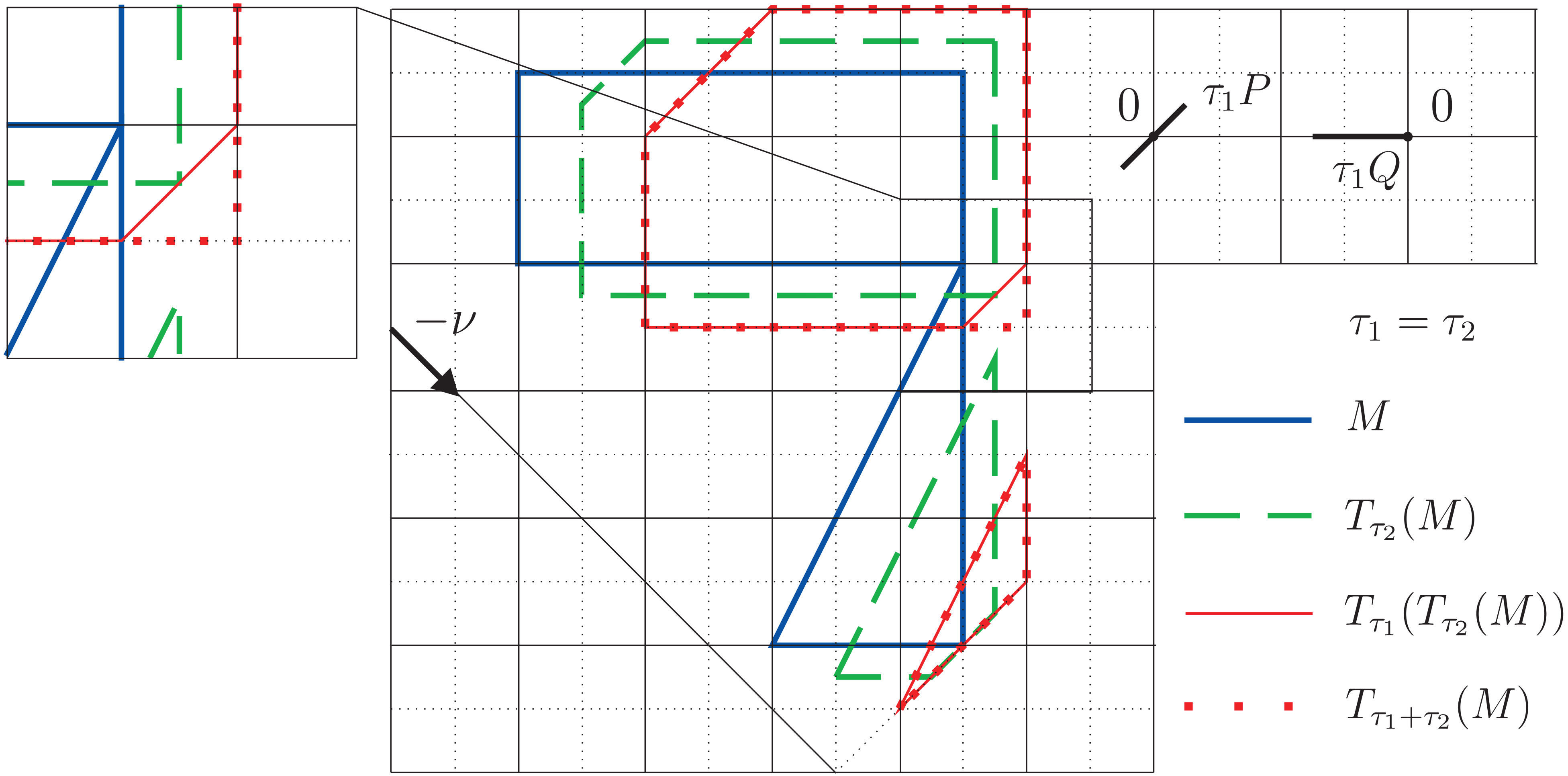
$$T_{\tau_1}(T_{\tau_2}(M)) = T_{\tau_1+\tau_2}(M), \quad \tau_2,\ \tau_1 + \tau_2 \in [0, \bar{\vartheta}].$$

(And, consequently, $W(t) = T_{\bar{\vartheta}-t}(M)$, $t \in [0, \bar{\vartheta}]$.)

3.3 Example 1: $\exists \nu \in \mathcal{V}\ \exists x \in M :\ \Pi_M(x, \nu)$ is not connected

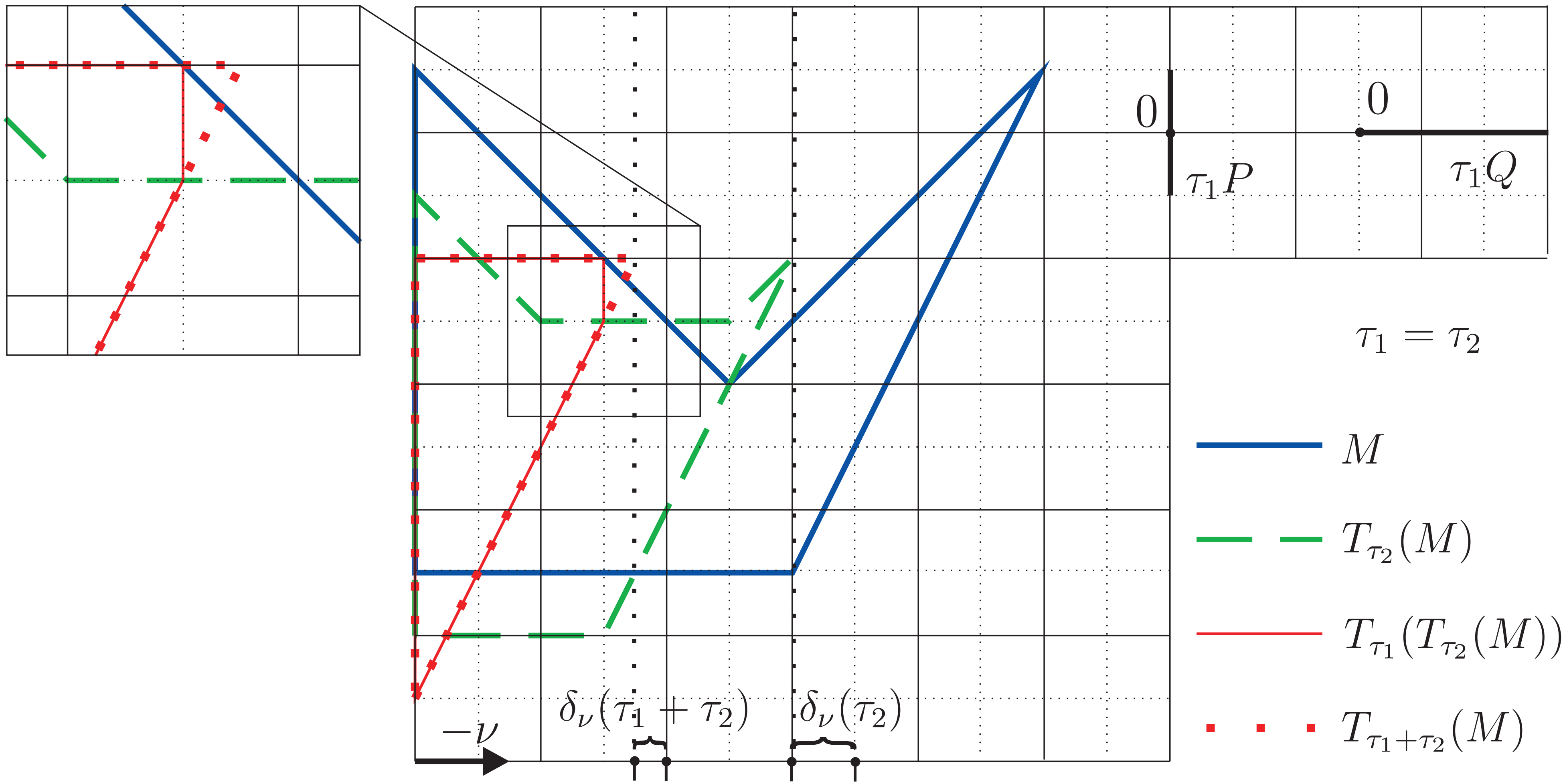


3.3 Example 2: $T_{\tau_2}(M)$ is not connected



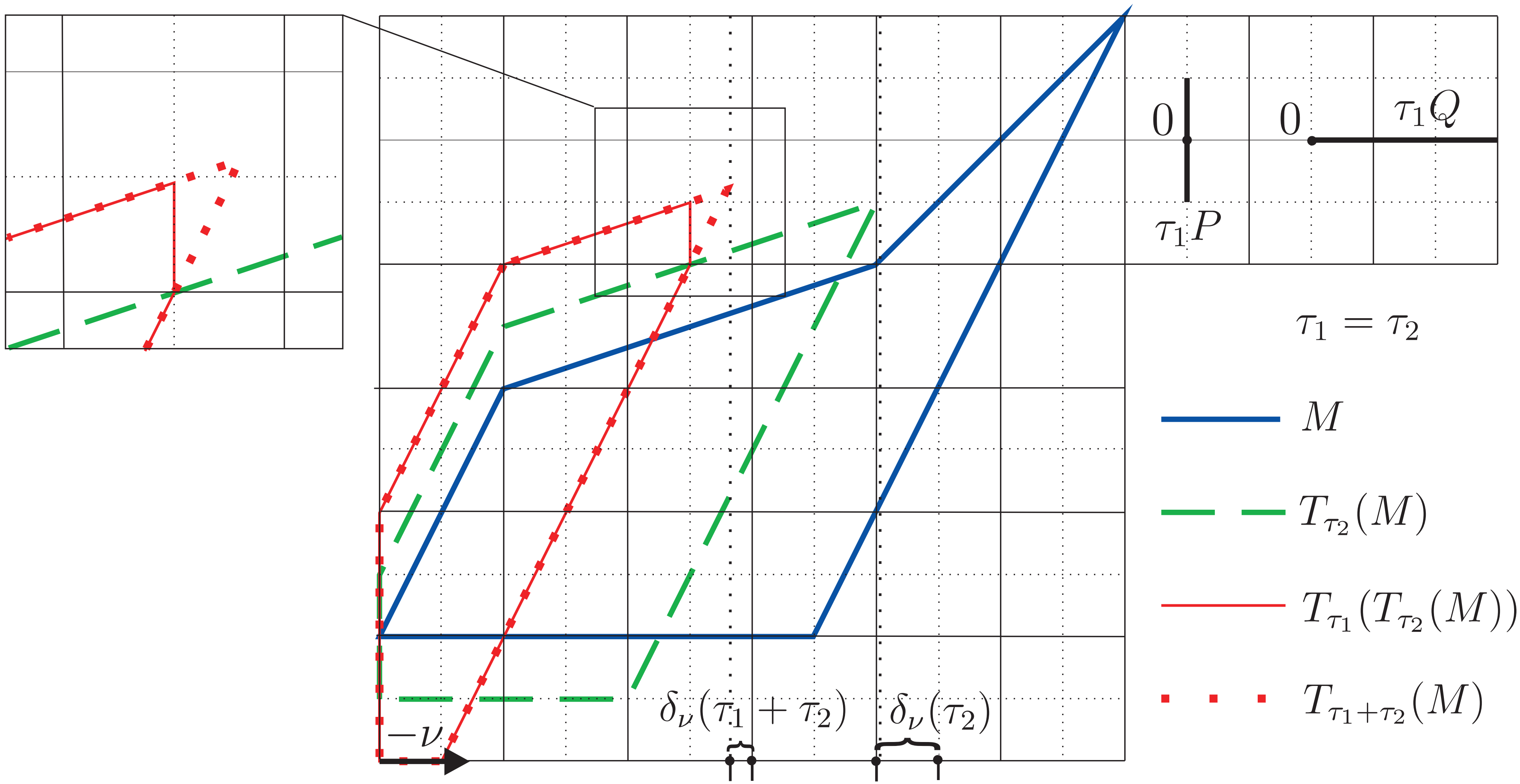
3.3 Example 3: $\delta_\nu(\tau_2) > \delta_\nu(\tau_1 + \tau_2)$

$$\delta_\nu(\tau) := \rho(-\nu; M) - \tau H(-\nu) - \rho(-\nu; T_\tau(M))$$



3.3 Example 4: $\delta_\nu(\tau_2) > \delta_\nu(\tau_1 + \tau_2)$

$$\delta_\nu(\tau) := \rho(-\nu; M) - \tau H(-\nu) - \rho(-\nu; T_\tau(M))$$



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