ment holds for ν. The statement that (5.18)–(5.20) hold for the one-sided limits now follows.

To establish (5.23) we first note that from (5.18) we have

$$H_y Y_x^* = -\nu K_y Y_x^*, \qquad H_z Z_x^* = -\mu R_z Z_x^*. \tag{5.24}$$

Since $\hat{R}(t, \phi^*(t), z^*(t)) = 0$ and since $\hat{R}(t, x, Z^*(t, x)) \geqq 0$, it follows that each component of $\hat{R}$ has a relative minimum at $(t, \phi^*(t))$. Hence $\hat{R}_x + \hat{R}_z Z_x^* = 0$, and so $\hat{\mu}\hat{R}_x + \hat{\mu}\hat{R}_z Z_x^* = 0$. Since those components of μ not in $\hat{\mu}$ are zero on $t_{i,k-1} \leqq t \leqq t_{ik}$, we have

$$\mu R_x + \mu R_z Z_x^* = 0.$$

Similarly, we obtain

$$\nu K_x + \nu K_y Y_x^* = 0.$$

Combining these last two equations with (5.24) and then substituting the result into (5.17), we get (5.23).

Remark. It is clear from the proof that the assumption that on an interval $(t_{i,k-1}, t_{ik})$ the components R^i of R such that $R^i = 0$ do not change, can be replaced by the assumption that there are a finite number of changes in the components of R.

We conclude by calling attention to one further necessary condition, namely Theorem 6 of [2]. We refer the reader to [2] for the theorem, and leave its proof in the present context for the reader. We also refer the reader to [2] for a sufficiency theorem.

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APPLICATIONS OF FUNCTIONAL ANALYSIS TO THE THEORY OF OPTIMAL PROCESSES*

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1. Introduction. In the years of development of optimal control theory, powerful general methods were created, based on the now widely known "maximum principle" and "optimality principle." The maximum principle is the most convenient method for solving problems of optimal programmed control; a detailed exposition is found in the fundamental work of L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko [1]. The methods of dynamic programming are given for a large class of problems of control, together with computational schemes for solving functional equations, by R. Bellman and his colleagues [2]. Parallel to the development in these and other directions (see [3], [4]) starting in 1956, attempts have been made to introduce methods of functional analysis into the study of optimal control problems.

At first it seemed that the methods of functional analysis applied only to a very restricted class of problems. But in spite of this, the number of studies using the ideas of functional analysis has increased. This is apparently explained by the fact that, in the solution of optimal control problems, with the help of the maximum principle, or by reduction to the Euler equations, there remains an indeterminate last step: as is known, these methods do not show how to select the initial condition for solving the adjoint system. The methods of dynamic programming and the approach that leads to the Hamilton-Jacobi equations do not have this deficiency. However, the solution of functional equations, to which both paths lead, is not an easy problem, and the advantages of the equation $u = u(x)$ over $u = u(x(t_0), t)$ are open to dispute if the control system is subject to the influence of perturbing forces or is nonstationary.

The functional approach to problems of optimal programmed control, which is described below, reduces variational problems to operations with functions of a finite number of variables. These are, as a rule, convex or concave functions, and determination of their extrema completes the solution of the optimal control problem. Game situations arise in many cases (statistical problems, pursuit problems, etc.), and the problem of determining the control functions reduces to the solution of a game whose players have finite-dimensional vectors as their strategies.

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The method using the ideas of functional analysis turned out to be very fruitful for a complete study of a wide class of problems in the theory of optimal processes (deterministic, statistical, adaptive), whether time-optimal, or optimal in the sense of terminal error or other criteria, as well as for two-point problems, both with variable endpoints and free endpoints. One of the typical features of the method is that it yields necessary and sufficient conditions for the existence of solutions. This fact makes it possible to study qualitative aspects of optimal processes: questions of controllability, existence and uniqueness of optimal controls, continuous dependence of solutions on initial data or parameters, mutual dependence of solutions of problems with various types of restrictions, etc.

Below we give a survey of certain works where the methods of functional analysis are used in solving problems in the theory of optimal processes. We describe methods for reducing variational (infinite-dimensional) problems to operations with functions of a finite number of variables, and we give computational algorithms; in addition we give conditions for controllability and existence of solutions for certain optimal problems.

It is not the author's intention to give a complete exposition of the problem of applications of functional analysis to the theory of optimal processes. In particular, she does not touch on the work of A. Ya. Dubovitskii and A. A. Milyutin, where the application of functional analysis to optimal control problems is treated differently. The main concern is with results obtained by the author and her colleagues in the Scientific Research Department of Power and Automation of the Ural Polytechnic Institute. In passing, results are also given from the work of other authors relating to the compatibility of the approach to the study of optimal processes.

We discuss basically optimal processes for objects described by ordinary linear differential equations, although the methods of solution may be extended to integrodifferential equations, partial differential equations, and certain nonlinear systems. We note that, from the point of view of concrete computations, generalizations to nonlinear equations are usually not as efficient as in the linear case. But since computation of optimal processes in nonlinear systems is often based on successive linearizations or piecewise-linear approximations, and leads to solution of the corresponding problems for linear systems, a complete study of linear systems from the functional analysis viewpoint is of interest, also, as a first step in analogous problems for nonlinear systems.

After describing in §2 the control systems on which the study is based, we discuss the following questions in §§3–10:

1. The aspects of functional analysis applicable to the theory of optimal processes (§3).
2. Problems of controllability of linear systems (§4).

3. Examples of the reduction of control problems to functional problems. Basic relations in optimal systems (§5).
4. Certain statistical problems of optimal control (§6).
5. Continuous dependence of solutions of optimal control problems on initial data and parameters (§7).
6. Problems of numerical solution (§7).
7. Application of functional analysis to certain problems of pursuit (§9).
8. Possible generalizations (§10).

2. Basic control systems. Consider the vector equations:

$$\frac{dx}{dt} = A(t)x + C(t)u + f(t) \tag{1}$$

and

$$\frac{dx}{dt} = A(t)x(t) + B(t)x(t-h) + C(t)u(t), \qquad h > 0' \tag{2}$$

where $x = (x_1, \cdots, x_n)$, $x \in X$, X is an n-dimensional space, $A(t)$, $B(t)$, $C(t)$ are matrices of dimension $n \times n$, $n \times n$, $n \times r$, respectively, $f(t) = (f_1(t), \cdots, f_n(t))$, $f_i(t)$ are external perturbations, u is an r-dimensional control function, h is a constant delay.

The solution of (1) at time $t = T$ for a given function $u = u(\tau)$ can be written

$$x(u, T, t_0) = x(T) = F(T, t_0)x(t_0) + \int_{t_0}^{T} F(T, \tau)\,[C(\tau)u(\tau) + f(\tau)]\,d\tau,$$

where t_0 is the initial moment and the matrix $F(t, \tau)$ satisfies the conditions

$$\frac{\partial F(t, \tau)}{\partial t} = A(t)F(t, \tau), \qquad F(t_0, t_0) = E,$$

E is the identity matrix.

Letting

$$\int_{t_0}^{T} F(T, \tau)C(\tau)u(\tau)\,d\tau = Su,$$

$$F(T, t_0)x(t_0) = c_1,$$

$$\int_{t_0}^{T} F(T, \tau)f(\tau)\,d\tau = c_2,$$

we arrive at an operator form of the solution to (1), which we shall also use later:

$$x(T) = x = Su + c, \qquad c = c_1 + c_2. \tag{3}$$

We consider next (2). Suppose we are given the function $\phi(\tau)$, and $x(\tau) \equiv \phi(\tau)$, $t_0 - h \leqq \tau < t_0$. We can show that

$$x = S_1 u + c^1, \tag{4}$$

where S_1 is a linear operator, c^1 is a constant vector. If (2) is a stationary system, then

$$S_1 u = \int_{t_0}^{T} F(T, \tau) C u(\tau)\, d\tau,$$

$$\frac{\partial F(t, \tau)}{\partial t} = AF(t, \tau) + BF(t - h, \tau), \qquad F(t, \tau) \equiv 0 \quad \text{for} \quad \tau > t,$$

$$\lim_{\tau \to t+0} F(t, \tau) = 0, \qquad \lim_{\tau \to t-0} F(t, \tau) = E,$$

$$c^1 = \int_{t_0 - h}^{t_0} F(T, \tau + h) B\phi(\tau)\, d\tau + F(T, t_0) x(t_0).$$

Let W be a finite-dimensional normed linear space. We give a few well-known definitions.

Definition 1. The function $\gamma = \gamma(w)$ is said to be *convex* in the convex region Δ, $\Delta \subset W$, if for all $w_1, w_2 \in \Delta$ and $\alpha \in [0, 1]$ we have the inequality

$$\gamma(\alpha w_1 + (1 - \alpha) w_2) \leqq \alpha \gamma(w_1) + (1 - \alpha)\gamma(w_2).$$

Definition 2. The function $\beta = \beta(w)$, $w \in \Delta$, is *quasi-convex* in w if for each δ the set $\Phi(w) = \{w : \beta(w) \leqq \delta\}$ is convex.

Definition 3. The hyperplane $H = \{w \in W : f(w) = 0\}$ is said to *support* the set M at the point $w_0 \in M$ if M lies on one side of H and $w_0 \in H$.

Definition 4. Let X be a normed linear space. Let X^* denote the space adjoint to X. If $X^{**} = X$, then we say that the space X is *reflexive*.

3. The aspects of functional analysis applicable to the theory of optimal processes. For the exposition of control problems we state some theorems from functional analysis [5]–[8] which are basic to the solution of optimal control problems.

3.1 Theorem on the separability of closed convex sets.

Theorem 1. *Let M_1 and M_2 be disjoint closed convex subsets of the reflexive Banach space X, and let one of them be bounded. Then the sets M_1 and M_2 can be separated by a hyperplane.*

Thus the theorem asserts the existence of a linear functional f, $f \in X^*$, and a number α, such that

$$f(h) \leqq \alpha \quad \text{for} \quad h \in M_1 \quad \text{and} \quad f(h) > \alpha \quad \text{for} \quad h \in M_2.$$

This theorem was used in 1956 by R. Bellman, I. Glicksberg and O. Gross in a two-point problem of time-optimal control. The same approach was used in 1963 by H. A. Antosiewicz [10] on a problem with variable right endpoint. R. Gabasov and the author showed the possibility of using a theorem on separability of closed convex sets for other control problems [11], [12], and they discovered a clear connection between control problems and the theory of linear inequalities [13]. We will indicate still more problems to which this theorem can be applied.

Let $p_i(z)$ be quasi-convex functions of z.

Problem (a). Minimization of a quasi-convex function of the final state of system (1). Given the duration of the process, $\tau = T - t_0$, determine a function $u^0(t)$ such that

$$p_1(x)|_{t=t_0} \leqq 0, \qquad p_2(x(u^0, T, t_0)) = \min_{u \in U} p_2(x(u, T, t_0)) = \delta^0,$$

where $U = \{u : p_3(u) \leqq 0\}$.

Problem (b). Transfer of an object from a given set $p_1(x)|_{t=t_0} \leqq 0$ in minimum time $\tau = T^0 - t_0$ to the set $p_2(x)|_{t=T^0} \leqq 0$, with $u \in U$.

Problem (c). Time-optimal control for systems with delayed argument. Suppose we are given (2) and we know that $x(\tau) \equiv \phi(\tau)$ for $t_0 - h \leqq \tau < t_0$, $x(t_0) = x_0$, where $\phi(\tau)$ is given a piecewise continuous function, and x_0 is a known vector. We wish to determine a control $u(t)$, $\| u \| \leqq 1$, where the condition

$$x(t) \equiv 0, \qquad t \geqq T,$$

is guaranteed in the minimal possible time T.

In these problems, the theorem on separability of closed convex sets is used to derive sufficient conditions for existence of solutions. Thus, for example, Problem (a) is handled as follows. Let δ be a positive number. Let $\Gamma(w) = \{w : p_2(w) \leqq \delta\}$, and define the set of admissibility

$$\Delta(x) = \{x : x = Su + c,\ u \in U,\ p_1(x)|_{t=t_0} \leqq 0\}.$$

The sets $\Gamma(w)$, $\Delta(w)$ are convex. If $\Gamma(w)$, $\Delta(w)$ are closed, then for $\delta < \delta^0$ the conditions are such that the above theorem is applicable. The analytic form of the condition of separability also leads to sufficient conditions (see §5) for existence of an admissible control, in which the functional being minimized takes on a given finite value.

A complete investigation of Problem (c) has been done by S. V. Churakova.

3.2. Existence theorem for a supporting plane to a convex surface.

Theorem 2. *Let $\gamma = \gamma(w)$ be a bounded function, convex in the region Δ, $\Delta \subset W$. Then at each point $(w, \gamma(w))$ we can construct a supporting plane.*

As an example of the application of this theorem, we consider the following problem for (1).

Given the numbers t_0, T, $T > t_0$, Δ_1, Δ_2, and points a_1, a_2 in X, find a control $u(t)$, $u \in U$, which minimizes the functional $\phi(x, u)$, convex on the set of elements (x, u), where the inequalities

$$\| x(t_0) - a_1 \| \leqq \Delta_1, \qquad \| x(T) - a_2 \| \leqq \Delta_2$$

must be satisfied.

In this case the set

$$\Omega(z) = \{z : z = (x, y), x = Su + c, u \in U, y = \phi(x, u)\}$$

need not be convex, but the surface $\delta(x) = \min_{x = Su + c, u \in U} \phi(x, u)$ is convex. An analogous situation occurs in control problems with certain restrictions on the function $u(t)$. The latter problems are dealt with in [14].

3.3. The L-problem in an abstract normed linear space. Given a normed linear space X of functions $h(\tau)$, $t_1 \leqq \tau \leqq t_2$, functions $h_i(\tau)$ from X, $i = 1, \cdots, n$, and numbers c_i, L, $L > 0$, find a linear functional f over X such that

$$f(h_i) = c_i, \qquad \| f \| \leqq L.$$

Conditions for solvability of this problem were obtained by M. G. Krein [7]. N. N. Krasovskii [15] was the first to use the L-problem for solving a problem of time-optimal control with fixed end conditions The class of controls was determined by the condition $\| u \| \leqq L$, where the norm for $u(t)$ is given by one of the standard relations:

$$(5) \qquad \| u \| = \operatorname{ess\,max}_{\tau} \max_j | u_j(\tau) |, \qquad t_1 \leqq \tau \leqq t_2,$$

$$(6) \qquad \| u \| = \operatorname{ess\,max}_{\tau} \left(\sum_{i=1}^{n} u_i^2(\tau) \right)^{1/2},$$

$$(7) \qquad \| u \| = \max_j \left(\int_{t_1}^{t_2} | u_j(\tau) |^p \, d\tau \right)^{1/p}, \qquad p \geqq 1.$$

We let the symbol $[Q]_j$ denote the jth row of the matrix Q. The coordinate form of (3) leads to the equalities

$$x_i(T) - c_i = \int_{t_0}^{T} ([F(T, \tau)C(\tau)]_i, u(\tau)) \, d\tau,$$

which can be treated as the values $x_i(T) - c_i$ of a linear functional generated by the function $u(\tau)$, $t_0 \leqq \tau \leqq T$, with bounded norm, $\| u \| \leqq L$, on the elements $[F(T, \tau)C(\tau)]_i$. This is also a typical formulation of the L-problem.

Using this approach, the authors of [16]–[18] studied limit passages from

the solution of time-optimal problems under restrictions (6), (7) as $p \to \infty$, $[C^*]_i \to 0$, $i > 2$, to the solutions of the analogous problems subject to the restrictions (5).

Subsequently the scope of problems which could be solved with the help of the L-problem was extended. The basic approach was construction of special spaces in which the norms corresponded to the type of restrictions given. Thus discrete systems with cyclic restrictions on the controlling functions were investigated in [19]. One such problem can be stated as follows.

Given an integer π, $\pi > 0$, determine N from the condition $N\pi \leqq K \leqq (N + 1)\pi$, where K is the duration of the process. Let $x(n + 1) = Ax(n) + Bu(n)$, $x(0) = x_0$, where n is discrete time, and A, B are constant matrices. For controls from the class

$$\max_{0 \leqq s \leqq N} \sum_{i = s\pi}^{(s+1)\pi - 1} | u_j(i) | \leqq 1,$$

we wish to determine the minimal K such that $x(K) = 0$.

The investigation of I. A. Litovchenko [20], who considers, from a functional analysis viewpoint, optimal processes with stepwise restrictions on controlling influences, is closely related to the cited work of R. Gabasov. The latter problems allow an interesting physical interpretation.

The idea of the solution of time-optimal problems with the help of the L-problem is generalized to more complicated problems in [21]. Many other problems can be reduced to the L-problem if certain transformations are made of relation (3). R. Gabasov and the author applied such reductions to problems with bounded phase coordinates, to systems connected by controls, and to systems with inertial regulators. The same approach was used in [22] in minimizing mean square error of a system.

3.4. Problem of imbeddability of convex bodies. This type is represented by the following problem: Given a normed linear space X of functions $h(\tau)$, $t_1 \leqq \tau \leqq t_2$, functions $h_i(\tau)$ from X, $i = 1, \cdots, n$, and numbers L, Δ, $L > 0$, $\Delta > 0$, find a linear functional f over X and an element c in n-dimensional space such that

$$f(h_i) = c_i, \qquad \| c \| \geqq \Delta, \qquad \| f \| \leqq L.$$

The present situation arises, for example, when a point x moving along the trajectory of (1) is to be transferred from a given convex region to the boundary of another, also convex, region containing the first [12].

3.5. Reduction of a variational problem to a game. We consider one such problem.

Let $x(s) = Su + c(s)$, where s is a parameter, $s \in \theta$, $u \in U$, U is a normed linear space, $\| u \| \leqq L$, L is a positive constant, S is a linear transformation from U to W, $x(s)$, $c(s)$ for fixed s are elements of the finite-dimensional space W.

Consider the problem of minimizing the function $f(x(\cdot))$, $x(\cdot) = \{x(s), s \in \theta\}$. Letting $x(\cdot) = c(\cdot) - y$, we restrict consideration to the case where $f(c(\cdot) - y)$ is quasi-convex in y. If

$$\min_{w \in W} f(c(\cdot) - w) = f(c(\cdot) - e) = d$$

and $\max_{\|g\| \leq 1} \{(g, e) - L \| S^*g \|\} = \Delta(e) \geqq 0$, then the point $y = e$ does not belong [12] to the interior of the region $\Delta(y) = \{y: y = -Su, \| u \| \leqq L\}$. Therefore the minimum δ^0 for $f(c(\cdot) - y)$ is attained on the boundary of the region $\Delta(y)$.

Let $\{g^0, y^0\}$ be a saddle point [23] of the following game:

$$\min_y \max_g \{f(c(\cdot) - y) + (g, y) - L \| S^*g \|\} \tag{8}$$
$$= \max_g \min_y \{f(c(\cdot) - y) + (g, y) - L \| S^*g \|\} = \alpha^0.$$

Then $(g^0, y^0) - L \| S^*g^0 \| = \max_g \{(g, y^0) - L \| S^*g \|\} = 0$. Therefore

$$f(c(\cdot) - y^0) \leqq f(c(\cdot) - y) + \max_g \{(g, y) - L \| S^*g \|\}.$$

If $\min_{y=-Su, \|u\| \leq L} f(c(\cdot) - y) = f(c(\cdot) - \tilde{y})$, clearly $\tilde{y} = y^0$, $\alpha^0 = \delta^0$.

Thus the problem of minimizing the quasi-convex function $f = f(c(\cdot) - y), y + Su = 0, \| u \| \leqq L$, reduces to the game (8). A more detailed description of control problems which reduce to games will be given in §6.

4. Problem of controllability of linear systems. An important problem in the theory of optimal processes and its applications is the problem of finding a control $u(x_0, x_1, t)$ which guarantees passage of a system from the initial state x_0 to a given state x_1. From the functional analysis point of view this two-point problem for systems (1), (2) can be interpreted as a problem of finding some linear functional (operator). The latter approach makes it possible to obtain effective conditions which guarantee existence of a control $u(x_0, x_1, t)$.

We give some definitions (cf. [24]). Let Z be the space of states of a dynamic system, U the set of control functions, $z = z(z_0, u, t)$ the state of the system at time t associated with the initial condition z_0, $z_0 \in Z$, $z_0 = z|_{t=t_0}$ and the control u, $u \in U$. Let X denote a subspace of Z, and $x = x(z_0, u, t)$ denote the projection of the state z on X. Let θ be the zero element in Z.

Definition 5. The state z_0 is called *controlled* in the class U (*controlled state*) if there exist a control[1] $u = u_{z_0}$, $u \in U$, and a number T, $t_0 \leqq T < +\infty$, such that $z(z_0, u, T) = \theta$.

Definition 6. The state z_0 is called *controlled* in the class U *with respect to a given set* X (*with respect to a controlled state*) if there exist a control $u = u_{z_0}, u \in U$, and a number $T, t_0 \leqq T < +\infty$, such that $x(z_0, u, T) = \theta_x$ (θ_x is the projection of θ on X).

Definition 7. If each state z_0, $z_0 \in Z$, of a dynamic system is controlled, then we say that the system is *completely controlled*. By a *relatively controlled system* we mean a dynamic system each state z_0 of which is relatively controlled.

Consider (1) where $f(t) \equiv 0$, $x(t_0) = x_0$. Assume that in Definitions 6 and 7 the subspace X is n-dimensional. Clearly the concept of "relatively controlled state" is equivalent to the known [24] term "controlled state." The properties of completely controlled systems can be obtained from the following considerations.

Let L be a positive constant. For (1) we find $\delta^0 = \min \| x(T) \|, \| u \| \leqq L$. From the definition of norm and the minimax theorem [23] we have:

$$\delta^0 = \min_{\|u\| \leq L} \| x(T) \| = \min_{\|u\| \leq L} \max_g \frac{(g, Su + c)}{\| g \|} = \max_g \min_{\|u\| \leq L} \frac{(g, Su + c)}{\| g \|}.$$

Thus

$$\delta^0 = \max_{\|g\| \leq 1} \{(g, c_1) - L \| S^*g \|\}, \qquad c_1 = F(T, t_0)x_0. \tag{9}$$

The assertion follows.

Lemma 1. *In order for system* (1) *to be completely controlled, it is necessary and sufficient that* $\| S^*g \| \neq 0$ *for arbitrary* $g \in X^*$, $\| g \| \neq 0$.

Necessity. Suppose system (1) is completely controlled, but there exists a vector g^0, $\| g^0 \| \neq 0$, such that $\| S^*g^0 \| = 0$. Consider the set $\omega(x_0) = \{x_0 : (g^0, c_1) > 0\}$. If $x_0' \in \omega(x_0)$, then

$$\max_{\|g\| \leq 1} \{(g, F(T, t_0)x_0') - L \| S^*g \|\} \geqq (g^0, F(T, t_0)x_0'), \qquad L > 0.$$

Thus $\delta^0 > 0$ for all $L > 0$, which contradicts the hypothesis.

Sufficiency. Consider (9). For each x_0 the function δ^0 is continuous in L and, for sufficiently large L, negative. Therefore for some $L = L(x_0)$ the quantity δ^0 is equal to zero.

The assertion is proved.

Thus system (1) is completely controlled only if $\| S^*g \| \neq 0, g \in X^*$,

[1] Below we let $U = \{u: \| u \| \leqq L, L < +\infty\}$.

$\| g \| \neq 0$. In the case of stationary systems,

$$\frac{dx}{dt} = Ax + Cu, \qquad x = (x_1, \cdots, x_n),$$

this condition turns into the requirement of linear independence of the vectors $C, AC, \cdots, A^{n-1}C$, where C is the vector obtained by R. V. Gamkrelidze [25]. If C is a matrix, then, as was shown by J. P. Lasalle [26], we must require that

$$\text{rank}\,\{C, AC, \cdots, A^{n-1}C\} = n. \tag{10}$$

Effective conditions for controllability of nonstationary systems can be obtained from Lemma 1.

Now consider the equation with delay (2). The space of states for (2) is the set of vector-valued functions

$$\{x(\tau), t - h \leqq \tau < t\}. \tag{11}$$

The initial state z_0 of system (2) is determined by the conditions

$$z_0 = \{x_0(\tau), x_0(\tau) \equiv \phi(\tau), t_0 - h \leqq \tau < t_0, x(t_0) = x_0\}. \tag{12}$$

The space of vectors x is a subspace of Z. The state $z = z(z_0, u, t)$ of system (2) in the space Z at time t is determined by the segment of the trajectory (11) from the space X.

Below we assume that motions of system (2) take place ($t \geqq t_0$) in the space of continuous functions; A, B, C are constant matrices, U is the set of piecewise continuous functions, and $t_0 = 0$.

According to Definitions 5–7, the state (12) of system (2) is controlled if there exists a control u, $u \in U$, such that $x(t) \equiv 0$, $T - h \leqq t \leqq T$ for $T < +\infty$.

The state (12) of system (2) is controlled with respect to X if there exists a control u, $u \in U$, such that $x(T) = 0$ for $T < +\infty$.

It follows from (4), (9) that system (2) is controlled with respect to X if and only if $\| S_1^* g \| \neq 0$ for $g \in X^*$, $\| g \| \neq 0$ (analogue of Lemma 1).

Effective necessary and sufficient conditions for relative controllability are available for this situation and can be stated as the following theorem.

THEOREM 3. *In order for system (2) to be relatively controlled, it is necessary that the rank of the matrix*

$$\{P_1^1, P_1^2, P_2^2, \cdots, P_{2^n-1}^n\}, \tag{13}$$

where $P_1^1 = C$, $P_{2l-1}^{k+1} = AP_l^k$, $P_{2l}^{k+1} = BP_l^k$, $l = 1, \cdots, 2^{k-1}$, $k = 1, \cdots, n-1$, *be equal to* n.

THEOREM 4. *In order for system (2) to be relatively controlled, it is necessary and sufficient that the rank of the matrix*

$$\{Q_1^1, Q_1^2, Q_2^2, \cdots, Q_n^n\}, \tag{14}$$

where $Q_1^1 = C$, $Q_l^{s+1} = BQ_{l-1}^s + AQ_l^s$, $l = 1, \cdots, k$, $k = 0, 1, \cdots, n-1$, $Q_l^s = 0$ *for* $l = 0$ *and* $l > k$, *be equal to* n.

In the case of differential equations without delay, sequences (13) and (14) coincide, and the conditions of Theorems 3, 4 reduce to condition (10).

The results relating to controllability described above often make it possible to study completely the problem of controllability of system (2). Consider, for example, the equation

$$\frac{dx}{dt} = Bx(t - h) + Cu(t). \tag{15}$$

The following assertion is true.

THEOREM 5. *In order for system (15) to be completely controlled it is necessary and sufficient that it be relatively controlled.*

System (2) is completely controlled if the matrix C is nonsingular. The problem is solved analogously for the equation

$$x^{(n)} + \sum_{i=1}^{n} (a_i x^{(n-i)}(t) + b_i x^{(n-i)}(t - h)) = cu(t),$$

which is always completely controlled.

5. Examples of reduction of control problems to functional problems. Basic relations in optimal systems. Problems of functional analysis arise in constructing admissible controls satisfying some boundary conditions (without the requirement of optimality with respect to a definite criterion). We consider several of them, with the goal of obtaining necessary and sufficient conditions for existence of solutions. We shall first show that investigation of existence and uniqueness reduces to the study of finite-dimensional extremal problems dual to those of [27].

The problem of minimizing the norm of the final state of trajectories of (1). Suppose we are given points a_1, a_2, $a_1 \in X$, $a_2 \in X$. For given t_0, $T > t_0$, $\Delta_1 > 0$, we wish to find a control $u^0(t)$, $\| u^0 \| \leqq 1$, such that

$$\| x(t_0) - a_1 \| \leqq \Delta_1, \quad \| x(u^0, T, t_0) - a_2 \| = \min_{\|u\| \leqq 1} \| x(u, T, t_0) - a_2 \| = \Delta_2^0.$$

We choose a number Δ_2 and find conditions for existence of an admissible control for which

$$\| x(t_0) - a_1 \| \leqq \Delta_1, \qquad \| x(T) - a_2 \| \leqq \Delta_2. \tag{16}$$

This is a functional analysis problem. Conditions for its solvability can be obtained by using the theorem on separability of convex closed sets (see §3). We give them in the form of a theorem.

THEOREM 6. *In order for problem* (16) *to have a solution, it is necessary and sufficient that the following inequality be satisfied:*

$$(17)\quad \max_{\|g\|=1} \Lambda(g, \Delta_2) = \max_{\|g\|=1} \{(g, F(T, t_0)a_1 + c_2 - a_2) - \Delta_1 \| F^*(T, t_0)g \| - \Delta_2 \| g \| - \| S^*g \| \} \leqq 0,$$
$$c_2 = \int_{t_0}^{T} F(T, \tau)f(\tau)\, d\tau.$$

The finding of optimal controls appears as the following step in the approach which uses methods of functional analysis.

In the present case we proceed as follows. The function $\Lambda(g, \Delta_2)$ is strictly monotone (decreasing) in Δ_2 ; therefore,

$$(18)\quad \Delta_2^0 = \max_{\|g\|\leqq 1} \{(g, F(T, t_0)a_1 + c_2 - a_2) - \Delta_1 \| F^*(T, t_0)g \| - \| S^*g \|\}.$$

Thus to determine Δ_2^0 we must solve the problem (18). If g^0 is a vector furnishing the solution (18), then, as follows from (17), the optimal control u^0 satisfies the condition

$$(19)\quad (S^*g^0, u^0) = \min_{\|u\|\leqq 1} (S^*g^0, u).$$

Thus Theorem 6 contains necessary and sufficient conditions for existence of a solution to control problem (16), the maximum principle (19) and relation (18), making it possible to find the vector g^0 (initial condition ψ_0 for the equation conjugate to (1) for $u \equiv 0$ equal to $\{-F^*(\tau)g^0\}$). The analogous conclusion can be drawn in the following problems.

Problem of minimizing the mean square error of system (1). Suppose we want to minimize

$$J(u) = \int_{t_0}^{T} \left(\sum_{i=1}^{n} \alpha_i\, x_i^2(\tau) + \sum_{j=1}^{r} \beta_j\, u_j^2(\tau) \right) d\tau, \qquad \alpha_i \geqq 0, \quad \beta_j \geqq 0, \quad \| u \| \leqq 1,$$

subject to (16), where α_i, β_j are given.

As above, we begin by considering the problem

$$(20)\quad J(u) \leqq \delta, \qquad \| x(t_0) - a_1 \| \leqq \Delta_1, \qquad \| x(T) - a_2 \| \leqq \Delta_2,$$

where conditions for solvability can be obtained by using the approach of the first theorems of §3.

THEOREM 7. *In order for problem* (20) *to have a solution, it is necessary and sufficient that*[2]

$$\max_{f>0, \|g\|=1} \{(g, F(T, t_0)a_1 + c_2 - a_2) - f\delta - \Delta_1 \| F^*(T, t_0)g \| - \Delta_2 \| g \| + \min_{\|u\|\leqq 1} [(S^*g, u) + fJ(u)]\} \leqq 0.$$

For the problem of optimal control we have the following result:

$$J(u^0) = \max_{\|g\|\leqq 1} \{(g, F(T, t_0)a_1 + c_2 - a_2) - \Delta_1 \| F^*(T, t_0)g \| - \Delta_2 \| g \| + \min_{\|u\|\leqq 1} [(S^*g, u) + J(u)]\},$$

where the function u^0 is determined from the condition

$$(21)\quad \min_{\|u\|\leqq 1} [(S^*g^0, u) + J(u)] = (S^*g^0, u^0) + J(u^0).$$

It is clear that we can pass from (17) and (20) to problems of time-optimal control subject to conditions (16) and (20), respectively. Here the optimal time of the process is the smallest number satisfying the inequality in the conditions for solvability.

We go on to a discussion of problems related to the L-problem. The latter has been used for a long time in the study of time-optimal control, in the following formulation:

Given points $x(t_0)$, $x_1 = 0$, transfer the trajectory (1) from the point $x(t_0)$ to $x_1 = 0$ in the least possible time, subject to $\| u \| \leqq 1$.

Suppose the norm of $u(t)$ is given by (5). First consider the problem of transferring from the point $x(t_0)$ to the point $x_1 = 0$. Its analytic form is

$$(22)\quad -x(t_0) - \int_{t_0}^{T} F^{-1}(\tau)f(\tau)\, d\tau = \xi = \int_{t_0}^{T} F^{-1}(\tau)C(\tau)u(\tau)\, d\tau.$$

As is known [7], problem (22) has a solution if and only if

$$(23)\quad \Lambda(T) = \min_{(g,\xi)=-1} \Lambda(g, T) = \min_{(g,\xi)=-1} \int_{t_0}^{T} \sum_{j=1}^{r} |(g, [C^*(\tau)(F^{-1}(\tau))^*]_j)|\, d\tau \geqq 1.$$

Suppose system (1) is completely controlled. For homogeneous systems the function $\Lambda(T)$ is continuous and strictly increasing [15] in T, and therefore the least $T = T^0$ is found from the condition

$$T^0 = \max \{T : \Lambda(g, T) = 1\}.$$

In other cases (inhomogeneous system, $x_1 \neq 0$) the solution $T = T^0$ gives the smallest root of the equation $\Lambda(T) = 1$. The optimal control satisfies

[2] $J(u)$ is computed from (3).

the condition

$$(24)\qquad \int_{t_0}^{T} (C^*(\tau)(F^{-1}(\tau))^* g^0, u^0(\tau))\, d\tau = -1,$$

where g^0 is the solution of problem (23) for $T = T^0$.

Now consider the following problem. Given numbers T, Δ, σ, $\sigma > \Delta$, find an equation for u, $\| u \| \leqq 1$, such that

$$(25)\qquad \| x(t_0) \| \leqq \Delta, \qquad \| x(T) \| \geqq \sigma.$$

The solution is carried out according to the scheme of §3.4.

THEOREM 8. *In order for problem* (25) *to have a solution, it is necessary and sufficient that the following inequality be satisfied:*

$$(26)\qquad \min_{\|g\|=1} \{\sigma \| g \| - \Delta \| F^*(T, t_0)g \| - \| S^* g \| - (g, c_2)\} \leqq 0.$$

Problem (25) is related to the following problem of optimal processes.

Problem of maximizing the norm of the final state of the trajectories of (1). Suppose we are given the numbers Δ and $\| x(t_0) \| \leqq \Delta$. We wish to choose the control u^0, $\| u^0 \| \leqq 1$, such that

$$\| x(u^0, T, t_0) \| = \max_{\|u\| \leqq 1} \| x(u, T, t_0) \| = \sigma^0.$$

From (26) we have

$$\sigma^0 = \max_{\|g\| \leqq 1} \{(g, c_2) + \Delta_1 \| F^*(T, t_0)g \| + \| S^* g \|\}.$$

From the solution g^0 of this problem we determine the optimal control, since

$$(27)\qquad (S^* g^0, u^0) = \max_{\|u\| \leqq 1} (S^* g^0, u).$$

We emphasize once more that Theorems 6, 7, 8, and (23) contain the maximum principle (see (19), (21), (24), (27)), existence theorems for admissible controls satisfying certain boundary conditions, existence theorems for optimal controls, and conditions (18), (23), etc., for determining the quantity g^0. These theorems also make it possible to find conditions under which the solutions of optimal control problems are continuous in the initial data and the parameters.

Remark 1. Up to now we have been concerned with controls constrained by conditions (5), (6), (7). Nonsymmetric restrictions on controls $u_j(\tau)$ of the form

$$d_j^{(1)} \leqq u_j(\tau) \leqq d_j^{(2)}, \qquad d_j^{(1)}, d_j^{(2)} \text{ const.},$$

can be investigated by introducing a nonsymmetric norm.

Remark 2. The results obtained allow a natural passage to a discrete model. However, in connection with the fact that the corresponding functions $\Delta(T)$ (in time-optimal problems) change their values by jumps, in an investigation of existence problems uniqueness of controls gives rise to peculiarities, which were noted and studied by R. Gabasov [28].

6. Some statistical problems of optimal control. Now we discuss a stochastic model of a controlled process, considering the effect of random factors of various kinds with known probabilistic characteristics.

Suppose the random vector $\tilde{x}$ of phase coordinates at a fixed moment of time $t = T$ has the form

$$(28)\qquad \tilde{x} = \tilde{x}(u, T, t_0) = Su + \tilde{c}, \qquad \tilde{x}, \tilde{c} \in X.$$

Here S is a linear operator, $\tilde{c}$ is a random vector—the value of some operator S_1 given on the space of initial conditions, external perturbations and characteristics of another process y. For example, if $\tilde{x}$ satisfies (1), then

$$\tilde{c} = F(T, t_0)x(t_0) + \int_{t_0}^{T} F(T, \tau) f(\tau)\, d\tau.$$

Let $f(z)$ be a positive function of z, and $Mf(\tilde{x})$ be the mathematical expectation of $f(\tilde{x})$.

DEFINITION 8. We say that the vector e, $e \in X$, is the *average* [29] *value* (average) of the random vector $\tilde{c}$, and the number d is the *measure of dispersion* (dispersion) if the following relation is satisfied:

$$Mf(\tilde{c} - e) = \min_z Mf(\tilde{c} - z) = d.$$

We restrict consideration to the case where $f(\tilde{x}) = \| \tilde{x} \|$. Let $\Phi_{\tilde{c}}(s)$, $s \in X$, be the distribution function of $\tilde{c}$.

Problem A. Find a control u^0, $\| u^0 \| \leqq 1$, achieving a minimum for the functional

$$M \| \tilde{x} \| = \int_X \| \tilde{x} \|\, d\Phi_{\tilde{c}}(s).$$

On the basis of the results of §3.5, in the general case we have

$$(29)\qquad \min_{\|u\| \leqq 1} M \| \tilde{x} \| = \delta^0 = \max_g \min_z \{M \| \tilde{c} - z \| + (g, z) - \| S^* g \|\}.$$

The optimal control is found from the condition

$$(S^* g^0, u^0) = \min_{\|u\| \leqq 1} (S^* g^0, u),$$

where g^0 is an element of the saddle point $\{g^0, z^0\}$ of the game (29).

Thus the problem of minimizing the mathematical expectation of the norm of the finite state (28) reduces to the solution of the game (29). It can be shown that for the minimal value of the functional we have the estimate

$$d \leqq \delta^0 \leqq d + \Delta(e),$$

where e is the average of the vector $\tilde{c}$, $\Delta(e) = \max_{\|g\| \leq 1} \{(g, e) - \| S^*g \|\}$, d is dispersion. Here for completely controlled systems $(g^0, z^0) > 0$, $(g^0, e) > 0$. Thus, if the object is one-dimensional (more precisely, if we are minimizing the mathematical expectation of the absolute value of one coordinate), then the optimal control is completely determined by the average e.

In certain cases Problem A is considerably simplified. Suppose, for example,

$$f(\tilde{x}) = \left(\int_X (\tilde{x}, \tilde{x}) \, d\Phi_{\tilde{c}}(s) \right)^{1/2};$$

then

$$(\delta^0)^2 = d^2 + \Delta^2(e), \qquad e = M\tilde{c}.$$

Thus in the latter case the optimal control is completely characterized by the vector $M\tilde{c}$:

$$(S^*g^0, u^0) = \min_{\|u\| \leq 1} (S^*g^0, u),$$

$$(g^0, M\tilde{c}) - \| S^*g^0 \| = \max_{\|g\| \leq 1} \{ (g, M\tilde{c}) - \| S^*g \|\}.$$

We now describe another problem encountered in applications, whose mathematical formulation leads to equations different from (28).

Suppose the action of control on an object may be discontinued at time t_1 with probability p_1, at time t_2 with probability p_2, etc. We are given a time $t = T$, a point $x = x_1$, and wish to minimize the mathematical expectation $M\| \tilde{x} - x_1 \|$.

One of the possible ways to solve this problem is the following. In (1), assume the matrix $C(t)$ equals $\alpha(t)E(t)$, where $E(t)$ is a matrix and $\alpha(t)$ is a random process of the following special form: in the intervals $(t_k, t_{k+1}]$ the function $\alpha(t)$ can take on only two values 0 or 1 with probabilities which are related in an obvious fashion to the quantities p_k. For such a stochastic model we can use the results of §3.5 (see [29]).

Now suppose that in the phase space X we are given the point x_1 and a neighborhood of it: $\| x - x_1 \| \leqq \epsilon$. A cross-section of the random process $\tilde{x}$ at a fixed moment gives a collection of random vectors $\tilde{x}(T)$ for which the quantity $\chi(u) = \text{Probability } \{\| x - x_1 \| \leqq \epsilon\} = P\{\| x - x_1 \| \leqq \epsilon\}$ is defined.

Problem B. Find a control u^0, $\| u^0 \| \leqq 1$, such that

$$\chi(u^0) = \max_{\|u\| \leq 1} \chi(u).$$

In the example of minimizing the mathematical expectation of the norm of the final state given above, we illustrated the way to reduce an infinite-dimensional variational problem to a game with two players whose strategies are finite-dimensional vectors. This method can also be applied (with certain restrictions) to Problem B. Here it turns out to be possible to obtain estimates for the maximally admissible probability without computing controls; the optimal control in the given statistical problem coincides with the optimal control in the deterministic problem:

$$x(T) = Su + x^0, \qquad \min_{\|u\| \leq 1} \| Su + x^0 \| = \| Su^0 + x^0 \|,$$

where x^0 is an element of the saddle point of a certain finite-dimensional game. Problem B is clearly related to the determination of the minimal radius ϵ (for fixed β), where

$$P\{\| x - x_1 \| \leqq \epsilon\} \geqq \beta,$$

whose solution we will not deal with.

7. Continuous dependence of solutions of optimal control problems on initial data and parameters. The discussion of these problems is based on the work of N. N. Krasovskii [16], [30] and the author [31]. Interesting properties of optimal controls, as functions of initial states x_0 and the parameter μ, arise in problems subject to the restrictions (5).

First we treat the problem of time-optimal control. The specific nature of the problem, whose solution—controls u—is a set of discontinuous functions, leads us to the following.

DEFINITION 9. The optimal solution $T^0(x_0, \mu)$, $u^0(x_0, \mu, t)$ is said to be *continuous* in the initial data x_0 and the parameter μ at the point (x_0^0, μ) if for each $\epsilon > 0$ there exists a $\delta > 0$ such that the inequalities

$$| T^0(x_0^0, \mu^0) - T(x_0, \mu) | < \epsilon,$$

$$\text{Meas } (E_j \mid u_j^0(x_0^0, \mu^0, t) - u_j^0(x_0, \mu, t) \mid \geqq \sigma) < \epsilon, \qquad \sigma > 0,$$

are satisfied, since only $\| x_0^0 - x_0 \| + | \mu^0 - \mu | < \delta$.

Continuous dependence of the solutions $T^0(x_0, \mu)$, $u^0(x_0, \mu, t)$ on the initial data and the parameter was first proved for linear homogeneous systems [31]. The proof was based on the property of monotonicity [15] in T of the function $\Lambda(T)$ (see (23)). The function Λ may lose this property if transfer occurs not to the origin but into some fixed point $x_1 = a$ of the space X. However in this case also we may obtain effective conditions guaranteeing continuous dependence of solutions on the initial data and the parameter. If the system is stationary, then one of these conditions (necessary and sufficient) is that the origin belong to the region $V = \{v: v = Aa + Cu, \| u \| \leq 1\}$.

Assume that the control system has the form

$$(30) \qquad \frac{dx}{dt} = A(t, \mu)x + C(t)u + f(t, \mu), \qquad \| u \| \leq 1,$$

where μ is a parameter, $\mu_1 \leq \mu \leq \mu_2$.

Suppose the minimal possible passage time of the trajectory (30) from the point (x_0, μ) to the point $(0, \mu)$ is $T^0(x_0, \mu)$.

THEOREM 9. *The optimal solution $T^0(x_0, \mu)$, $u^0(x_0, \mu, t)$ for system* (30) *is continuous in the initial data x_0 and the parameter μ if and only if for each positive number ν, we can construct a neighborhood $\Delta(x)$ of the point* $(0, \mu)$, *for points of which there exists a control u_x, $\| u_x \| \leq 1$, transferring points x into the point $x = 0$ in time $t \leq \nu$.*

In [17] the author established, for an optimal high-speed problem, existence of an optimal Lyapunov function (optimal time $T^0(x_0, \mu)$) which has continuous partial derivatives of any order in x_{i0} and the parameter μ. This fact made it possible to prove [31] that the function $T = T^0(x_0, \mu)$, subject to the restrictions (5), has continuous partial derivatives of any order in the coordinates x_{i0}, $\| x_0 \| \neq 0$, and the parameter at any point which is not a control switching point.

Continuous dependence of solutions on initial data and a parameter was studied, and the appropriate sufficient conditions were also established for nonlinear systems [31].

We note that the regular properties of solutions of problems of time-optimal control, with respect to initial data in the entire space X of states of the system and the parameter μ, are inherent in problems with "smooth" restrictions, for example, of type (6).

Consequently for such problems of optimal high-speed the heuristic principle of R. Bellman can be considered strictly justified.

In considering optimal control problems with criteria other than high speed, in many cases (minimization concerning the final state, convex control function), due to the fact that the functionals allow an explicit representation (see, for example, (18)), properties of functionals, such as continuous dependence and differentiability in initial data and parameters are easily proved. In more complicated problems, conditions guaranteeing continuous dependence of solutions of optimal control problems on initial data and parameters can be obtained on the basis of necessary and sufficient conditions for existence of solutions, examples of which are given in §5.

8. Problem of numerical solution. As was shown in §5, the methods of functional analysis applied to optimal control problems lead to additional conditions (compared to Euler equations or the maximum principle), which, as a rule, facilitate the problem of determining the initial condition for the adjoint system. Essentially this is the gradient method with large steps [32].

Thus, suppose we minimize the quantity $\| x(T) \| = (x(T), x(T))^{1/2}$ on the trajectories of (1), where $C(t) = b(t)$, $b(t)$ is the vector for the control $u(t)$ satisfying the condition $| u(t) | \leq 1$. We first show that the problem of finding the gradient reduces to integration of the original system for some specially chosen control.

It follows from (9) that

$$(31) \qquad \min_{|u| \leq 1} \| x(T) \| = \Delta^0 = \max_{(g,g) \leq 1} \left\{ (g, c) - \int_{t_0}^{T} |(g, F(T, \tau)b(\tau))| \, d\tau \right\}.$$

We note once again that the vector g^0 solving problem (31) is related as follows to the initial condition ψ_0 of the equation $\psi_0 = -F^*(T)g^0$ conjugate to the homogeneous one for (1).

We let $\lambda(g)$ denote the expression under the max in (31). Let g^1 be some vector, $\| g^1 \| = 1$, and let u^1 be the control for which

$$\int_{t_0}^{T} (g^1, F(T, \tau)bu^1(\tau)) \, d\tau = \min_{|u| \leq 1} \int_{t_0}^{T} (g^1, F(T, \tau)bu(\tau)) \, d\tau.$$

We determine the point $x^1 = Su^1 + c$. It can be shown that

$$\operatorname{grad} \frac{\lambda(g)}{\| g \|} \bigg|_{g=g^1} = x^1 - \lambda(g^1)g^1, \qquad \lambda(g^1) = (g^1, x^1).$$

Thus the problem of determining the gradient at each step reduces to finding x^1 and consequently to integrating (1) for $u = u^1$. This operation takes considerable time in solving problem (31), and therefore gradient methods with small step have little application here. Since it is possible to obtain information on the position of the maximum for $\lambda(g)$ in the direction of the gradient, we can use the method of steepest ascent. We shall discuss the latter for problem (31).

Suppose the vector g^1 satisfies the condition $(g^1, c) > 0$. We find u^1, x^1 as described above:

$$(S^*g^1, u^1) = \min_{|u| \leq 1} (S^*g^1, u), \qquad x^1 = Su^1 + c.$$

If ϵ is a given number characterizing the accuracy of computing δ^0, then for $\| x^1 \| - (g^1, x^1) = \epsilon^1 > \epsilon$ we proceed as follows:

Assume that the process is at the kth step. Let $g^k = g^{k-1}(\alpha^{k-1})$ and find u^k, x^k such that $(S^*g^k, u^k) = \min_{|u| \leq 1} (S^*g^k, u)$, $x^k = Su^k + c$. We introduce the element $g^k(\alpha) = (1-\alpha)g^k + \alpha x^k$ and construct the function $\mu^k(\alpha) = \lambda(g^k(\alpha))/\| g^k(\alpha) \|$. Let $\bar{g}^k = x^k/\| x^k \|$ and $\min_{|u|\leq 1}(S^*\bar{g}^k, u) = (S^*\bar{g}^k, u^k)$, $\bar{x}^k = Su^k + c$. Compute

$$\mu^k(0), \quad \mu^k(1), \quad \left.\frac{d\mu^k}{d\alpha}\right|_{\alpha=0}, \quad \left.\frac{d\mu^k}{d\alpha}\right|_{\alpha=1},$$

so that then we approximate $\mu(\alpha)$ by another function. We have

$$\mu^k(0) = (g^k, x^k), \qquad \mu^k(1) = (\bar{x}^k, x^k)/\| x^k \|,$$

$$\left.\frac{d\mu^k}{d\alpha}\right|_{\alpha=0} = \| x^k \|^2 - (x^k, g^k), \qquad \left.\frac{d\mu^k}{d\alpha}\right|_{\alpha=1} = \left(\bar{x}^k, \frac{(g^k, x^k)x^k - (x^k, x^k)g^k}{(x^k, x^k)^{3/2}}\right).$$

Assume that $\beta(\alpha)$ is the approximating function. We compute α^k from the condition $\beta(\alpha^k) = \max_{0\leq\alpha\leq 1}\beta(\alpha)$. If $\| x^{k+1} \| - (g^{k+1}, x^{k+1}) = \epsilon^{k+1} \leq \epsilon$, then the process stops. We note that these operations are sufficient for computing second derivatives for $\mu(\alpha)$ at the points $\alpha = 1$, $\alpha = 0$.

The rate of convergence of the proposed algorithm essentially depends on the method of introducing the parameter α, and on the coordinate system (λ, α).

Successive approximation methods for other problems are described in [14], [33].

As was shown in §5, §6, with the approach described the variational (infinite-dimensional) problem reduces to operations with convex (concave), convex-concave functions of a finite number of variables. Here one also sees the clear connection of the theory of optimal processes with nonlinear convex programming [34], [35]. Problems (18), (23) and others from §5, §6 involve convex programming, and numerical algorithms of the latter can be used to construct optimal controls.

9. Application of functional analysis to problems of pursuit. We consider only problems of programmed pursuit and discuss the possibility of constructing strategies of a pursuing point [36]–[38].

Suppose that two points, x and y, are moving in n-dimensional phase space, y being pursued by x. The controls u, v, represented by points, are subject to the conditions

$$\| u \| \leq l, \qquad \| v \| \leq m, \qquad l, m \text{ const.} > 0.$$

The equations of motion have the form

$$\dot{x} = A(t)x + C(t)u + f^1(t), \qquad x(t_0) = x_0,$$
$$\dot{y} = E(t)y + D(t)v + f^2(t), \qquad y(t_0) = y_0, \tag{32}$$

where $A(t)$, $C(t)$, $D(t)$, $E(t)$ are known matrices and $f_i^1(t)$, $f_i^2(t)$ are given functions.

Problem A. Find controls $u^0 = u^0(t_0, t)$, $v^0 = v^0(t_0, t)$ such that

$$\max_{\|v\|\leq m}\min_{\|u\|\leq l} T_\epsilon(u, v) = T_\epsilon(u^0, v^0) = T^0,$$

where $T_\epsilon(u, v)$ is the time required for the point x with control u to reach an ϵ-neighborhood of the point y, using the control v.

Problem B. Given the instants of time t_0, $t = T$, choose $u^1 = u^1(t_0, t)$, $v^1 = v^1(t_0, t)$ such that

$$\max_{\|v\|\leq m}\min_{\|u\|\leq l} \| x(u, T, t_0) - y(v, T, t_0) \| = \| x(u^1, T, t_0) - y(v^1, T, t_0) \|.$$

Equations (32) are linear, and therefore, under the condition that motion is considered from the instant $t = t_0$, we have the representation

$$z = Su + Qv + c$$

for the vector $z = x - y$ at the moment $t = T$. Here $S = S(T, t_0)$, $Q = Q(T, t_0)$ are linear operators and $c = c(T, t_0)$ is a vector (cf. (3)).

It follows from Theorem 4 that the point x reaches an ϵ-neighborhood of the point y, using the control v, only when

$$\lambda(v, T, t_0) = \max_{\|g\|=1}\{(g, c + Qv) - \epsilon\| g \| - l \| S^*g \|\} \leq 0. \tag{33}$$

The least $T = T(v)$ satisfying (33) is equal to the minimal time of pursuit of the point y with control v: $T(v) = \min_{\|u\|\leq l} T_\epsilon(u, v)$. Therefore for T^0 we have

$$T^0 = \max_{\|v\|\leq m} T(v) = T(v^0).$$

The control v^0, substituted into (33), determines u^0: $(S^*g^0, u^0) = \min_{\|u\|\leq l}(S^*g^0, u)$, where g^0 is the solution of problem (33) with $v = v^0$, $T = T^0$.

We introduce the functions

$$\Lambda(T, t_0) = \max_{\|v\|\leq m}\lambda(v, T, t_0)$$
$$= \max_{\|g\|=1}\{(g, c) - \epsilon \| g \| - l \| S^*g \| + m \| Q^*g \|\}, \tag{34}$$

$$\Lambda^+(T, t_0) = \begin{cases} \Lambda(T, t_0) & \text{for } \Lambda(T, t_0) > 0, \\ 0 & \text{for } \Lambda(T, t_0) \leq 0. \end{cases}$$

Let $\lambda = \lambda(v, T, t_0)$, $\| v \| \leq m$, be continuous from the right in T.

The smallest of the numbers θ satisfying the condition $\Lambda^+(\theta, t_0) = \min_{T\geq t_0}\Lambda^+(T, t_0)$ is denoted by T^θ, and the vector g solving (34) for $T = T^\theta$ is denoted by g^θ.

DEFINITION 10. The functions $u^\theta(t_0, s)$, $v^\theta(t_0, s)$, $t_0 \leq s \leq T^0$ constructed using the relations

$$(S^*g^\theta, u^\theta) = \min_{\|u\| \leq l} (S^*g^\theta, u), \qquad (Q^*g^\theta, v^\theta) = \max_{\|v\| \leq m} (Q^*g^\theta, v),$$

are called *θ-optimal controls.*

It is easier to find θ-optimal controls in the sense defined than to determine the functions $u^0(t_0, t)$, $v^0(t_0, t)$. Therefore it is of interest to find the relationship between the numbers T^0, T^θ and the controls $u^0, u^\theta, v^0, v^\theta$.

From the definition of the numbers T^0, T^θ it follows that $T^\theta \geqq T^0$. Let $\Lambda(T^\theta, t_0) = 0$.

THEOREM 10. *If* $\lambda(v^\theta, T, t_0) > 0$, $t_0 \leqq T < T^\theta$, *then*

$$T^\theta = T^0, \quad u^\theta(t_0, s) = u^0(t_0, s), \quad v^\theta(t_0, s) = v^0(t_0, s), \quad t_0 \leqq s \leqq T^0.$$

If $T = \theta^1$ *is the smallest number for which* $\lambda(v^\theta, T, t_0) \leqq 0$, $\theta^1 < T^\theta$, *and there does not exist* v, $\| v \| \leqq m$, *with the properties* $\lambda(v, T, t_0) > 0$, $t_0 \leqq T \leqq \theta^1$, *then* $T^0 = \theta^1$, $u^{\theta^1}(t_0, s) = u^0(t_0, s)$, $v^{\theta^1}(t_0, s) = v^0(t_0, s)$, $t_0 \leqq s \leqq \theta^1$.

The strategy $u = u^\theta(s_0, s)$ for the point x guarantees transfer into an ε-neighborhood of the point y for any choice of control v in the time $t \leqq T^\theta - t_0$.

If the situation is such that the point x knows the position of the point y at time t and the equation of motion (32), and according to these data a strategy is to be constructed which guarantees reaching an ε-neighborhood of y in the least possible time, then on the basis of the above we conclude that for $\Lambda(T^\theta, t_0) = 0$, and the conditions of Theorem 10, optimal pursuit can be achieved with θ-optimal controls.

Note that if the conditions of Theorem 10 are not satisfied, then it makes sense to release the point y from the θ-optimal control, since then $T(v^\theta) < T(v^0)$.

We proceed to Problem B. For given $v(s)$, $t_0 \leqq s \leqq T$, the minimal distance $\delta(v, T, t_0)$ which the point x can approach at time $s = T$ is

$$\delta(v, T, t_0) = \max_{\|g\| \leq 1} \{(g, c + Qv) - l \| S^*g \|\}$$

and

$$\max_{\|v\| \leq m} \delta(v, T, t_0) = \max_{\|g\| \leq 1} \{(g, c) - l \| S^*g \| + m \| Q^*g \|\}.$$

Thus the θ-optimal controls constructed according to the relations (33), (34) coincide with the controls $u^1(t_0, t)$, $v^1(t_0, t)$ for Problem B.

We let $u(t, s)$ denote the optimal control for Problem B with initial time t, computed at time s.

If the point x knows the technical capabilities of y (the system of equations of motion) and the position of y at each moment s, then the strategy $u = u(t, t)$ guarantees optimal pursuit in the sense of minimal motion away from y.

Other optimal pursuit problems are the subject of a special exposition.

10. On possible generalizations. It was already mentioned in the introduction that the methods of investigation described are applicable to a wide range of problems. Above, in the presentation of the basic ideas, we chose the simplest systems. Now we describe several possible ways to generalize the results obtained.

(i) In studying optimal processes for ordinary differential equations we first introduced (3). But this relation is a general property of linear systems (partial differential, integrodifferential, integral and other equations), and one can usually arrive at it with the help of Green's functions or other analogous means. Therefore the methods of functional analysis described are also applicable here.

(ii) Also for simplicity of presentation, the class of functions was defined by the condition $\| u \| \leqq 1$. This restriction can be removed, choosing as a restriction on the controls any bounded convex closed set in the space of vector-functions $u(t)$. (Such problems were stated in §3.)

(iii) Restrictions on the controls can also be weakened in another direction. Namely, instead of (1), consider the equation

$$\frac{dx}{dt} = A(t)x + \phi(u, t), \qquad u \in U, \tag{35}$$

where $\phi(u, t)$ is a vector-function continuous in u, t, and U is a bounded closed set.

It is possible to extend the given results to (35) because the set of admissibility for it is convex and closed. The latter fact was proved in work by H. Halkin [39], L. W. Neustadt [40].

(iv) The reasoning by which the results described for linear systems are extended to nonlinear systems,

$$\frac{dx}{dt} = f(x, u, t), \tag{36}$$

is presented in work by N. N. Krasovskii [30] and the author [31].

Unfortunately, generalizations in this direction are less effective, since one of the advantages of the methods of functional analysis (related to the determination of g^0) disappears in this case, although it is possible to investigate qualitative problems in the theory of optimal processes for (36), [30], [31].

(5) Generalizations involving passage to the infinite-dimensional case subject to (6) (or (7), $p > 1$), as is known, are not difficult, since the

theorems of functional analysis presented in §3 remain valid in this case [5]–[7]. But the effectiveness of the solutions is decreased, since the infinite-dimensional (variational) problem again reduces to the finite-dimensional. However, in certain cases the methods of functional analysis from the above point of view may also be of interest in the infinite-dimensional case [41].

It is natural that the considerations presented above relate to deterministic and stochastic systems as well as to systems with adaptation. In the latter case the necessary computations increase extraordinarily.

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AN EXTENSION OF AN INFORMATION-THEORETIC DERIVATION OF CERTAIN LIMIT RELATIONS FOR A MARKOV CHAIN*

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In [1] a limit relation for the transition probabilities of a stationary Markov chain with a countable number of states was derived by the use of certain properties of information measures. In this paper we shall use essentially the same techniques to derive a limit relation for a Markov chain with a countable number of states but with constant transition probabilities only. We shall assume that the reader is familiar with [1] and shall therefore omit certain details.

Consider a Markov chain with constant transition probabilities

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \cdot & \cdot & \cdots \\ p_{n1} & p_{n2} & \cdots \\ \cdot & \cdot & \cdots \end{bmatrix}, \tag{1}$$

where

$$\sum_j p_{ij} = 1, \qquad i = 1, 2, \cdots, \quad p_{ij} > 0, \tag{2}$$

with the absolute distributions

$$\mu_j^{(m+1)} = \sum_i \mu_i^{(m)} p_{ij}, \qquad j = 1, 2, \cdots, \quad \sum_j \mu_j^{(m)} = 1, \tag{3}$$

and with the m-step transition probabilities

$$p_{jk}^{(m+1)} = \sum_h p_{jh} p_{hk}^{(m)} = \sum_h p_{jh}^{(m)} p_{hk}, \qquad j, k = 1, 2, \cdots, \tag{4}$$

$$\sum_k p_{jk}^{(m)} = 1, \qquad j = 1, 2, \cdots . \tag{5}$$

We now prove the following theorem.

THEOREM. *For a Markov chain with a countable number of states and constant transition probabilities*, $\lim_{m\to\infty} (p_{hk}^{(m)}/\mu_k^{(m)}) = 1$.

Consider the discrimination information between the systems of probabilities (see [1])

$$P_i^{(m)}: \{p_{i1}^{(m)}, p_{i2}^{(m)}, \cdots\}, \qquad U^{(m)}: \{\mu_1^{(m)}, \mu_2^{(m)}, \cdots\}, \tag{6}$$

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