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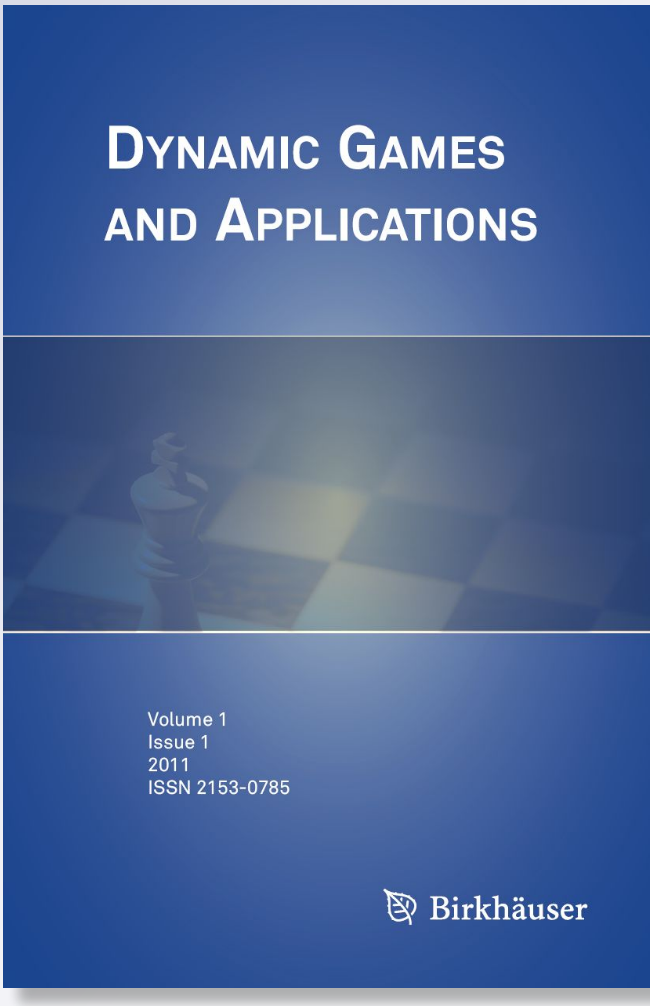

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Model Problem in a Line with Two Pursuers and One Evader

Sergey A. Ganebny · Sergey S. Kumkov · Stéphane Le Ménec · Valerii S. Patsko

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Abstract An antagonistic differential game is considered where motion occurs in a straight line. Deviations between the first and second pursuers and the evader are computed at the instants T_1 and T_2, respectively. The pursuers act in coordination. Their aim is to minimize the resultant miss, which is equal to the minimum of the deviations happened at the instants T_1 and T_2. Numerical study of value function level sets (Lebesgue sets) for qualitatively different cases is given. A method for constructing optimal feedback controls is suggested on the basis of switching lines. The results of a numerical simulation are shown.

Keywords Pursuit-evasion differential game · Linear dynamics · Value function · Optimal feedback control

1 Introduction and Problem Formulation

1. In the paper, a model differential game with two pursuers and one evader is studied. Three inertial objects move in the straight line. The dynamics for pursuers P_1 and P_2 is

$$\begin{aligned} &\ddot{z}_{P_1} = a_{P_1}, && \ddot{z}_{P_2} = a_{P_2}, \\ &\dot{a}_{P_1} = (u_1 - a_{P_1})/l_{P_1}, && \dot{a}_{P_2} = (u_2 - a_{P_2})/l_{P_2}, \\ &|u_1| \le \mu_1, && |u_2| \le \mu_2, \\ &a_{P_1}(t_0) = 0, && a_{P_2}(t_0) = 0. \end{aligned} \tag{1}$$

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Here, z_{P_1} and z_{P_2} are the geometric coordinates of the pursuers, a_{P_1} and a_{P_2} are their accelerations generated by the controls u_1 and u_2. The time constants l_{P_1} and l_{P_2} define how fast the controls affect the systems.

The dynamics of the evader E is similar:

$$\ddot{z}_E = a_E, \qquad \dot{a}_E = (v - a_E)/l_E, \qquad |v| \le \nu, \qquad a_E(t_0) = 0. \tag{2}$$

Let us fix some instants T_1 and T_2. At the instant T_1, the miss of the first pursuer with respect to the evader is computed, and at the instant T_2, the miss of the second one is computed:

$$r_{P_1,E}(T_1) = \big|z_E(T_1) - z_{P_1,E}(T_1)\big|, \qquad r_{P_2,E}(T_2) = \big|z_E(T_2) - z_{P_2,E}(T_2)\big|. \tag{3}$$

Assume that the pursuers act in coordination. This means that we can join them into one player P (which will be called the *first player*). This player governs the vector control $u = (u_1, u_2)$. The evader is regarded as the *second player*. The resultant miss is the following value:

$$\varphi = \min\big\{r_{P_1,E}(T_1),\ r_{P_2,E}(T_2)\big\}. \tag{4}$$

At any instant t, both players know the exact values of all state coordinates z_{P_1}, $\dot{z}_{P_1}$, a_{P_1}, z_{P_2}, $\dot{z}_{P_2}$, a_{P_2}, z_E, $\dot{z}_E$, a_E. The vector composed of these components is denoted as z. The first player choosing its feedback control minimizes the miss φ, the second one maximizes it.

Relations (1)–(4) define a standard antagonistic differential game. One needs to construct the value function $(t, z) \mapsto \mathcal{V}(t, z)$ of this game and optimal (or quasioptimal) strategies of the players.

2. Up to now, there are a lot of publications dealing with differential games where one group of objects pursues another group; see, for example, the following works [1, 2, 5, 7, 8, 16, 20–23, 27]. The problem under consideration has two pursuers and one evader. So, from the point of view of number of objects, it is the simplest one. On the other hand, strict mathematical studies of problems "group-on-group" usually include quite strong assumptions onto the dynamics of objects, dimension of the state vector, and conditions of termination. Conversely, this paper considers the problem without any assumptions of these types. Solution of the problem can be interesting for the group differential games.

3. Let us describe a practical problem, whose reasonable simplification gives the model game (1)–(4). Suppose that two pursuing objects attack the evading one on collision courses. They can be rockets or aircrafts in the horizontal plane. A nominal motion of the first pursuer is chosen such that at the instant T_1 the exact capture occurs. In the same way, a nominal motion of the second pursuer is chosen (the capture is at the instant T_2). But indeed, the real positions of the objects differ from the nominal ones. Moreover, the evader using its control can change its trajectory in comparison with the nominal one (but not principally, without sharp turns). Correcting coordinated efforts of the pursuers are computed during the process by the feedback method to minimize the result miss, which is the minimum of absolute values of deviations at the instants T_1 and T_2 from the first and second pursuers, respectively, to the evader.

The passage from the original non-linear dynamics to a dynamics, which is linearized with respect to the nominal motions, gives [24, 26] the problem under consideration.

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2 Passage to Two-Dimensional Differential Game

At first, let us pass to relative geometric coordinates

$$y_1 = z_E - z_{P_1}, \qquad y_2 = z_E - z_{P_2} \tag{5}$$

in dynamics (1), (2) and payoff function (4). After this, we have the following notation:

$$\begin{aligned} &\ddot{y}_1 = a_E - a_{P_1}, && \ddot{y}_2 = a_E - a_{P_2}, \\ &\dot{a}_{P_1} = (u_1 - a_{P_1})/l_{P_1}, && \dot{a}_{P_2} = (u_2 - a_{P_2})/l_{P_2}, \\ &\dot{a}_E = (v - a_E)/l_{P_1}, && |u_2| \le \mu_2, \\ &|u_1| \le \mu_1, \ |v| \le \nu, && \varphi = \min\{|y_1(T_1)|, |y_2(T_2)|\}. \end{aligned} \tag{6}$$

State variables of system (6) are $y_1, \dot{y}_1, a_{P_1}, y_2, \dot{y}_2, a_{P_2}, a_E$; u_1 and u_2 are controls of the first player; v is the control of the second one. The payoff function φ depends on the coordinate y_1 at the instant T_1 and on the coordinate y_2 at the instant T_2. From a general point of view (existence of the value function, positional type of the optimal strategies), the differential game (6) is a particular case of a differential game with a positional functional [11].

A standard approach to study linear differential games with fixed terminal instant and payoff function depending on some state coordinates at the terminal instant is to pass to new state coordinates (see, for example, [12, 13]) that can be treated as values of the target coordinates forecast to the terminal instant under zero controls. Often, these coordinates are called the *zero effort miss coordinates* [24–26]. In our case, we have two instants T_1 and T_2, but coordinates computed at these instants are independent; namely, at the instant T_1, we should take into account $y_1(T_1)$ only, and at the instant T_2, we use the value $y_2(T_2)$. This fact allows us to use the mentioned approach when solving the differential game (6). Thus, we pass to new state coordinates x_1 and x_2 where $x_1(t)$ is the value of y_1 forecast to the instant T_1 and $x_2(t)$ is the value of y_2 forecast to the instant T_2.

The forecast values are computed by the formula

$$x_i = y_i + \dot{y}_i \tau_i - a_{P_i} l_{P_i}^2 h(\tau_i / l_{P_i}) + a_E l_E^2 h(\tau_i / l_E), \quad i = 1, 2. \tag{7}$$

Here, x_i, y_i, $\dot{y}_i$, a_{P_i}, and a_E depend on t; $\tau_i = T_i - t$. Function h is described by the relation

$$h(\alpha) = e^{-\alpha} + \alpha - 1.$$

Emphasize that the values τ_1 and τ_2 are connected to each other by the relation $\tau_1 - \tau_2 = \text{const} = T_1 - T_2$. It is very important that $x_i(T_i) = y_i(T_i)$. Let $X(t, z)$ be a two-dimensional vector composed of the variables x_1, x_2 defined by formulas (5), (7).

The dynamics in the new coordinates x_1, x_2 is the following [15]:

$$\begin{aligned} &\dot{x}_1 = -l_{P_1} h(\tau_1 / l_{P_1}) u_1 + l_E h(\tau_1 / l_E) v, \\ &\dot{x}_2 = -l_{P_2} h(\tau_2 / l_{P_2}) u_2 + l_E h(\tau_2 / l_E) v, \\ &|u_1| \le \mu_1, \qquad |u_2| \le \mu_2, \qquad |v| \le \nu, \\ &\varphi(x_1(T_1), x_2(T_2)) = \min\{|x_1(T_1)|, |x_2(T_2)|\}. \end{aligned} \tag{8}$$

The first player governs the controls u_1, u_2 and minimizes the payoff φ; the second one has the control v and maximizes φ.

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Note that the control u_1 (u_2) affects only the horizontal (vertical) component $\dot{x}_1$ ($\dot{x}_2$) of the velocity vector $\dot{x} = (\dot{x}_1, \dot{x}_2)^{\mathrm{T}}$. When $T_1 = T_2$, the second summand in dynamics (8) is the same for $\dot{x}_1$ and $\dot{x}_2$. Thus, the component of the velocity vector $\dot{x}$ depending on the second player control is directed at any instant t along the bisectrix of the first and third quadrants of the plane x_1, x_2. When $v = +\nu$, the angle between the axis x_1 and the velocity vector of the second player is 45°; when $v = -\nu$, the angle is 225°. This property simplifies the dynamics in comparison with the case $T_1 \neq T_2$.

Let $x = (x_1, x_2)^{\mathrm{T}}$ and $V(t, x)$ be the value of the value function of game (8) at the position (t, x). From general results of the theory of differential games, it follows that

$$\mathcal{V}(t, z) = V\big(t, X(t, z)\big). \tag{9}$$

Relation (9) allows to compute the value function of the original game (1)–(4) using the value function for game (8). The transformation $(t, z) \mapsto x = X(t, z)$ helps also to map the feedback controls of the first and second players in game (8) found as functions depending on (t, x) to corresponding controls in game (1)–(4), which are functions of (t, z).

For any $c \geq 0$, a level set (a Lebesgue set)

$$W_c = \big\{(t, x) : V(t, x) \leq c\big\}$$

of the value function in game (8) can be treated as the solvability set for the considered game with the result not greater than c, that is, for a differential game with dynamics (8) and the terminal set

$$M_c = \big\{(t, x) : t = T_1,\ |x_1| \leq c\big\} \cup \big\{(t, x) : t = T_2,\ |x_2| \leq c\big\}.$$

When $c = 0$, one has the situation of the exact capture. The exact capture means equality to zero of, at least, one of $x_1(T_1)$ and $x_2(T_2)$.

Let

$$W_c(t) = \big\{x : (t, x) \in W_c\big\}$$

be the time section (t-section) of the set W_c at the instant t. Similarly, let $M_c(t)$ for $t = T_1$ and $t = T_2$ be the t-section of the set M_c at the instant t.

Comparing dynamics capabilities of each of pursuers P_1 and P_2 and the evader E, one can introduce the parameters [15, 26]

$$\eta_i = \mu_i/\nu, \qquad \varepsilon_i = l_E/l_{P_i}, \quad i = 1, 2.$$

They define the shape of the solvability sets in the individual games P_1 against E and P_2 against E.

Namely, depending on values of η_i and $\eta_i\varepsilon_i$ (which are not equal to 1 simultaneously), there are 4 cases [26] of the solvability set evolution (see Fig. 1):

- expansion in the backward time (a strong pursuer);
- contraction in the backward time (a weak pursuer);
- expansion until some backward time instant and further contraction;
- contraction until some backward time instant and further expansion (if the solvability set still has not broken).

Respectively, given combinations of pursuers' capabilities in individual games and durations T_1, T_2 (equal/different), there are significant number of variants for the problem with two pursuers and one evader.

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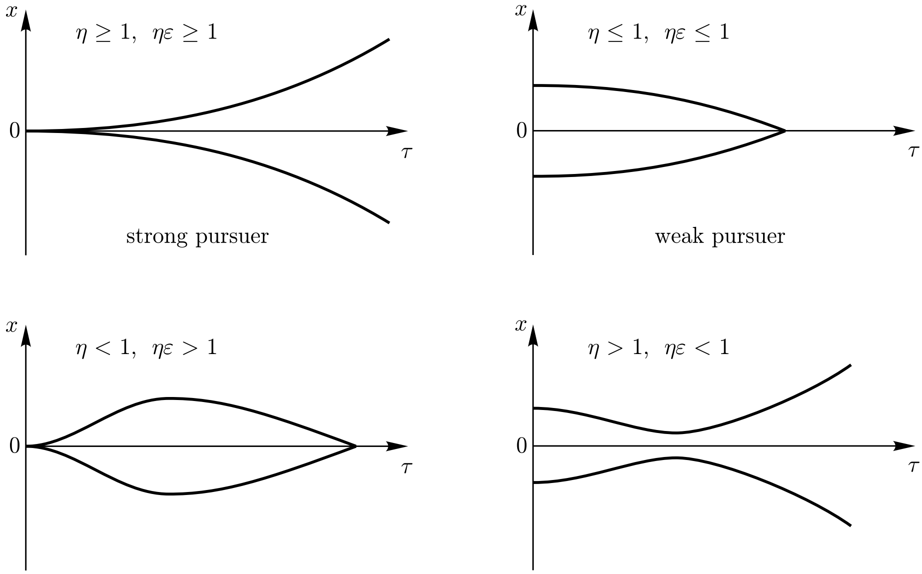

Fig. 1 Variants of the solvability set evolution in an individual game

We use the following ideology for solving the problem. Choose the parameters η_i, ε_i, and also the instants T_i, $i = 1, 2$; then, using some quite fine grid of values of c, compute level sets W_c of the value function. After that, process them to obtain optimal (or quasioptimal) controls of the first and second player close to the optimal ones and defined by switching lines. Having the controls, one can compute trajectories of the system.

Up to now, different workgroups suggested many algorithms for numeric solution of differential games of quite general type (see, for example, [4, 6, 17, 29]). Problem (8) is of the second order in the phase variable and can be rewritten as

$$\begin{aligned} &\dot{x} = \mathcal{D}_1(t)u_1 + \mathcal{D}_2(t)u_2 + \mathcal{E}(t)v, \\ &|u_1| \le \mu_1, \qquad |u_2| \le \mu_2, \qquad |v| \le \nu. \end{aligned} \tag{10}$$

Here, $x = (x_1, x_2)^{\mathrm{T}}$; vectors $\mathcal{D}_1(t)$, $\mathcal{D}_2(t)$, and $\mathcal{E}(t)$ are defined as

$$\begin{aligned} &\mathcal{D}_1(t) = \bigl(-l_{P_1} h\bigl((T_1 - t)/l_{P_1}\bigr)^{\mathrm{T}}, 0\bigr), \qquad \mathcal{D}_2(t) = \bigl(0, -l_{P_2} h\bigl((T_2 - t)/l_{P_2}\bigr)\bigr)^{\mathrm{T}}, \\ &\mathcal{E}(t) = \bigl(l_E h\bigl((T_1 - t)/l_E\bigr), l_E h\bigl((T_2 - t)/l_E\bigr)\bigr)^{\mathrm{T}}. \end{aligned}$$

The control of the first player has two independent components u_1 and u_2. The vector $\mathcal{D}_1(t)$ ($\mathcal{D}_2(t)$) is directed along the horizontal (vertical) axis. The second player's control v is scalar. When $T_1 = T_2$, the angle between the axis x_1 and the vector $\mathcal{E}(t)$ equals 45°; when $T_1 \ne T_2$, the angle changes with time.

Due to specificity of our problem, we use special methods for constructing level sets of the value function and optimal strategies. This allows us to make very fast computations of many variants of the game.

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3 Maximal Stable Bridge: Control with Discrimination of Opponent. The Main Idea of Numerical Construction

A level set W_c of the value function V is a maximal stable bridge (MSB), breaking on the terminal set M_c [12, 13].

Let $T_1 = T_2$. Denote $T_f = T_1$. Using the concept of MSB from [12, 13], we can say that W_c is the set maximal by inclusion in the space $(t \leq T_f, x)$ such that $W_c(T_f) = M_c(T_f)$ and the *stability* property holds: for any position $(t_*, x_*) \in W_c(t_*)$, $t_* < T_f$, any instant $t^* > t_*$, $t^* \leq T_f$, any constant control v of the second player, which obeys the constraint $|v| \leq \nu$, there is a measurable control $t \to (u_1(t), u_2(t))^{\mathrm{T}}$ of the first player, $t \in [t_*, t^*)$, $|u_1(t)| \leq \mu_1$, $|u_2(t)| \leq \mu_2$, guiding system (8) from the state x_* to the set $W_c(t^*)$ at the instant t^*.

The stability property assumes discrimination of the second player by the first one: the choice of the first player's control in the interval $[t_*, t^*)$ is made after the second player announces his control in this interval.

It is known (see [12, 13]) that any MSB is close. The set

$$W_c^{(2)}(t) = \mathrm{cl}\left(R^2 \setminus W_c(t)\right)$$

(here, the symbol cl denotes the operation of closure) is the time section of MSB $W_c^{(2)}$ for the second player at the instant t. The bridge terminates at the instant T_f on the set $M_c^{(2)}(T_f) = \mathrm{cl}(R^2 \setminus M_c(T_f))$. If the initial position of system (8) is in $W_c^{(2)}$ and if the first player is discriminated by the second one, then the second player is able to guide the motion of the system to the set $M_c^{(2)}(T_f)$ at the instant T_f. Thus, $\partial W_c = \partial W_c^{(2)}$. It is proved that for any initial position $(t_0, x_0) \in \partial W_c$, the value c is the best guaranteed result for the first (second) player in the class of feedback controls.

Presence of an idealized element (the discrimination of the opponent) allowed to create effective numerical methods for backward construction of MSBs (see [30]). Linearity of the dynamics and two-dimensionality of the state variable sufficiently simplify the algorithms.

The algorithm, which is suggested by the authors for constructing the approximating sets $\widetilde{W}_c(t)$, uses a time grid in the interval $[0, T_f]$: $t_N = T_f$, t_{N-1}, t_{n-2}, $\ldots$. For any instant t_k from the taken grid, the set $\widetilde{W}_c(t_k)$ is built on the basis of the previous set $\widetilde{W}_c(t_{k+1})$ and a dynamics obtained from (8) by fixing its value at the instant t_{k+1}. So dynamics (8), which varies in the interval $(t_k, t_{k+1}]$, is changed by a dynamics with simple motions [9]. The set $\widetilde{W}_c(t_k)$ is regarded as a collection of all positions at the instant t_k where from the first player guarantees guiding the system to the set $\widetilde{W}_c(t_{k+1})$ under "frozen" dynamics (8) and discrimination of the second player. The corresponding formula has the form

$$\widetilde{W}_c(t_k) = \left(\widetilde{W}_c(t_{k+1}) - (t_{k+1} - t_k)\mathcal{D}(t_{k+1}) \cdot P\right) \overset{*}{-} (t_{k+1} - t_k)\mathcal{E}(t_{k+1}) \cdot Q. \qquad (11)$$

Here, $\mathcal{D}(t_{k+1})$ is a matrix composed of columns $\mathcal{D}_1(t_{k+1})$ and $\mathcal{D}_2(t_{k+1})$ of system (10); the sets P and Q are

$$P = \left\{(u_1, u_2) : |u_1| \leq \mu_1,\ |u_2| \leq \mu_2\right\}, \qquad Q = \left\{v : |v| \leq \nu\right\}.$$

The symbol $\overset{*}{-}$ denotes the geometric difference (Minkowski difference) of two sets:

$$\mathcal{A} \overset{*}{-} \mathcal{B} = \bigcap_{b \in \mathcal{B}} (\mathcal{A} - b).$$

The boundary condition for the recursive computations (11) is assumed to be $\widetilde{W}_c(t_N) = M_c(T_f)$.

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Due to symmetry of dynamics (8) and the set $W_c(T_f)$ with respect to the origin, one gets that for any $t \le T_f$ the time section $W_c(t)$ is symmetric also.

If $T_1 \ne T_2$, then there is no appreciable complication in constructing MSBs for the problem considered in this paper in comparison with the case $T_1 = T_2$. Indeed, let $T_1 > T_2$. Then in the interval $(T_2, T_1]$ in (8), we take into account only the dynamics of the variable x_1 when building the bridge W_c backwardly from the instant T_1. Thus, the terminal set at the instant T_1 is taken as $M_c(T_1) = \{(x_1, x_2) : |x_1| \le c\}$. When the constructions are made up to the instant T_2, we add the set $M_c(T_2)$, that is, we take

$$W_c(T_2) = W_c(T_2 + 0) \cup \big\{(x_1, x_2) : |x_2| \le c\big\},$$

and further constructions are made on the basis of this set.

So, our tool for finding a level set of the value function in game (8) corresponding to a number c is the backward procedure for constructing a MSB with the terminal set M_c.

The solvability set with the index equal to c in the individual game $P1$–E ($P2$–E) is the maximal stable bridge built in the coordinates t, x_1 (t, x_2) and terminating at the instant T_1 (T_2) on the set $|x_1| \le c$ ($|x_2| \le c$). Its t-section, if it is non-empty, is a segment in the axis x_1 (x_2) symmetric with respect to the origin. In the plane x_1, x_2, this segment corresponds to a vertical (horizontal) strip of the same width near the axis x_2 (x_1). It is evident that when $t \le T_1$ ($t \le T_2$) such a strip is contained in the section $W_c(t)$ of MSB W_c of game (8) with the terminal set M_c.

4 Case of Strong Pursuers

4.1 Constructing Level Sets of the Value Function

In the case of two strong pursuers, the t-sections of MSBs in individual games $P1$–E and $P2$–E grow with increasing the backward time. This gives that for any $c \ge 0$ and any $t \le \bar{t} = \min\{T_1, T_2\}$ the set $W_c(t)$ includes a cross near the axes x_1, x_2, which expands with decreasing t. Therefore, the set

$$W_c^{(2)}(t) = \mathrm{cl}\big(R^2 \backslash W_c(t)\big),$$

which is a t-section of MSB $W_c^{(2)}$ for the second player, can be regarded as consisting of four subsets $W_c^{(2),i}(t)$, $i = \overline{1,4}$. Each of them does not intersect with others and is located in one of four quadrants of the plane x_1, x_2. At the instant $\bar{t}$, the set $W_c^{(2),i}(\bar{t})$ is unbounded convex set, namely, a right angle with sides parallel to the axes x_1, x_2. Going backward in time from such a set, the second player tries to expand it, and the first one tries to contract. For any $t \le \bar{t}$, the set $W_c^{(2),i}(t)$ is convex. The approximating set $\widetilde{W}_c^{(2),i}(t_k)$ is computed by the formula

$$\widetilde{W}_c^{(2),i}(t_k) = \big(\widetilde{W}_c^{(2),i}(t_{k+1}) - (t_{k+1} - t_k)\mathcal{E}(t_{k+1}) \cdot Q\big) \overset{*}{-} (t_{k+1} - t_k)\mathcal{D}(t_{k+1}) \cdot P.$$

The convexity simplifies [10, 14] the algorithm for constructing the geometric difference. The error of numeric constructions of the sections $W_c^{(2),i}(t)$ is determined, in fact, almost only by discretizations on t and by "freezing" dynamics in each interval of the discrete time. Inside any interval, a game with simple motions is considered and these constructions are exact.

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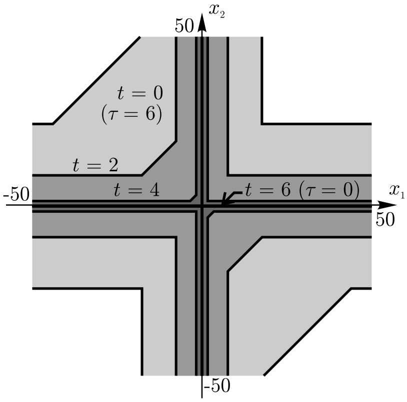

Fig. 2 Two strong pursuers, equal terminal instants: time sections of the maximal stable bridge W_0

Having made independent computations in each quadrant (due to the central symmetry, it is sufficient to make the constructions in the I and II quadrants only), one gets the sets $W_c^{(2),i}(t)$, $i = \overline{1,4}$, and, therefore, the set

$$W_c(t) = \mathrm{cl}\left(R^2 \setminus \bigcup_{i=\overline{1,4}} W_c^{(2),i}(t)\right).$$

A proof of convergence of algorithm and its estimations for the convex case are given in [3]. From results of this work, convergence of the method suggested here for constructing level sets W_c of the value function follows for the case of strong pursuers.

Let us give results of constructing t-sections $W_c(t)$ for the following values of the game parameters:

$$\mu_1 = 2, \qquad \mu_2 = 3, \qquad \nu = 1, \qquad l_{P_1} = 1/2, \qquad l_{P_2} = 1/0.857, \qquad l_E = 1. \qquad (12)$$

Equal Terminal Instants Let $T_1 = T_2 = 6$. Figure 2 shows results of constructing the set W_0 (that is, with $c = 0$). In the figure, one can see several time sections $W_0(t)$ of this set. The bridge has a quite simple structure. At the initial instant $\tau = 0$ of the backward time (when $t = 6$), its section coincides with the target set, which is the union of two coordinate axes. Further, at the instants $t = 4$, 2, 0, the cross thickens, and two triangles are added to it. The widths of the vertical and horizontal parts of the cross correspond to sizes of MSBs in the individual games with the first and second pursuers. These triangles are located in the II and IV quadrants (where the signs of x_1 and x_2 are different, in other words, when the evader is between the pursuers). They give the zone where the capture is possible only under collective actions of both pursuers.

Time sections $W_c(t)$ of other bridges W_c, $c > 0$, have a shape similar to $W_0(t)$. In Fig. 3, one can see the sections $W_c(t)$ at $t = 2$ $(\tau = 4)$ for a collection $\{W_c\}$ corresponding to some series of values of the parameter c. For other instants t, the structure of the sections $W_c(t)$ is similar.

Different Terminal Instants Let $T_1 = 7$, $T_2 = 5$. Results of constructing the set W_0 are given in Fig. 4. When $t < 5$, time sections $W_0(t)$ grow both horizontally and vertically; two additional triangles appear, but in this case they are curvilinear.

Total structure of the sections $W_c(t)$ at $t = 2$ is shown in Fig. 5.

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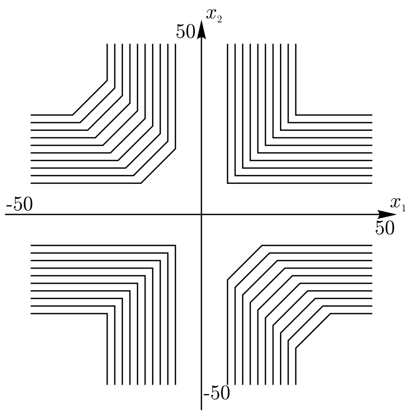

Fig. 3 Two strong pursuers, equal terminal instants: level sets of the value function, $t = 2$

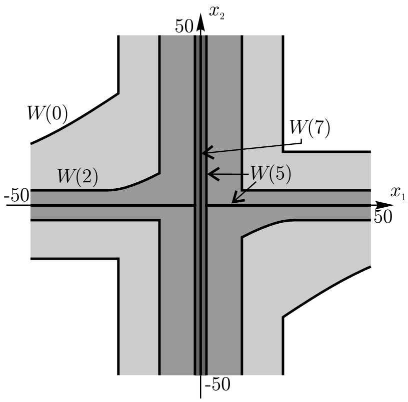

Fig. 4 Two strong pursuers, different terminal instants: time sections of the maximal stable bridge W_0

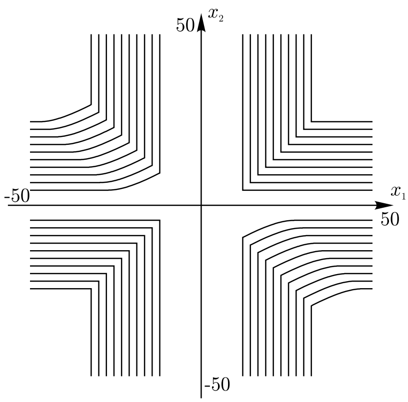

Fig. 5 Two strong pursuers, different terminal instants: level sets of the value function, $t = 2$

In Fig. 6, the set W_0 is shown in the three-dimensional space t, x_1, x_2.

The given results are typical for the case of strong pursuers. For other variants of the parameters in the framework of this case, MSBs and the value function are similar. When $T_1 = T_2$, the sets $W_c(t)$ can be described analytically. This was done in paper [15]. Also,

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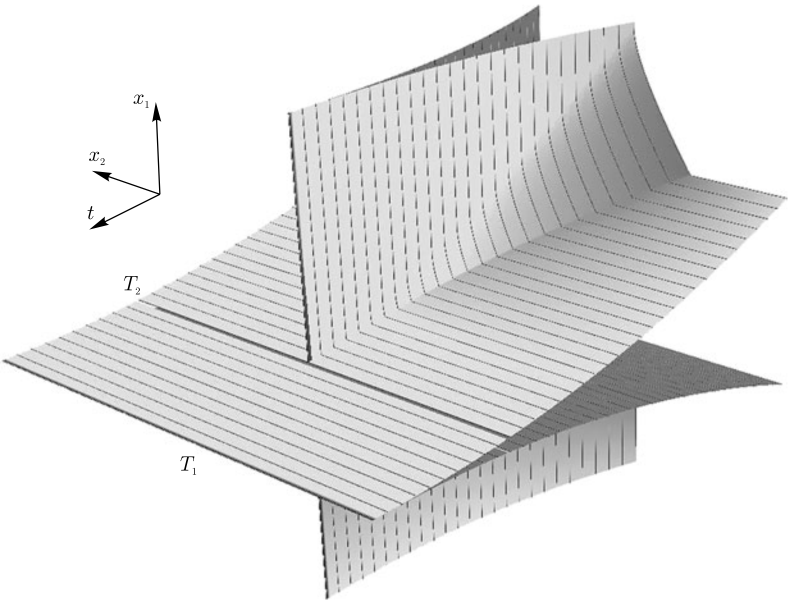

Fig. 6 Strong pursuers, different terminal instants: a three-dimensional view of the set W_0

there the case $T_1 \neq T_2$ was studied. But for it, only an upper approximations of the sets $W_c(t)$ was obtained.

4.2 Switching Lines in the Case of Strong Pursuers

Feedback Control of the First Player Analyzing the change of the value function along a horizontal line in the plane x_1, x_2 for a fixed instant t, one can conclude that the minimum of the function is reached in the segment of intersection of this line and the set $W_0(t)$. The function is monotonic at both sides of the segment. For points at the right (at the left) from the segment, the control $u_1^* = \mu_1$ ($u_1^* = -\mu_1$) directs the vector $\mathcal{D}_1(t)u_1$ to the minimum.

Splitting the plane into horizontal lines and extracting for each line the segment of minimum of the value function, one can gather these segments into a set in the plane and draw a switching line through this set, which separates the plane into two parts at the instant t. At the right from this switching line, we choose the control $u_1^* = \mu_1$, and at the left the control is $u_1^* = -\mu_1$. On the switching line, the control u_1^* can be arbitrary obeying the constraint $|u_1^*| \leq \mu_1$. The easiest way is to take the vertical axis x_2 as the switching line.

In the same way, using the vector $\mathcal{D}_2(t)$, we can conclude that the horizontal axis x_1 can be taken as the switching line for the control u_2.

Thus,

$$u_i^*(t, x) = \begin{cases} \mu_i, & \text{if } x_i > 0, \\ -\mu_i, & \text{if } x_i < 0, \\ \text{any } u_i \in [-\mu_i, \mu_i], & \text{if } x_i = 0. \end{cases} \tag{13}$$

Note that if $T_1 \neq T_2$ then for $i = 1$ formula (13) is used in the interval $[0, T_1]$, and for $i = 2$ it is applied in the interval $[0, T_2]$.

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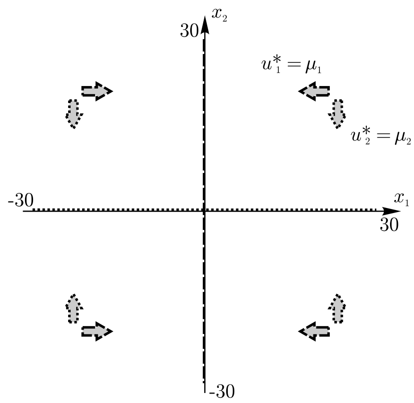

Fig. 7 Two strong pursuers, equal terminal instants: switching lines for the first player

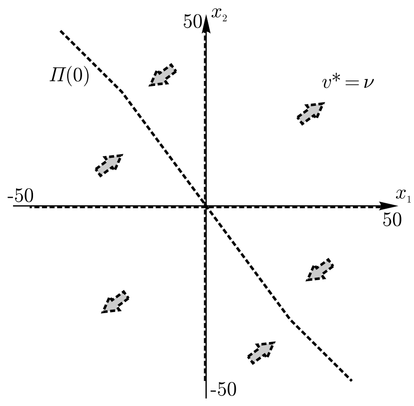

Fig. 8 Two strong pursuers, equal terminal instants: switching lines for the second player, $t = 0$

The switching lines (the coordinate axes) at any t divide the plane x_1, x_2 into 4 cells. In each of these cells, the control $u^* = (u_1^*, u_2^*)^{\mathrm{T}}$ of the first player is constant. The synthesis of the first player's control is the same for all time instants and is shown in Fig. 7. Arrows denote the direction of the vectors $\mathcal{D}_i(t)u_i^*$, $i = 1, 2$.

Feedback Control of the Second Player For a fixed instant t, consider a split of the plane x_1, x_2 into lines parallel to the vector $\mathcal{E}(t)$. Take the segments of local minimum and local maximum of the value function on all lines. One can easily see that for any line (except lines passing near the origin), there are two segments of local minimum and one of local maximum located between them. The segments of minimum appear by intersection of the line with the set $W_0(t)$. The segment of maximum for the case $T_1 = T_2$ coincides with the rectilinear part of the boundary of some set $W_c(t)$ and has slope angle equal to 45°. If $T_1 \neq T_2$, then the segment of maximum degenerates to a point coinciding with the corner point of a curvilinear triangle. For any point in the line outside all the segments, the control v is chosen in such a way that the vector $\mathcal{E}(t)v$ is oriented to the direction of growth of the value function. So, there are two parts of the line, where $v^* = \nu$, and two parts, where $v^* = -\nu$.

For a fixed instant t, $t \leq \min\{T_1, T_2\}$, the switching lines for the second player comprise of the coordinate axes and some line $\Pi(t)$, which passes through the segments of local

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minimum (for simplicity, through middles of these segments) if $T_1 = T_2$, and through the corner points of curvilinear triangles if $T_1 \neq T_2$.

Inside each of 6 cells, into which the plane is separated by the switching lines of the second player, the optimal control is taken certain: either $v^* = \nu$ or $v^* = -\nu$. In each of these switching lines, an arbitrary value $v^* \in [-\nu, \nu]$ can be taken.

If $T_1 \neq T_2$, then for $t > \min\{T_1, T_2\}$ the switching line for the second player's control is just one corresponding coordinate axis.

The second player optimal synthesis for the case $T_1 = 7$, $T_2 = 5$ is shown in Fig. 8 for $t = 0$. Arrows denote direction of the vectors $\mathcal{E}(t)v^*$.

4.3 Generating Feedback Controls. Discrete Scheme of Control

Switching lines are built as a result of processing the boundary of the sets $W_c(t)$. Some grid of instants t_s with the step Δ is introduced, where the t-sections $W_{c_j}(t_s)$ of the maximal stable bridges W_{c_j} are constructed by the backward procedure. The values c_j are also taken in some grid with the step Δc. For any instant t_s, approximating switching lines are stored as polygonal lines in the memory of a computer.

Having a position $x(t_s)$ at the instant t_s, it is possible to compute the controls $u_1^*(t_s, x(t_s))$ and $u_2^*(t_s, x(t_s))$ analyzing location of the point $x(t_s)$ with respect to the switching lines for u_1 and u_2. The vectors $\mathcal{D}_1(t_s)$ and $\mathcal{D}_2(t_s)$ are used for this. The values of u_1^* and u_2^* are defined by formula (13). Drawing a ray from the point $x(t_s)$ with the directing vector $\mathcal{D}_i(t_s)$, one can decide whether it crosses a switching line corresponding to the index i. If it does not, then $u_i^*(t_s, x(t_s)) = -\mu_i$; if it crosses, then $u_i^*(t_s, x(t_s)) = \mu_i$.

The first player control chosen at the instant t_s is kept until the instant $t_{s+1} = t_s + \Delta$. At the position $(t_{s+1}, x(t_{s+1}))$, a new control value is chosen, etc. So, the feedback control generated by the switching lines is used in a discrete control scheme [12, 13].

To construct $v^*(t_s, x(t_s))$, we use the vector $\mathcal{E}(t_s)$. Compute how many times (even or odd) a ray with the beginning at the point $x(t_s)$ and the directing vector $\mathcal{E}(t_s)$ crosses the second player switching lines. If the number of crosses is even (absence of crosses means that the number equals zero and is even), then we take $v^*(t_s, x(t_s)) = +\nu$; otherwise, $v^*(t_s, x(t_s)) = -\nu$. The chosen control is kept until the next instant t_{s+1}. In the position $(t_{s+1}, x(t_{s+1}))$, a new control is built, etc.

This synthesis for the first (second) player is optimal. The feedback control of the first (second) player built on the basis of switching lines guarantees the limit of result as $\Delta \to 0$ and $\Delta c \to 0$, which is not greater (not less) than $V(t_0, x_0)$ for any initial position (t_0, x_0). Moreover, the suggested players' controls are stable with respect to small error of measurements of the phase state and small inaccuracies of the numerical constructions of the switching lines.

The property of stability near each certain switching line is stipulated by directions of the arrows of corresponding controls on both sides of the line. If the arrows are directed to the switching line, the stability with respect to the numerical and informational errors takes place. From Fig. 8, one can see that the switching lines for the second player's control v, which coincide with the coordinate axes, do not possess this property: the arrows of the vectors $\mathcal{E}(t)v^*$ are directed from the switching lines. But in the considered case, these lines are located inside the set $W_0(t)$ where the value function is constant and equal to 0. Thus, errors of computing the control v within the set $W_0(t)$ are not important. So, there is the stability near these lines, too.

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4.4 Justifying Optimality and Stability of Players' Controls

Let the time step of the backward construction of the level sets W_c of the value function (MSBs) be so small that we can ignore the inaccuracies of the constructions. Also, let the step Δc of the collection $\{W_c\}$ be quite small too. So, using the denotation V, we do not distinguish the "ideal" value function and the one constructed numerically.

Let us prove the optimality of the first and second players' controls suggested above and their stability with respect to informational errors.

At first, we consider the case when $T_1 = T_2$. Denote $T_f = T_1$.

Let $\varepsilon \geq 0$ and the set

$$\mathcal{M}_\varepsilon = \big\{(x_1, x_2) : |x_1| \leq \varepsilon\big\} \cup \big\{(x_1, x_2) : |x_2| \leq \varepsilon\big\}$$

be a cross of semiwidth equal to ε near the axes x_1 and x_2.

Note that the Lipschitz constant $L(t)$ of the function $x \mapsto V(t, x)$ for any $t \leq T_f$ equals 1. This follows from a well-known statement [28, pp. 110–111] that the Lipschitz constant of the value function on the phase variable in a game of type (10) with fixed terminal instant coincides with the Lipschitz constant of the terminal payoff function. In our case, the terminal payoff function at the instant T_f is $\min\{|x_1|, |x_2|\}$, and its Lipschitz constant is 1.

Denote by K the maximal magnitude of the absolute value of the velocity of system (10) in the time interval $[0, T_f]$.

Optimality and Stability of the First Player's Control Let the first player apply his strategy in a discrete scheme of control with the time step Δ. Suppose that the first player errs in computing the phase state of system (10) with the error not greater than $\varepsilon \geq 0$. This means that if the true position of system (10) at some instant t is $x(t)$, then the measuring unit gives to the player some position $x(t) + \zeta(t)$ where $|\zeta(t)| \leq \varepsilon$. Due to these errors, the first player can choose incorrect values u_1^* and u_2^* if $x(t) \in \mathcal{M}_\varepsilon$. If $x(t) \notin \mathcal{M}_\varepsilon$, then $\operatorname{sign}(x_i(t) + \zeta_i(t)) = \operatorname{sign} x_i(t)$, $i = 1, 2$, and the choice is correct for any error of this type.

Taking into account the results of constructing level sets of the value function in the case of strong pursuers, we can conclude that for any $t \leq T_f$ the function $x \mapsto V(t, x)$ has non-positive directional derivative along the vectors $\mathcal{D}_1(t)u_1^*(t, x)$ and $\mathcal{D}_2(t)u_2^*(t, x)$ at any point x beyond the axes x_1 and x_2. Thus, if $x(t) \notin \mathcal{M}_\varepsilon$ for some time interval $[\underline{t}, \bar{t}]$ and constant controls u_1 and u_2 are chosen as $u_1^*(x(\underline{t}) + \zeta(\underline{t}))$ and $u_2^*(x(\underline{t}) + \zeta(\underline{t}))$, then $V(\bar{t}, x(\bar{t})) \leq V(\underline{t}, x(\underline{t}))$ for any feasible open-loop control $v(\cdot)$ of the second player in this interval. (We count this fact as evident.)

Consider some motion in the interval $[t_0, T_f]$, $t_0 \in [0, T_f]$.

If $x(t) \notin \mathcal{M}_\varepsilon$ for any t, then $V(T_f, x(T_f)) \leq V(t_0, x(t_0))$.

Let $x(\hat{t}) \in \mathcal{M}_\varepsilon$ for some $\hat{t} \in [t_0, T_f]$. Denote by $\tilde{t}$ the maximum of such instants. Then in the interval $(\tilde{t}, T_f]$, the point $x(t)$ is outside of the set $\mathcal{M}_\varepsilon$. Consider a time step $[t_s, t_s + \Delta)$ of the discrete control scheme, where s is such that $\tilde{t} \in [t_s, t_s + \Delta)$. For $t \in (\tilde{t}, T_f]$, the chosen control $u(t) = (u_1(t), u_2(t))^{\mathrm{T}}$ can differ from $u^* = (u_1^*, u_2^*)^{\mathrm{T}}$ in the interval $(\tilde{t}, t_s + \Delta)$ only. This gives that

$$V\big(T_f, x(T_f)\big) \leq V\big(t_s + \Delta, x(t_s + \Delta)\big).$$

Taking into account monotonic growth of the sets $W_c(t)$ with decreasing t, we have

$$V\big(t_s + \Delta, x(t_s + \Delta)\big) \leq V\big(\tilde{t}, x(t_s + \Delta)\big).$$

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Using the Lipschitz constant $L(\tilde{t}) = 1$, we get

$$V\big(\tilde{t}, x(t_s + \Delta)\big) \leq V\big(\tilde{t}, x(\tilde{t})\big) + L(\tilde{t}) \cdot \Delta \cdot K = V\big(\tilde{t}, x(\tilde{t})\big) + \Delta \cdot K.$$

Since $x(\tilde{t}) \in \mathcal{M}_\varepsilon$ then $V(\tilde{t}, x(\tilde{t})) \leq \varepsilon$. Thus,

$$V\big(T_f, x(T_f)\big) \leq \varepsilon + \Delta \cdot K.$$

Gathering the cases when the motion $x(t)$ is outside $\mathcal{M}_\varepsilon$ and when it hits the set $\mathcal{M}_\varepsilon$, we obtain

$$V\big(T_f, x(T_f)\big) \leq \max\big\{V\big(t_0, x(t_0)\big),\ \varepsilon + \Delta \cdot K\big\}. \tag{14}$$

The following proposition is true:

Proposition 1 *Let the first player apply his strategy u^* in a discrete control scheme with the time step $\Delta > 0$ under errors of measurements bounded by a value $\varepsilon \geq 0$. Then for any initial position $(t_0, x(t_0))$, $t_0 \in [0, T_f]$, and for any feasible realization $v(\cdot)$ of the second player's control*, *estimation* (14) *is held.*

This statement characterizes the optimality of the strategy u^* and its stability with respect to small inaccuracies of measurements of the phase state of system (10).

Optimality and Stability of the Second Player's Control Construct a strip of semiwidth r along the direction of the vector $\mathcal{E}(t)$ near the switching line $\Pi(t)$ (which is located in the I and III quadrants). Denote the strip by $\Pi_r(t)$.

Fix a $r \geq 0$ and choose $\varepsilon \geq 0$ such that the closed ε-neighborhood of the line $\Pi(t)$ belongs to $\Pi_r(t)$ for any $t \in [0, T_f]$. Such a limitation for ε is called *consistency condition* of ε with r.

Let the second player get a measurement $x(t) + \zeta(t)$, where $|\zeta(t)| \leq \varepsilon$, instead of the exact position $x(t)$, $t \in [t_0, T_f]$.

Assume that the motion $x(t)$ in $[t_0, T_f]$ does not hit the set $\mathcal{M}_\varepsilon$. Then it stays in the same quadrant, and the informational error can affect the correctness of the choice of the control only when the motions is inside the strip $\Pi_r(t)$ (if the motion reaches it). The function $x \mapsto -V(t, x)$ is convex inside any quadrant. So, for any $t \in [t_0, T_f]$, the estimation is true [18, 19]:

$$-V\big(t, x(t)\big) \leq -V\big(t_0, x(t_0)\big) + \Lambda(t - t_0, r, \Delta),$$

$$\Lambda(t - t_0, r, \Delta) = 2\sqrt{(2K\nu\Delta + r)\beta\nu}(t - t_0) + 4K\nu\Delta + r.$$

Here, β is the Lipschitz constant of the function $t \mapsto \mathcal{E}(t)$ in the interval $[0, T_f]$. Since $V(t, x(t)) \geq 0$, we get

$$V\big(t, x(t)\big) \geq \max\big\{0, V\big(t_0, x(t_0)\big) - \Lambda(t - t_0, r, \Delta)\big\}, \quad t \in [t_0, T_f]. \tag{15}$$

(1) Let the initial position $(t_0, x(t_0))$ be such that

$$V\big(t_0, x(t_0)\big) \geq \Lambda(T_f, r, \Delta) + \varepsilon.$$

Then $x(t_0) \notin \mathcal{M}_\varepsilon$ and in the interval $[t_0, T_f]$, due to (15), the motion $x(t)$ does not go to the set $\mathcal{M}_\varepsilon$. Thus,

$$V\big(T_f, x(T_f)\big) \geq V\big(t_0, x(t_0)\big) - \Lambda(t - t_0, r, \Delta), \quad t \in [t_0, T_f].$$

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(2) Let $V(t_0, x(t_0)) < \Lambda(T_f, r, \Delta) + \varepsilon$. In this case, we write a trivial estimation $V(t, x(t)) \geq 0$.

Gathering the cases (1) and (2), we obtain the following estimation:

$$V\big(T_f, x(T_f)\big) \geq \max\big\{0, V\big(t_0, x(t_0)\big) - \Lambda(T_f, r, \Delta)\big\}. \tag{16}$$

The following proposition is true:

Proposition 2 *Fix an arbitrary $r \geq 0$. Suppose that some $\varepsilon \geq 0$ obeys the consistency condition with r.*

Let the second player apply his strategy v^* in a discrete control scheme with the time step $\Delta > 0$ under informational errors bounded by the value ε. Then for any initial position $(t_0, x(t_0))$, $t_0 \in [0, T_f]$, and any feasible realization $u(\cdot)$ of the first player's control, estimation (16) is held.

This statement characterizes the optimality of the strategy v^* and its stability with respect to small inaccuracies of measurements of the phase state of system (10).

Comparing the structure of Propositions 1 and 2, pay attention to the fact that in Proposition 1, the values ε, Δ and the initial position $(t_0, x(t_0))$ can be arbitrary. At the same time, in Proposition 2, the values r, Δ and ε (which depends on r) are taken according to the value $V(t_0, x(t_0))$ of the value function at the initial position to provide small inaccuracy Λ and, therefore, non-triviality of estimation (16).

Now let $T_1 \neq T_2$. Consider the case $T_1 > T_2$ (the other case is similar). The Lipschitz constant $L(t)$ of the function $x \mapsto V(t, x)$ for any $t \leq T_1$ equals 1. Indeed, when $t \in (T_2, T_1]$, the value $V(t, x)$ is defined by the payoff $|x_1(T_1)|$. Its Lipschitz constant is 1. Therefore, $L(t) = 1$ if $t \in (T_2, T_1]$. When $t \leq T_2$, the value $V(t, x)$ is computed as the value of the value function in the game with fixed terminal instant T_2 and payoff function $\min\{V(T_2 + 0, x), |x_2|\}$. Since the Lipschitz constant of this payoff is 1, then $L(t) = 1$ when $t \leq T_2$.

Define the set $\mathcal{M}_\varepsilon$ as a strip of semiwidth ε near the axis x_2 for $t \in (T_2, T_1]$ and as a cross (as was done above) for $t \leq T_2$.

Let K be the maximal magnitude of the velocity of system (10) in the interval $[0, T_1]$, and β be the Lipschitz constant of the function $t \mapsto \mathcal{E}(t)$ in the interval $[0, T_2]$.

In the case $T_1 > T_2$, statements analogous to Propositions 1 and 2 are true. The difference is that in the interval $(T_2, T_1]$ only the control u_1^* of the first player works and the switching line $\Pi(t)$ for the second player disappears. The evaluations are similar to those in the proofs of Propositions 1 and 2. At first, they are written for the interval $[0, T_2]$ and further made for the interval $[T_2, T_1]$.

4.5 Simulation Results

Let the pursuers P_1, P_2, and the evader E move in the plane. This plane is called the *original geometric space*. At the initial instant t_0, velocities of all objects are parallel to the horizontal axis (Fig. 9) and sufficiently greater than the possible changes of the lateral velocity components. Velocity of each object has a constant component parallel to the horizontal axis. Magnitudes of these components are such that the rendezvous of the objects P_1 and E happens at the instant T_1, and the objects P_2 and E encounter at the instant T_2. The dynamics of lateral motion is described by relations (1), (2); the resultant miss is given by formula (4).

Here and in the simulations below, the initial lateral velocities and accelerations are assumed to be zero:

$$\dot{z}_{P_1}^0 = \dot{z}_{P_2}^0 = \dot{z}_E^0 = 0, \qquad a_{P_1}^0 = a_{P_2}^0 = a_E^0 = 0.$$

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Fig. 9 Schematic initial positions of the pursuers and evader

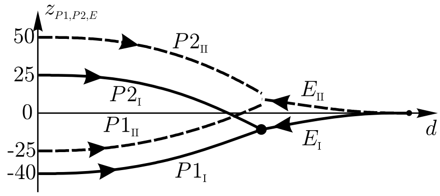

Fig. 10 Two strong pursuers, equal termination instants: trajectories in the original geometric space

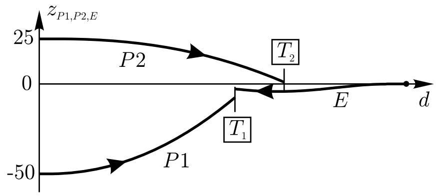

Fig. 11 Two strong pursuers, different termination instants: trajectories in the original geometric space

In the following figures, the horizontal axis is denoted by the symbol d. So, the coordinate d shows the longitudinal position of the objects. Controls of the objects affect the vertical (lateral) coordinate.

In Fig. 10, one can see the trajectories of the objects for the case of strong pursuers and equal terminal instants for the game parameters (12), $t_0 = 0$, and $T_1 = T_2 = 6$. The pursuers P_1, P_2, and the evader E act optimally. The trajectories drawn by solid lines correspond to the following initial data:

$$z^0_{P_1} = -40, \qquad z^0_{P_2} = 25, \qquad z^0_E = 0.$$

The dashed lines denote the trajectories for the initial positions

$$z^0_{P_1} = -25, \qquad z^0_{P_2} = 50, \qquad z^0_E = 0.$$

In the first case, the evader is successfully captured (at the terminal instant, the positions of both pursuers are the same as the position of the evader). In the second variant of initial positions, the evader escapes: at the terminal instant no one of the pursuers superposes with the evader. In this case, one can see that the evader aims itself to the middle between the terminal positions of the pursuers (this guarantees to him the maximum of the payoff function φ).

Figure 11 shows the optimal trajectories for the initial positions $z^0_{P_1} = -50$, $z^0_{P_2} = 25$, $z^0_E = 0$ at the instant $t_0 = 0$ and the terminal instants $T_1 = 7$, $T_2 = 5$. The symbol T_1 (T_2) is put near the longitudinal positions d where the rendezvous of P_1 and E (of P_2 and E) happens. To avoid misunderstanding of them as some coordinate marks, they are embraced by frames. One can see that at the beginning the evader escapes from the second pursuer and goes down, and after that, the evader's control is changed to escape from the first pursuer and the evader goes up.

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5 Case of Weak Pursuers

Since in the case of weak pursuers the t-sections of MSBs in individual games $P1$–E and $P2$–E contract with growth of the backward time and become empty at some instant, the set $W_c(t)$ for any $c \geq 0$ with decreasing of t loses infinite sizes along axes x_1 and x_2. So, during backward constructions of the sets $W_c(t)$, one cannot apply independent procedures in four quadrants of the plane x_1, x_2 as it was made in the case of strong pursuers. To compute the geometric difference in formula (11), one can use the relation

$$\mathcal{A} \overset{*}{-} \mathcal{B} = \bigcap_{b\in\mathcal{B}} (\mathcal{A} - b) = \Bigl(\bigcup_{b\in\mathcal{B}} (\mathcal{A} - b)'\Bigr)' = \Bigl(\bigcup_{b\in\mathcal{B}} (\mathcal{A}' - b)\Bigr)' = (\mathcal{A}' - \mathcal{B})'.$$

Here, the prime symbol denotes the complement operation: $\mathcal{A}' = R^2 \backslash \mathcal{A}$. As a result, the passage from the set $\widetilde{W}_c(t_{k+1})$ to the set $\widetilde{W}_c(t_k)$ in the backward procedure is performed as a number of application of algebraic sum operation (Minkowski sum) of a non-convex set and a segment. The specificity of the plane is such that during approximation of a boundary of a connected set by a polygonal line, this line can be locally considered either as convex, or as concave. This property allows to create an efficient algorithm for constructing the algebraic sum of a non-convex set and a segment.

During the numerical studies, it was discovered that the connected set $W_c(t)$ with decreasing of t loses connectedness and disjoins into two separate parts. This can happen both when $T_1 = T_2$, and when $T_1 \neq T_2$. Loss of connectedness complexifies the solution of the game and, in particular, constructing optimal strategies of players.

Take the parameters

$$\mu_1 = 0.9, \qquad \mu_2 = 0.8, \qquad \nu = 1, \qquad l_{P_1} = l_{P_2} = 1/0.7, \qquad l_E = 1.$$

Let us show results for the case of different terminal instants only: $T_1 = 9$, $T_2 = 7$.

Since in this variant the evader is more maneuverable than the pursuers, the first player cannot guarantee the exact capture.

Fix some level of the miss, namely, $|x_1(T_1)| \leq 2.0$, $|x_2(T_2)| \leq 2.0$. Time sections $W_{2.0}(t)$ of the corresponding MSB are shown in Fig. 12. The upper-left subfigure corresponds to the instant T_1 when the first player stops to pursue. The next subfigure shows the picture for the instant T_2 when the second pursuer finishes its pursuit. At this instant, the horizontal strip is added, which is a bit wider than the vertical one contracted during the passed period of the backward time. Then the bridge contracts both in horizontal and vertical directions, and two additional curvilinear triangles appear (see upper-right subfigure). The lower-left subfigure gives the view of the section when the vertical strip collapses, and the next subfigure shows the configuration just after the collapse of the horizontal strip. At this instant, the section loses connectivity and disjoins into two parts symmetrical with respect to the origin. Further, these parts continue to contract (as can be seen in the lower-right subfigure) and finally disappear.

Time sections $\{W_c(t)\}$ are given in Fig. 13 at the instant $t = 2$ $(\tau_1 = 7,\ \tau_2 = 5)$. The set W_c in the space t, x_1, x_2 for $c = 2.0$ is shown in Fig. 14. During evolution of the sections $W_{2.0}(t)$ in t, they change their structure at some instants. These places are marked by drops in the constructed surface of the set.

We process the sets $W_c(t)$ taken for some grids on c and t to get switching lines.

Switching lines of the first player are given in Fig. 15 at the instant $t = 2$ $(\tau_1 = 7$, $\tau_2 = 5)$. The dashed line is the switching line for the component u_1; the dotted one is for the component u_2. The switching lines are obtained as a result of the analysis of the func-

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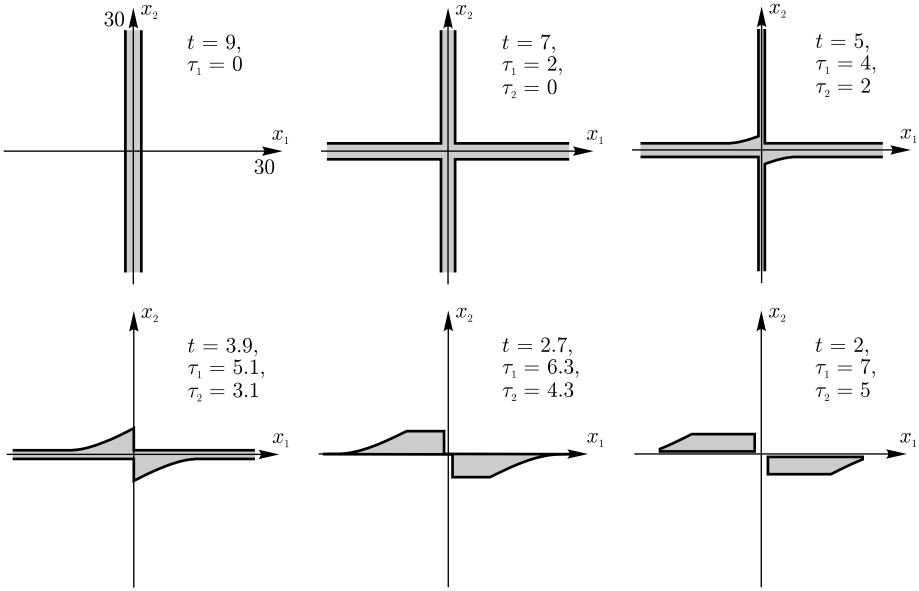
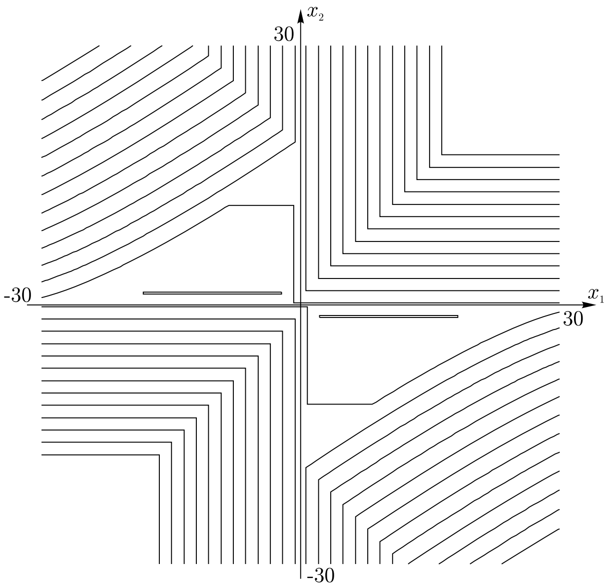


Fig. 12 Two weak pursuers, different termination instants: time sections of the maximal stable bridge $W_{2.0}$

Fig. 13 Two weak pursuers, different terminal instants: level sets of the value function, $t = 2$



tion $x \to V(t, x)$ in horizontal (for u_1 in accordance with the direction of the vector $\mathcal{D}_1(t)$) and vertical (for u_2 in accordance with the direction of the vector $\mathcal{D}_2(t)$) lines. If in the considered horizontal (vertical) line the minimum of the value function is attained in a segment, then the middle of such a segment is taken as a point for the switching line. Arrows show the directions of the vectors $\mathcal{D}_1(t)u_1^*$ and $\mathcal{D}_2(t)u_2^*$ in 4 cells. The "spikes" of the switching line for the control u_2^* near the origin arise due to discreteness of the collection of the

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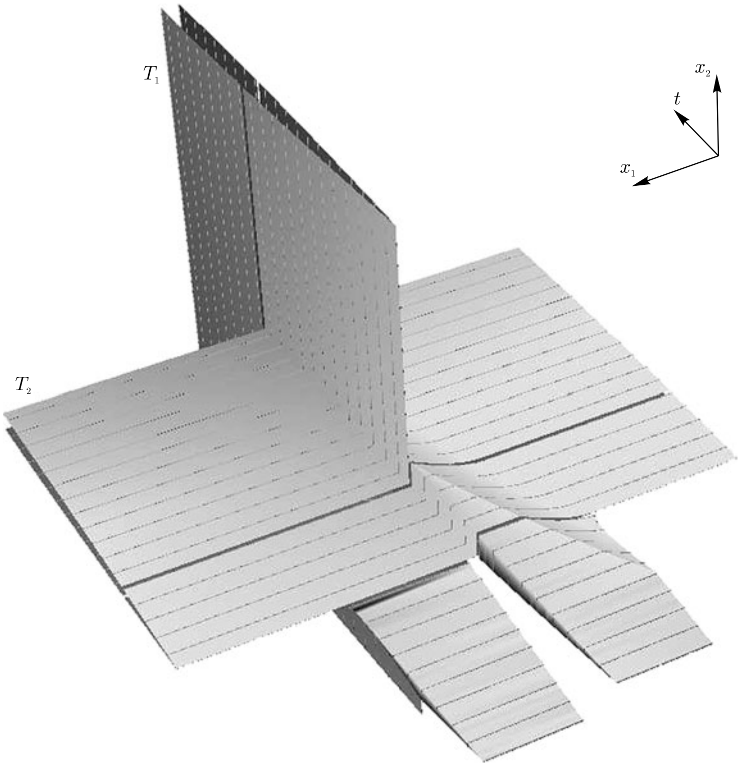
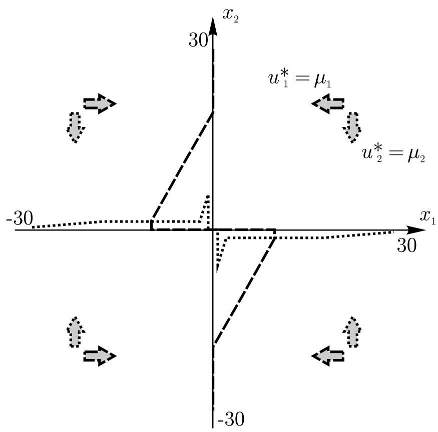


Fig. 14 Two weak pursuers, different terminal instants: a three-dimensional view of the set $W_{2.0}$

Fig. 15 Two weak pursuers, equal terminal instants: switching lines for the first player, $t = 2$



sets $W_c(t)$ involved for constructing switching lines. As the step on c goes to zero, these spikes disappear.

In Fig. 16 switching lines and the directions of the vectors $\mathcal{E}(t)v^*$ are shown for $t = 2$. In this picture, we have 4 cells with constant values of the second player control.

The suggested methods for the players' controls, which use the switching lines, in the case of weak pursuers need additional theoretical foundation to justify their closeness to the

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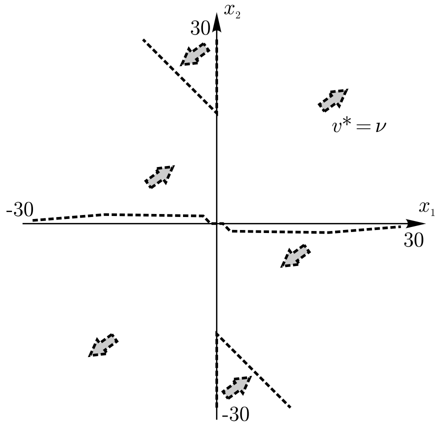

Fig. 16 Two weak pursuers, equal terminal instants: switching lines for the second player, $t = 2$

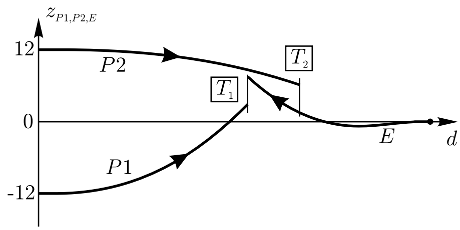

Fig. 17 Two weak pursuers, different termination instants: trajectories of the objects in the original space, optimal control of the second player

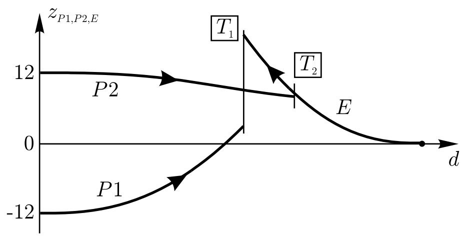

Fig. 18 Two weak pursuers, different termination instants: trajectories of the objects in the original geometric space, constant control of the second player $v = +\nu$

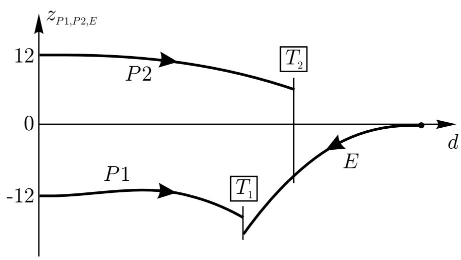

Fig. 19 Two weak pursuers, different termination instants: trajectories of the objects in the original geometric space, constant control of the second player $v = -\nu$

optimal ones and to analyze their stability. Corresponding proofs, in the authors' opinion, are more difficult than the proofs of Propositions 1 and 2 for the case of strong pursuers. Now, the authors regard the strategies constructed numerically as "practically optimal". For example, to cause the non-optimality of the first player's control, the second player should behave himself very sophisticatedly near the switching lines of the first player.

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Figures 17, 18, and 19 show trajectories in the original geometric space (as in Sect. 4.5) for the case of weak pursuers and different terminal instants. The initial positions at the instant $t_0 = 2$ are taken as follows:

$$z_{P_1}^0 = -12, \qquad z_{P_2}^0 = 12, \qquad z_E^0 = 0.$$

Trajectories in Fig. 17 are built for the optimal controls of all objects. At the beginning of the pursuit, the evader closes to the first (lower) pursuer. It is done to increase the miss from the second (upper) pursuer at the instant T_2. Further closing is not reasonable, and the evader switches its control to increase the miss from the first pursuer at the instant T_1.

Figure 18 gives the trajectories when the pursuers use their optimal feedback controls generated by switching lines, but the evader applies a constant control $v \equiv \nu$ escaping from P_1 and ignoring P_2. In Fig. 19, the situation is given when the evader, vice versa, keeps control $v \equiv -\nu$ escaping from P_2 and ignoring P_1. In these both situations, the payoff is less than in the case when the second player uses the optimal control. If a constant control $v = +\nu$ is applied, the miss to the second pursuer at the instant T_2 is less; if the second player keeps $v = -\nu$, the miss to the first pursuer at the instant T_1 decreases.

6 One Strong and One Weak Pursuers

Let us change parameters of the second pursuer in (12) and take the following parameters of the game:

$$\mu_1 = 2, \qquad \mu_2 = 1, \qquad \nu = 1, \qquad l_{P_1} = 1/2, \qquad l_{P_2} = 1/0.3, \qquad l_E = 1.$$

Now the evader is more maneuverable than the second pursuer, and an exact capture by this pursuer is unavailable. Assume $T_1 = 5$, $T_2 = 7$.

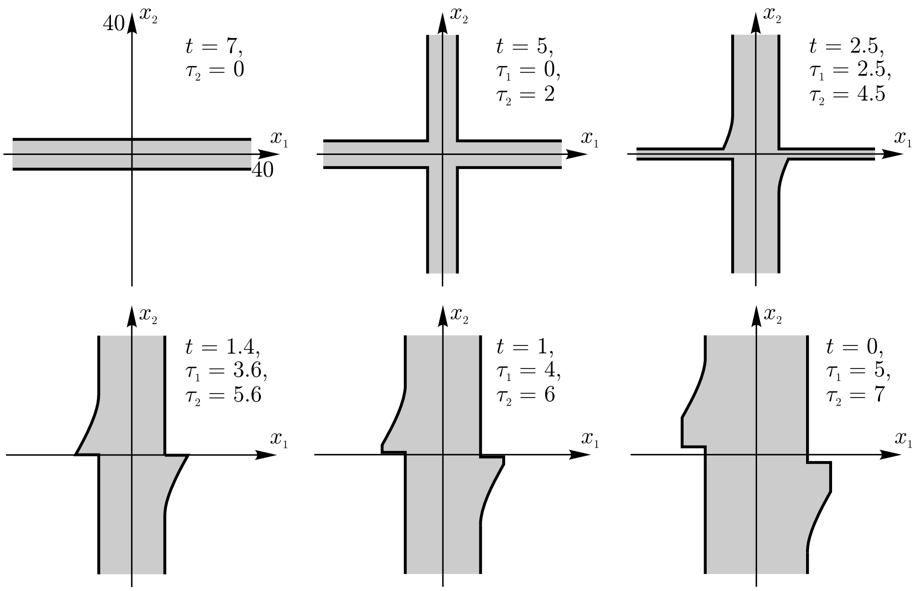

Fig. 20 One strong and one weak pursuers, different termination instants: time sections of the maximal stable bridge $W_{5.0}$

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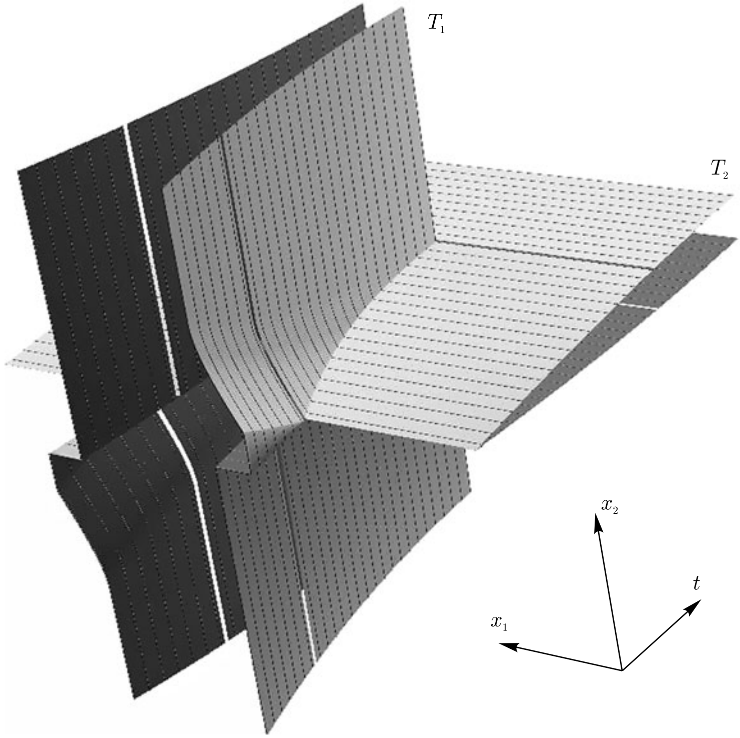

Fig. 21 One strong and one weak pursuers, different termination instants: a three-dimensional view of the set $W_{5.0}$

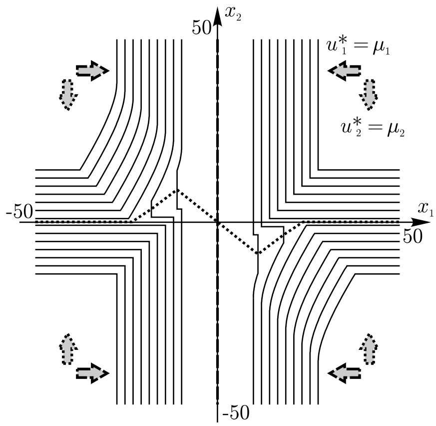

Fig. 22 One strong and one weak pursuers, different termination instants: switching lines and optimal controls for the first player (the pursuers), $t = 1$

In Fig. 20, there are sections of MSB $W_{5.0}$ (that is, $c = 5.0$) for 6 instants: $t = 7.0$, 5.0, 2.5, 1.4, 1.0, 0.0. The horizontal part of its time section $W_{5.0}(\tau)$ decreases with growth of τ, and breaks further. The vertical part grows. After breaking the individual stable bridge of the second pursuer (and respective collapse of the horizontal part of the cross), there is the vertical strip only with two additional parts determined by the joint actions of both pursuers.

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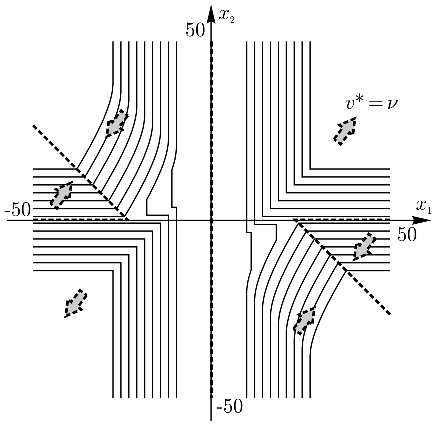

Fig. 23 One strong and one weak pursuers, different termination instants: switching lines and optimal controls for the second player (the evader), $t = 1$

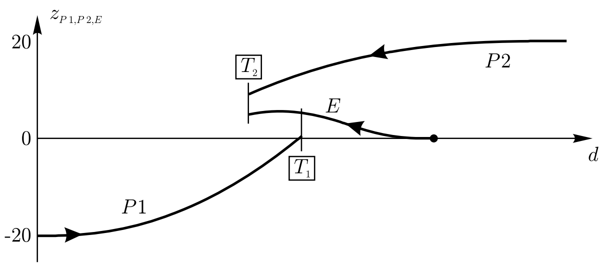

Fig. 24 One strong and one weak pursuers, different termination instants: trajectories of the objects in the original geometric space

A three-dimensional view of the set $W_{5.0}$ in coordinates t, x_1, x_2 is given in Fig. 21.

Switching lines of the first and second players for the instant $t = 1$ are given in Figs. 22 and 23. These lines are obtained by processing collection $\{W_c(t = 1)\}$ computed for different values of c. In comparison with the previous case of two weak pursuers, the switching lines for the first player have simpler structure.

In Fig. 24, one can see the optimal trajectories in the original geometric space. For simulations, the initial lateral deviations at the instant $t_0 = 0$ are taken as $z^0_{P_1} = -20$, $z^0_{P_2} = 20$, $z^0_E = 0$. Longitudinal components of the initial positions and velocities are such that the evader moves toward one pursuer, but from another.

7 Varying Advantage of Pursuers

7.1 Variant 1

Let us pass to the case of varying advantage of pursuers. Consider a variant when both pursuers P_1 and P_2 are equal, with that at the beginning of the backward time, the bridges in the individual games contract and further expand. Choose the game parameters in such a way that for some c the section $W_c(t)$ of MSB W_c with decreasing of t disjoins into two parts, which join back with further decreasing of t.

Parameters of the game are taken as follows:

$$\mu_1 = \mu_2 = 1.1, \qquad \nu = 1, \qquad l_{P_1} = l_{P_2} = 1/0.6, \qquad l_E = 1.$$

Termination instants are equal: $T_1 = T_2 = 20$.

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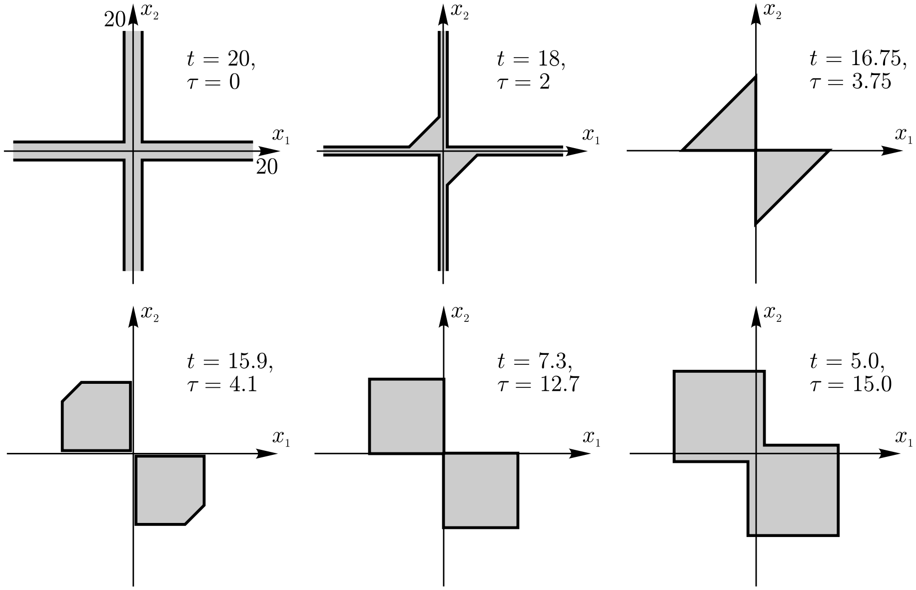

Fig. 25 Varying advantage of the pursuers, variant 1: time sections of the maximal stable bridge $W_{0.526}$

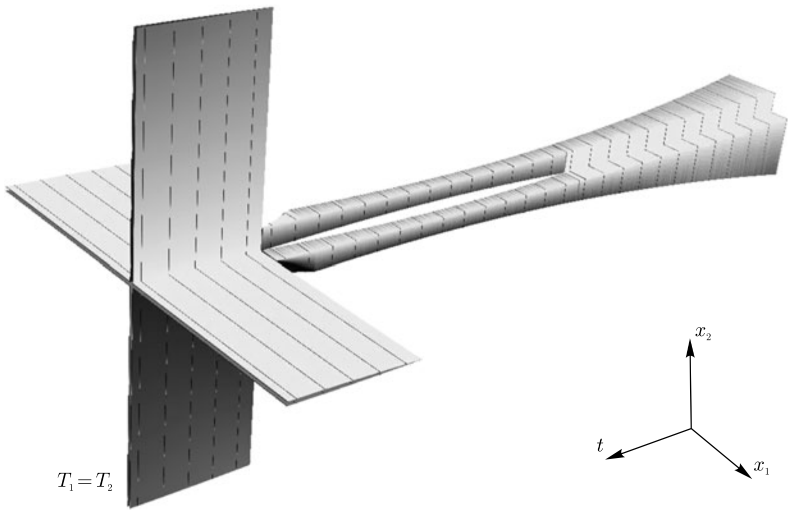

Fig. 26 Varying advantage of the pursuers, variant 1, equal termination instants: a three-dimensional view of the maximal stable bridge $W_{0.526}$

In Fig. 25, the time sections of MSB $W_{0.526}$ are shown for 6 instants: $t = 20.0$, 18.0, 16.75, 15.9, 7.5, 5.0. At the termination instant, the terminal set is taken as a cross (the upper-left subfigure).

At the beginning of backward time, the structure of the bridge is similar to the case of two weak players: widths of both vertical and horizontal strips of the "cross" decreases, and

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Fig. 27 Varying advantage of the pursuers, variant 1, equal termination instants: switching lines and optimal controls for the first player (the pursuers), $t = 12.5$

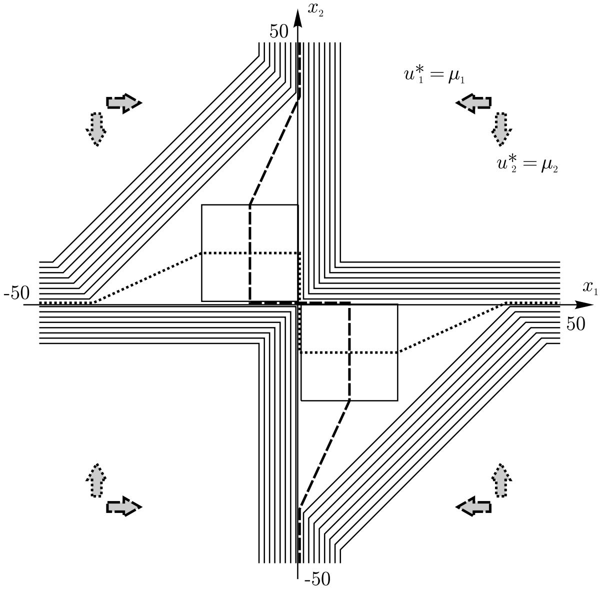
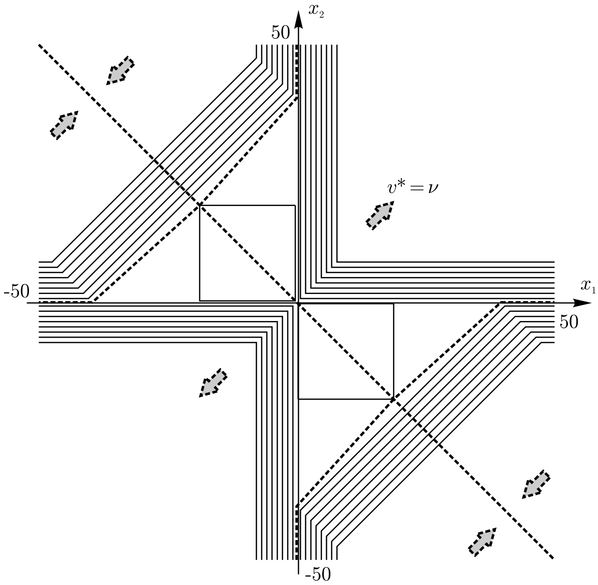


Fig. 28 Varying advantage of the pursuers, variant 1, equal termination instants: switching lines and optimal controls for the second player (the evader), $t = 12.5$



two straight-linear additional triangles of joint capture zone appear (the upper-middle subfigure). Then at some instant, both strips collapse, and only the triangles constitute the time section of the bridge (the upper-right subfigure). Further, the triangles continue to contract, so they become two pentagons separated by an empty space near the origin (the lower-left subfigure). Transformation to pentagons can be explained in the following way: the first player using its controls expands the triangles vertically and horizontally, and the second

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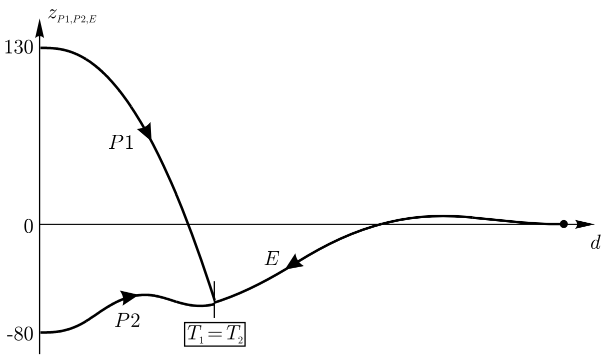

Fig. 29 Varying advantage of the pursuers, variant 1, equal termination instants: trajectories of the objects in the original geometric space

player contracts them in diagonal direction. So, vertical and horizontal edges appear, but the diagonal part becomes shorter. Also, in general, size of each figure decreases slowly.

Due to action of the second player, the diagonal disappears and the pentagons convert to squares at some instant (this is not shown in Fig. 25). After that, the pursuers have advantage, and total contraction is changed by growth: the squares start to enlarge. After some time passes, the squares touch each other at the origin due to the growth (the lower-middle subfigure). Since the enlargement continues, their sizes grow, and the squares start to overlap forming one "eight-like" shape (the lower-right subfigure).

A three-dimensional view of MSB W_c corresponding to $c = 0.526$ is shown in Fig. 26.

Figures 27 and 28 show time sections of a collection of MSBs and switching lines for the first and second players, respectively, for the instant $t = 12.5$.

Note that near the central part of the switching line of the second player passing through the origin, the arrows of the vectors $\mathcal{E}(t)v^*$ are directed from the switching line. In this case at any point in the switching line, one of values $v^* = \pm\nu$ should be chosen. Choice of some intermediate value $v \in (-\nu, \nu)$ is not optimal. A consequence of this is instability of the second player's control with respect to informational disturbances. Erring in analysis of the location of the current position with respect to the switching line, the second player, generally speaking, allows a sliding regime along the switching line, which changes in time. The value function can decrease along the trajectory.

To construct optimal trajectories in the original geometric space, we choose the following initial positions at the instant $t_0 = 0$:

$$z^0_{P_1} = 130, \qquad z^0_{P_2} = -80, \qquad z_E = 0.$$

The resultant miss at the instant $T_1 = T_2 = 20$ is about 1.1 and almost cannot be seen in Fig. 29.

7.2 Variant 2

Let now MSBs in the individual game $P1$–E expand at the beginning of the backward time and further contract ($\eta_1 < 1$, $\eta_1\varepsilon_1 > 1$). Vice versa, in the individual game $P2$–E, let MSBs contract at first and expand further ($\eta_2 > 1$, $\eta_2\varepsilon_2 < 1$). Parameters of the game are taken as follows:

$$\mu_1 = 0.8, \qquad \mu_2 = 1.3, \qquad \nu = 1, \qquad l_{P_1} = 1/20, \qquad l_{P_2} = 1/0.5, \qquad l_E = 1.$$

Termination instants: $T_1 = 15$, $T_2 = 13.5$.

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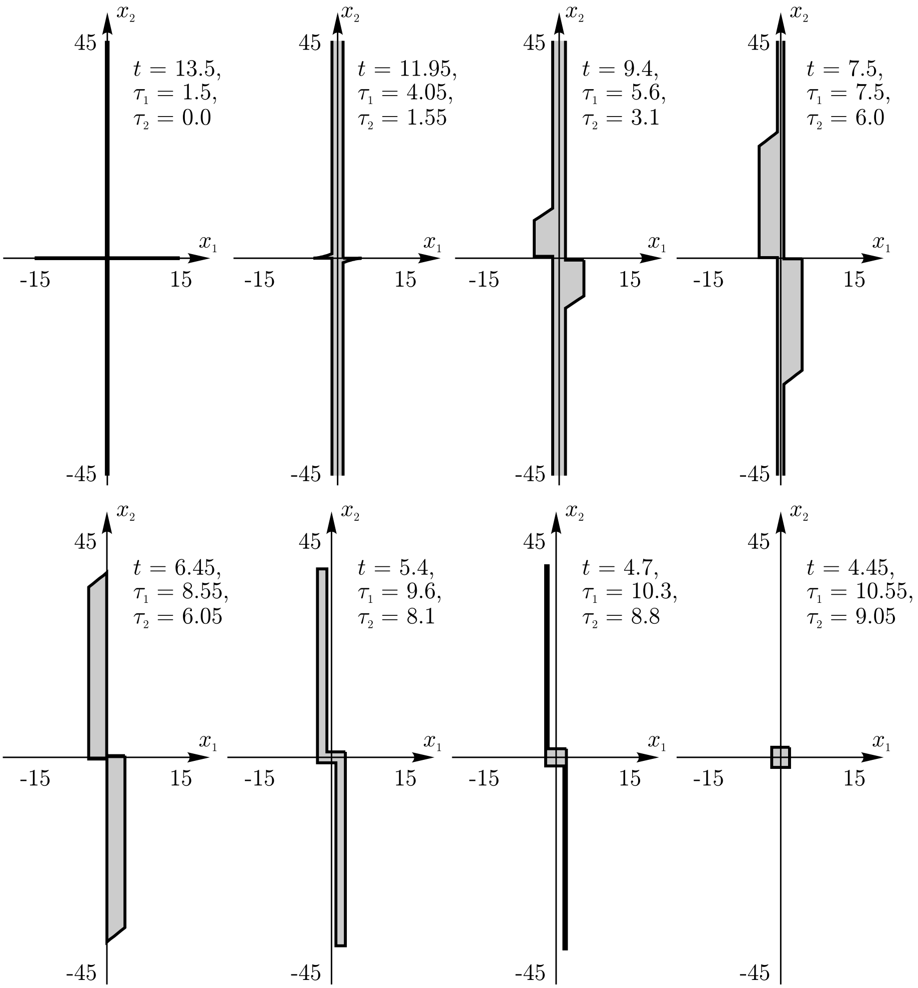

Fig. 30 Varying advantage of the pursuers, variant 2, different termination instants: t-sections of MSB $W_{0.263}$

In Fig. 30, t-sections of MSB $W_{0.263}$ are shown for eight instants: $t = 13.5, 11.95, 9.4, 7.5, 6.45, 5.4, 4.7, 4.45$. At the instant $t = T_1 = 15$, the terminal set is taken as a vertical strip with the half-width equal to 0.263.

At the beginning of the backward time, the t-section of MSB is a vertical strip and has growing width. At the instant $t = T_2 = 13.5$, a horizontal strip of half-width 0.263 is added to the vertical one, which is at that instant. With further growing of the backward time, additional curvilinear triangles appear in the II and IV quadrants. Outside them, the horizontal component of the set $W_{0.263}(t)$ contracts. At the instant $t = 11.95$, the infinite horizontal component vanishes. Then some growth in the horizontal direction takes place with high vertical expand of the knobs generated by the curvilinear triangles. Near the instant $t = 9.4$, horizontal increasing is changed by contraction. At the instant $t = 6.45$, the infinite vertical component disappears. Further with growing the backward time, horizontal contraction

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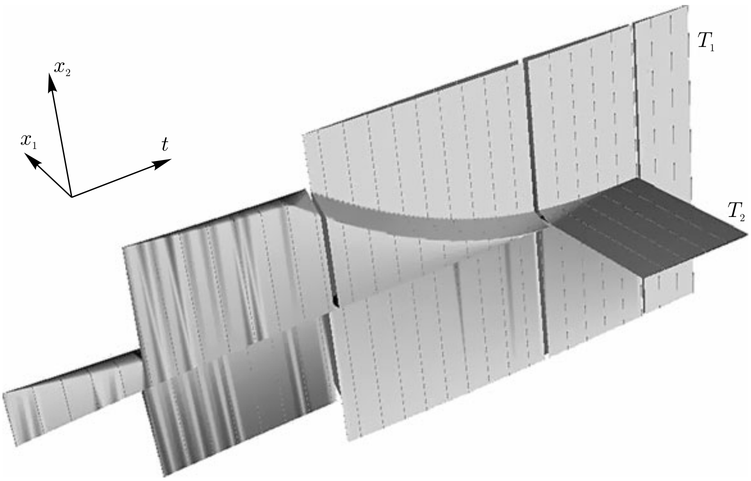

Fig. 31 Varying advantage of the pursuers, variant 2, different termination instants: a view of MSB $W_{0.263}$ in the three-dimensional space t, x_1, x_2

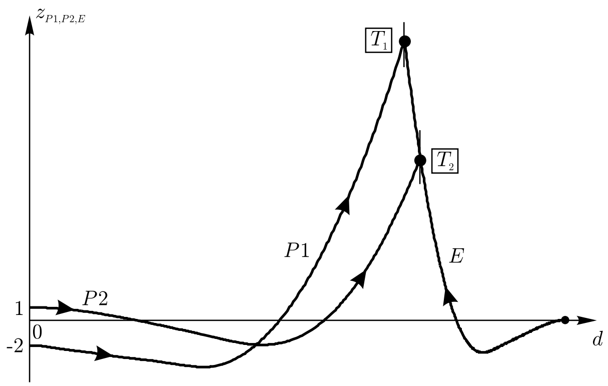

Fig. 32 Varying advantage of the pursuers, variant 2, different termination instants: trajectories of the objects in the original geometric space

and vertical dilatation have approximately equal speed. When $t \leq 5.4$, each t-section has two vertical protuberances, which collapse at some instant close to $t = 4.45$. After that, t-sections are rectangles which dilate in the vertical direction and constrict in the horizontal one. At the instant $t = 0.15$, MSB degenerates.

A three-dimensional view of the set $W_{0.263}$ can be seen in Fig. 31. One can see some dints on the surface, which are faults of the algorithm for construction of the three-dimensional object.

Figure 32 gives a view of the optimal trajectories in the original geometric space for the following initial positions taken at the instant $t_0 = 0$:

$$z_{P_1}^0 = -2, \qquad z_{P_2}^0 = 1, \qquad z_E = 0.$$

At the instants $t = T_1$ and $t = T_2$, there are almost zero deviations of the evader from corresponding pursuers.

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8 Conclusion

The paper deals with numerical investigation of a differential game with two pursuers and one evader. With the help of the standard change of variables, the problem is reduced to a two-dimensional antagonistic game. The difficulty of solution is connected to non-convexity of the terminal payoff function. For typical variants of the game parameters, an analysis of the level sets (Lebesgue sets) of the value function is done. Three-dimensional views of the level sets are given. Quasioptimal strategies of the players are suggested. They are based on usage of switching lines. Situations are emphasized when the strategies defined by the switching lines are optimal. In general, the question about optimality of the controls needs additional study.

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