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Pursuit-Evasion Games of High Speed Evader

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Abstract In this paper, we address pursuit-evasion games of high speed evader involving multiple pursuers and a single evader with holonomic constraints in an open domain. The existing work on this problem discussed the required formation and capture strategy for a group of pursuers. However, the formulation has mathematical errors and has raised concerns over the validity of the developed capture strategy. This paper uses the idea of Apollonius circle to develop an escape strategy for the high speed evader, resolving the shortfalls in the existing work. The strategy is built on a concept of perfectly encircled formation and the conditions required to construct the same are presented. The escape strategy contains two steps. Firstly, the evader employs a strategy that forces a gap in the formation against all the admissible strategies of a group of pursuers. In the second step, it uses this gap to escape. The strategy considers both direct and indirect gaps in the formations. The indirect gap is encountered when a group of three or four pursuers is employed to capture. The efficacy of the escape strategy is established using simulation results.

Keywords Pursuit-evasion games

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1 Introduction

In recent years, pursuit-evasion (PE) games have received meticulous attention among the researchers in the domain of decision and control. They cover a broad class of problems in the areas of missile guidance, unmanned aerial vehicles (UAVs), and robotics. Some of the recent experimental works on PE games using probabilistic theory have encouraged engineers to employ these solutions for surveillance and security using aerial robots. One of those solutions is a computationally feasible greedy pursuit scheme in a probabilistic theoretical framework which was implemented on UAVs and unmanned ground vehicles (UGVs) [23]. Another one being a randomized strategy developed for a single pursuer against an unpredictable evader in a connected domain using visibility-based approach [9]. Also, a discrete-time PE game of a single evader was studied with players having identical sensing and motion ranges [3].

Broadly, PE games can be classified into three types based on the speeds of the players involved; (i) a case where the speed of a pursuer is higher than the speed of an evader; (ii) a case where the speeds of both pursuer and evader are equal; and (iii) the case of a high speed evader where its speed is higher than the speed of a pursuer.

The first case is widely explored through various guidance schemes in missile guidance related research and development. The conventional methods of target inception and their variants, including line of sight (LOS) guidance and proportional navigation (PN) guidance, have proved to be effective for their simplicity and ease of implementation in real world applications. A more detailed study on these techniques can be found in [28]. The idea of capturability was developed using these classical guidance schemes where PE games with simple motions were modeled [17]. But, it can not be extended to players taking complex paths and maneuvers that any vehicle can perform. This particular case has also attracted solutions from the theory of optimal control. A lot of work has been carried out using optimal control techniques which can be found in [28]. However, the optimal control theory requires strategy of a moving target based on which the pursuer(s) develop guidance and control strategies for capturability. Therefore, the approach becomes ineffective in the case of an intelligent target which also tries to correct its path and thereby trying to escape from an approaching pursuer. This is the case where both a pursuer (missile) and an evader (target) have divergent objectives and try to employ optimal strategies at their disposal. This problem is tackled by formulating a zero-sum differential game, and is also viewed as a min-max problem in a game theoretical framework [1]. In this regard, variational techniques were also applied to solve differential games and to obtain conditions for capture and optimality [6].

A differential game formulation can be used to solve a PE game [5, 16]. The associated Hamilton-Jacobi-Isaacs (HJI) partial differential equation has to be solved in this regard to obtain optimal strategies for both the players. Though this method is known for its accurate results, it suffers from the *curse of dimensionality*. Hence, it can not be used for games involving three or more pursuers as it is computationally intensive. This approach was implemented in real-time conditions to analyze PE between two aircrafts using point-mass aircraft models in a horizontal plane [18]. It was also explored using differential dynamic programming technique [10]. The differential game formulation was then used to obtain a numerical solution for a three-dimensional PE game, the problem of intercepting an evasive spacecraft by a pursuing spacecraft [4]. Another interesting solution was obtained for a missile-aircraft pursuit-evasion problem, formulated using a three-dimensional linearized kinematic model with bounded control [20]. The unparalleled efficiency of this approach in solving a two player PE game can be seen in the aforementioned works.

The second case of a pursuer and an evader having equal speeds was best solved using the approach of safe-reachable area cooperative pursuit which is also computationally efficient. The problem was addressed in a closed domain with holonomic multiple pursuers and an evader with a decentralized strategy. The idea is that a group of pursuers tend to drive the safe-reachable area of an evader to zero in a finite time and guarantees successful capture [7]. This approach was later extended to the case of non-holonomic agents using the concept of Dubins distance [12]. This work also provided another cooperative strategy based on pure proportional navigation law for capturing an evader.

The final case of a high speed evader was first addressed using the approach of Apollonius circles by Isaacs which was used to study PE games with unequal speeds [8]. It is necessary to deploy multiple pursuers in an open domain to capture a high speed evader owing to its speed advantage. The capturability depends on the initial conditions and the strategies employed by the players. The exiting literature contains a formation geometry for a group of pursuers that was said to ensure successful capture with appropriate strategies [24]. Using the formation geometry, a decentralized approach was developed to capture a superior evader with jamming confrontation and in noisy environments [25–27]. A similar approach was followed by other researchers to develop strategies using the existing geometry as an initial condition [2, 11]. This particular formation requires minimum number of pursuers that can enclose an evader and is also called a *perfectly encircled formation* (PEF). This paper develops an escape strategy for the evader trapped in a PEF and contradicts the existing work using a mathematical framework constructed on the idea of Apollonius circle. The preliminary work for the same is presented in [19].

The success of safe-reachable area cooperative pursuit in capturing an evader with same speed has motivated us to explore the case of high speed evader. However, in this case, it is computationally arduous to obtain a closed form solution for the safe-reachable area when multiple pursuers are involved. In the case of a game with single pursuer and single evader, the safe-reachable area can be obtained using the Hamilton-Jacobi-Bellman (HJB) equation which also includes the region of the associated Apollonius circle [22]. The methodology of using Apollonius circle is justified as it keeps the problem tractable while dealing with a PEF. Another reason being the involvement of only the convex side of the actual safe-reachable area that overlaps with the Apollonius circle. A PEF can be obtained with a minimum of three pursuers. In this paper, we build on our previous work and develop strategies using both the direct and indirect gaps encountered in different formations. This paper generalizes an escape strategy to all the possible cases available for a PEF.

The rest of the paper is organized as follows. Section 2 formally describes the problem statement. Section 3 presents the mathematical preliminaries required to develop an escape strategy that includes the properties of Apollonius circle(s). Section 4 discusses the conditions required to create a PEF. Section 5 presents a mathematical proof of existence of an escape strategy for any general PEF formed using three or more pursuers. Simulation results for the proposed escape strategy are presented in Section 6 to show that an evader can escape from a PEF. The concluding remarks and the scope for future work are discussed in Section 7.

2 Problem Statement

Consider a PE game of N pursuers and one evader in an open 2-D domain with the pursuers in a perfectly encircled formation (PEF). The aim of the pursuers is to capture the evader by having one of the pursuers within the radius of capture of the evader, r_c, whereas the evader tries to escape. The initial distance of each pursuer from the evader is greater than its radius of capture. A constant speed model with heading control is considered for the players that are also holonomic. The equations of motion for the players involved in a game are given as follows,

$$\begin{aligned} \dot{x}_i &= V_i \cos \psi_i, \\ \dot{y}_i &= V_i \sin \psi_i, \\ \dot{\psi}_i &= \frac{u_i}{V_i}, \end{aligned} \tag{1}$$

where $i \in \{\mathbf{P}, e\}$, $\mathbf{P} = \{p_1, ..., p_N\}$ is the set of pursuers, and e denotes an evader. $[x_i, y_i] \in \mathbf{R}^2$ is the position of the i^{th} player. $V_i = V_p, \forall i \in \mathbf{P}$, and V_p, V_e are the speeds of pursuer and evader respectively i.e, all pursuers have equal speed. In the case of high speed evader, $V_p < V_e$. ψ_i is the heading angle, and u_i represents lateral acceleration which acts as a control input for the i^{th} player. There is no constraint acting on the control input which means that any agent can change its orientation instantaneously.

A group of pursuers in a PEF try to shrink the area around an evader while keeping the formation intact for the capture to occur. Whereas the evader tries to escape by creating a *gap* between any two Apollonius circles of neighboring pursuers. It is also understood that a PEF ensures *instantaneous nullification of escape path* for an evader, i.e., at that instant a straight path in any direction will intersect an Apollonius circle leading to capture. Creation of any gap will break the condition of instantaneous nullification of escape path and will aid in a successful evasion. If the evader creates a gap which makes its safe-reachable area unbounded and comes out of the PEF, then it is called a successful evasion.

Problem Given the initial configuration of a perfectly encircled formation of N pursuers, $N \geq 3$, find an escape strategy for an evader that will ensure successful escape from the formation.

Before going to develop an escape strategy, the required preliminaries for Apollonius circle are discussed in the following section.

3 Apollonius Circle

The idea of Apollonius circle in PE games was first discussed by Isaacs [8]. Consider a pursuer, $P[x_p, y_p]$, and an evader, $E[x_e, y_e]$, following the kinematics given in Eq. 1. The corresponding Apollonius circle is obtained by finding the locus of points X which take equal time for an evader and a pursuer to reach in the Euclidean sense. The locus can be obtained using the relation,

$$\frac{PX}{V_p} = \frac{EX}{V_e}$$

The locus is a circle with center,

$$O\left(\frac{x_p - \lambda^2 x_e}{1 - \lambda^2}, \frac{y_p - \lambda^2 y_e}{1 - \lambda^2}\right),$$

and radius,

$$r = \frac{\lambda\sqrt{(x_p - x_e)^2 + (y_p - y_e)^2}}{1 - \lambda^2},$$

where $\lambda = \frac{V_p}{V_e}$.

A typical sample is given in Fig. 1 along with the tangents from an evader. For the case of a high speed evader, the pursuer lies inside the corresponding Apollonius circle whereas the evader lies outside. From here onwards, a tangent point will refer to the point on an Apollonius circle when joined to an evader forms a tangent line to the circle. Any straight path of the evader that passes through the Apollonius circle will lead to its capture provided the pursuer moves toward the point of intersection. The two properties of Apollonius circle, which are going to be used extensively in this paper, are presented in the following lemmas.

Lemma 3.1 *The angle subtended at the evader by the line joining the center of Apollonius circle and any two of the tangent points is independent of the positions of a pursuer and an evader.*

Proof From Fig. 1, let the angle subtended be α. Then,

$$\sin\alpha = \frac{OT_1}{OE}$$

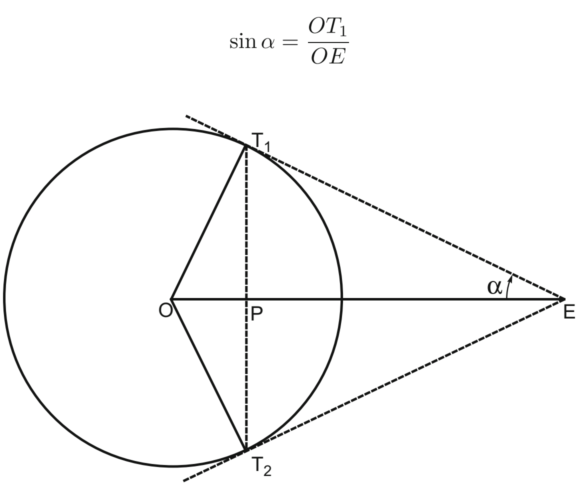


Fig. 1 Apollonius circle

Here, OT_1 is the radius of Apollonius circle and OE is the distance between the center of the circle and the evader.

$$\Rightarrow OE = \sqrt{\left(\frac{x_p - \lambda^2 x_e}{1-\lambda^2} - x_e\right)^2 + \left(\frac{y_p - \lambda^2 y_e}{1-\lambda^2} - y_e\right)^2},$$

$$= \sqrt{\left(\frac{x_p - x_e}{1-\lambda^2}\right)^2 + \left(\frac{y_p - y_e}{1-\lambda^2}\right)^2},$$

$$= \frac{r}{\lambda},$$

$$= \frac{OT_1}{\lambda}$$

$$\Rightarrow \frac{OT_1}{OE} = \lambda \Rightarrow \sin\alpha = \lambda$$

Hence, the angle depends only on the ratio of speeds of pursuer and evader and not on their positions. □

Lemma 3.2 *The line joining the tangent points on the Apollonius circle passes through the pursuer and is perpendicular to line joining the evader and the pursuer.*

Proof From Lemma 3.1, we know $\sin\alpha = \lambda$. Moreover the points on the Apollonius circle follow the relation, $\frac{PX}{V_p} = \frac{EX}{V_e} \Rightarrow \frac{PX}{EX} = \frac{V_p}{V_e} = \lambda$. Since the tangent points lie on the circle, for the tangent point T_1,

$$\frac{PT_1}{ET_1} = \lambda = \sin\alpha$$

$$\Rightarrow \angle EPT_1 = \frac{\pi}{2}$$

Also, $\triangle PET_1$ and $\triangle PET_2$ are congruent triangles, since $ET_1 = ET_2$, $\angle PET_1 = \angle PET_2 = \alpha$, and PE forms the common side. Therefore T_1T_2 is a straight line passing through P and is perpendicular to PE.

Using the above lemmas, the necessary conditions to form a PEF are derived in the following section. □

4 Perfectly Encircled Formation

A perfectly encircled formation (PEF) is the one in which a set of pursuers encircle an evader in such a way that the Apollonius circle of each pursuer will have common tangent points with its neighboring pursuers and all the tangent lines from the common tangent points pass through the evader. A typical example is given in Fig. 2. If an evader is trapped in a PEF, then it does not have an instantaneous escape path to get out of the formation. This is because for the evader at that instant, a straight path in any direction will intersect atleast one of the Apollonius circles, leading to capture. The conditions on the pursuers to create such a formation are discussed in Lemmas 4.1 and 4.2.

Lemma 4.1 *The speed of each pursuer required to create a perfectly encircled formation using N identical pursuers is* $\sin(\pi/N)$ *times the speed of evader.*

Proof From Lemma 3.1, the angle subtended at the evader by the line joining the center and a tangent point of the Apollonius circle is α. In Fig. 1, $\triangle OET_1$ and $\triangle OET_2$ are congruent triangles and hence, $\angle OET_1 = \angle OET_2 = \alpha$. Therefore, $\angle T_1ET_2 = 2\alpha$, i.e. the angle subtended at the evader by the line joining the two tangent points of an Apollonius circle is 2α and is independent of the positions of pursuer and evader. □

From Fig. 2, to encircle the evader with N pursuers, we have

$$2N\alpha = 2\pi$$
$$\Rightarrow \alpha = \pi/N$$

Since $\alpha = \arcsin(\lambda)$,

$$\lambda = \sin(\pi/N)$$
$$\Rightarrow V_p = V_e \sin(\pi/N)$$

Lemma 4.2 *The N pursuers required to form a perfectly encircled formation should lie equidistant from the evader.*

Proof From Fig. 2, consider $\triangle O_{i-1}T_{(i-1)i}E$ and $\triangle O_iT_{(i-1)i}E$ of the $(i-1)^{th}$ pursuer and i^{th} pursuer respectively.

As $\angle O_{i-1}T_{(i-1)i}E = \angle O_iT_{(i-1)i}E = \pi/2$, $\angle O_{i-1}ET_{(i-1)i} = \angle O_iET_{(i-1)i}$ and $T_{(i-1)i}E$ is the common side, both the triangles are congruent.

$$\Rightarrow O_{i-1}E = O_iE$$
$$\Rightarrow P_{i-1}E = P_iE$$

Therefore all the pursuers are equidistant from the evader. □

Remark The minimum number of pursuers required to arrive at a PEF that bounds a high speed evader instantaneously is three.

An escape strategy for a high speed evader from a PEF of N pursuers, $N \geq 3$ is presented in the following section.

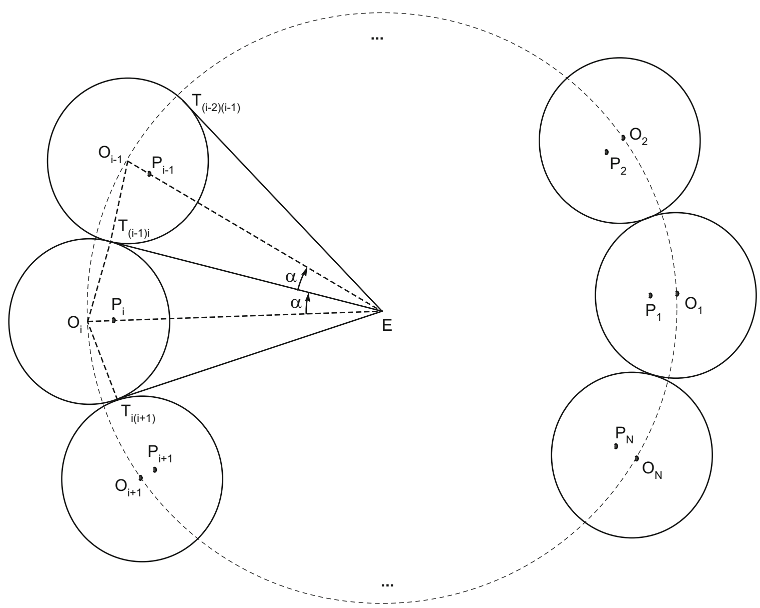


Fig. 2 Perfectly encircled formation of a group of pursuers around the evader

5 Pursuit-Evasion Game and Escape Strategy

A PEF ensures only an instantaneous nullification of escape paths for the evader. But it does not ensure successful capture. In order to capture, a group of pursuers has to make sure that the formation is maintained and the area bounded by it reduces to zero in finite time. While at the same time, the evader will also try to employ a strategy that forces a *gap* between the Apollonius circles of any two neighboring pursuers that can ensure a successful evasion. This is a min-max problem in terms of classical game theory.

5.1 PEF as a Regular Polygon

A PEF in Fig. 2 can be transformed into a regular polygon using the positions of the players, and the corresponding tangent points. Since all the pursuers are equidistant and so are their tangent points, by joining the successive tangent points around the evader, a regular polygon can be created with each pursuer lying at midpoints of sides of the polygon. The safe reachable area of evader will be less than the area of the polygon because the latter includes fraction of Apollonius circle of each pursuer. This can be observed in Fig. 3 for a case of four pursuers. For a general case of N pursuers, the PEF can be seen as an N sided regular polygon with the vertices representing the common tangent points of the Apollonius circles and the pursuers themselves residing at midpoint of each side with evader at the center of the polygon. The same is depicted in Fig. 4 for a case of four pursuers.

In terms of the proposed regular polygon of N sides, each pursuer should maintain common tangent points with its neighbors which ensures instantaneous nullification of escape path for an evader at any given time instant. The following lemma will aid in developing an escape strategy.

Lemma 5.1 *In a perfectly encircled formation, if an evader moves toward a common tangent point, then the corresponding pursuers should move toward the tangent point to overcome the possibility of a gap emerging between them.*

Proof From Fig. 5, consider a situation of evader moving toward a vertex of the regular polygon formed due to N pursuers, $N \geq 3$.

The evader moves to a new position E' along the line ET_{12} in a time t. Now, at least one of the pursuers of the corresponding vertex T_{12} should move toward

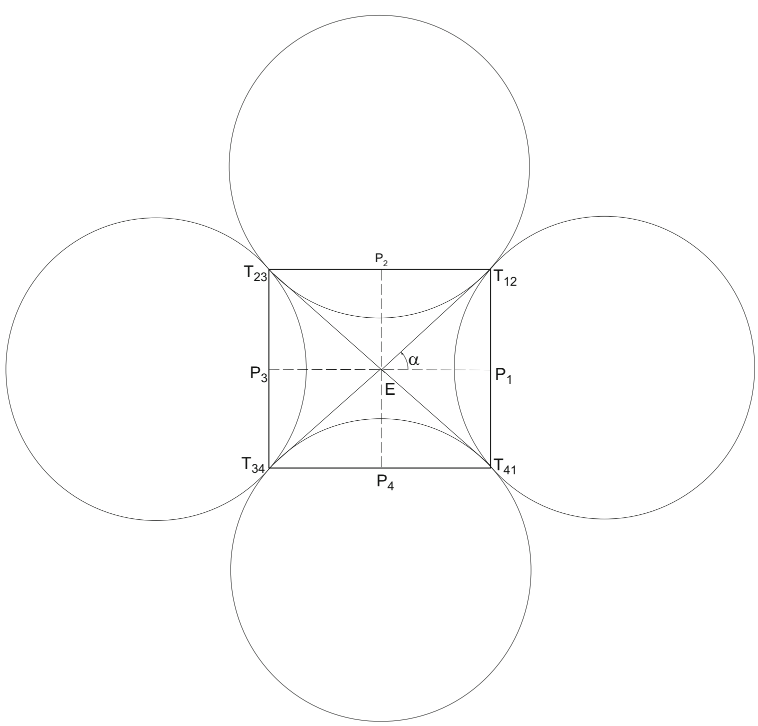


Fig. 3 Regular polygon for the case of four pursuers with Apollonius circles

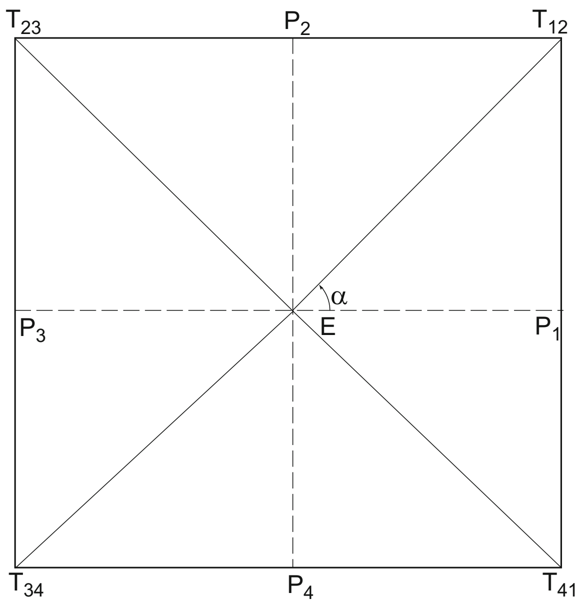


Fig. 4 Proposed regular polygon for the case of four pursuers

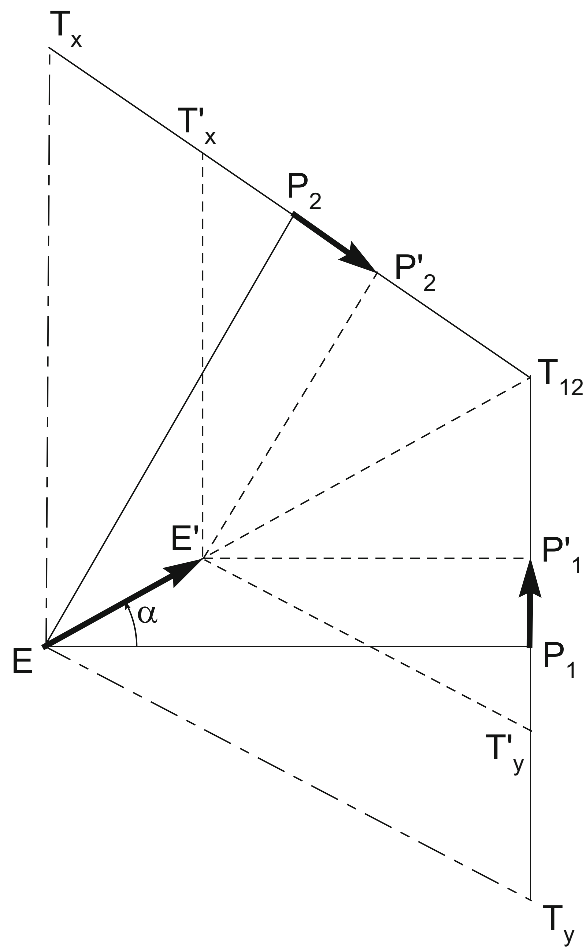


Fig. 5 PE scenario in the case of an evader moving toward a tangent point

the tangent point to capture the evader. Say, P_1 is moving toward the vertex. The triangle $\triangle EP_1T_{12}$ is same as a collision triangle that is used in missile guidance [21]. In the time t, P_1 will reach the new position P_1'. Now the line $E'P_1'$ will lie parallel to the line EP_1. From Lemmas 3.1 and 3.2, the tangent point T_{12} remains as it is. The second tangent point, T_y of the pursuer P_1 will move to T_y', $\angle P_1'E'T_y' = \alpha = \pi/N$.

Now consider the neighboring pursuer P_2. In the time t, the pursuer P_2 has to move in such a way that its new tangent point corresponding to the vertex T_{12} has to either remain at T_{12} or on the line joining the points E' and T_{12}. This will ensure the nullification of escape path for the evader in between P_1 and P_2. Here again, the triangle $\triangle EP_2T_{12}$ is a collision triangle [21]. Any other path will result in its corresponding tangent point at T_{12} falling away, creating a gap between P_1 and P_2, an escape path for the evader. Therefore the pursuer P_2 has to move along the line P_2T_{12}. Next, it is shown that with a PEF, it is not possible to capture a high speed evader. The following theorem establishes an escape strategy from a PEF of five or more pursuers. □

5.2 Case of Five or More Pursuers

Theorem 5.1 *In a PEF of five or more pursuers, the movement of evader toward any of the vertex of the resulting regular polygon will eventually create an instantaneous escape path that can ensure successful evasion under any strategy of the pursuers.*

Proof Consider a situation as shown in Fig. 6 in which an evader moving toward a vertex of the regular polygon formed due to N pursuers, $N \geq 3$.

The evader moved to E' along the line ET_{12} in a time t. From Lemma 4.1, since the evader is moving toward the common tangent point, the pursuers P_1 and P_2 must move toward T_{12}. Therefore, the new position of pursuer P_1 after time t is P_1' which lies on the line P_1T_{12} and the new tangent points are T_{12} and T_{31}'.

The objective of pursuer P_3 is to move in such a way that its tangent point corresponding to the vertex T_{31} will move either to the point T_{31}' or at least to

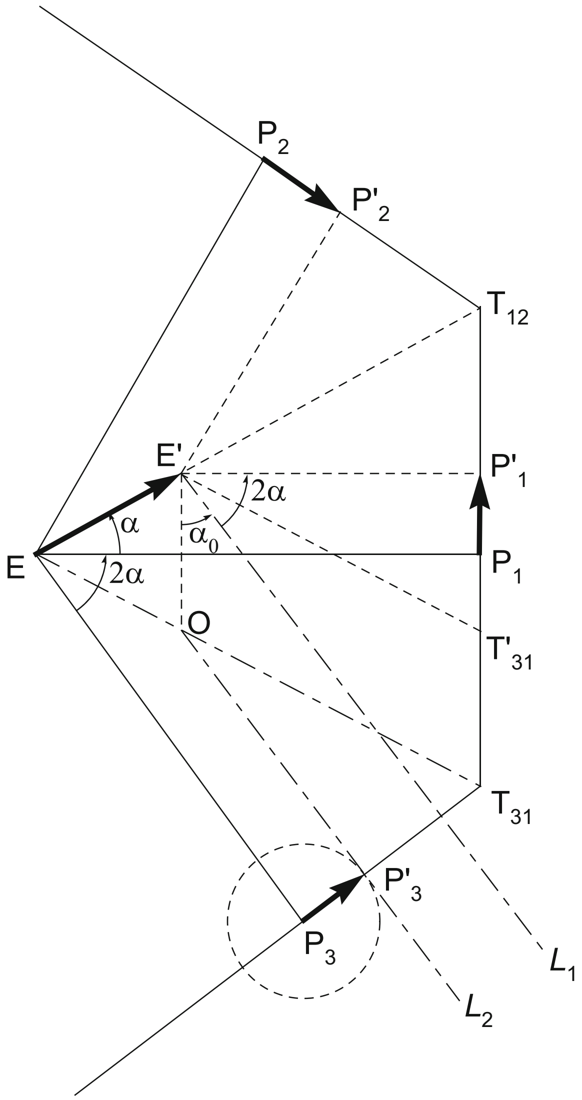


Fig. 6 PE scenario in the case of an evader moving toward a tangent point of Apollonius circle

any point on the line joining the points E' and T'_{31}. This will ensure nullification of an escape path for the evader in between P_1 and P_3. This means if the pursuer P_3 moves to a new position in time t, then the corresponding new tangent point, T''_{31}, must lie on the line $E'T'_{31}$. For this to happen, the new position of P_3 must lie on the line L_1 which makes an angle α with $E'T'_{31}$ as shown in Fig. 6. The possible positions that P_3 can take in time t can be seen as a circle C_1 around P_3 in Fig. 6. The circle intersects the line P_3T_{31} at P'_3.

The line perpendicular to EP_1 and passing through E' meets the line ET_{31} at O. The point O can be seen as the position, the evader would have taken if it had traveled along ET_{31} in the same time t. The line joining O and P'_3, L_2, lies perpendicular to P_3T_{31} and is tangent to the circle C_1 of pursuer P_3. The lines L_1 and L_2 are parallel lines, as they are at equal angle, α, with two parallel lines $E'T'_{31}$ and ET_{31}.

Since the points E, E' and T_{12} lie on a straight line, $\angle EE'T_{12} = \pi$. Let the angle created at E' by line L_1 and line $E'O$ be α_0. Now, from Fig. 6,

$$\begin{aligned} \alpha_0 + \angle EE'O + \angle L_1E'T_{12} &= \pi \\ \Rightarrow \alpha_0 + \pi/2 + 2\alpha &= \pi \\ \Rightarrow \alpha_0 + 2\alpha &= \pi/2 \\ \Rightarrow \alpha_0 &= \pi/2 - 2\pi/N \end{aligned}$$

For $N \geq 5$, α_0 is a positive value and hence the line L_1 cannot intersect the circle C_1. This results in the formation of a gap between pursuers P_3 and P_1 that can ensure a successful evasion.

For the case of four pursuers, α_0 is zero. Because both the pursuers are not equidistant, so are their corresponding tangent points. But the corresponding tangent points and the evader will lie on the same line. This will result in the formation of an indirect gap between the two pursuers. The case of three pursuers is similar to the case of four pursuers and the mathematical proof of existence of an escape strategy for both the cases is presented in the following subsections. □

5.3 Case of Four Pursuers

The indirect gap in this case, PEF of four pursuers, is shown in Fig. 7. The analysis carried out till now is based on a general assumption that each player knows instantaneous position and velocity of every other player. This means if an evader, after reaching E', shown in Fig. 7, takes a different path, then the pursuers can observe the change only at that instant and for which, they have to instantly change their paths to match the new strategy the evader is taking. Since there are two indirect gaps created, one between the pursuers P_2, P_3, and other between P_1, P_4, the evader can use either of them by moving in the direction as shown in Fig. 8.

Figure 8 depicts the pursuit-evasion scenario in the area between pursuers, P_2 and P_3. Consider a path along $E'E''$ that lies in between the lines $E'T''_{23}$ and

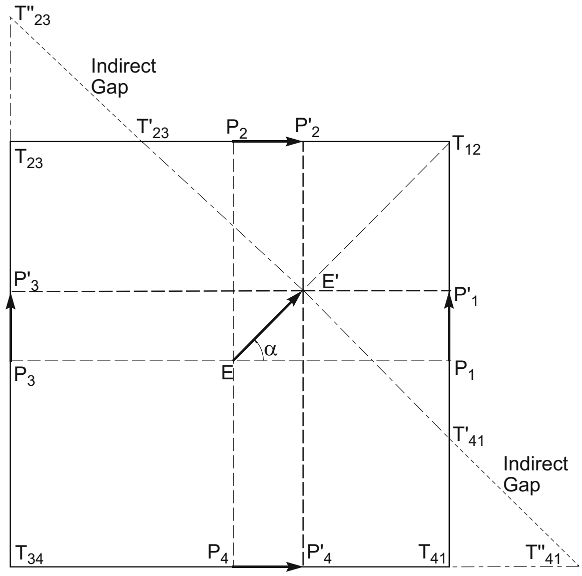


Fig. 7 PE scenario in the case of four pursuers when the evader creates an indirect gap

$E'P_3'$. Under these circumstances, the path of evader will intersect the Apollonius circle of pursuer P_3 at O. Hence pursuer P_3 has to move along the line $P_3'O$ as $\triangle P_3'OE'$ is a collision triangle. By the time evader moves to the new position E'', pursuer P_3 has to move to P_3'' where $P_3''E''$ lie parallel to $P_3'E'$. At P_3'', pursuer P_3 has a tangent point T_{DG} corresponding to the point T_{23}''. $P_3''T_{DG}$ will lie perpendicular to $P_3''E''$ and $\angle P_3''E''T_{DG} = \alpha = \pi/4$.

Now, the objective of pursuer P_2 is to move in such a way that its tangent point T_{23}' will move onto the line joining E'' and T_{DG} to ensure instantaneous nullification of escape path. For that to happen, from Lemma 3.1, it has to move onto the line L_1 which is perpendicular to $E''P_3''$ and passes through E''.

The perpendicular distance from point P_2' onto the line L_1 will be same as the distance between E' and D, D is the point of intersection of line L_1 with $E'P_3'$. Let us assume that pursuer P_2 achieves a new position P_2'' by traveling perpendicular to the line L_1.

Let $\angle T_{23}''E'E''$be $\theta_0, \theta_0 \in (0, \frac{\pi}{4}), \Rightarrow E'D = E'E''$ $\cos(\frac{\pi}{4} - \theta_0)$

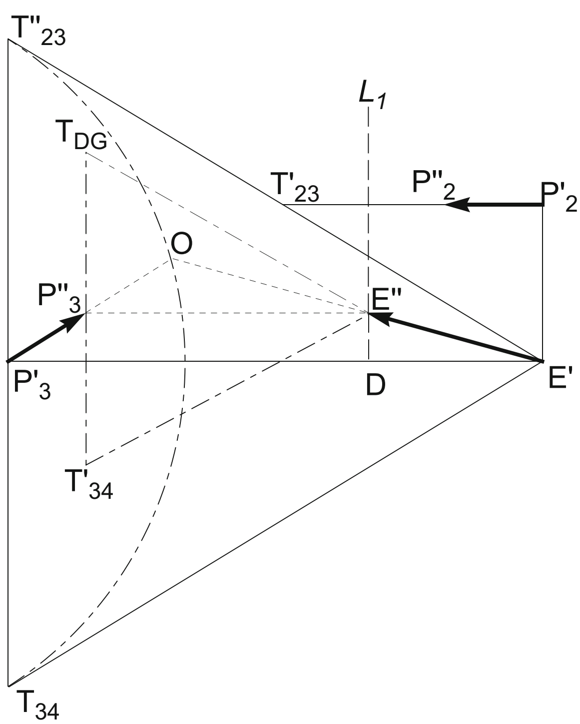


Fig. 8 Pursuit-Evasion scenario in the case of four pursuers when the evader changes its direction creating a direct gap

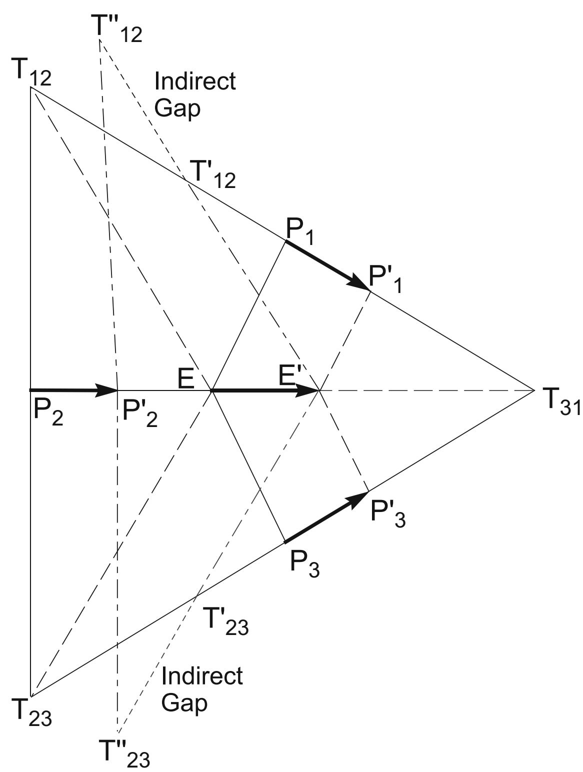


Fig. 9 PE scenario in the case of three pursuers when the evader creates an indirect gap

Since there are four pursuers,

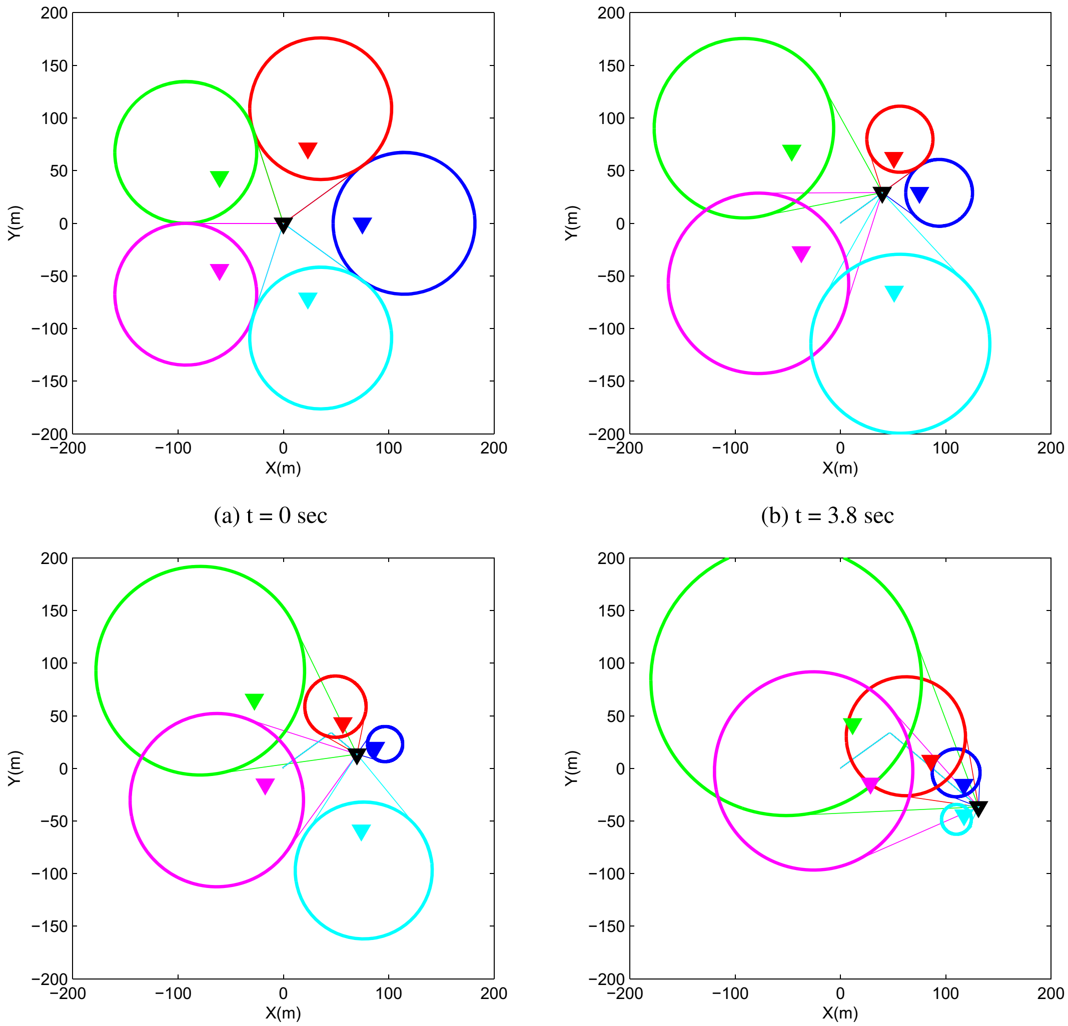
$$\lambda = \sin(\pi/4) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow P_2{}'P_2{}'' = \frac{E'E''}{\sqrt{2}} \Rightarrow P_2{}'P_2{}'' < E'D, \forall \theta_0 \in (0, \pi/4)$$

Therefore pursuer P_2 cannot reach the line L_1 and there is a gap created between the pursuers P_3 and P_2 that ensures successful evasion. For the sake of forming a gap, the evader has to changes its direction within the admissible range of angles given above.

5.4 Case of Three Pursuers

The case of three pursuers is similar to the case of four pursuers and the scenario of an indirect gap formation is shown in Fig. 9. The pursuer P_2 in the figure has no other choice than to go behind the evader as it is understood that moving in any other direction can cause a direct gap for the evader to escape. The movement of evader toward the point T_{31} creates an indirect gap between P_2 and P_3. And, the reason for the formation of an indirect gap is that the pursuer can not match the speed of the high speed evader in this case. Since the pursuers are not equidistant, they are not able to keep the PEF intact. Now, the scenario of P_2 and P_3



(a) t = 0 sec

(b) t = 3.8 sec

(c) t = 6.9 sec

(d) t = 12.9 sec

Fig. 10 PE scenarios with five pursuers under the proposed strategy at different time instants in this case, can be compared to that of P_2 and P_3 respectively in Fig. 8. Using a similar approach, from Fig. 8, the equations and angles involved are presented below.

$$\begin{aligned}\angle P_2'E'T_{23}' &= \angle P_3'E'T_{23}' = \pi/3,\\ \angle DE'T_{23}' &= \pi/6,\\ \angle E''E'T_{23}' &= \theta_0\\ \Rightarrow E'D &= E'E''\cos\left(\frac{\pi}{6}-\theta_0\right);\theta_0\in\left(0,\frac{\pi}{6}\right)\end{aligned}$$

Since there are three pursuers,

$$\begin{aligned}\lambda &= \sin(\pi/3) = \frac{\sqrt{3}}{2}\\ \Rightarrow P_2'P_2'' &= \frac{\sqrt{3}E'E''}{2}\\ \Rightarrow P_2'P_2'' &< E'D, \forall\theta_0\in(0,\pi/6)\end{aligned}$$

Therefore pursuer P_2 cannot reach the line L_1 and there is a gap created between the pursuers P_3 and P_2 that ensures successful evasion. Similar to the previous case of four pursuers, the evader has to change its orientation within the admissible range of angles as given above to create a direct gap. Now, the escape strategy is presented in the following subsection.

5.5 Escape Strategy

The escape strategy of an evader can be summarized in two elements. In case of an evader in a PEF of N pursuers, for $N \geq 5$, the evader will initially move toward any one of the tangent points. A gap will then be created between the corresponding pursuers that the evader uses to escape by changing its course toward the gap created.

In the case of $N = 3, 4$, the evader will first move toward any one of the tangents points that results in formation of an indirect gap between two sets of pursuers, each set has two pursuers. Now, it will select one of the two indirect gaps and changes its courses, to create a direct gap between the pursuers. Finally, it has to change its course toward the direct gap created to escape. In both the cases, the evader has to create a gap big enough depending on its radius of capture to escape successfully.

6 Simulation Results

In this section, the performance of the proposed strategies is demonstrated using two examples. In the first example, the PE game is played with a set of five pursuers in a PEF around an evader. In the second example, we set the number of pursuers to four to show a special case of indirect gap. The simulations are carried out in MATLAB and the time step is, $\Delta t = 0.1$ sec. The parameters kept constant for both the examples include the evader's radius of capture and its speed, and also the initial positions of the pursuers in a PEF. The radius of capture for the evader is considered to be 5 m. The evader's speed is fixed to, $V_e = 13$ m/s. The initial positions of a set of pursuers in a PEF are at a distance of 75 m from the evader.

The scenarios of the game at different time instants for the first example of five pursuers are shown in Fig. 10. A PEF of five pursuers is first presented. As $N = 5$, the speed of pursuers is, $V_p = V_e \sin(\pi/5) = 7.6$ m/s. The evader employs the proposed strategy and starts moving toward one of the tangent points. Due to this, two direct gaps are created between the pursuers that can be seen in Fig. 10 at the time instant, $t = 3.8$ sec. The evader takes advantage of this opportunity and deviates from its initial path. It takes an escape path by moving toward the gap which can be seen at

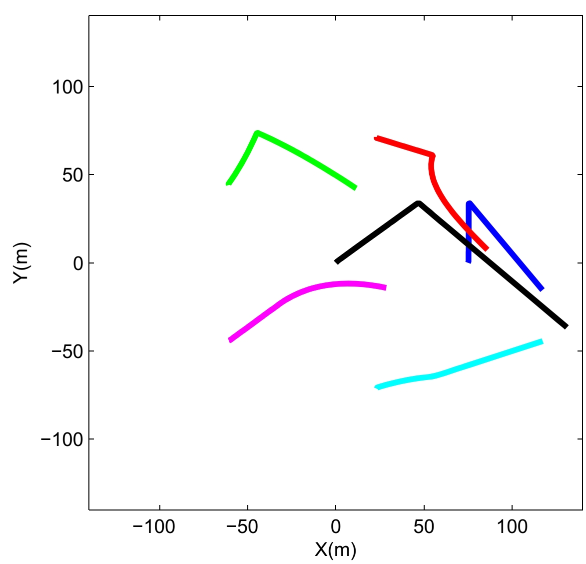
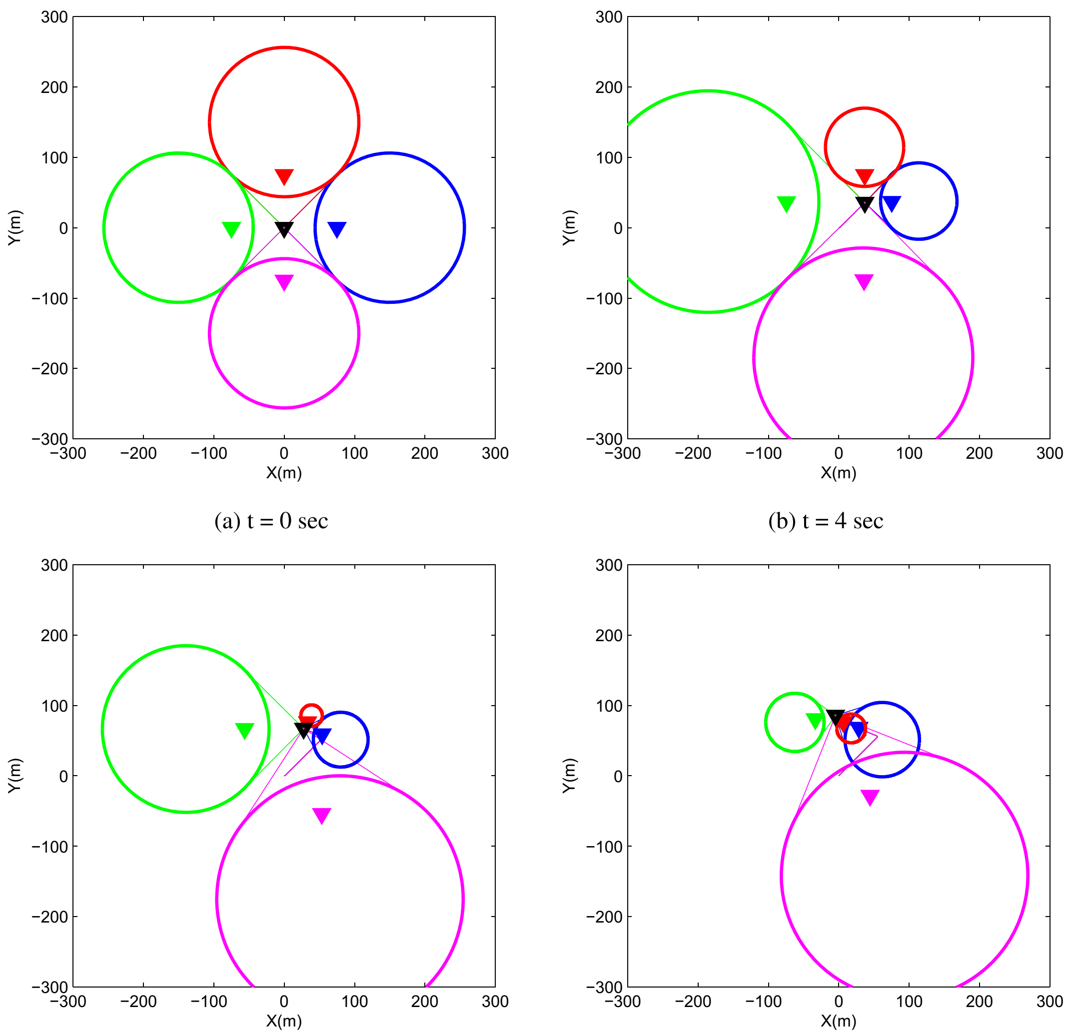


Fig. 11 Case of five pursuers: Paths traced by the players, each agent can be identified from its corresponding color given in Fig. 10

the time instant, $t = 6.9$ sec. This results in a successful evasion which is also shown in Fig. 10 at the time instant, $t = 12.9$ sec. The paths traced by the players are shown in Fig. 11.

Next, the special case of indirect gap formation in a PEF of four pursuers is presented. The scenarios of the game at different time instants for this case are shown in Fig. 12. A PEF of four pursuers is first presented. As $N = 4$, the speed of pursuers is, $V_p = V_e \sin(\pi/4) = 9.2$ m/s. The evader employs the proposed strategy and starts moving toward one of the tangent points. Due to this, two indirect gaps are created between the pursuers that can be seen in Fig. 12 at the time instant, $t = 4$ sec. The evader takes advantage of this opportunity and deviates from its initial path. It takes a path to create a direct gap which can be seen at the time instant, $t = 8.4$ sec. Finally, the evader uses the direct gap to escape which can be seen in Fig. 12 at the time instant, $t = 11.4$ sec. The paths traced by the players are shown in Fig. 13 and the successful evasion can be observed in this case. The videos demonstrating these simulations are published in Youtube [13, 14]. And, the video for the simulation demonstrating evasion from a PEF of six pursuers is also uploaded [15, 19].

It can be observed from the simulations that the proposed strategies guide an evader successfully to



(a) t = 0 sec

(b) t = 4 sec

(c) t = 8.4 sec

(d) t = 11.4 sec

Fig. 12 PE scenarios with four pursuers under the proposed strategy at different time instants

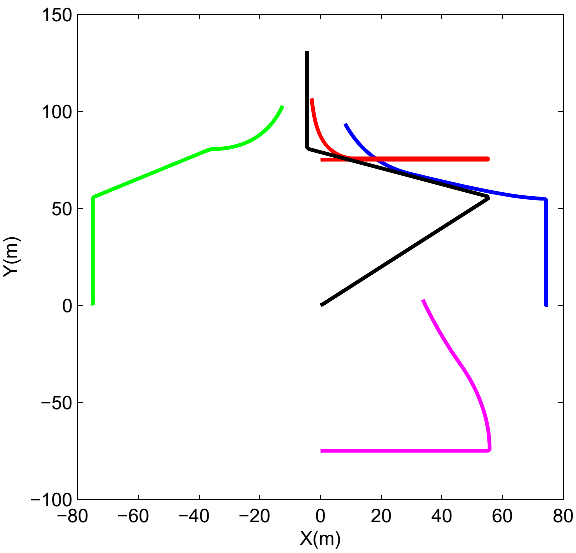


Fig. 13 Case of four pursuers: Paths traced by the players, each agent can be identified from its corresponding color given in Fig. 12

escape from the PEFs and is in line with our mathematical development.

7 Conclusions and Future Work

In this paper, we have studied pursuit-evasion games of multiple pursuers and a single high speed evader with perfectly encircled formation as an initial condition. The idea of Apollonius circle is used to describe the perfectly encircled formation. In this process, few properties of Apollonius circles and the required conditions to create a perfectly encircled formation are also established. It is shown that a perfectly encircled formation only ensures instantaneous nullification of an escape path. For successful capture, the formation should remain connected until the area bounded by it shrinks to zero. For such games, it is mathematically proved that the capture is not possible with the number of pursuers required to form a perfectly encircled formation. An escape strategy for the evader has also been identified using the properties of Apollonius circle and the formation geometry. The idea of perfectly encircled formation and the proposed escape strategy unlock an idea of analyzing a formation which includes overlapping of Apollonius circles. This may potentially reduce the possibility of direct and indirect gaps being created during the pursuit which will be considered in our future works. Also, it would be interesting to study pursuit-evasion games of this kind in closed domain.

References

1. Balakrishnan, S.N., Tsourdos, A., White, B.A.: Advances in Missile Guidance, Control, and Estimation. Automation and Control Engineering. CRC Press, 2012. Chapter 9, doi:10.1201/b12503-10
2. Bao-Fu, F., Qi-Shu, P., Bing-Rong, H., Lei, D., Qiu-Bo, Z., Zhaosheng, Z.: Research on high speed evader vs. multi lower speed pursuers in multi pursuit-evasion games. Inf. Technol. J. **11**(8), 989–997 (2012). doi:10.3923/itj.2012.989.997
3. Bopardikar, S.D., Bullo, F., Hespanha, J.P.: On discrete-time pursuit-evasion games with sensing limitations. IEEE Trans. Robot. **24**(6), 1429–1439 (2008). doi:10.1109/TRO.2008.2006721
4. Conway, B.A., Pontani, M.: Numerical solution of the three-dimensional orbital pursuit-evasion game. J. Guid. Control Dyn. **32**(2), 474–487 (2009). doi:10.2514/1.37962
5. Evans, L., Souganidis, P.: Differential games and representation formulas for solutions of hamilton-jacobi-isaacs equations. Indiana Univ. Math. J. **33**(5), 773–797 (1984). doi:10.1512/iumj.1984.33.33040
6. Ho, Y., Bryson, A., Baron, S.: Differential games and optimal pursuit-evasion strategies. IEEE Trans. Autom. Control **10**(4), 385–389 (1965). doi:10.1109/TAC.1965.1098197
7. Huang, H., Zhang, W., Ding, J., Stipanovic, D.M., Tomlin, C.J.: Guaranteed decentralized pursuit-evasion in the plane with multiple pursuers. In: Proceedings of the Fifieth IEEE Conference on Decision and Control and European Control Conference, pp. 4835–4840. IEEE, Orlando, FL (2011). doi:10.1109/CDC.2011.6161237
8. Isaacs, R.: Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization. Wiley, New York (1965). chapter 6
9. Isler, V., Kannan, S., Khanna, S.: Randomized pursuit-evasion in a polygonal environment. IEEE Trans. Robot. **21**(5), 875–884 (2005). doi:10.1109/TRO.2005.851373
10. Jarmark, B., Hillberg, C.: Pursuit-evasion between two realistic aircraft. J. Guid. Control Dyn. **7**(6), 690–694 (1984). doi:10.2514/3.19914
11. Jin, S., Qu, Z.: Pursuit-evasion games with multi-pursuer vs. one fast evader. In: Proceedings of the Eighth World Congress on Intelligent Control and Automation (WCICA), pp. 3184–3189. IEEE, Jinan, China (2010). doi:10.1109/WCICA.2010.5553770
12. Kothari, M., Manathara, J.G., Postlethwaite, I.: A cooperative pursuit-evasion game for nonholonomic systems. In: Proceedings of the Ninteenth IFAC World Congress, vol. 19, pp. 1977–1984, Cape Town, South Africa (2014). doi:10.3182/20140824-6-ZA-1003.01992
13. Makkapati, V.R.: Simulation: Evasion from a PEF of 4 pursuers
14. Makkapati, V.R.: Simulation: Evasion from a PEF of 5 pursuers

15. Makkapati, V.R.: Simulation: Evasion from a PEF of 6 pursuers
16. Mitchell, I.M., Bayen, R.M., Tomlin, C.J.: Bayen a time-dependent hamilton-jacobi formulation of reachable sets for continuous dynamic games. IEEE Trans. Autom. Control **50**(7), 947–957 (2005). doi:10.1109/TAC.2005.851439
17. Pachter, M., games, Y.Y.: Simple-motion pursuit-evasion differential part 1: Stroboscopic strategies in collision-course guidance and proportional navigation. J. Optim. Theory Appl. **51**(1), 95–127 (1986). doi:10.1007/BF00938604
18. Rajan, N., Prasad, U.R., Rao, N.J.: Pursuit-evasion of two aircraft in a horizontal plane. J. Guid. Control Dyn. **3**(3), 261–267 (1980). doi:10.2514/3.55982
19. Ramana, M.V., Kothari, M.: A cooperative pursuit-evasion game of a high speed evader. In: Proceedings of the IEEE Fifty-Fourth Annual Conference on Decision and Control (CDC), pp. 2969–2974. IEEE, Osaka, Japan (2015). doi:10.1109/CDC.2015.7402668
20. Shinar, J., Gutman, S.: Three-dimensional optimal pursuit and evasion with bounded controls. IEEE Trans. Autom. Control **25**(3), 492–496 (1980). doi:10.1109/TAC.1980.1102372
21. Shneydor, N.A.: Missile Guidance and Pursuit: Kinematics, Dynamics and Control. Horwood Series in Engineering Science. Horwood Publishing Limited, Chichester, England (1998). Chapter 4
22. Takei, R., Huang, H., Ding, J., Tomlin, C.J.: Time-optimal multi-stage motion planning with guaranteed collision avoidance via an open-loop game formulation. In: Proceedings of the IEEE International Conference on Robotics and Automation (ICRA), pp. 323–329. IEEE, Saint Paul, MN (2012). doi:10.1109/ICRA.2012.6225074
23. Vidal, R., Shakernia, O., Jin Kim, H., Shim, D.H., Sastry, S.: Probabilistic pursuit-evasion games: Theory, implementation, and experimental evaluation. IEEE Trans. Robot. Autom. **18**(5), 662–669 (2002). doi:10.1109/TRA.2002.804040
24. Wang, X., Cruz Jr, J.B., Chen, G., Pham, K., Blasch, E.: Formation control in multi-player pursuit evasion game with superior evaders. In: Proceedings of the Defense Transformation and Net-Centric Systems, volume 6578. International Society for Optics and Photonics (2007). doi:10.1117/12.723300
25. Mo, W., Chen, G., Cruz, J.B., Haynes, L., Pham, K., Blasch, E.: Multi-pursuer multievader pursuit-evasion games with jamming confrontation. J. Aerosp. Comput. Inform. Commun. **4**(3), 693–706 (2007). doi:10.2514/1.25329
26. Wei, M., Chen, G., Cruz, J.B., Haynes, L.S., Chang, M.-H., Blasch, E.: A decentralized approach to pursuer-evader games with multiple superior evaders in noisy environments. In: Proccedings of the IEEE Aerospace Conference, pp. 1–10. IEEE, Big Sky, MT (2007). doi:10.1109/AERO.2007.353051
27. Wei, M., Chen, G., Cruz Jr, J.B., Hayes, L., Chang, M.-H.: A decentralized approach to pursuer-evader games with multiple superior evaders. In: IEEE Intelligent Transportation Systems Conference, pp. 1586–1591. IEEE (2006). doi:10.1109/ITSC.2006.1707450
28. Zarchan, P.: Tactical and Strategic Missile Guidance, volume 239 of Progress in Astronautics and Aeronautics, 6th edn. American Institute of Aernautics and Astronautics, Inc., Reston, VA (2012). Chapters 2, 8

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