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Optimal Evasive Maneuvers in Maritime Collision Avoidance

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Abstract

EVASIVE MANEUVERS are determined for each of two potentially colliding ships, such that their miss distance is maximized. These maneuvers are frequently contrary to the established rules of the road. It is assumed that the ship speeds are constant during the encounter, and that the turn-rates of the ships are bounded between symmetrical limits, corresponding to hard right and hard left turns. The optimal turn directions are found to be explicit functions of the range, bearing and heading between the two ships. The cooperative case, when both ships maneuver, and the non-cooperative case, when only one ship maneuvers, are both analyzed. Examples of the optimal maneuvers for two identical ships are presented in detail.

Introduction

Some seven per cent of the world's maritime fleet was involved in a two-ship collision in 1970 (Ref. 1). As traffic densities and ship dimensions increase, we can expect this alarming figure to rise proportionally, unless substantial improvements are made in both the international collision avoidance regulations and the associated equipment. While the regulations are due for revision in 1972, it is expected that these rules will continue to deal principally with the responsibilities for maneuvering, rather than with the collision-avoidance maneuvers themselves.

The maritime regulations for collision prevention relate to many aspects of the two-ship encounter (Ref. 2). The specification of evasive maneuvers, however, has generally been based on ignoring the ship dynamics and on applying intuitive reasoning to specific relative geometries (Refs. 3–7). While these maneuvers can provide safe clearance when the initial range is large, the resulting miss distance may be unacceptably small for other initial geometries. In fact, cases can be found in which the recommended maneuvers actually lead to a collision when more realistic ship dynamics are assumed.

For example, consider two ships having equal speeds and maximum turn rates, located relative to each other as shown in Fig. 1 (a). Assume that the two identical ships have minimum turn radii equal to 4000 ft., and suppose that they are first aware of each other when ship B is 1600 ft. to the left and 3280 ft. ahead of ship A, and headed to A's right. Reference 5 cites the following regulation governing this situation:

> "When two vessels are proceeding in such directions as to involve risk of collision, unless one is a hampered vessel, each shall alter course or speed or both so as to cause the line of sight to the other to rotate in an anti-clockwise direction."

Assuming that the speeds cannot be significantly altered, the above regulation requires that both ships turn hard right, in order to cause a counterclockwise rotation of both lines of sight. As shown in Fig. 1(b), the resulting motion may lead to a collision.

On the other hand, the evasive maneuvers determined by the method presented in this paper require both ships to turn left, which produces the motion of Fig. 1(c). Here the lines of sight rotate

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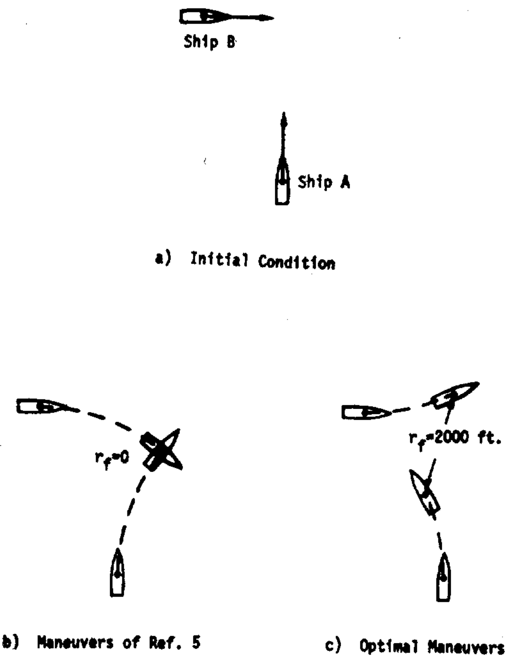

Fig. 1—Ship Motion for Two Sets of Maneuvers.

clockwise, and the minimum value of the range is 2000 ft. This dramatic example, which is by no means artificial or contrived, illustrates the practical implications of the present mathematical approach to the determination of the "optimal" or "best" collision-avoidance maneuvers.

The present analysis models the transient portion of the two-ship encounter, and emphasis is placed on determining the turn directions which should be used by the ships in order to best avoid a collision. Here the "best" maneuvers are defined as those which maximize the miss-distance, or the separation at minimum range. This simple performance index permits a quantitative comparison of any two sets of maneuvers. It is used to determine those maneuvers which are "optimal", in the sense that the miss-distance is maximized. When a collision is truly imminent, this index accurately reflects the concerns and interests of the personnel aboard both ships.

Method of Analysis

The "optimal" evasive maneuvers can be determined in at least two ways, which are briefly described as follows:

1. The equations of motion for each ship (Ref. 8) including realistic transient effects caused by the rudder deflections (controls), can be used in a trial-and-error procedure with different relative initial conditions. Each trial uses a specific set of controls and the resulting miss-distance is found by integrating forward the two sets of equations of motion until the range between the ships is a minimum. The best control sequences for each initial condition are those which yield the greatest miss-distance.
2. The dynamic equations of the ships are simplified so as to include only the principal effects of the controls, and the equations of motion are expressed in an axis system which gives the motion of one ship *relative to* the other. For this simplified model, analytical methods are used to find the controls of both ships at the time of minimum range (i.e., at termination of the evasive maneuvers). When the equations of relative motion are integrated *backwards* in time, using these controls, a path is determined along which the terminal relative condition could have originated.

The first approach has at least two disadvantages. The computational load quickly becomes very great for multiple initial conditions even when attention is restricted to specific ships. Furthermore, it is possible to overlook certain sets of maneuvers which might yield greater miss distances than those resulting from the assumed maneuvers. The second approach, however, leads to conclusions which are as valid as the equations used to approximate the motion. If the simplified equations provide an adequate approximation to the actual motion, it is safe to conclude that the derived maneuvers are nearly optimal in a practical sense. Thus, the second approach based on simplified dynamics has been chosen as the basis for the analysis of this paper.

The method of determining the optimal collision avoidance maneuvers is based on "optimal control theory." In recent years, techniques based on this theory (Ref. 9) have been developed for analyzing problems having the following two characteristics:

1. The time-variation of the variables (i.e.,

the "state") defining the dynamic system is implied by the equations of motion. These differential equations have forcing functions, or "controls", to be determined.

2. The performance of the system in response to any control variation is measured by a given "payoff" criterion, which is an index to be maximized by the choice of control. Constraints on the state or the controls are accounted for in this maximization.

In the collision-avoidance problem, as is presently explained, the controls are the turn-rates of the two ships (due to rudder deflections). The payoff is the miss-distance separating the ships at the time of closest approach.

A detailed mathematical description of the motion of a typical ship in response to changes in the rudder setting is very complex. For present purposes, however, the short-term motion of the ship can be represented by a constant-speed model, for which lateral accelerations are the only means of control (Ref. 3). Speed changes are assumed to be negligible, and therefore each ship can maneuver only by changing its heading. These assumptions reflect the fact that normal forces acting on a ship in a turn are typically much larger than the available axial forces which would cause changes in speed. The turn rates of both ships are assumed to be bounded between symmetrical limits, corresponding to hard left or hard right turns. A ship's path corresponding to a specific constant turn rate is therefore a circular arc, and the path itself is smooth, even when the rudder switches from hard left to hard right.

The motivation for the above choice of dynamic model is that the relative motion is described by only three variables, which are the range, bearing and heading of one ship relative to the other defined in (Fig. 2). Despite the simplicity of this model, it is found that the optimal maneuvers must be determined by numerical computation for a specific pair of ships.

The differential equations of relative motion give the time derivatives of the position and heading of one ship relative to the other, in terms of the turn rates of the two ships. These equations are presented in the Appendix. The equations are simplified by normalizing the units of length and time; i.e., so that the faster ship (Ship A) has

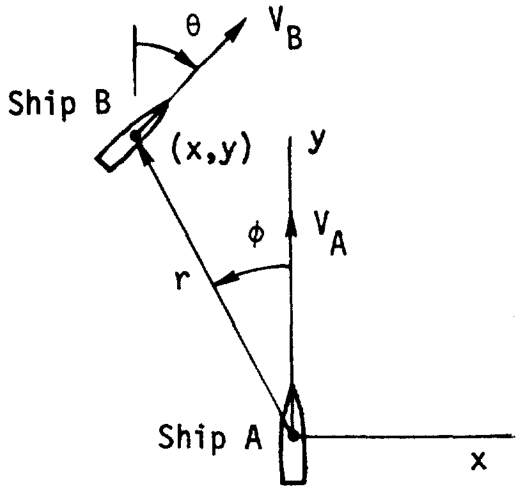

Fig. 2—Geometry of the Two-Ship Encounter.

unit speed and unit maximum turn rate. The slower ship (Ship B) then has the dimensionless speed $\gamma = V_B/V_A \leq 1$, and a maximum turn rate equal to ω. The maximum miss-distance achievable from a given relative position and heading is denoted by r_f, and this has also been normalized by the minimum turn radius of the faster ship.

In the "cooperative" collision avoidance situation, *both* ships maneuver so as to maximize the miss-distance, and for this problem the turn rates of both ships are considered as available controls. In the "non-cooperative" problem, it is assumed that only one of the ships is capable of evasive maneuvers, while the other ship follows a straight path. A significant result of this study is a demonstration of the effect of cooperation on the avoidance maneuvers and the resulting miss-distance which can be achieved from a given initial relative position and heading. In the cooperative case, the normalized controls (turn rates) to be determined are σ_A and σ_B, which have magnitudes no greater than unity. Thus, for example, $\sigma_A = +1$ corresponds to a hard right turn for ship A. The non-cooperative case is analyzed under the assumption that ship B does not maneuver during the encounter, and therefore $\sigma_B = 0$.

As shown in the Appendix, the turn directions of the ships at the time of closest approach can be easily determined mathematically, and these maneuvers are exactly as would be expected intuitively. The retrograde solutions* to the equations of motion are then used with these terminal controls to find where any specific terminal condition could have originated.

Numerical Results for Identical Ships

Quantitative results for this model of the collision-avoidance problem require that the speeds and maximum turn-rates of the ships be specified numerically. For purposes of illustration, these parameters are chosen to be those of two identical ships: i.e., $\gamma = \omega = 1$. Results of the analysis are now given as optimal maneuver strategies for both ships, in the cooperative case, and for one ship, in the noncooperative case. These results were obtained using the optimal control technique just discussed. The mathematical details are presented in the Appendix.

Cooperative Case

In the cooperative case, both ships maneuver so as to maximize the final miss-distance. As shown in the Appendix, both ships turn hard right or hard left until the range-rate is zero. The dependence of the maneuvers on the relative position is shown in Fig. 3, for several values of the relative heading θ in the range 30° to 180°. The evasive turn directions of the ships are indicated by the subscripts "R" and "L". That is, a typical notation in this diagram is "$A_R B_L$". This notation indicates that Ship A (located at the origin of this relative axis system) is to turn hard right while B turns hard left, whenever ship B is located in this region relative to Ship A. Relative positions to the right of the diagonal line through the origin are those for which the range-rate is positive, and for which maneuvers are therefore unnecessary.

The contours of constant r_f shown in Fig. 3 are the normalized maximum miss-distances which can be obtained when both ships maneuver as indicated; for example, if ship B is initially located on the contour $r_f = 1$, the final miss-distance can be no greater than one minimum turn-radius, and will be less than this value if either ship deviates from its optimal turn maneuver.

It is seen that for each relative heading (except $\theta = 0°$ and 180°), the plane of positions of ship B relative to ship A may be divided into three regions. These will be referred to as "maneuver regions," since the optimal maneuver for each ship depends on the region in which B is located relative to A. At the intersection of these regions, the same miss-distance results from use of any of the three maneuver strategies. This intersection is referred to as a dispersal point (Ref. 10). Possible paths from the "dispersal point" are illustrated in Fig. 4 for the initial heading $\theta = 120°$ (the initial position can be read from Fig. 3(d)).

Non-Cooperative Case

The non-cooperative case is analyzed by assuming that ship B does not maneuver, but instead travels in a straight path on the surface of the sea, with $\sigma_B = 0$. Relative to the cooperative case, it is found that for some positions, a different evasive maneuver may be indicated for ship A. The optimal collision avoidance maneuver for one ship then depends upon whether or not the other ship is cooperative.

Numerical results for the non-cooperative case are shown in Fig. 5. Evasive maneuvers for ship A are seen to be nearly independent of whether or not B cooperates. That is, the line separating the maneuver regions for ship A has approximately the same location regardless of B's cooperation. However, the increase in miss distance due to B's cooperation can be appreciable, as shown by comparing the r_f contours of Fig. 5 (a) to those of Fig. 3 (b). For example if ship B is initially located near the point $x = -.9$, $y = .5$, $\theta = 60°$, the miss-distance for the cooperative case is given by Fig. 3(b) as $r_f \simeq .8$. If B does *not* maneuver from this initial relative position, Fig. 5 (a) shows that the non-cooperative miss-distance is approximately $r_f = .5$, or about 60% of the value for the cooperative case.

The constant-heading diagrams of Figs. 3 and 5 can also be used to determine the miss-distance which results if neither ship maneuvers. In this case, the heading remains fixed and B's relative motion is a straight path, perpendicular to the line $\dot{r} = 0$, which passes diagonally through the

* "Retrograde" is used to mean "backwards-time", i.e., the independent variable of the differential equations is the "time-to-go" until the range is a minimum.

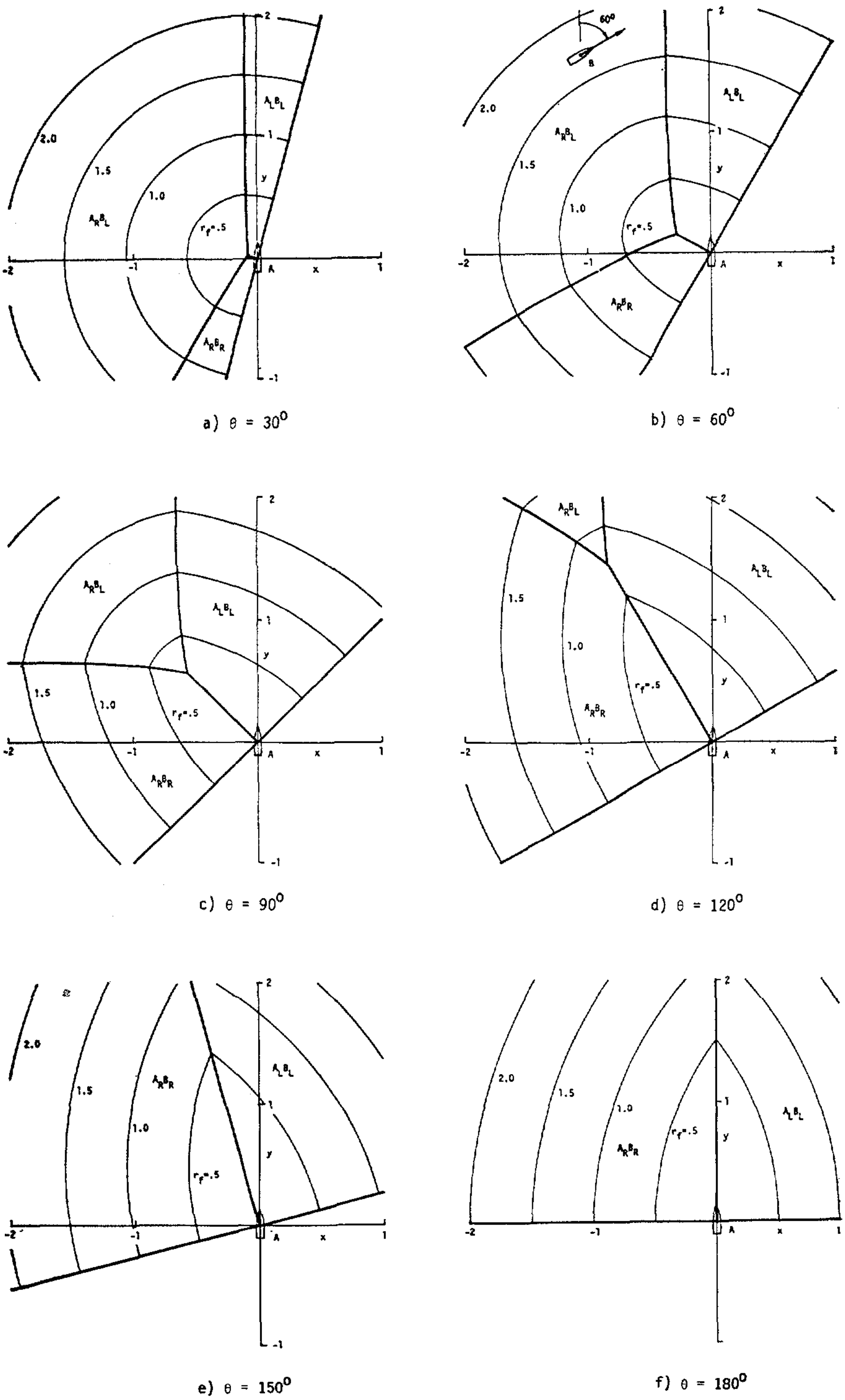

Fig. 3—Optimal Maneuvers and Miss-Distances for Identical Ships, Cooperative Case.

origin in these figures. In particular, the straight line segment of Fig. 3 which separates the region "$A_R B_R$" from the region "$A_L B_L$" is a locus of initial conditions leading to a collision if neither ship turns, i.e., if $\sigma_A = \sigma_B = 0$.

Summary of Results

Optimal collision avoidance maneuvers have been found for a simplified mathematical model

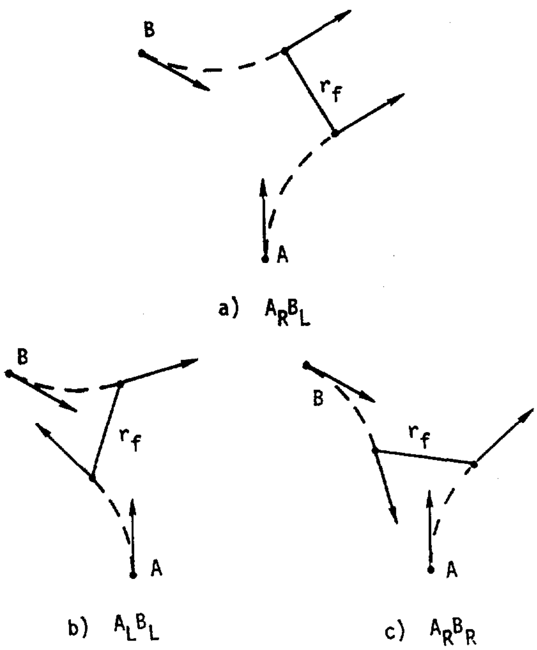

Fig. 4—Dispersal Point Trajectories, Cooperative Case ($\theta_0 = 120°$).

of the two-ship encounter. The criterion maximized by the choice of turn directions was the miss-distance. Both cooperative and non-cooperative cases have been examined, with results presented for the special case of identical ships. The analytical methods used here modeled the relative *motion*, or the set of *future* relative positions of the ships. This dynamic approach is in contrast to the intuitive maneuvers based on the *present* relative position, and given in the references as recommended "Rules of the Road".

The avoidance maneuvers recommended in Ref. 2–6 were specified as functions only of the bearing of the threatening ship. We have shown here, however, that the optimal evasive maneuvers also depend upon the relative range and heading. Normalized diagrams were presented for the case of identical ships which show the optimal turn maneuvers of the ships and the resulting miss-distance.* It was demonstrated that initial conditions exist for which a collision can be avoided *only* if turns in the directions of the optimal maneuvers are used. It was also found that these maneuvers are not necessarily unique. That is, certain relative positions exist for which more than one set of maneuvers is optimal. These multiple-maneuver or dispersal points help to explain the underlying tactical

* While results are given here only for two identical ships, it is known that the evasive maneuvers also depend on each ship's speed and maximum turn rate.

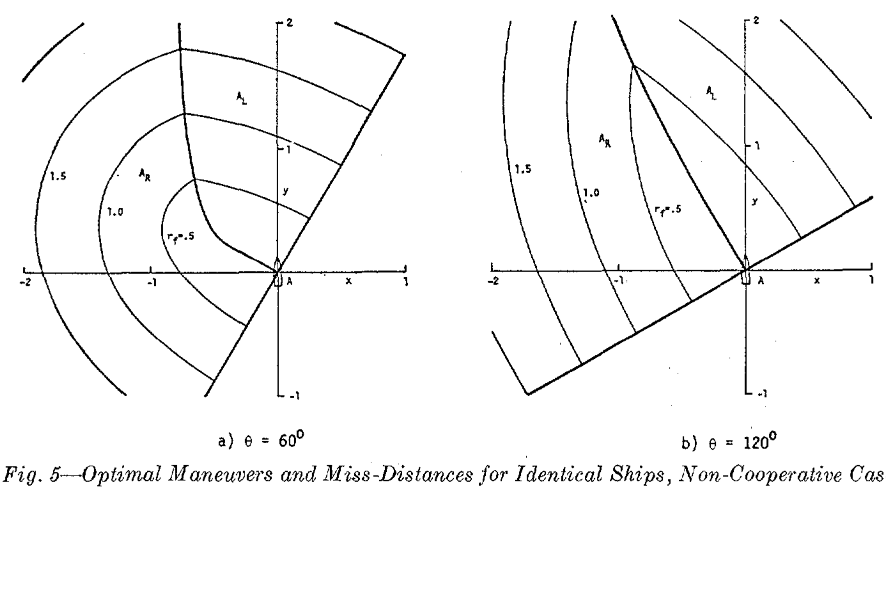

Fig. 5—Optimal Maneuvers and Miss-Distances for Identical Ships, Non-Cooperative Case.

difficulties in solving the collision avoidance problem.

Appendix

Equations for Relative Ship Motion and Terminal Conditions

Under the assumptions given in the body of the paper, the position and heading of ship B relative to ship A obey the following normalized differential equations:

$$\begin{aligned} \dot{x} &= -\sigma_A y + \gamma \sin\theta \\ \dot{y} &= -1 + \sigma_A x + \gamma \cos\theta \\ \dot{\theta} &= -\sigma_A + \omega\sigma_B \end{aligned} \tag{1}$$

Here, x, y, and θ are the position and heading variables shown in Fig. 2. The slower ship B has the speed $\gamma \leq 1$ and a maximum turn rate equal to ω. The controls σ_A and $\omega\sigma_B$ are respectively the normalized turn rates of A and B, which are bounded in magnitude; i.e., $-1 \leq \sigma_A, \sigma_B \leq +1$. The position equations can also be written in polar coordinates as

$$\begin{aligned} \dot{r} &= -\cos\phi + \gamma\cos(\theta - \phi) \\ \dot{\phi} &= -\sigma_A + [\sin\phi + \gamma\sin(\theta - \phi)]/r \end{aligned} \tag{2}$$

For the simplest version of the collision avoidance problem, the quantity to be maximized is the miss-distance, $r(t_f) = r_f$. The time of closest approach, t_f, is given implicitly by Eq. (2) as

$$\dot{r}(t_f) = -\cos\phi_f + \gamma\cos(\theta_f - \phi_f) = 0$$

That is, when the range-rate is zero, each terminal bearing is associated with two values of relative heading, which are

$$\theta_f = \phi_f \pm \cos^{-1}\left[\frac{\cos\phi_f}{\gamma}\right] \tag{3}$$

For the case of identical ships, $\gamma = 1$ and $\theta_f = 0$ or $\theta_f = 2\phi_f$.

Necessary Conditions for Optimal Trajectories

The problem of maximizing the miss-distance can be posed as a free-time, terminal-payoff type (Ref. 9), with either or both turn rates as the governing controls. The methods of optimal control theory may be used to define the "Hamiltonian" for the problem as the total time derivative of the payoff along an optimal path. The Hamiltonian can be expressed in polar coordinates as

$$\max_{\sigma_A,\ \sigma_B} H = \max_{\sigma_A,\ \sigma_B} [\lambda_r\dot{r} + \lambda_\phi\dot{\phi} + \lambda_\theta\dot{\theta}] = 0 \tag{4}$$

This fundamental equation provides an implicit description of the optimal maneuvers, σ_A and σ_B. Substituting into this equation from Eqs. (1) and (2) gives the controls for ships A and B in terms of $\lambda^T = [\lambda_r, \lambda_\phi, \lambda_\theta]$, as follows*:

$$\begin{aligned} \sigma_A &= -\text{sign}(\lambda_\phi + \lambda_\theta) \\ \sigma_B &= \text{sign}\,\lambda_\theta \end{aligned} \tag{5}$$

The adjoint vector (sometimes called a Lagrange multiplier) $\lambda(t)$ can be shown to satisfy the equation

$$\dot{\lambda}^T = -\partial H/\partial x \tag{6}$$

where the state vector is $x^T = [r, \phi, \theta]$. That is, $\dot{\lambda}_r = -\partial H/\partial r$, $\dot{\lambda}_\phi = -\partial H/\partial\phi$ and $\dot{\lambda}_\theta = -\partial H/\partial\theta$. The terminal boundary conditions for this equation are most easily expressed in polar coordinates as

$$\lambda^T(t_f) = [\lambda_r, \lambda_\phi, \lambda_\theta] = [1, 0, 0], \tag{7}$$

since the performance criterion is $r(t_f) = r_f$, which is independent of the bearing and heading angles. Therefore, the arguments in Eq. (5) are equal to zero when the time-to-go, τ, is zero. The retrograde time derivatives are then needed to determine the maneuvers immediately before $\dot{r} = 0$. Using Eqs. (6) and (7), we find

$$\begin{aligned} \mathring{\lambda}_\phi(t_f) &= \sin\phi_f + \gamma\sin(\theta_f - \phi_f) \\ \mathring{\lambda}_\theta(t_f) &= -\gamma\sin(\theta_f - \phi_f) \end{aligned} \tag{8}$$

where the superscript circle denotes a derivative with respect to the time-to-go, τ. That is, for example, $\mathring{x} = dx/d\tau = -dx/dt$, since $\tau = t_f - t$. The two terminal conditions of Eq. (3) are associated with the following strategies:

$$\begin{aligned} &\theta_f = \phi_f - \cos^{-1}\left(\frac{\cos\phi_f}{\gamma}\right): \\ &\sigma_A = \sigma_B = -\text{sign}\,\phi_f \\ &\theta_f = \phi_f + \cos^{-1}\left(\frac{\cos\phi_f}{\gamma}\right): \\ &\sigma_A = -\sigma_B = -\text{sign}\,\phi_f, \end{aligned} \tag{9}$$

* The signum function is defined as $\sigma = \text{sign}\ a = a/|a| = \pm 1$, unless $a = 0$, in which case it is undefined.

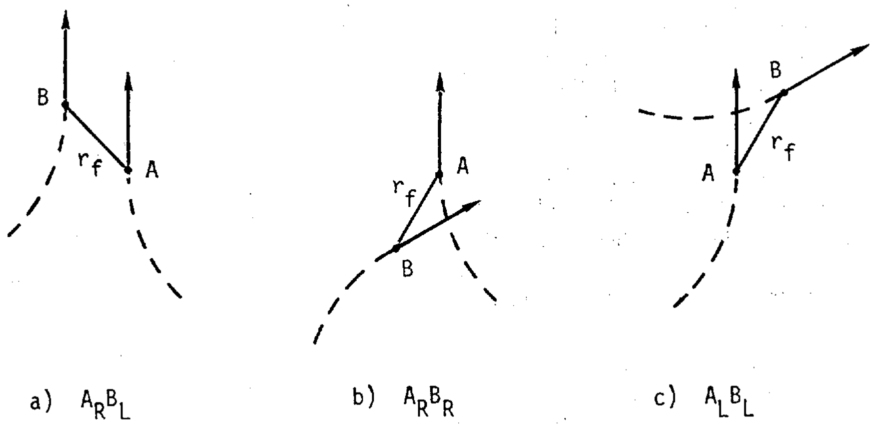

Fig. 6—Optimal Terminal Maneuvers for Identical Ships.

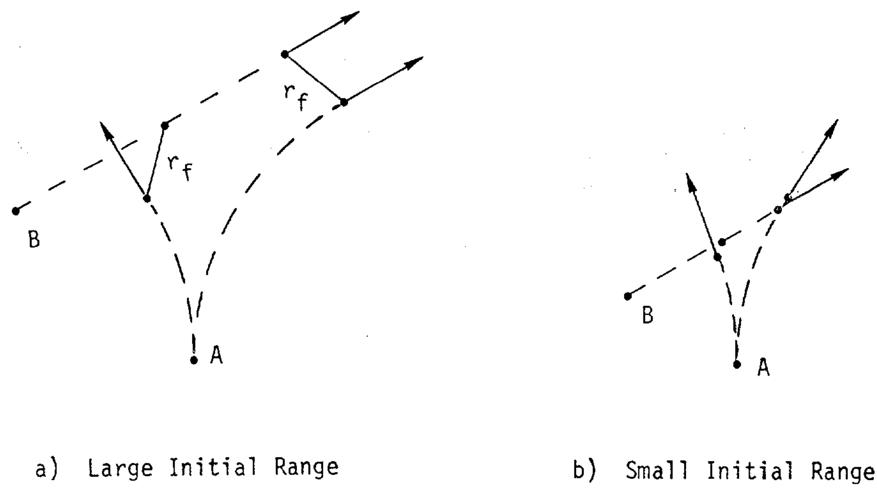

Fig. 7.—Dispersal Point Trajectories, Non-Cooperative Case (θ_0 = 60°).

unless $\cos \phi_f = \gamma < 1$. In this case, it is easy to prove that B's optimal strategy may be $\sigma_B = 0$, corresponding to straight-line motion.

The terminal maneuvers for identical ships are easily stated: Each ship is *turning away from the other* at the time of minimum range. When the solutions are expressed in retrograde time, it is possible to learn where any terminal condition must have originated. In the non-cooperative case, $\sigma_B = 0$ by assumption, and here A is turning away from B when $t = t_f$.

Maneuver Regions, Identical Ships

When the ships are identical, it follows that $\gamma = \omega = 1$, and the only optimal maneuvers are sharp right or left turns, according to Eq. (9). For a specific choice of initial heading angle, the lines separating the various regions can be determined by using the solutions to Eq. (1), with σ_A and σ_B as determined by Eq. (9). Thus, if $\theta_f \neq 0$, the turn rates are the same, as shown in Fig. 6(b) and 6(c). When $\theta_f = 0$, the turn rates are opposite, as illustrated in Fig. 6(a).

For any choice of terminal range, the solutions to Eqs. (1) are expressible in terms of x_f, y_f, θ_f and τ. The two geometric constraints $\sqrt{x_f^2 + y_f^2} = r_f$ and $\theta_f(\phi_f)$ are then imposed on these relations. The parameter τ can be eliminated, so that when θ is fixed, loci of the form $f(x, y, r_f) = 0$ are determined. These loci have the geometric form shown in Fig. 3, where all of the maneuver combinations of Fig. 6 are used. The heavy lines of Fig. 3 denote the so-called "dispersal" lines, for which two different sets of maneuvers are optimal.

In the non-cooperative case, a similar analysis leads to the determination of a dispersal line for ship A, at a specific value of the heading. That is, a terminal range r_f is chosen, and the heading angle θ is fixed. The relative position is then expressible parametrically as $x(\tau)$, $y(\tau)$, for a given value of the terminal bearing. As in the cooperative case, when ship B is located where two of these loci intersect, the ship A can turn either way. This is illustrated in Fig. 7 for an initial heading of 60°. The initial relative positions here can be read from Fig. 5(a).

Acknowledgments

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