International Game Theory Review, Vol. 12, No. 1 (2010) 1–17

DOI: 10.1142/S0219198910002489

GUARANTEED STRATEGIES FOR NONLINEAR MULTI-PLAYER PURSUIT-EVASION GAMES*

DUŠAN M. STIPANOVIĆ[†], ARIK MELIKYAN
and NAIRA HOVAKIMYAN[‡]

[†]*Department of Industrial and Enterprise Systems
Engineering and the Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
Urbana, IL 61801, USA
dusan@illinois.edu*

[‡]*Department of Mechanical Science and Engineering
University of Illinois at Urbana-Champaign
Urbana, IL 61801, USA
nhovakim@illinois.edu.*

In this paper, we provide a methodology to design strategies for either guaranteed capture or guaranteed evasion in the case of pursuit-evasion games with multiple players which are represented by nonlinear dynamic models. This methodology is based on the continuously differentiable upper and lower approximations of the minimum and maximum function of an arbitrary number of arguments, comparison principle, and differential inequalities.

Keywords: Pursuit-evasion games; differential inequalities; multi-player dynamic games; Liapunov analysis.

1. Introduction

Consideration of optimal strategies for dynamic pursuit-evasion games dates back to the original work of Isaacs (1965). The problem of "pursuing a moving object with another controlled object" was formulated as a stochastic optimal control problem in Pontryagin *et al.* (1962). It is important to point out a significant contribution in the theory of pursuit-evasion games based on extremal aiming method that was introduced and developed by Krasovskii and Subbotin (see Krasovskii and Subbotin (1988), Subbotin (1995)). In order to deal with the nondifferentiability of solutions of the Hamilton-Jacobi partial differential equations, viscosity solutions [Crandall and Lions (1983), Bardi and Capuzzo-Dolcetta (1997)] and so called "minmax" solutions Subbotin (1984), were independently introduced. Numerical approximations

*This paper appears almost a year and half after the untimely death of Arik Melikyan. D. M. Stipanović and N. Hovakimyan would like to dedicate this paper to his memory.
[†]Corresponding author.

based on viscosity solutions were provided in Falcone and Ferretti (1994), Bardi and Capuzzo-Dolcetta (1997). Some particular strategies for the players in various multi-player pursuit-evasion games were presented in Hagedorn and Breakwell (1976), Breakwell and Hagedorn (1979), Melikyan (1981), Pashkov and Terekhov (1987), Levchenkov and Pashkov (1990), Petrosjan (1993), Petrov (1994), Vagin and Petrov (2002), Petrov (2003). For an application of generalized characteristics of partial differential equations to differential pursuit-evasion games we refer to the results reported in Melikyan (1981, 1998).

Another interesting and important scenario when the players obtain information at discrete time instances was first considered in Melikyan (1973) for the case of one pursuer and one evader. This result was later generalized using a cost of information for more complex models for the pursuer and the evader in Olsder and Pourtallier (1995). Finally, the case of pursuit-evasion games with several pursuers and one evader with discrete observations that is based on a comparison with solutions to differential games with continuous observations available to the players, was studied in Melikyan and Pourtallier (1996). The problem in which one of the two players is provided with the delayed observations was considered in Chernousko and Melikyan (1975).

In this paper we follow another approach to define strategies in the pursuit-evasion games which is based on the Liapunov type of analysis [Stipanović *et al.* (2004)]. Instead of solving a Hamilton-Jacobi-Isaacs partial differential equation for a value function, a specific function of the norms of relative distances between pursuers and evaders is considered. This function is appropriately chosen from the set of functions that are continuously differentiable and represent approximations of the minimum and maximum function. Strategies are then formulated by either maximizing or minimizing the growth, that is, the time derivative of the corresponding differentiable Liapunov-like function. One of the most important features of this approach is that the methodology is applicable to a wide class of linear multi-player pursuit-evasion games. These results were later extended in Stipanović *et al.* (2009) to include more complicated nonlinear models that are affine in control strategies for the players in the game. The pursuit-evasion games considered, were restricted to the case of two pursuers and two evaders. In this paper we generalize these results to multiple pursuit-evasion games by introducing convergent approximations of the minimum and maximum function of an arbitrary number of arguments and by using less restrictive comparison results.

The organization of the paper is as follows. In Sec. 2 we introduce functions that represent convergent lower and upper approximations of both the minimum and the maximum function. Some of the most general comparison results are recalled in Sec. 3. Guaranteed strategies for either capture or evasion of the evaders are provided in Sec. 4. Finally, as an illustration of the proposed methodology, we consider pursuit-evasion games with nonidentical nonholonomic players described by the unicycle model in Sec. 5.

2. Properties of the Minimum and the Maximum Function Approximations

In this section we study generalized functions introduced in Stipanović *et al.* (2009) that approximate minimum and maximum of two arguments for the case of an arbitrary number of arguments. These functions will be later used to establish sufficient conditions for either guaranteed capture or evasion of all or some of the evaders. As a starting point, let us assume that we are given N positive numbers a_i, $i \in \mathbf{N}$, where $\mathbf{N} = \{1, \ldots, N\}$. In order to approximate the minimum function from below, we consider the following function:

$$\underline{\sigma}_\delta(a_1, \ldots, a_N) = \sqrt[\delta]{\frac{1}{\sum_{i=1}^{N} a_i^{-\delta}}}, \quad \delta > 0 \tag{1}$$

and similarly for the approximation from above, we consider the following function:

$$\overline{\sigma}_\delta(a_1, \ldots, a_N) = \sqrt[\delta]{\frac{N}{\sum_{i=1}^{N} a_i^{-\delta}}}, \quad \delta > 0. \tag{2}$$

Let us denote $a_m = \min_{i \in \mathbf{N}}\{a_i\}$ and define m as a variable taking integer value j representing the index of a minimal a_j, that is $m = j$. Notice that if the minimum value is achieved by more than one argument, we can choose any of the corresponding indices without any loss of generality. Now, we can state the following theorem:

Theorem 2.1. *The minimum approximation functions satisfy the following properties*:

$$\underline{\sigma}_\delta \leq a_m \leq \overline{\sigma}_\delta, \quad \forall\, \delta > 0, \tag{3}$$

$$\lim_{\delta \to \infty} \underline{\sigma}_\delta = \lim_{\delta \to \infty} \overline{\sigma}_\delta = a_m. \tag{4}$$

Proof. First notice that the approximation functions may be written as:

$$\begin{aligned} \underline{\sigma}_\delta &= \frac{a_m}{\sqrt[\delta]{1 + \sum_{i \neq m} (a_m/a_i)^\delta}}, \\ \overline{\sigma}_\delta &= \frac{a_m \sqrt[\delta]{N}}{\sqrt[\delta]{1 + \sum_{i \neq m} (a_m/a_i)^\delta}}. \end{aligned} \tag{5}$$

Also,

$$\lim_{\delta \to \infty} \sqrt[\delta]{1 + \sum_{i=1}^{N} c_i^\delta} = 1 \quad \text{if } (\forall\, i \in \{1, \ldots, N\}) \quad (c_i \in [0, 1]) \tag{6}$$

which is true due to the following:

$$1 \leq \sqrt[\delta]{1 + \sum_{i=1}^{N} c_i^{\delta}} \leq \sqrt[\delta]{1+N} \quad \text{if } (c_i \in [0,1]) \text{ and } \lim_{\delta\to\infty} \sqrt[\delta]{1+N} = 1. \tag{7}$$

Finally, since $a_m/a_i \leq 1$ for all $i \in \{1,\ldots,N\}$ and $N \geq 1 + \sum_{i\neq m}(a_m/a_i)^{\delta}$ we conclude that the statements of the theorem are true. □

Another interesting feature that is easy to show is that the minimum approximation functions behave well for any finite positive δ when the minimum approaches zero, that is,

$$\lim_{a_m\to 0} \underline{\sigma}_{\delta} = \lim_{a_m\to 0} \overline{\sigma}_{\delta} = 0 \tag{8}$$

which is a direct consequence of the Eqs. (5) and (6).

In order to approximate the maximum function from below, we introduce the following function:

$$\underline{\rho}_{\delta}(a_1,\ldots,a_N) = \sqrt[\delta]{\frac{\sum_{i=1}^{N} a_i^{\delta}}{N}}, \quad \delta > 0 \tag{9}$$

and similarly for the approximation from above, we propose to use

$$\overline{\rho}_{\delta}(a_1,\ldots,a_N) = \sqrt[\delta]{\sum_{i=1}^{N} a_i^{\delta}}, \quad \delta > 0. \tag{10}$$

Let us denote $a_M = \max_{i\in\mathbf{N}}\{a_i\}$ and define M as a variable taking integer value of the index of a maximal a_j, that is $M = j$. Again, notice that if the set of maximal variables has more than one element we can choose any one of them without any loss of generality. Now, analogously to the case of approximating the minimum, we formulate the following theorem:

Theorem 2.2. *The convergent maximum approximation functions satisfy the following properties*:

$$\underline{\rho}_{\delta} \leq a_M \leq \overline{\rho}_{\delta}, \quad \forall\, \delta > 0, \tag{11}$$

$$\lim_{\delta\to\infty} \underline{\rho}_{\delta} = \lim_{\delta\to\infty} \overline{\rho}_{\delta} = a_M. \tag{12}$$

Proof. Notice that the approximation functions can be rewritten as:

$$\begin{aligned} \underline{\rho}_{\delta} &= a_M \frac{\sqrt[\delta]{1+\sum_{i\neq M}(a_i/a_M)^{\delta}}}{\sqrt[\delta]{N}}, \\ \overline{\rho}_{\delta} &= a_M \sqrt[\delta]{1 + \sum_{i\neq M}(a_i/a_M)^{\delta}}. \end{aligned} \tag{13}$$

Again, using the following property:

$$\lim_{\delta\to\infty} \sqrt[\delta]{1+\sum_{i=1}^{N} c_i^{\delta}} = 1 \quad \text{if } (\forall\, i \in \{1,\dots,N\}) \quad (c_i \in [0,1]), \tag{14}$$

the fact that $a_i/a_M \leq 1$ for all $i \in \{1,\dots,N\}$, and $N \geq 1 + \sum_{i\neq M}(a_i/a_M)^{\delta}$, we conclude that the statements of the theorem are true. □

Finally, it is interesting to note that the lower and the upper convergent approximations of both the minimum and the maximum function may be linked to the constant elasticity of substitution (CES) functions with particular coefficients, multiplying the arguments, that are either 1 or $1/N$, respectively (for more details see Luenberger (1995)).

3. Comparison Principle Theorems

In this section we recall the comparison principle theorems to be used for proving that the strategies of the players would guarantee either capture of all evaders or their evasion from the pursuers. The following theorem [Lakshmikantham *et al.* (1989)] will be used in proving guaranteed capture results:

Theorem 3.1. *Let $v \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ such that $v(t,x)$ is locally Lipschitzian in x. Assume that $G \in C[\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}]$ and for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$,*

$$Dv(t,x) \leq G(t,x,v(t,x)). \tag{15}$$

Let $x(t) = x(t,t_0,x_0)$ be any solution of $\dot{x} = f(t,x), x(t_0) = x_0, t_0 \in \mathbb{R}_+$ where $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, existing on $[t_0,\infty)$. Also, let us assume that $r(t,t_0,x_0,u_0)$ is the maximal solution of

$$\dot{u} = G(t,x(t),u), \quad u(t_0) = u_0 \tag{16}$$

existing for $t \geq t_0$. Then $v(t_0,x_0) \leq u_0$ implies

$$v(t,x(t)) \leq r(t,t_0,x_0,u_0), \quad t \geq t_0. \tag{17}$$

In the formulation of Theorem 3.1, D represents any Dini derivative [Bainov and Simeonov (1992)] yet we assume that $v(t,x)$ is continuously differentiable in the domains of interest so that all Dini derivatives coincide with the standard total time derivative d/dt. Also, we use $dx/dt = \dot{x}$ when function $x(\cdot)$ is only a function of time, that is, $x \equiv x(t)$. Finally, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0,\infty)$, and $C[\mathbb{D}_1, \mathbb{D}_2]$ denotes the set of all continuous functions with domain $\mathbb{D}_1$ and codomain $\mathbb{D}_2$ [Lakshmikantham and Leela (1969)]. For more details on notation and comparison results we refer to Lakshmikantham and Leela (1969, 1989) and Bainov and Simeonov (1992).

In order to establish guaranteed evasion results we need the following straightforward modification of Theorem 3.1 which is justified along the lines of the basic arguments provided in Lakshmikantham and Leela (1969) and Bainov and Simeonov (1992).

Theorem 3.2. *Let $v \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ such that $v(t,x)$ is locally Lipschitzian in x. Assume that $g \in C[\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}]$ and for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$,*

$$Dv(t,x) \geq g(t,x,v(t,x)). \tag{18}$$

Let $x(t) = x(t,t_0,x_0)$ be any solution of $\dot{x} = f(t,x), x(t_0) = x_0, t_0 \in \mathbb{R}_+$ where $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, existing on $[t_0, \infty)$. Also, let us assume that $z(t,t_0,x_0,u_0)$ is the minimal solution of

$$\dot{q} = g(t,x(t),q), \quad q(t_0) = q_0 \tag{19}$$

existing for $t \geq t_0$. Then $v(t_0,x_0) \geq q_0$ implies

$$v(t,x(t)) \geq z(t,t_0,x_0,q_0), \quad t \geq t_0. \tag{20}$$

4. Differential Inequalities and Pursuit-Evasion Games

Let us assign $e_i \in \mathbb{R}^{n_i}, i \in \{1,\ldots,N_e\}$, to be a vector of all state variables corresponding to the i-th evader where N_e denotes the total number of evaders. Similarly, let us assign $p_j \in \mathbb{R}^{n_j}, j \in \{1,\ldots,N_p\}$, to be a vector of all state variables corresponding to the j-th pursuer where N_p denotes the total number of pursuers. In order to simplify the notation let us concatenate all the individual state vectors into the vectors $e = [e_1^T,\ldots,e_{N_e}^T]^T$ and $p = [p_1^T,\ldots,p_{N_p}^T]^T$. These two vectors are of dimensions defined by the dimensions of players' individual state vectors. Let us assume that the evaders' dynamics are given in its compact form as

$$\dot{e} = f_e(e,u_e) \tag{21}$$

and similarly that the pursuers' dynamics are given by

$$\dot{p} = f_p(e,u_p) \tag{22}$$

where u_e and u_p represent evaders' and pursuers' input strategies, respectively. In order to generalize results presented in Stipanović *et al.* (2009) we start by considering the following function:

$$\phi_\delta^i(e_i,p) = \overline{\sigma}_\delta(\|\overline{e}_i - \overline{p}_1\|,\ldots,\|\overline{e}_i - \overline{p}_{N_p}\|) \tag{23}$$

where $\overline{e}_i$ and $\overline{p}_j$ represent n-dimensional rectangular coordinates in the corresponding n-dimensional space for the i-th evader and the j-th pursuer, respectively. Obviously the state variables in $\overline{e}_i$ and $\overline{p}_j$ are subsets of the state variables in e_i and p_j, respectively. Without loss of generality and to simplify the notation, we introduce functions $\phi_\delta^i(\cdot,\cdot)$, $i \in \{1,\ldots,N_e\}$, as functions of e_i and p. Furthermore, we define

$$\pi_\delta(e,p) = \overline{\rho}_\delta(\phi_\delta^1(e_1,p),\ldots,\phi_\delta^{N_e}(e_{N_e},p)). \tag{24}$$

Now, we can define the following function:

$$v(e,p) = \pi_\delta(e,p) \tag{25}$$

and compute the corresponding strategies as follows:

$$\begin{aligned} \hat{u}_e(e,p) &= \arg\max_{u_e \in \mathcal{U}_e} \left\{ \frac{dv}{dt} \right\} = \arg\max_{u_e \in \mathcal{U}_e} \left\{ \frac{\partial v(e,p)}{\partial e} f_e(e,u_e) \right\} \\ \hat{u}_p(e,p) &= \arg\min_{u_p \in \mathcal{U}_p} \left\{ \frac{dv}{dt} \right\} = \arg\min_{u_p \in \mathcal{U}_p} \left\{ \frac{\partial v(e,p)}{\partial p} f_p(p,u_p) \right\} \end{aligned} \tag{26}$$

where $\mathcal{U}_e$ and $\mathcal{U}_p$ represent admissible classes of functions for the evaders' and pursuers' strategies, respectively. To streamline our presentation we assume that the classes of admissible functions are such that the objective function that is either minimized or maximized determines the arguments for the solution function. Therefore, the fact that $\frac{\partial v(e,p)}{\partial e} f_e(e,u_e)$ depends on e and p implies that the solution $\hat{u}_e(\cdot)$ is also a function of e and p. One of the most general examples is a class of piecewise continuous functions that are norm bounded. The construction of strategies follows the main ideas of the design of controllers based on Liapunov functions (for more details see [Khalil (2002), Bacciotti and Rosier (2005), Blanchini and Miani (2008)]). In order to use comparison principle we approximate from above the total time derivative of function $v(e,p)$ as:

$$\begin{aligned} \frac{dv(e,p)}{dt} &= \frac{\partial v(e,p)}{\partial e} f_e(e,\hat{u}_e(e,p)) + \frac{\partial v(e,p)}{\partial p} f_p(p,\hat{u}_p(e,p)) \\ &\le G(e,p,v(e,p)) \end{aligned} \tag{27}$$

where $G(\cdot,\cdot,\cdot)$ is a scalar continuous function of its arguments and $\hat{u}_e(e,p)$ and $\hat{u}_p(e,p)$ are respectively collections of the evaders' and pursuers' strategies that maximize or minimize the time derivative, that is the growth, of the function $v(e,p)$. Again, $f_j(j,\hat{u}_j(e,p)), j \in \{e,p\}$ represent collective dynamics of the evaders (when $j=e$) and the pursuers (when $j=p$) for the previously defined collective strategies.

By defining the capture of an evader to be accomplished whenever its Euclidean distance to any of the pursuers becomes less than a prescribed positive number R (also known as the "soft capture") we state the following theorem:

Theorem 4.1. *Assume that the initial conditions $e_0 = e(t_0)$ and $p_0 = p(t_0)$ at the initial time t_0 are such that the players are outside of the capture regions defined by a positive number R, and that the maximal solution (as defined in Lakshmikantham and Leela (1969)) of the following differential equation:*

$$\frac{dw}{dt} = G(e(t),p(t),w), \quad w_0 = v_0(e_0,p_0) \tag{28}$$

is denoted as $\bar{w}(t,t_0,e_0,p_0,w_0)$ along the trajectories of the players' dynamic systems for the collections of their strategies $\hat{u}_e(e,p)$ and $\hat{u}_p(e,p)$. Then the capture of all evaders is guaranteed when pursuers use collective strategies provided in a vector *form as $\hat{u}_p(e,p)$ within a finite time interval $T - t_0$ after the initial time for any feedback strategies of the evaders if $\bar{w}(T, t_0, e_0, p_0, w_0) < R$.*

Proof. So the assumption of the theorem is that the pursuers choose their strategies to be $\hat{u}_p(e,p)$. Then, if the evaders choose any strategy $\bar{u}_e(e,p) \in \mathcal{U}_e$, from (26) and (27) it follows that:

$$\left.\frac{dv(e,p)}{dt}\right|_{\substack{u_e=\bar{u}_e(e,p)\\ u_p=\hat{u}_p(e,p)}} \leq \left.\frac{dv(e,p)}{dt}\right|_{\substack{u_e=\hat{u}_e(e,p)\\ u_p=\hat{u}_p(e,p)}} \leq G(e,p,v(e,p)). \tag{29}$$

Thus, for any strategy $\bar{u}_e(e,p) \in \mathcal{U}_e$, we obtain the differential inequality (29) and using the comparison principle we obtain:

$$v(e(t), p(t)) \leq \bar{w}(t, t_0, e_0, p_0, w_0), \quad t \geq t_0. \tag{30}$$

From (11) and (24) it follows that:

$$\max\{\phi_\delta^1(e_1(t), p(t)), \ldots, \phi_\delta^{N_e}(e_{N_e}(t), p(t))\} \leq v(e(t), p(t)) \tag{31}$$

which implies

$$\phi_\delta^i(e_i(t), p(t)) \leq v(e(t), p(t)) \tag{32}$$

for all $i \in \{1, \ldots, N_e\}$. Then, using Eqs. (3) and (23) we obtain

$$\min\{\|\overline{e}_i(t) - \overline{p}_1(t)\|, \ldots, \|\overline{e}_i(t) - \overline{p}_{N_p}(t)\|\} \leq v(e(t), p(t)), \quad t \geq t_0, \tag{33}$$

for all $i \in \{1, \ldots, N_e\}$. Finally, from inequalities (33) and an assumption of the theorem, we obtain

$$\min\{\|\overline{e}_i(T) - \overline{p}_1(T)\|, \ldots, \|\overline{e}_i(T) - \overline{p}_{N_p}(T)\|\} < R \tag{34}$$

for all $i \in \{1, \ldots, N_e\}$ which is a guarantee that all evaders will be captured before or at time T and thus the theorem is proved. □

Now, let us first consider a problem of evasion of a single evader $i, i \in \{1, \ldots, N_e\}$, from the pursuers. In order to do so first we assume that the i-th evader's dynamics is given by

$$\dot{e}_i = f_e^i(e_i, u_e^i). \tag{35}$$

To obtain strategies for the guaranteed evasion we consider the following approximation function

$$\eta_\delta^i(e_i, p) = \underline{\sigma}_\delta(\|\overline{e}_i - \overline{p}_1\|, \ldots, \|\overline{e}_i - \overline{p}_{N_p}\|) \tag{36}$$

and define $v_i(\cdot,\cdot)$ as

$$v_i(e_i, p) = \eta_\delta^i(e_i, p). \tag{37}$$

Similarly to the case of guaranteed capture, let us again bound the total time derivative of $v_i(e_i, p)$ yet in this case this bound will be from below as

$$\begin{aligned}\frac{dv_i(e_i,p)}{dt} &= \frac{\partial v_i(e_i,p)}{\partial e_i} f_e^i(e_i, \tilde{u}_e^i(e_i,p)) + \frac{\partial v_i(e_i,p)}{\partial p} f_p(p, \tilde{u}_p^i(e_i,p)) \\ &\geq g_i(e_i, p, v_i(e_i,p))\end{aligned} \tag{38}$$

where the strategies are computed from:

$$\begin{aligned}\tilde{u}_e^i(e_i,p) &= \arg\max_{u_e^i \in \mathcal{U}_e^i} \left\{\frac{dv_i}{dt}\right\} = \arg\max_{u_e^i \in \mathcal{U}_e^i} \left\{\frac{\partial v_i(e_i,p)}{\partial e_i} f_e^i(e_i, u_e^i)\right\} \\ \tilde{u}_p^i(e_i,p) &= \arg\min_{u_p \in \mathcal{U}_p} \left\{\frac{dv_i}{dt}\right\} = \arg\min_{u_p \in \mathcal{U}_p} \left\{\frac{\partial v_i(e_i,p)}{\partial p} f_p(p, u_p)\right\}\end{aligned} \tag{39}$$

and $g_i(\cdot,\cdot,\cdot)$ is a scalar continuous function of its arguments.

Theorem 4.2. *Assume that the initial conditions* $e_{i0} = e_i(t_0)$ *and* $p_0 = p(t_0)$ *at the initial time* t_0 *are such that the players are outside of the region defined by the set* $\{(e_i, p) : v_i(e_i, p) \geq R\}$ (*notice that this is an over or outer approximation of the soft capture set*) *and that the minimal solution* (*as defined in* [*Lakshmikantham and Leela* (1969)]) *of the following differential equation*:

$$\frac{dz_i}{dt} = g_i(e_i(t), p(t), z_i), \quad z_i(t_0) = z_{i0} = v_i(e_{i0}, p_0) \tag{40}$$

is denoted as $\underline{z}_i(t, t_0, e_{i0}, p_0, z_{i0})$ *along the trajectories of the players' dynamic systems for the collections of their strategies* $\tilde{u}_e^i(e_i, p)$ *and* $\tilde{u}_p(e_i, p)$. *Then the evasion of the i-th evader is guaranteed for any admissible composite strategy of the pursuers if* $\underline{z}_i(t, t_0, e_{i0}, p_0, z_{i0}) > R$, *for all* $t \geq t_0$.

Proof. So, the assumption of the theorem is that the i-th evader chooses its strategy to be $\tilde{u}_e^i(e_i, p)$. Then, if the pursuers choose any collective strategy $\bar{u}_p(e_i, p) \in \mathcal{U}_p$, from (38) and (39) it follows that:

$$\left.\frac{dv_i(e_i,p)}{dt}\right|_{\substack{u_e^i = \tilde{u}_e^i(e_i,p) \\ u_p = \bar{u}_p(e_i,p)}} \geq \left.\frac{dv_i(e_i,p)}{dt}\right|_{\substack{u_e^i = \tilde{u}_e^i(e_i,p) \\ u_p = \tilde{u}_p(e_i,p)}} \geq g_i(e_i, p, v_i(e_i,p)). \tag{41}$$

Thus, for any strategy $\bar{u}_p(e, p) \in \mathcal{U}_p$, we obtain the differential inequality (41) and using the comparison principle we obtain:

$$v_i(e_i(t), p(t)) \geq \underline{z}(t, t_0, e_{i0}, p_0, z_{i0}), \quad t \geq t_0. \tag{42}$$

From (3), (36) and (37), it follows that:

$$\min\{\|\overline{e}_i(t) - \overline{p}_1(t)\|, \ldots, \|\overline{e}_i(t) - \overline{p}_{N_p}(t)\|\} \geq v_i(e_i(t), p(t)), \quad t \geq t_0 \tag{43}$$

for all $i \in \{1, \ldots, N_e\}$. Finally, using Eqs. (42) and (43), and an assumption of the theorem, we obtain

$$\min\{\|\overline{e}_i(t) - \overline{p}_1(t)\|, \ldots, \|\overline{e}_i(t) - \overline{p}_{N_p}(t)\|\} \geq \underline{z}(t, t_0, e_{i0}, p_0, z_{i0}) > R, \quad t \geq t_0 \tag{44}$$

which is a guarantee that the i-th evader will never be captured by the pursuers. This completes the proof. □

Now, let us construct cooperative strategies for the evaders that would guarantee that none of them will be captured by the pursuers. For the cooperative case we propose the following function:

$$\gamma_\delta(e,p) = \underline{\sigma}_\delta(\eta_\delta^i(e_1,p),\ldots,\eta_\delta^i(e_{N_e},p)) \tag{45}$$

and define

$$v(e,p) = \gamma_\delta(e,p). \tag{46}$$

This is a cooperative case since $v(e,p)$ considers all evaders at the same time. Notice that due to the assumption that the players' dynamics are independent, the cooperation is described through the function $v(e,p)$. Then, we proceed by defining the strategies for the players as:

$$\begin{aligned}\tilde{u}_e(e,p) &= \arg\max_{u_e\in\mathcal{U}_e}\left\{\frac{dv}{dt}\right\} = \arg\max_{u_e\in\mathcal{U}_e}\left\{\frac{\partial v(e,p)}{\partial e}f_e(e,u_e)\right\}\\ \tilde{u}_p(e,p) &= \arg\min_{u_p\in\mathcal{U}_p}\left\{\frac{dv}{dt}\right\} = \arg\min_{u_p\in\mathcal{U}_p}\left\{\frac{\partial v(e,p)}{\partial p}f_p(p,u_p)\right\}\end{aligned} \tag{47}$$

where again $\mathcal{U}_e$ and $\mathcal{U}_p$ represent admissible classes of functions for the evaders' and pursuers' strategies, respectively. In order to use the comparison principle, we approximate from below the total time derivative of function $v(e,p)$ as

$$\begin{aligned}\frac{dv(e,p)}{dt} &= \frac{\partial v(e,p)}{\partial e}f_e(e,\tilde{u}_e(e,p)) + \frac{\partial v(e,p)}{\partial p}f_p(p,\tilde{u}_p(e,p))\\ &\geq g(e,p,v(e,p)).\end{aligned} \tag{48}$$

where $g(\cdot,\cdot,\cdot)$ is a scalar continuous function of its arguments.

Now, we are ready to formulate a theorem on the collective evasion as follows:

Theorem 4.3. *Assume that the initial conditions* $e_0 = e(t_0)$ *and* $p_0 = p(t_0)$ *at the initial time* t_0 *are such that the players are outside of the region defined by the set* $\{(e,p) : v(e,p) \geq R\}$ (*notice that this is an over or outer approximation of the soft capture set for any evader*) *and that the minimal solution* (*as defined in* [*Lakshmikantham and Leela* (1969)]) *of the following differential equation*:

$$\frac{dz}{dt} = g(e(t),p(t),z), \quad z(t_0) = z_0 = v(e_0,p_0) \tag{49}$$

is denoted as $\underline{z}(t,t_0,e_0,p_0,z_0)$ *along the trajectories of the players' dynamic systems for the collections of their strategies* $\tilde{u}_e(e,p)$ *and* $\tilde{u}_p(e,p)$. *Then the evasion of all evaders is guaranteed for any admissible composite strategy of the pursuers if* $\underline{z}(t,t_0,e_0,p_0,z_0) > R$, *for all* $t \geq t_0$.

Proof. So, an assumption of the theorem is that the evaders choose their collective strategy to be $\tilde{u}_e^i(e_i, p)$. Then, if the pursuers choose any collective strategy $\bar{u}_p(e_i, p) \in \mathcal{U}_p$, from (47) and (48) it follows that:

$$\left.\frac{dv(e,p)}{dt}\right|_{\substack{u_e\,=\,\tilde{u}_e(e,p)\\ u_p\,=\,\bar{u}_p(e,p)}} \geq \left.\frac{dv(e,p)}{dt}\right|_{\substack{u_e\,=\,\tilde{u}_e(e,p)\\ u_p\,=\,\tilde{u}_p(e,p)}} \geq g(e, p, v(e,p)). \tag{50}$$

Thus, for any strategy $\bar{u}_p(e, p) \in \mathcal{U}_p$, differential inequality (50) is valid, and using the comparison principle we obtain

$$v(e(t), p(t)) \geq \underline{z}(t, t_0, e_0, p_0, z_0), \quad t \geq t_0. \tag{51}$$

From (3), (45) and (46) it follows that:

$$\min\{\eta_\delta^i(e_1(t), p(t)), \ldots, \eta_\delta^i(e_{N_e}(t), p(t))\} \geq v(e(t), p(t)), \quad t \geq t_0 \tag{52}$$

which implies

$$\eta_\delta^i(e_1(t), p(t)) \geq v(e(t), p(t)), \quad t \geq t_0, \tag{53}$$

for all $i \in \{1, \ldots, N_e\}$. Then, from Eqs. (3) and (36) we obtain

$$\min\{\|\overline{e}_i(t) - \overline{p}_1(t)\|, \ldots, \|\overline{e}_i(t) - \overline{p}_{N_p}(t)\|\} \geq v(e(t), p(t)), \quad t \geq t_0 \tag{54}$$

for all $i \in \{1, \ldots, N_e\}$. Inequalities (51) and (54) imply that for all $i \in \{1, \ldots, N_e\}$,

$$\min\{\|\overline{e}_i(t) - \overline{p}_1(t)\|, \ldots, \|\overline{e}_i(t) - \overline{p}_{N_p}(t)\|\} \geq \underline{z}(t, t_0, e_0, p_0, z_0) > R, \quad t \geq t_0 \tag{55}$$

which is a guarantee that all evaders will never be captured by any of the pursuers. This completes the proof. □

Notice that by defining goals of the players in terms of trajectories of the system either reaching or not reaching the target sets, we can generalize the proposed methodology to be applicable to a larger class of dynamic games rather than pursuit-evasion games only.

5. Unicycle Players

In order to provide an illustration for designing players' strategies using the proposed approach we consider pursuit-evasion games where the players are modelled using a nonholonomic nonlinear model also known as the unicycle [Spong *et al.* (2005)]. Let us assume that each player i, where the total number of players is denoted by N, that is $i \in \mathbf{N} = \{1, \ldots, N\}$, is modelled using the unicycle model, that is,

$$\begin{aligned} \dot{X}_i &= s_i \cos(\varphi_i) \\ \dot{Y}_i &= s_i \sin(\varphi_i) \\ \dot{\varphi}_i &= \omega_i \end{aligned} \tag{56}$$

where X_i and Y_i represent planar rectangular coordinates and φ_i represents the heading angle of the i-th player. Velocity, denoted by s_i, and angular velocity, denoted by ω_i, are assumed to be norm bounded by μ_i and ν_i, respectively.

Assuming that the Liapunov-like function for the i-th player is given by:

$$\upsilon_i \equiv \upsilon_i(P_i, P^i), \quad P_i = [X_i, Y_i]^T, \;\; P^i = \{P_j : j \neq i,\; j \in \mathcal{N}_i\} \tag{57}$$

where $\mathcal{N}_i$ is a *set of players of interest* to player i. A *set of players of interest* is defined based on the individual role of the player. For example, if a player is an evader then the set of interest would include indices of all pursuers. A Liapunov-like function candidate associated with the i-th player denoted by $\upsilon_i(\cdot,\cdot)$ is an appropriate approximation of the minimum or the maximum function depending whether the i-th player is a pursuer or an evader. Then, the control strategies for the pursuers, that is, velocity and angular velocity are given by the following expression:

$$\begin{aligned}\begin{bmatrix}\hat{s}_i \\ \hat{\omega}_i\end{bmatrix} &= \underset{\|s_i\|\le\mu_i,\|\omega_i\|\le\nu_i}{\arg\min} \frac{d\upsilon_i}{dt} \\ &= \underset{\|s_i\|\le\mu_i,\|\omega_i\|\le\nu_i}{\arg\min} \left\{\left(\frac{\partial\upsilon_i}{\partial X_i}\cos(\varphi_i) + \frac{\partial\upsilon_i}{\partial Y_i}\sin(\varphi_i)\right)s_i\right\}.\end{aligned} \tag{58}$$

From Eq. (58) it follows that the *closed-form solution* for the pursuers' velocities are given by the following formula:

$$\begin{aligned}\hat{s}_i &= -\mu_i \mathrm{sign}\left(\frac{\partial\upsilon_i}{\partial X_i}\cos(\varphi_i) + \frac{\partial\upsilon_i}{\partial Y_i}\sin(\varphi_i)\right) \\ &= -\mu_i \mathrm{sign}\left(\sin\left(\varphi_i + \arctan\left(\frac{\partial\upsilon_i/\partial X_i}{\partial\upsilon_i/\partial Y_i}\right)\right)\right),\end{aligned} \tag{59}$$

where $\mathrm{sign}(\cdot)$ denotes the standard sign function. Similarly, for the evaders we consider the $\arg\max$ function in (58) and obtain the following formula:

$$\begin{aligned}\hat{s}_i &= \mu_i \mathrm{sign}\left(\frac{\partial\upsilon_i}{\partial X_i}\cos(\varphi_i) + \frac{\partial\upsilon_i}{\partial Y_i}\sin(\varphi_i)\right) \\ &= \mu_i \mathrm{sign}\left(\sin\left(\varphi_i + \arctan\left(\frac{\partial\upsilon_i/\partial X_i}{\partial\upsilon_i/\partial Y_i}\right)\right)\right).\end{aligned} \tag{60}$$

From Eqs. (59) and (60) it can be easily shown that the maximal and minimal velocities will be achieved if the heading angle is $\pi/2$ (that is, in both cases) which implies that the desired heading angle is given by

$$\varphi_i^{\mathrm{des}} = \pi/2 - \arctan\left(\frac{\partial\upsilon_i/\partial X_i}{\partial\upsilon_i/\partial Y_i}\right). \tag{61}$$

Then, one possible norm bounded solution for the angular velocity strategy of the i-the player is to guide the player toward the desired heading angle as,

$$\hat{\omega}_i = -\nu_i \mathrm{sign}(\varphi_i - \varphi_i^{\mathrm{des}}). \tag{62}$$

Notice that the differential equation $\dot{\varphi}_i = \hat{\omega}_i$ is finite-time stable if φ_i^{des} is a constant. It is interesting to note that the functional forms for the angular velocities are the same for all the players.

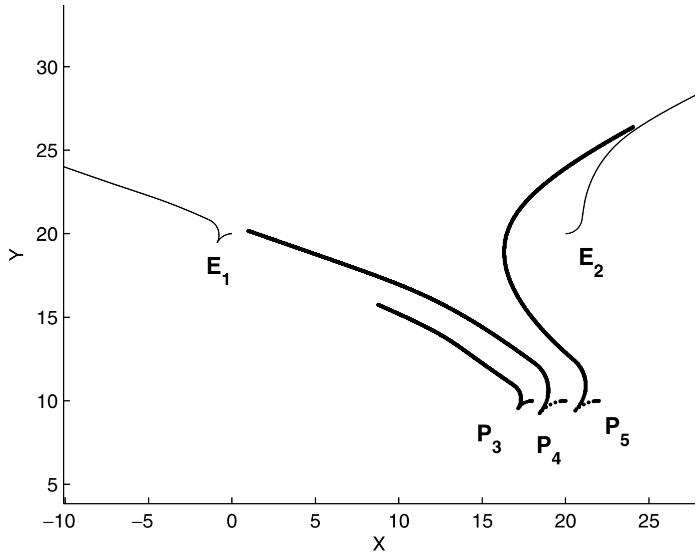

Fig. 1. Pursuit-evasion game for the first set of initial conditions with five nonidentical unicycle players.

Finally, in order to illustrate the proposed design of players' strategies let us consider a pursuit-evasion game with three pursuers and two evaders. The first scenario depicted in Fig. 1 shows two evaders with the initial conditions in the upper part of the figure and three pursuers with the initial conditions in the lower part of the figure. So, initial Y-coordinates of evaders and pursuers are the same, respectively. The trajectories for the evaders appear as thinner lines while the trajectories for the pursuers appear as thicker lines. We will refer to the evaders as players one and two and denote them as $\mathbf{E}_1$ and $\mathbf{E}_2$ (as shown in Fig. 1). Similarly, we will refer to the three pursuers as players three, four and five and denote them as $\mathbf{P}_3$, $\mathbf{P}_4$ and $\mathbf{P}_5$, respectively. This is also depicted in Fig. 1 by placing the corresponding labels next to the players' trajectories. By doing so, we assign subscript indices 1 and 2 to players which are evaders and indices 3, 4 and 5 to players which are pursuers. In the normalized units the bounds for the velocities of the players are $\mu_1 = \mu_2 = \mu_3 = 1, \mu_4 = \mu_5 = 2$ and $\nu_i = 1$, for all $i \in \{1, 2, 3, 4, 5\}$. The strategies are computed using equation (26), that is, Eqs. (58)–(62), and the composite function (25) (which is the same for all players) with $\delta = 3$. In Fig. 1, we can see that the players three and four (that is, pursuers $\mathbf{P}_3$ and $\mathbf{P}_4$) pursue player one (that is, evader $\mathbf{E}_1$) and that player five (that is, pursuer $\mathbf{P}_5$) pursues player two (that is, evader $\mathbf{E}_2$). Since players four and five are the fastest they will capture players one and two. We do not provide closed-form solutions for the players' strategies and the time derivative of the composite function since the derivation is straightforward using Eqs. (25), (58)–(62), and the equations are long and cumbersome.

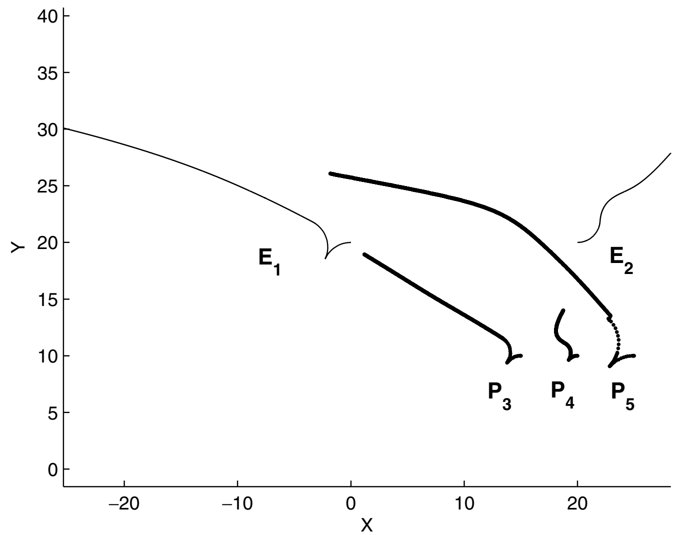

Fig. 2. Pursuit-evasion game for the second set of initial conditions with five nonidentical unicycle players: trajectories at the beginning of the pursuit.

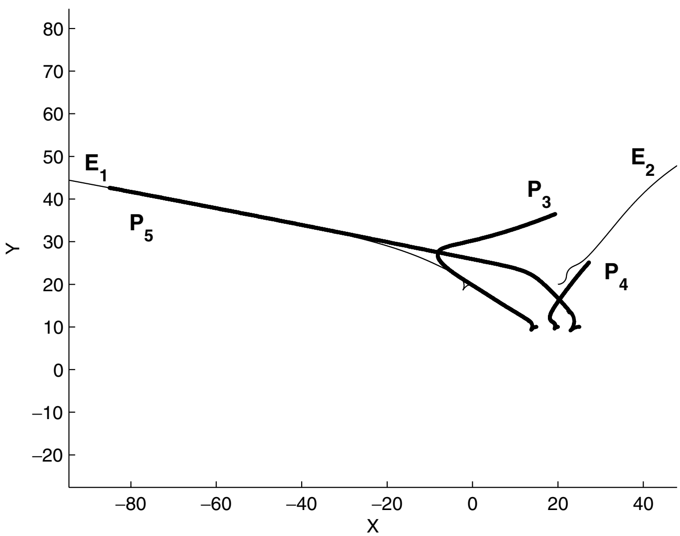

Fig. 3. Pursuit-evasion game for the second set of initial conditions with five nonidentical unicycle players: trajectories over the whole time horizon.

A more complex example is depicted in Figs. 2 and 3 where we only changed initial conditions slightly (not to change the numbering order of the players) and the bounds on the velocities as $\mu_1 = 5, \mu_2 = 2, \mu_3 = 3, \mu_4 = 1, \mu_5 = 6, \nu_1 = \nu_3 = \nu_5 = 2$, and $\nu_2 = \nu_4 = 1$. Parameter $\delta = 3$ remained unchanged as well as all the equations used to compute players' strategies. It is interesting that the initial strategy of player three (that is, pursuer $\mathbf{P}_3$) is to pursue player one (that is, evader $\mathbf{E}_1$) as depicted in Fig. 2 yet after a while it turns and starts pursuing player two (that is, evader $\mathbf{E}_2$) as shown in Fig. 3 since player four (that is, pursuer $\mathbf{P}_4$) is too slow for player two. Player five (that is, pursuer $\mathbf{P}_5$), as the fastest player, pursues from the very beginning player one who is the fastest among the two evaders.

These initial simulation results show capabilities of the strategies that are proposed yet many open questions still remain to be addressed in our future work. Some of the immediate issues to be considered are to include delays and study robustness properties of the methodology as well as the possibility that the players update their information only at discrete-time instances.

6. Conclusion

In this paper we provide a methodology of designing strategies for the players that guarantee either capture or evasion of all or some evaders in multi-player pursuit-evasion games. The players' dynamics are represented by nonlinear models and the sufficient conditions are formulated using a Liapunov type of analysis based on the comparison principle and differential inequalities by considering differentiable functions that are convergent approximations of the minimum and the maximum function.

Acknowledgments

This work has been supported by the Alexander von Humboldt Foundation, Bonn, Germany. The first author would also like to thank Mr. Juan S. Mejía for his help in producing the Matlab code used to obtain simulation results provided in the paper.

References

Bacciotti, A. and Rosier, L. [2005] *Liapunov Functions and Stability in Control Theory*, 2nd edn., Springer-Verlag, Berlin, Germany.

Bainov, D. and Simeonov, P. [1992] Integral inequalities and applications, *Mathematics and Its Applications*, Vol. 57, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Bardi, M. and Capuzzo-Dolcetta, I. [1997] *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, Boston, MA.

Blanchini, F. and Miani, S. [2008] *Set-Theoretic Methods in Control*, Birkhäuser, Boston, MA.

Breakwell, J. V. and Hagedorn, P. [1979] Point capture of two evaders in succession, *Journal of Optimization Theory and Applications* **27**, 89–97.

Chernousko, F. L. and Melikyan, A. A. [1975] Some differential games with incomplete information, Lecture Notes in Computer Science, Vol. 27, Springer-Verlag, Berlin, pp. 445–450.

Crandall, M. G. and Lions, P.-L. [1983] Viscosity solutions of Hamilton-Jacobi equations, *Transactions of American Mathematical Society* **277**, 1–42.

Falcone, M. and Ferretti, R. [1994] Discrete time high-order schemes for viscosity solutions of Hamilton-Jacobi-Bellman equations, *Numerische Mathematik* **67**, 315–344.

Hagedorn, P. and Breakwell, J. V. [1976] A differential game with two pursuers and one evader, *Journal of Optimization Theory and Applications* **18**, 15–29.

Isaacs, R. [1965] *Differential Games: A Mathematical Theorey With Applications to Warfare and Pursuit, Control and Optimization*, John Wiley and Sons, Inc., New York, NY.

Khalil, H. K. [2002] *Nonlinear Systems*, 3rd edn., Prentice Hall, Upper Saddle River, NJ.

Krasovskii, N. N. and Subbotin, A. I. [1988] *Game-Theoretical Control Problems*, Springer-Verlag, New York, NY.

Lakshmikantham, V. and Leela, S. [1969] Differential and integral inequalities: Theory and applications, Mathematics in Science and Engineering, Vol. 55, Academic Press, New York, NY.

Lakshmikantham, V., Leela, S. and Martinyuk, A. A. [1989] *Stability Analysis of Nonlinear Systems*, Marcel Dekker, New York, NY.

Levchenkov, A. Y. and Pashkov, A. G. [1990] Differential game of optimal approach of two inertial pursuers to a noninertial evader, *Journal of Optimization Theory and Applications* **65**, 501–518.

Luenberger, D. G. [1995] *Microeconomic Theory*, McGraw-Hill, Inc., New York, NY.

Melikyan, A. A. [1973] *On Minimal Observations in a Game of Encounter* (in Russian) PMM **37**(3), 407–414.

Melikyan, A. A. [1981] Optimal interaction of two pursuers in a game problem (in Russian) *Tekhnicheskaya Kibernetika* (2), 49–56.

Melikyan, A. A. [1998] Generalized characteristics of first order PDEs, Birkhäuser, Boston, MA.

Melikyan, A. A., Hovakimyan, N. H. and Harutyunyan, L. [1998] Games of simple pursuit and approach on two-dimensional cone, *Journal of Optimization, Theory and Applications* **98**(3), 515–543.

Melikyan, A. A. and Pourtallier, A. [1996] *Games with Several Pursuers and One Evader with Discrete Observations*, Game Theory and Applications (Petrosjan, L. A. and Mazalov, V. V. eds.), Vol. 2, Nova Science Publishers, New York, NY, pp. 169–184.

Olsder, G. J. and Pourtallier, O. [1995] Optimal selection of observation times in a costly information game, *New Trends in Dynamic Games and Applications* (Olsder, G. J. ed.), *Annals of the International Society of Dynamic Games*, Vol. 3, Birkhäuser, Boston, MA, pp. 227–246.

Pashkov, A. G. and Terekhov, S. D. [1987] A differential game of approach with two pursuers and one evader, *Journal of Optimization Theory and Applications* **55**, 303–311.

Petrosjan, L. A. [1993], *Differential Games of Pursuit*, Series on Optimization, Vol. 2, World Scientific, Singapore.

Petrov, N. N. [1994] Existence of the value of a many-person game of pursuit, *Journal of Applied Mathematics and Mechanics* **58**(4), 593–600.

Petrov, N. N. [2003] "Soft" capture in Pontryagin's example with many participants, *Journal of Applied Mathematics and Mechanics* **67**(5), 671–680.

Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F. [1962] *The Mathematical Theory of Optimal Processes*, Interscience Publishers, New York, NY.

Spong, M. W., Hutchinson, S. and Vidyasagar, M. [2005] *Robot Modeling and Control*, John Wiley & Sons, Hoboken, NJ.

Stipanović, D. M., Melikyan, A. and Hovakimyan, N. [2009] Some sufficient conditions for multi-player pursuit-evasion games with continuous and discrete observations, *Annals of Dynamic Games* **10**, 133–145.

Stipanović, D. M., Sriram and Tomlin, C. J. [2004] Strategies for agents in multi-player pursuit-evasion games, *Proceedings of the Eleventh International Symposium on Dynamic Games and Applications* (Tucson, Arizona).

Subbotin, A. [1984] Generalization of the main equation of differential game theory, *Journal of Optimization Theory and Applications* **43**, 103–133.

Subbotin, A. I. [1995] *Generalized Solutions of First-Order PDEs: The Dynamical Optimization Prospective*, Birkhäuser, Boston, MA.

Vagin, D. A. and Petrov, N. N. [2002] A problem of group pursuit with phase constraints, *Journal of Applied Mathematics and Mechanics* **66**(2), 225–232.