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Advances in Dynamic Games

Theory, Applications, and Numerical Methods for Differential and Stochastic Games

Dedicated to the Memory of Arik A. Melikyan

Birkhäuser

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This volume of the Annals of Dynamic Games is dedicated to the memory of

Arik Artavazdovich Melikyan

Foreword

While these Annals were in preparation, on April 6th, 2009 in Moscow, Arik Artavazdovich Melikyan left us.

An outstanding mathematician, member of the International Society of Dynamic Games and best friend, he has very unexpectedly left us in a deep mourning.

His basic contribution to the theory of pursuit and evasion games lies in new approaches for finding the solutions of Bellman–Isaacs equations and constructing new singular surfaces.

His scientific achievements were highly regarded in Russia, which he always considered as his motherland, and where he was elected a corresponding member of the Russian Academy of Sciences. In his second mother country, Armenia, he was also a foreign member of the National Academy of Sciences.

He was a good friend, an optimistic and cheerful person, and as such he will remain in our thoughts.

St. Petersburg, April 2009 *Leon Petrosyan*

Preface

Modern game theory has evolved enormously since its inception in the 1920s from the work of Borel (1921, 1924, 1927) and von Neumann (1928). Dynamic game theory branched from the pioneering work on differential games by R. Isaacs, L.S. Pontryagin and his school, and from seminal papers on extensive form games by Kuhn and on stochastic games by Shapley.

Since these early development days, dynamic game theory has had a significant impact in such diverse disciplines as applied mathematics, economics, systems theory, engineering, operations research, biology, ecology, and the environmental sciences. Modern dynamic game theory now relies on wide-ranging mathematical and computational methods, and possible applications are rich and challenging.

This edited volume focuses on various aspects of dynamic game theory, providing state-of-the-art information on recent theoretical developments and new application domains and examples. Most of the selected papers are based on presentations at the 13th International Symposium on Dynamic Games and Applications held in Wrocław, Lower Silesia, Poland at the beginning of Summer 2008. The symposium is held every two years under the auspices of the International Society of Dynamic Games (ISDG).

The papers selected to appear in the Annals cover a variety of topics ranging from theory to applications in biology, ecology, engineering, economics, and finance. The list of contributors consists of both well-known and young researchers from all over the world. Every paper has gone through a stringent reviewing process.

While we were in the middle of the review process, our fellow editor and good friend Arik Artavazdovich Melikyan suddenly passed away. This volume is dedicated to him as a tribute to his contribution in the field of dynamic games.

The volume is divided into five parts. The first part contains eight papers devoted to theoretical developments in differential games and general dynamic games, including new numerical methods. Part II contains five papers on pursuit/evasion games, including an historical perspective on the homicidal chauffeur game, collision avoidance and search games, and guidance problems. Part III is devoted to evolutionary games, with four papers on stable strategies, social interactions, mating, and telecommunication. Part IV contains two papers on cooperative games, addressing the problems of dynamic consistency and imputations when the horizon of the game is different or not known with certainty. Finally, Part V contains

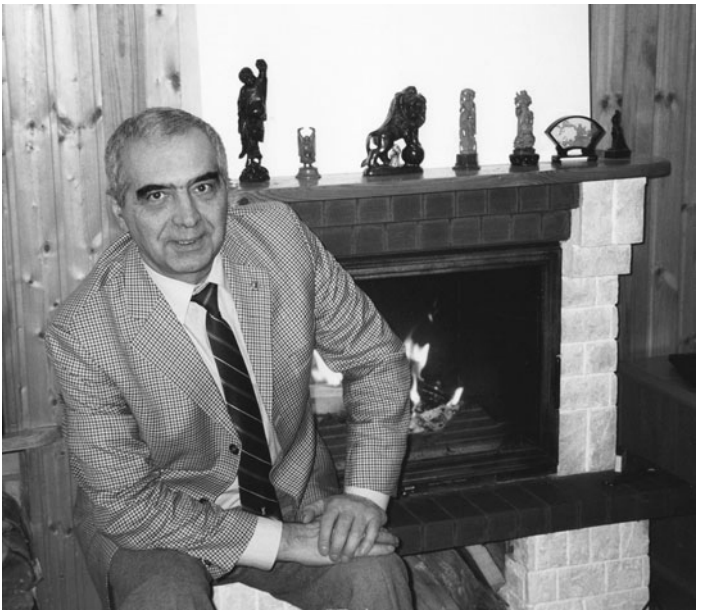
nine papers devoted to various applications of dynamic games, covering modeling, solutions, and numerical approaches. Applications range from management of fisheries and environmental agreements to insurance, option pricing and taxation, supply chain management, and channel allocation.

The editors are indebted to many colleagues involved in the editorial process. Special thanks go to Valerii S. Patsko, Andrei R. Akhmetzhanov, and Naira Hovakimyan, who helped us recover the editorial work of Arik after his death, put together his work, and collected testimonies from his friends.

Our warmest thanks go to the large number of referees for the papers submitted to the Annals. Without their important contribution this volume would not have been possible.

HEC Montréal, Canada, WUT, Wrocław, Poland *Michèle Breton*
June 2010 *Krzysztof Szajowski*

A Tribute to Arik Artavazdovich Melikyan



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3. Publications
4. *Pierre Bernhard*, The Long History of Arik's Conjecture
5. *Arik A. Melikyan and Fumiaki Imado*, Optimal Control Problem for a Dynamical System on an Ellipsoid

Testimonies

Arik Artavazdovich Melikyan died suddenly on April 6, 2009, in the Institute for Problems in Mechanics of the Russian Academy of Science (IPMech), where he worked for almost 40 years.

Having graduated with honors from Moscow Institute of Physics and Technology, Arik began to work at IPMech. Here he received his Ph.D. and Dr. of Sciences degrees, was awarded the State Prize of Russia for his research, was elected a Corresponding Member of the Russian Academy of Sciences, and became Head of the Laboratory of Control of Mechanical Systems.

I became acquainted with Arik when he was a bright student vividly interested in the theory of differential games, which was, at that time, a new and fascinating field of research. I was a supervisor of his Ph.D. thesis devoted to the new topic of differential games with incomplete information. Later, I worked with Arik for a long time, sometimes more tightly, sometimes on different subjects, but we always kept in close professional and human contact.

Arik was, of course, a brilliant scientist who obtained original and important results in differential games, the theory of the Hamilton–Jacobi equation, and optimal control.

He was also a remarkable personality: very kind and humane, who was loved and respected by everybody who knew him. We will never forget Arik.

Professor Felix Chernousko
Academician of the Russian Academy of Sciences
Director of the Institute for Problems in Mechanics
Russian Academy of Sciences, Moscow

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Arik Melikyan was my Ph.D. advisor from 1988 to 1992 in Moscow. During those years I used to interact quite often with his family both in Moscow and in Yerevan. We had a lot of cozy tea evenings with his wife Karina, his children Zara and Artem in their Moscow apartment; and also warm interactions with his mother Shushanik, his brother Gagik, and his sister Gayane in Yerevan. Ever since then we maintained a close friendship with Arik and his family members. So, my memories of Arik are rich, diverse, and colorful.

As his former Ph.D. student, in this small write-up I would like to place the emphasis on the wisdom that Arik shared with me and most probably with his other graduate students as well. He had a sense of humor and a sharp wit very unique to him. I was always left with an impression that there was no challenge for him — he knew the best way out of every situation. He could easily build relationships between math and life, between science and music, between art and politics. He could lend comfort and calmness, be on your side and the opposite side simultaneously, find the right path out, and the right words to say. Arik was not just a great advisor; that might have been a lot easier to achieve. With his serious attitude to the quality of technical work and the significance of foundational work and open problems in science, he always made hard fundamental problems look much simpler. In fact, only after many years of completing my thesis with him, I realized the significance of my Ph.D. contribution to the canvas of fundamental mathematics. According to Arik, we were having fun with "pretty mathematics." Prettiness was the major driver for him in every pursuit. Arik always seemed to be happy and content with life. He never complained. He knew how to appreciate every minute of life, how to cherish friendship, and how to be a friend. Arik's viewpoint of the world and his philosophy of life are still an integral part of my thinking in my everyday routine.

God bless his soul.

Naira Hovakimyan
Professor and Schaller faculty scholar
Mechanical Science and Engineering
University of Illinois at Urbana-Champaign

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Using the philosophy of antagonistic differential games, A.A. Melikyan developed his method of singular characteristics for nonlinear first-order partial differential equations. Within the framework of differential games, the singular characteristics are regarded as "optimal" motions by virtue of some control actions of players.

Having spoken to experts in mathematical physics, Arik always looked for the opportunity of a "non-game" interpretation of the singular characteristics. For example, equivocal singular surfaces are specific for differential games. But are such surfaces known and met in problems describing physical processes related to liquid or gas flows? If yes, then what is the physical interpretation of such singular surfaces?

I think that Arik's book on singular characteristics essentially extends the classic theory and will find many interested readers in the future.

Valerii S. Patsko
Institute of Mathematics and Mechanics
Ural Branch of Russian Academy of Sciences
Ekaterinburg, Russia

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In autumn of 1971 Felix Chernousko from the Soviet Academy of Sciences asked me to be an examiner for the Ph.D. candidacy thesis of Arik Melikyan called "On control problems and differential pursuit games with incomplete information." The basic idea was to define a discrete set of observations necessary to solve the control or pursuit problem of a particular kind. This was a new problem setting, which seemed to also have practical importance. And I agreed. This was the start of our friendly and fruitful relationship. I often visited the seminar of Chernousko, where Arik was also very active, and Arik visited St. Petersburg many times to present his new results in differential games in our university seminar. Since at that time I was interested in simple pursuit games with geometrical constraints, we had many topics for discussion. Arik was open minded, friendly, optimistic, and had a very kind personality. He was also religious, which was uncommon for Soviet times. He regularly visited the Armenian Apostolic Church of Surb Harutyun in Moscow and, when in St. Petersburg, the Church of Surb Katarine on Nevsky Prospekt. He truly loved his native country the Soviet Union (later Russia), and of course his motherland, Armenia. I remember an episode. I think it was in 1991. I went to Firenze for a Game Theory Conference and on the way there stopped in Venice for six hours to visit the St. Lazars Island, where in 1717 the Armenian monastery was founded by Abate Mchitar. I was very much impressed by this visit and, meeting Arik during the conference, told him about it. At that time the financial situation in the Soviet Union was very far from good and especially scientists were in the worst condition. But in spite of that, Arik changed his return plans in order to visit the monastery on his way back to Moscow. In winter of 2008 Arik and I were elected foreign members of the National Academy of Sciences of the Republic of Armenia. I had hoped to meet him during the General Assembly of the Academy in April 2009, but things went in a different way. My family and I will always remember Arik as an honest scientist and good friend.

Leon Petrosjan

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Arik was my colleague and friend for more than 30 years. For many years we shared one room in the Institute for Problems in Mechanics of the Russian Academy of Sciences. I had the pleasure to work with him, to enjoy his exclusive benevolence, sincere interest in scientific ideas and achievements of his colleagues, and his rare ability to be sincerely glad about success and depressed about the failure of other persons. He had a healthy sense of humor, loved and knew literature and the arts. He loved life in all its diversity.

Nikolai Bolotnik

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Warm, gentle, caring, kind, a great mathematician, and always the same are the words that I would use to describe Arik. I certainly lost somebody who had become a dear friend.

Dušan M. Stipanović

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One of the last times I met Arik, I told him I was worried that all of Armenia's population will soon leave the country, given the well-known difficulties. He smiled and told me that this would probably happen if the world were linear, but possibly not if it were nonlinear. I shall miss my meetings with him.

Berç Rustem

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It was really sad to hear the news of Arik's sudden death at his laboratory. He was almost the same age as I am, and I am afraid he overworked himself. Although we had communicated with each other since 1990, we became close friends when we were invited to Kiev Technical University by Prof. Arkadii Chikrii in June 2002. At the conference I was very interested in his talk "Differential Games on Curved Surfaces." I invited him to Shinshu University in April 2006. During his five weeks' stay, I brought him to many universities and sightseeing spots. I introduced him to many professors, who were all impressed by his deep knowledge and activities in many fields. He was particularly pleased with the varieties of Japanese cooking and the blooming cherry blossoms. We soaked in a hot spring together, drank, and sang Russian songs in a bar. He often said he would come to Japan again with his wife. Alas, it has become an impossible dream. Now I can only pray for his peaceful rest.

Professor Fumiaki Imado
Shinshu University, Japan

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Arik Melikyan became, to me, not simply a supervisor of my studies, but a great teacher. I was always surprised at his wisdom and irrepressible energy.

Georgy Naumov
Former student of Arik Melikyan

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Arik had a very valuable skill: the ability to influence people. We first met while organizing the 13th ISDG symposium in Wrocław. He brought a different perspective to our discussions, and we learned quickly to respect each other's views. He was always quiet, and in appearance was perhaps unremarkable, yet he had an incredible strength of character. He gave us all so much more than he took, sharing his experience, skills, and knowledge.

I last saw Arik in Delft after the retirement party for G.J. Olsder. He was walking down the old streets, commenting on the habits of local students, planning future meetings, and clearly enjoying life. No one expected that we would never again

talk to him, see him, or hear his voice over Skype... We have lost an exceptional person who will be remembered with great warmth and respect by me and by all his colleagues.

Krzysztof Szajowski

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I became a student of Arik Melikyan in 2004, during the last year of my undergraduate studies. I knew Arik for five years, and I thank my fate that it gave me a chance to work under the supervision of this great man. Arik was a kind person, always very polite; he was a gentleman with his own style of life. For me, work with Arik was like a game, very joyful and interesting. Unfortunately, God decided to take him away from us very early.

Andrei Akhmetzhanov
Former student of Arik Melikyan

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Arik Melikyan was not just a great scientist, but also a good senior friend of mine. His contributions and ideas influenced and broadened my research interests, and also educated a wide audience of young generation of Armenian scientists. During the last few years, Arik Melikyan visited his home country, Armenia, more often than in prior years, and we had the opportunity of numerous meetings, discussions, and exchanges of ideas. Arik Melikyan was well respected in Armenia: recently he was elected as a foreign member of the National Academy of Sciences of Armenia. We were looking forward to hosting him and honoring him here during the week of April 18–25 as a part of the celebrations in honor of foreign members of the Academy, but an unexpected event took away this opportunity. We will always cherish our memories of Arik as a close friend, a sincere patriot, and a great scientist.

Vahan Avetisyan
Leading research scientist
Institute of Mechanics
Armenian Academy of Sciences

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Arik was my great friend for many years. We met at FizTech (Moscow Institute of Physics and Technology) in 1966, and from the very first meeting I remember a strong impression of a unique mixture of kindness, humor, and fairness that made me deeply interested in becoming Arik's friend. We met many times during my six years of studies in Dolgoprudny. Yet later in 1972, when I started my Ph.D. studies at the Institute of Problems in Mechanics, we became colleagues in a rather small group conducting research on optimal control of uncertain dynamical systems and became close friends. No wonder, Arik was the witness at

my wedding in 1976. Since then we could be separated by a long distance, but our friendship was as strong as it was in our young years. In twenty years of living and working in Atlanta, I made no visit to Moscow without meeting Arik and his wonderful family. Needless to say, our daughters became friends as well. Many times Arik came to Atlanta as a Visiting Professor at the School of Mathematics at Georgia Tech. Our common interest in information and control led to the Fogarty Award, a US grant to a joint USA-Russian team for developing algorithms for analysis of DNA sequences carrying genetic information. Through the years, Arik was constantly present in my world; I felt his support, saw his smile, heard his wise and warm words. I cannot believe that Arik is no longer with us. I will always remember him as my dear friend.

Mark Borodovsky, Ph.D.
Director, Center for Bioinformatics and Computational Genomics at Georgia Tech
Regents' Professor, Wallace H. Coulter Department of Biomedical Engineering
Georgia Tech and Emory University
Georgia Tech Division of Computational Science and Engineering
Atlanta, Georgia, USA

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I met Arik Melikyan for the first time at the Institute of Problems in Mechanics of the Russian Academy of Sciences in 1969. My first impression was of an extremely charming and handsome young man, who had biblical, noble, unbelievable beauty and charm, reminding me of old Assyrian bas-reliefs. My first impression was true: Arik Melikyan was indeed noble and handsome all his life. He was a great friend. With Arik you could share the most sacred secrets. He was always very patient, diplomatic, balanced; he never made hasty conclusions. He had a purity that was inspiring to others.

The first problem that Arik tackled was the problem of two boats chasing each other around an island. It was a well-known problem from Isaacs' book. In some sense, this defined the subject of his research interests throughout his life. Arik enjoyed dealing with real-life problems that were difficult from a mathematical point of view and without trivial solutions. Committing himself to that path, Arik made several fundamental discoveries about the structure of the solutions to the Hamilton–Jacobi–Bellman–Isaacs equation, which constitutes the basis for the modern theory of differential games. His successful career is the result of sincere dedication to science and tremendous diligence.

The death of a person presents a property of reverse perspective, in the sense that all the private and minor points tend to disappear, while the main features surface and have the potential of lasting forever. The main feature for Arik was his unparalleled nobleness and dignity.

Alexander Bratus

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Arik A. Melikyan (10.05.1944–04.06.2009)

Arik Melikyan, a prominent scientist in the area of differential games, optimal control, and related areas of the theory of differential equations, a Corresponding Member of the Russian Academy of Sciences, a recipient of the Russian State Prize, a foreign member of Armenian National Academy of Sciences, died on April 6, 2009. He was only 64 years old.

Arik Melikyan is well known for his fundamental and long lasting contributions to the theory of optimal control and differential games. He generalized the classical concept of regular differential games and extended the mathematical tools to a substantially broader class of game problems, calling these problems regular in a broader sense. Arik is well known for generalization of the method of regular characteristics for first-order PDEs to handle non-smooth value functions and/or initial conditions. In differential games, quite often the value function of the underlying Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation is not smooth. In such cases, the phase portrait for the complete state space cannot be covered by optimal paths using the regular method of characteristics. Arik Melikyan was the first to obtain the algorithms for construction of optimal phase portraits in differential games with non-smooth value functions by exploring the differential geometric structure of the phase space associated with non-uniqueness of minimal geodesic lines connecting the players. This led to analytical description of a large class of singular surfaces and provided the opportunity for extending the classical method of characteristics to games with non-smooth value functions. The Method of Singular Characteristics by Arik Melikyan is widely used today not just in the analysis of pursuit-evasion games in the presence of obstacles, but also in the investigation of propagation of tangential discontinuities in some physical media and in processing visual information for the purpose of feedback. Arik also significantly contributed to the theory of control of uncertain systems and developed game-theoretic type algorithms for optimal search for objects moving in a bounded area using a computer vision system with a restricted aperture. Arik Melikyan is the author of two monographs and more than 150 papers in Russian and international scientific journals.

Arik Melikyan was born on October 5, 1944 in Yerevan, Armenia. His parents – Artavazd and Shushanik – were high school biology teachers. His brother Gagik is an engineer, and his sister Gayane is a chemist. Arik graduated from high school

in 1961, Magna cum Laude (Golden Medal in Russia), and was accepted to the Department of Cybernetics in Yerevan Polytechnic Institute. Being an outstanding student, he sought the highest possible quality of education available at the time in the former USSR. In 1963 he moved to Moscow to continue his education in the Department of Aeromechanics and Applied Mathematics at the Moscow Institute of Physics and Technology (MIPT). Arik graduated from MIPT, Magna cum Laude, in 1969 as a physicist-engineer, specializing in flight dynamics and control. From his early childhood he proved to be an extremely bright person. Arik was not just a gifted mathematician; as a young student he was one of the top athlete-runners in Armenia. He studied music and played several instruments (piano, accordion). He enjoyed learning languages and spoke several of them. Along with his major interest in mathematics he had strong interests in journalism, literature, and the arts in general. He also took classes in the Department of Journalism at the Lomonosov Moscow State University, publishing interviews with prominent figures of the time. He also wrote short stories for a major Russian literary newspaper. He was actively involved in student life – organizing concerts of well-known singers and actors for the students and professors of MIPT, and writing for the student newspaper. He was a very generous and kind person: as a student and later as a mature professional, he was always taking time to help friends, colleagues, and students. His friends and colleagues liked him for his easy and joyful character, his respect for others, his great intelligence and sense of humor, and his love of people and his profession.

Upon his graduation from the Moscow Institute of Physics and Technology, Arik became a Ph.D. student at this institute and continued his research in mathematical theory of control under the scientific supervision of Felix Chernousko. In 1972 he defended his Ph.D. thesis on differential games with incomplete information. After that he worked at the Institute for Problems in Mechanics of the USSR Academy of Sciences (currently the A.Yu. Ishlinskii Institute for Problems in Mechanics of the Russian Academy of Sciences). His entire scientific career was associated with this institute; he grew from a junior research fellow to become the head of a laboratory. In 1986, he defended the thesis "Game Problems of Dynamics" and received his Doctor-of-Science (habilitation) degree. In 1998, he was awarded the State Prize of the Russian Federation in the Field of Science and Technology. In 2003, he was elected a Corresponding Member of the Russian Academy of Sciences.

Arik Melikyan performed great organizational work for the community: he was a member of the Russian National Committee on Theoretical and Applied Mechanics and of the Executive Committee of the International Society of Dynamic Games, a member of the editorial boards of the *Journal of Computer and Systems Science International* and *International Game Theory Review.* He was a member of various Program Committees and Organizational Committees of numerous Russian and international conferences on the mathematical theory of control and differential games. As a member of the Editorial Board of the *Journal of Computer and Systems Science International,* Arik Melikyan was noted for his benevolent and attentive attitude to authors and his high standards with regard to the quality and language of papers accepted for publication.

In addition to being an outstanding scientist, Arik was a great teacher and mentor. He always considered teaching on par with his research and enjoyed his professorship at MIPT. Arik was a great friend of his graduate students. He used to work with them long hours, and he maintained his friendship with his students throughout his life. He was a professor of the Moscow Institute of Physics and Technology, where he delivered original courses on the calculus of variations, optimal control, and differential games. Arik was also a member of the Educational and Methodological Council of the Department of Aerophysics and Space Exploration. His lectures were always a great success among students.

Arik was not only an exceptional scientist and a gifted teacher but a wonderful person. He was exclusively benevolent and responsive, helped others to find the way out of complicated situations. He was optimistic with a good sense of humor, able to solve difficult life problems. Arik Melikyan will forever live in the memory of all people who knew him.

He is survived by his loving wife Karina, daughter Zara, and son Artem.

Bibliography of Works by Arik A. Melikyan

Books

1. Chernousko, F.L., Melikyan, A.A.: Igrovye zadachi upravleniya i poiska (Game-Theoretical Control and Search Problems). Nauka, Moscow (1978) (in Russian)
2. Melikyan, A.A.: Generalized Characteristics of First Order PDEs. Applications in Optimal Control and Differential Games. Birkhäuser, Boston (1998)

Theses

3. Melikyan, A.A.: Some Problems of Optimal Control and Differential Games with Incomplete Information. PhD Thesis, Acad. Sci. USSR, Institute for Problems in Mechanics, Moscow (1972) (in Russian)
4. Melikyan, A.A.: Some Problems of Optimal Control and Differential Games with Incomplete Information. Author's summary of PhD Thesis, Moscow Institute of Physics and Technology, Moscow (1972) (in Russian)
5. Melikyan, A.A.: Construction of Optimal Motions in Game Problems of Dynamics. Doctoral Thesis, Acad. Sci. USSR, Institute for Problems in Mechanics, Moscow (1985) (in Russian)
6. Melikyan, A.A.: Construction of Optimal Motions in Game Problems of Dynamics. Author's summary of Doctoral Thesis, Acad. Sci. USSR, Ural Sci. Center, Institute of Mathematics and Mechanics, Sverdlovsk (1985) (in Russian)

Papers in Peer-Reviewed Journals

7. Melikian, A.A., Chernousko, F.L.: Certain minimax control problems with incomplete information. J. Appl. Math. Mech. **35(6)**, 907–916 (1971). Translated from Prikl. Mat. Mekh. **35(6)**, 952–961 (1971) (in Russian)
8. Melikjan, A.A.: Differential games with information gaps. Izv. Akad. Nauk Armjan. SSR, Ser. Matematika **7(4)**, 300–308 (1972) (in Russian)
9. Melikyan, A.A., Chernousko, F.L.: On differential games with variable conditions of information. Dokl. Akad. Nauk SSSR **203(1)**, 46–49 (1972) (in Russian)
10. Melikian, A.A.: On minimal observations in a game of encounter. J. Appl. Math. Mech. **37(3)**, 407–414 (1973). Translated from Prikl. Mat. Mekh. **37(3)**, 426–433 (1973) (in Russian)

The bibliography was collected by Valerii S. Patsko and Alexander E. Utemov.

11. Melikjan, A.A.: The minimal number of moments of observations in a model approach game. Izv. Akad. Nauk Armjan. SSR, Ser. Matematika **9(3)**, 242–247 (1974) (in Russian)
12. Melikian, A.A.: On optimum selection of noise intervals in differential games of encounter. J. Appl. Math. Mech. **39(2)**, 195–203 (1975). Translated from Prikl. Mat. Mekh. **39(2)**, 207–215 (1975) (in Russian)
13. Korneev, V.A., Melikian, A.A.: Differential game with impulse control of one player. J. Appl. Math. Mech. **41(2)**, 345–349 (1977). Translated from Prikl. Mat. Mekh. **41(2)**, 354–357 (1977) (in Russian)
14. Melikjan, A.A.: Game value in a linear differential approach game. Dokl. Akad. Nauk SSSR **237(3)**, 521–524 (1977) (in Russian)
15. Korneev, V.A., Melikian, A.A.: Impulse-differential encounter game. J. Appl. Math. Mech. **42(2)**, 220–228 (1978). Translated from Prikl. Mat. Mekh. **42(2)**, 210–218 (1978) (in Russian)
16. Chernousko, F.L., Melikjan, A.A.: Minimax problems of control, observation and search. Teoret. Prilozhna Mekh. **9(3)**, 16–24 (1978) (in Russian)
17. Melikyan, A.A., Chernousko, F.L.: The choice of instants of observation in a linear game of approach. Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibernet. **1**, 11–16 (1979) (in Russian)
18. Melikian, A.A.: Game problem of the design of a multi-impulse motion correction. J. Appl. Math. Mech. **44(5)**, 555–560 (1980). Translated from Prikl. Mat. Mekh. **44(5)**, 783–791 (1980) (in Russian)
19. Anchev, A., Melikyan, A.A.: Optimal reorientation of a satellite on a circular orbit. Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela **6**, 37–42 (1980) (in Russian)
20. Melikyan, A.A.: Optimal interaction of two pursuers in a game problem. Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibernet. **2**, 49–56 (1981) (in Russian)
21. Melikyan, A.A.: The Cauchy problem with an unknown boundary for a first-order partial differential equation. Soviet Math. Dokl. **24(2)**, 268–273 (1981). Translated from Dokl. Akad. Nauk SSSR **260(2)**, 288–292 (1981) (in Russian)
22. Melikyan, A.A.: Necessary conditions for optimality on a surface of discontinuity of a certain type in a differential game. Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibernet. **4**, 10–18 (1981) (in Russian)
23. Korneyev, V.A., Melikyan, A.A.: Optimal synthesis in a game problem of pursuit with a small parameter. Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibernet. **3**, 52–58 (1982) (in Russian)
24. Vishnevetskii, L.S., Melikian, A.A.: Optimal pursuit on a plane in the presence of an obstacle. J. Appl. Math. Mech. **46(4)**, 485–490 (1982). Translated from Prikl. Mat. Mekh. **46(4)**, 613–620 (1982) (in Russian)
25. Melikyan, A.A.: Construction of weak discontinuities in optimal control problems and differential games. Engrg. Cybernetics **22(1)**, 129–133 (1984). Translated from Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibernet. **1**, 45–50 (1984) (in Russian)
26. Melikyan, A.A., Ovseevich, A.I.: Hamiltonian systems with a specified invariant manifold and some of their applications. J. Appl. Math. Mech. **48(2)**, 140–145 (1984). Translated from Prikl. Mat. Mekh. **48(2)**, 205–213 (1984) (in Russian)
27. Melikyan, A.A., Sokolov, B.N.: The construction of universal surfaces in extremal problems of dynamics. J. Appl. Math. Mech. **49(2)**, 144–149 (1985). Translated from Prikl. Mat. Mekh. **49(2)**, 191–198 (1985) (in Russian)
28. Korneev, V.A., Melikyan, A.A., Titovskiy, I.N.: Stabilization of aircraft glide path in wind disturbances in the minimax formulation. Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibernet. **3**, 132–139 (1985) (in Russian)
29. Melikyan, A.A., Prosyanyuk, V.V., Sokolov, B.N., Yakovlev, V.Ya.: Optimization of the arrangement of forming protrusions on the surface of a turning roller. Mashinovedenie **4**, 64–70 (1986) (in Russian)
30. Grudev, A.I., Melikyan, A.A.: Modelling of dynamics of a processing section. Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibernet. **3**, 32–40 (1987) (in Russian)
31. Grudev, A.I., Melikyan, A.A.: Simulation modelling of flexible automated systems. Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibernet. **4**, 119–130 (1988) (in Russian)

32. Melikyan, A.A.: Minimax control problem in the case of incomplete information about target point position. Soviet J. Comput. Systems Sci. **27(5)**, 121–127 (1989). Translated from Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibernet. **2**, 111–118 (1989) (in Russian)
33. Melikyan, A.A.: The problem of time-optimal control with search for a target point. J. Appl. Math. Mech. **54(1)**, 1–7 (1990). Translated from Prikl. Mat. Mekh. **54(1)**, 3–11 (1990) (in Russian)
34. Melikyan, A.A.: A positional method for the analysis of singular motions in extremal problems in dynamics. Soviet Math. Dokl. **41(2)**, 205–209 (1990). Translated from Dokl. Akad. Nauk SSSR **311(1)**, 22–26 (1990) (in Russian)
35. Melikyan, A.A., Ovakimyan, N.V.: Singular trajectories in a problem of simple pursuit on a manifold. J. Appl. Math. Mech. **55(1)**, 42–48 (1991). Translated from Prikl. Mat. Mekh. **55(1)**, 54–62 (1991) (in Russian)
36. Melikyan, A.A., Ovakimyan, N.V.: A simple pursuit-and-evasion game on a two-dimensional cone. J. Appl. Math. Mech. **55(5)**, 607–618 (1991). Translated from Prikl. Mat. Mekh. **55(5)**, 741–751 (1991) (in Russian)
37. Melikyan, A.A.: Optimal guaranteeing control of a dynamical system with search for the target point. Soviet Math. Dokl. **43(1)**, 193–197 (1991). Translated from Dokl. Akad. Nauk SSSR **316(4)**, 815–819 (1991) (in Russian)
38. Bocharov, V.Yu., Melikyan, A.A.: Guaranteed speed problem with target point search in a polygonal uncertainty domain. Soviet J. Comput. Systems Sci. **30(1)**, 61–69 (1992). Translated from Izv. Akad. Nauk SSSR, Ser. Tekhn. Kibernet. **1**, 82–92 (1992) (in Russian)
39. Melikyan, A.A., Ovakimyan, N.V.: A differential game of simple approach on manifolds. J. Appl. Math. Mech. **57(1)**, 47–57 (1993). Translated from Prikl. Mat. Mekh. **57(1)**, 41–51 (1993) (in Russian)
40. Melikyan, A.A.: Necessary conditions for optimality of a singular surface in the form of synthesis for a nonlinear controlled system. J. Comput. Systems Sci. Internat. **32(2)**, 134–139 (1994). Translated from Izv. Akad. Nauk, Ser. Tekhn. Kibernet. **1**, 158–163 (1993) (in Russian)
41. Melikyan, A.A.: Necessary optimality conditions for a singular surface in the form of synthesis. J. Optim. Theory Appl. **82(2)**, 203–217 (1994)
42. Melikyan, A.A.: Games of simple pursuit and approach on the manifolds. Dynam. Control **4(4)**, 395–405 (1994)
43. Golovko, A.S., Melikyan, A.A.: The problem of the optimal speed of response when searching for the target point in a circular region of uncertainty. J. Comput. Systems Sci. Internat. **33(5)**, 42–48 (1995). Translated from Izv. Akad. Nauk, Ser. Tekhn. Kibernet. **4**, 40–47 (1994) (in Russian)
44. Korneev, V.A., Melikyan, A.A.: Construction of generalized solution to two-dimensional Hamilton–Jacobi equation by the method of characteristics. Izv. Akad. Nauk, Teor. Sist. Upr. **6**, 168–177 (1995) (in Russian)
45. Melikyan, A.A.: Singular characteristics of first-order partial differential equations. Russian Acad. Sci. Dokl. Math. **54(3)**, 831–834 (1996). Translated from Dokl. Akad. Nauk **351(1)**, 24–28 (1996) (in Russian)
46. Melikyan, A., Pourtallier, O.: Games with several pursuers and one evader with discrete observations. Nova J. Math. Game Theory Algebra **6(1)**, 81–95 (1996)
47. Klepfish, B.R., Melikyan, A.A.: Optimization of the shape of a disk cam in a planar mechanism. Mechanics of Solids **31(5)**, 9–13 (1996). Translated from Izv. Akad. Nauk, Mekh. Tverd. Tela **5**, 13–18 (1996) (in Russian)
48. Melikyan, A.A., Ovakimyan, N.V., Harutiunian, L.L.: Games of simple pursuit and approach on a two-dimensional cone. J. Optim. Theory Appl. **98(3)**, 515–543 (1998)
49. Shmatkov, A.M., Melikyan, A.A., Chernousko, F.L., Borodovsky, M.: Finding prokariotic genes by the "frame-by-frame" algorithm: targeting gene starts and overlapping genes. Bioinformatics **15(11)**, 874–886 (1999)
50. Borodovsky, M.Yu., Melikyan, A.A., Chernousko, F.L., Shmatkov, A.M.: Application of a hidden Markov model and dynamic programming for gene recognition in DNAs of bacteria

and viruses. J. Comput. Systems Sci. Internat. **39(1)**, 155–161 (2000). Translated from Izv. Akad. Nauk, Teor. Sist. Upr. **1**, 163–169 (2000) (in Russian)
51. Bolotnik, N.N., Melikyan, A.A.: Optimal control of a second order dynamic system with impacts. J. Comput. Systems Sci. Internat. **39(6)**, 63–73 (2000). Translated from Izv. Akad. Nauk, Teor. Sist. Upr. **6**, 47–57 (2000) (in Russian)
52. Hovakimyan, N., Melikyan, A.: Geometry of pursuit-evasion on second order rotation surfaces. Dynam. Control **10(3)**, 297–312 (2000)
53. Melikyan, A.A.: Propagation of weak discontinuities of the solutions to the multiple integral variational problem. In: Proc. Steklov Inst. Math., suppl. 2: Problems of Control in Dynamic Systems, S113–S124 (2000). Translated from Trudy Instituta Matematiki i Mekhaniki UrO RAN **6(2)**, 446–459 (2000) (in Russian)
54. Aldakimov, Yu.V., Melikyan, A.A., Naumov, G.V.: Mode reconstruction in a one-parameter family of optimal control problems. J. Appl. Math. Mech. **65(3)**, 387–394 (2001). Translated from Prikl. Mat. Mekh. **65(3)**, 400–407 (2001) (in Russian)
55. Melikyan, A.A.: Equations of motion on the phase constraint surface of an optimal control problem in the form of singular characteristics. Differential Equations **38(11)**, 1581–1588 (2002). Translated from Differents. Uravneniya **38(11)**, 1490–1497 (2002) (in Russian)
56. Melikyan, A.A.: Structure of the value function in pursuit-evasion games on the surfaces of revolution. Cybernet. Systems Anal. **38(3)**, 444–452 (2002). See also Kibernet. Sistem. Anal. **3**, 155–163 (2002)
57. Korneev, V.A., Melikyan, A.A.: Construction of weak discontinuities of the solution of a variational problem in a domain. Izv. Akad. Nauk, Teor. Sist. Upr. **1**, 47–60 (2003) (in Russian)
58. Melikyan, A.: Some properties of the Bellman–Isaacs equation for the games on surfaces of revolution. Int. Game Theory Rev. **6(1)**, 171–179 (2004)
59. Melikyan, A.A.: Primary strategies of simple pursuit in differential games on two-sided plane figures. J. Appl. Math. Mech. **68(4)**, 545–554 (2004). Translated from Prikl. Mat. Mekh. **68(4)**, 611–622 (2004) (in Russian)
60. Melikyan, A., Bernhard, P.: Geometry of optimal paths around focal singular surfaces in differential games. Appl. Math. Optim. **52(1)**, 23–37 (2005)
61. Melikyan, A.A.: Singular characteristics in the boundary conditions of the Hamilton–Jacobi equation. Doklady Mathematics **72(2)**, 723–727 (2005). Translated from Dokl. Akad. Nauk **404(4)**, 451–455 (2005) (in Russian)
62. Korneev, V.A., Melikyan, A.A.: Numerical investigation of weak waves in the solution to the variational wave equation. J. Comput. Systems Sci. Internat. **44(1)**, 58–69 (2005). Translated from Izv. Akad. Nauk, Teor. Sist. Upr. **1**, 63–74 (2005) (in Russian)
63. Zhelnin, Yu.N., Melikyan, A.A., Utemov, A.Ye., Chernousko, F.L.: Bringing a non-linear manoeuvring object to the optimal position in the shortest time. J. Appl. Math. Mech. **69(2)**, 163–172 (2005). Translated from Prikl. Mat. Mekh. **69(2)**, 179–190 (2005) (in Russian)
64. Melikyan, A.A.: Necessary optimality conditions for different phase portraits in a neighborhood of a singular arc. In: Proc. Steklov Inst. Math., Vol. 255, suppl. 2: Control, Stability, and Inverse Problems of Dynamics, S126–S139 (2006). Translated from Trudy Instituta Matematiki i Mekhaniki UrO RAN **12(2)**, 129–141 (2006) (in Russian)
65. Akhmetzhanov, A.R., Melikyan, A.A.: A linear differential game on a plane with a minimum-type functional. J. Appl. Math. Mech. **71(2)**, 234–246 (2007). Translated from Prikl. Mat. Mekh. **71(2)**, 264–276 (2007) (in Russian)
66. Melikyan, A., Akhmetzhanov, A., Hovakimyan, N.: On initial value and terminal value problems for Hamilton–Jacobi equation. Systems Control Lett. **56(11–12)**, 714–721 (2007)
67. Akhmetzhanov, A.R., Melikyan, A.A., Fradkov, A.L.: The optimality of the velocity-gradient method in the problem of controlling the escape from a potential well. J. Appl. Math. Mech. **71(6)**, 809–818 (2007). Translated from Prikl. Mat. Mekh. **71(6)**, 899–909 (2007) (in Russian)

68. Day, M.V., Melikyan, A.A.: Simple singularities for Hamilton–Jacobi equations with max-concave Hamiltonians and generalized characteristics. J. Optim. Theory Appl. **138(2)**, 155–174 (2008)
69. Melikyan, A., Olsder, G.J., Akhmetzhanov, A.: Parameter analysis of a zero-sum differential game approximating a nonzero-sum game. Int. Game Theory Rev. **10(4)**, 437–459 (2008)
70. Melikyan, A., Olsder, G.J.: Boundary singularities and characteristics of Hamilton–Jacobi equation. Journal of Dynamical and Control Systems **16(1)**, 77–99 (2010)
71. Melikyan, A.A.: Singular boundary characteristics of Hamilton–Jacobi equation. J. Appl. Math. Mech. **74(2)**, 144–153 (2010). Translated from Prikl. Mat. Mekh. **74(2)**, 202–215 (2010) (in Russian)
72. Stipanović, D.M., Melikyan, A., Hovakimyan, N.: Guaranteed strategies for nonlinear multi-player pursuit-evasion games. Int. Game Theory Rev. **12(1)**, 1–17 (2010)

Papers in Peer-Reviewed Collections

73. Chernousko, F.L., Melikyan, A.A.: Some differential games with incomplete information. In: Marchuk, G.I. (ed.) Optimization Techniques (IFIP Technical Conference, July 1-7, 1974, Novosibirsk, USSR), Lect. Notes in Computer Sci., Vol. 27, pp. 445–450. Springer-Verlag, Berlin (1974)
74. Melikyan, A.A.: Invariant manifolds of Hamiltonian systems in the theory of singularities of the Bellman–Isaacs equation. In: Matrosov, V.M., Ertegov, V.D. (eds.) Ustoichivost' dvizheniya (Stability of motion), pp. 227–230. Nauka, Sibirsk. Otdel., Novosibirsk (1985) (in Russian)
75. Melikyan, A.A.: The problem of time-optimal control with the search of the goal point. In: Sebastian, H.-J., Tammer, K. (eds.) System Modelling and Optimization, Lect. Notes in Control and Inform. Sci., Vol. 143, pp. 348–357. Springer, Berlin (1990)
76. Melikyan, A.A.: The method of characteristics for constructing singular paths and manifolds in optimal control and differential games. In: Hamalainen, R.P., Ehtamo H.K. (eds.) Differential Games – Developments in Modelling and Computation, Lect. Notes in Control and Inform. Sci., Vol. 156, pp. 81–90. Springer, Berlin (1991)
77. Melikyan, A.A.: Time-optimal control problem with goal point uncertainty. In: Kurzhanski, A.B., Veliov, V.M. (eds.) Modeling Techniques for Uncertain Systems, Progr. Systems Control Theory, Vol. 18, pp. 205–215. Birkhäuser, Boston (1994)
78. Melikyan, A.A.: Singular paths in differential games with simple motion. In: Basar, T., Haurie, A. (eds.) Advances in Dynamic Games and Applications, Ann. Internat. Soc. Dynam. Games, Vol. 1, pp. 125–135. Birkhäuser, Boston (1994)
79. Melikyan, A.A., Pourtallier, O.: Games with several pursuers and one evader with discrete observations. In: Petrosyan, L.A., Mazalov, V.V. (eds.) Game Theory and Applications, Vol. 2, pp. 169–184. Nova Sci. Publ., New York (1996)
80. Melikyan, A.A., Shinar, J.: Identification and construction of singular surfaces in pursuit-evasion games. In: Filar, J.A., Gaitsgory, V., Mizukami, K. (eds.) Advances in Dynamic Games and Applications, Ann. Internat. Soc. Dynam. Games, Vol. 5, pp. 151–175. Birkhäuser, Boston (2000)
81. Melikyan, A.A.: Singular characteristics of first-order PDEs in optimal control and differential games. J. Mat. Sci. **103(6)**, 745–755 (2001). Translated from Itogi Nauki i Tekhniki, Ser. Sovrem. Mat. Pril., Temat. Obz., Vol. 64, pp. 179–196. VINITI, Moscow (1999) (in Russian)
82. Melikyan, A.A.: Geometry of pursuit-evasion games on two-dimensional manifolds. In: Jorgensen, S., Quincampoix, M., Vincent, T.L. (eds.) Advances in Dynamic Games Theory: Numerical Methods, Algorithms, and Applications to Ecology and Economics, Ann. Internat. Soc. Dynam. Games, Vol. 9, pp. 173–194. Birkhäuser, Boston (2007)
83. Stipanović, D.M., Melikyan, A., Hovakimyan, N.: Some sufficient conditions for multi-player pursuit-evasion games with continuous and discrete observations. In: Bernhard, P., Gaitsgory, V., Pourtallier, O. (eds.) Advances in Dynamic Games and Their Applications. Analitical and Numerical Developments, Ann. Internat. Soc. Dynam. Games, Vol. 10, pp. 133–145. Birkhäuser, Boston (2009)

Scientific Reports

84. Melikyan, A.A., Ovakimyan, N.V.: Game problem of simple pursuit on the manifolds. Preprint no. 489, Institute for Problems in Mechanics, USSR Academy of Sciences, Moscow, 1991. 44 pp. (in Russian)
85. Melikyan, A.A., Ovakimyan, N.V.: Differential games of simple pursuit and approach on manifolds. Preprint, Institute of Mechanics, Armenian Academy of Sciences, Yerevan, 1993. 44 pp.
86. Melikyan, A.A., Botkin, N.D., Turova, V.L.: Propagation of disturbances in inhomogeneous anisotropic media. Preprint no. 181, Stiftung Caesar, Bonn, 2004. 25 pp.

Preprints and Proceedings of Conferences

87. Melikyan, A.: Singular paths in differential games with simple motion. In: Preprint Volume of the 5th International Symposium on Dynamic Games and Applications, 10 pp. Université de Genève (1992)
88. Melikyan, A.A., Shinar, J.: Identification and construction of singular surfaces in pursuit-evasion games. In: Breton, M., Zaccour, G. (eds.) Preprint Volume of the 6th International Symposium on Dynamic Games and Applications, pp. 517–529. GERAD, Ecole des H.E.C. (1994)
89. Melikyan, A.A.: What is Kelley condition from the point of view of synthesis. In: Shinar, J., Kirillova, F.M. (eds.) A Postprint Volume of the 10th IFAC Workshop "Control Applications of Optimization", pp. 47–50. Pergamon Press, London (1996)
90. Melikyan, A.A.: Equations for singular focal paths in dynamic games. In: Filar, J.A., Gaitsgory, V., Imado, F. (eds.) Proceedings of the 7th International Symposium on Dynamic Games and Applications, Vol. 2, pp. 690–695. Kanagawa (1996)
91. Melikyan, A.A., Pourtallier, O.: Games with several pursuers and one evader with discrete observations. In: Filar, J.A., Gaitsgory, V., Imado, F. (eds.) Proceedings of the 7th International Symposium on Dynamic Games and Applications, Vol. 2, pp. 672–689. Kanagawa (1996)
92. Melikyan, A.A.: Singular characteristics in optimal control and differential games. In: Huijberts, H.J.C., Nijmeijer, H., van der Schaft, A.J., Scherpen, J.M.A. (eds.) Proceedings of the 4th IFAC Symposium on Nonlinear Control Systems Design, Vol. 1, pp. 100–102. Elsevier-Pergamon, Oxford (1998)
93. Melikyan, A.A.: Games of pursuit on two-sided plane with a hole. In: Preprint Volume of the 8th International Symposium on Dynamic Games and Applications, pp. 394–396. Maastricht (1998)
94. Melikyan, A.A., Korneev, V.A.: Construction algorithm for non-smooth game value using singular characteristics. In: Preprint Volume of the 8th International Symposium on Dynamic Games and Applications, pp. 397–401. Maastricht (1998)
95. Melikyan, A.A.: Singular characteristics in dynamic programming. In: Proceedings of American Control Conference, Vol. 4, pp. 2857–2862. San Diego (1999)
96. Bolotnik, N.N., Melikyan, A.A.: Time-optimal control of a vibroimpact system with one degree of freedom. In: Chernousko, F.L., Fradkov, A.L. (eds.) Proceedings 2nd Intern. Conf. "Control of Oscillations and Chaos", Vol. 2, pp. 249–252. St. Petersburg (2000)
97. Hovakimyan, N., Melikyan, A.: Bifurcations in pursuit-evasion games on one-parameter family of surfaces. In: Nowak, A.S., Szajowski, K. (eds.) Proceedings of the Ninth International Symposium on Dynamic Games and Applications, Vol. 1, pp. 247–262. University of South Australia, Adelaide (2000)
98. Melikyan, A.A., Naumov, G.V.: Phase portrait bifurcation in one-parameter family of optimal control problems. In: Preprints of the 5th IFAC Symposium "Nonlinear Control Systems", pp. 1286–1291. St. Petersburg (2001)
99. Bolotnik, N.N., Melikyan, A.A.: Optimal control of a mechanical system with impacts. In: Preprints of the 5th IFAC Symposium "Nonlinear Control Systems", pp. 1590–1594. St. Petersburg (2001)

100. Melikyan, A.A., Naumov, G.V.: Phase portrait bifurcation in one-parameter family of optimal control problems. In: Kurzhanski, A.B., Fradkov, A.L. (eds.) Proceedings of the 5th IFAC Symposium "Nonlinear Control Systems", Vol. 3, pp. 1205–1210. Elsevier, Oxford (2002)
101. Bolotnik, N.N., Melikyan, A.A.: Optimal control of a mechanical system with impacts. In: Kurzhanski, A.B., Fradkov, A.L. (eds.) Proceedings of the 5th IFAC Symposium "Nonlinear Control Systems", Vol. 3, pp. 1505–1510. Elsevier, Oxford (2002)
102. Melikyan, A.A.: Construction of singular surfaces in multiple integral variational problem. In: Camacho, E.F., Basanez, L., de la Puente, J.A. (eds.) Proceedings of the 15th Triennial World Congress of the IFAC, Vol. C: Stability and Nonlinear Systems, pp. 465–470. Barcelona (2002)
103. Melikyan, A.A.: Pursuit-evasion games on the surfaces of revolution. In: Petrosjan, L.A., Zenkevich, N.A. (eds.) Proceedings of the Tenth International Symposium on Dynamic Games and Applications, Vol. 2: International Society of Dynamic Games, pp. 617–622. Saint-Petersburg State University, Inst. Chem., St. Petersburg (2002)
104. Melikyan, A., Hovakimyan, N.: Some variation calculus problems in dynamic games on 2D surfaces. In: Proc. Vol. Intern. Congr. of Math., Satellite Conf. on Game Theory and Appl., pp. 513–522. Quingdao Publ. House, Quingdao (2002)
105. Melikyan, A., Hovakimyan, N.: Some variational calculus problems in dynamic games on 2D surfaces. In: Petrosyan, L., Yeung, D. (eds.) Proceedings ICM Millennium Lectures on Games, pp. 287–296. Birkhäuser, Berlin, Heidelberg, New York (2003)
106. Melikyan, A., Hovakimyan, N., Ikeda, Y.: Dynamic programming approach to a minimum distance optimal control problem. In: Proceedings of the 42nd IEEE Conference on Decision and Control, pp. 239–244. Maui, Hawaii (2003)
107. Melikyan, A.: Geometry of pursuit-evasion games on 2D manifolds. In: Vincent, T.L. (ed.) Proceedings of the 11th International Symposium on Dynamic Games and Applications, Vol. 2, pp. 650–669. University of Arizona, Tucson (2004)
108. Melikyan, A., Akhmetzhanov, A.: A linear 2D differential game with a minimum-type cost function. In: Vincent, T.L. (ed.) Proceedings of the 11th International Symposium on Dynamic Games and Applications, Vol. 2, pp. 637–649. University of Arizona, Tucson (2004)
109. Melikyan, A., Akhmetzhanov, A., Hovakimyan, N.: A remark on IVP and TVP non-smooth viscosity solutions to Hamilton–Jacobi equations. In: Proceedings of American Control Conference, Vol. 2, pp. 864–869. Portland (2005)
110. Melikyan, A., Botkin, N., Turova, V.: Modelling of disturbance propagation in anisotropic media using differential game methods. In: Horacek, P., Simandl, M., Zitek, P. (eds.) Preprints of the 16 IFAC World Congress, 6 pp. Prague (2005)
111. Melikyan, A.A., Botkin, N.D., Turova, V.L.: Singularity phenomena in ray propagation in anisotropic media. In: Fradkov, A.L., Churilov, A.N. (eds.) Proceedings of the International Conference "Physics and Control", pp. 373–378. St. Petersburg (2005)
112. Stipanović, D.M., Melikyan, A., Hovakimyan, N.: Some sufficient conditions for multiplayer pursuit-evasion games with continuous and discrete observations. In: Proceedings of the 12th International Symposium on Dynamic Games and Applications, 11 pp. Sophia Antipolis (2006)
113. Melikyan, A., Hovakimyan, N., Akhmetzhanov, A.: Application of singular characteristics for image processing. In: Proceedings of the 12th International Symposium on Dynamic Games and Applications, 7 pp. Sophia Antipolis (2006)
114. Melikyan, A.A., Olsder, G.J.: Feedback solution of a non-zero-sum game via reduction to a zero-sum game. In: Proceedings of the 12th International Symposium on Dynamic Games and Applications, 16 pp. Sophia Antipolis (2006)
115. Melikyan, A.A.: Some applications of boundary characteristics of Hamilton–Jacobi equation. In: Proceedings of the IX International Chetayev Conference "Analytical Mechanics, Stability and Control of Motion", Vol. 1, pp. 114–118. Institute for System Dynamics and Control Theory of Siberian Branch of the Russian Academy of Sciences, Irkutsk (2007)

116. Stipanović, D.M., Melikyan, A., Hovakimyan, N.: Guaranteed strategies for nonholonomic players in pursuit-evasion games. In: Melikyan, A.A., Nowak, A.S., Szajowski, K.J. (eds.) 13th International Symposium on Dynamic Games and Applications, 3 pp. Wroclaw (2008)
117. Melikyan, A., Olsder, G.J., Akhmetzhanov, A.: A sufficient condition for the reduction of a nonzero-sum game to a zero-sum one. In: Melikyan, A.A., Nowak, A.S., Szajowski, K.J. (eds.) 13th International Symposium on Dynamic Games and Applications, 14 pp. Wroclaw (2008)

The Long History of Arik's Conjecture

Pierre Bernhard

Long ago, when both Arik Melikyan and I were young – we are of the same age – at an international conference, I was giving a talk on my constructive theories for singular surfaces of two-person zero-sum differential games. I had constructive equations for all the classical menagerie: dispersal lines, universal lines, equivocal lines, switch envelopes... but one piece was missing: I could only account for one-dimensional focal lines in two-dimensional D.G.'s. For higher-dimensional focal manifolds, I could only offer a conjecture, which, if true, would let me propose a constructive theory.

Soon after my talk, Arik came up to me, telling me that he did not believe my conjecture. Contrary to my account, he conjectured that such hyper-surfaces should be traversed by *two non-colinear fields of trajectories*. If he was right, as he soon convinced me, we were left with an intriguing phenomenon, but still no constructive theory. And we lacked an example.

We continued discussing this issue by mail and each time we met for years. Some years later, he had examples of singularities similar to his conjecture in mathematical physics. But we still had no example in D.G.'s, and no certified way of constructing one. We decided to try and reverse engineer a D.G. from one of his examples. But this failed.

In January 2002, we met at the Lyapunov Institute in Moscow. I had in hand a complicated and uncertain example of what might have been an instance of Arik's conjecture. I remember that I had written him a couple of weeks before, telling him that I had a pair of coupled, first-order PDEs, and that his first answer had been, "But nobody does that!" I showed him my proposed example. He quickly agreed that it was indeed an instance of what we had been after for all those years.

Following this, he was able to get a scholarship from the Lyapunov Institute to come and spend some time in Sophia Antipolis, where we would work together. There, he quickly understood why I had these two PDEs, and made a general

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theory of higher-dimensional focal hyper-surfaces, including a constructive theory involving a pair of first-order PDEs. This theory was his, even though it was published in a joint paper [1], because the original example was mine.

Let me add that the original exposition of this example – in mathematical finance – which can be found in a paper at the 2002 ISDG conference [2], was really difficult to understand. With Arik's theory, in our joint paper, it takes less than one page. Subsequently my Ph.D. student Stéphane Thiery showed that by using Arik's conjecture – now, theorem – he could derive the focal surface in that example in a few lines.

Over time, we became good friends, as Arik was such a kind person. I spontaneously turned to him for any question about characteristic theory of Isaacs' equation and many other topics in differential games. We also started a collaboration with his Ph.D. student, Andrei Akhmetzhanov. I was seeking, through Andrei, Arik's opinion on our ongoing work on singular surfaces in Nash equilibria of nonzero-sum games when, on the morning of April 7, I learned from Valerii Patsko of his sudden death. This was a shock and left me, like all his many friends, quite saddened.

References

1. Arik Melikyan and Pierre Bernhard. Geometry of optimal trajectories around a focal singular surface in differential games. *Applied Mathematics and Optimization*, 52(1):23–37, 2005.
2. Pierre Bernhard, Naïma El Farouq, and Stéphane Thiery. An impulsive differential game arising in finance with interesting singularities. In Alain Haurie, Shigeo Muto, Leon A. Petrosjan, and T.E.S. Raghavan, editors, *10th ISDG International Symposium on Dynamic Games and Applications, Saint Petersburg, 2002*, volume 8 of *Annals of the ISDG: Advances in Dynamic Games*, pages 335–363. 2006.

Optimal Control Problem for a Dynamical System on an Ellipsoid*

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Abstract A point mass is considered whose motion is controlled in three-dimensional space by a force of a bounded absolute value. The presence of gravity is assumed. The task of the control is to bring the point from a given initial position to a terminal position with minimal time, while the trajectory is required to be on the surface of the given ellipsoid. Different terminal conditions are considered. The problem is investigated using a state-constraint maximum principle.

1 Introduction

Differential games and optimal control problems for dynamical systems on the surfaces and manifolds are considered in [1–4].

2 Dynamics of a Point Mass on an Ellipsoid

The dynamics are given by the differential equations and control constraints:

$$\ddot{x} = u_1, \quad \ddot{y} = u_2, \quad \ddot{z} = u_3 + g \tag{1}$$
$$u_1^2 + u_2^2 + u_3^2 \le \mu^2, \quad \mu = \frac{F_*}{m}.$$

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Here, m is the mass of the considered point, F_* is the maximal absolute value of the control force, u_i are normalized force components (control parameters), μ is the normalized absolute value of the control vector, and g is the gravity constant. The point mass is restricted to move on the surface of the ellipsoid:

$$R(x, y, z) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0. \tag{2}$$

Differentiating twice the function $R(x, y, z)$ along the solutions of the system (1) (until the control parameters explicitly arise, see [2]), one can get the following three necessary conditions:

$$R(x, y, z) = 0, \quad \dot{R} = \langle n, v \rangle = k_1 x\dot{x} + k_2 y\dot{y} + k_3 z\dot{z} = 0 \tag{3}$$

$$\ddot{R} = k_1\dot{x}^2 + k_2\dot{y}^2 + k_3\dot{z}^2 + k_1 x u_1 + k_2 y u_2 + k_3 z(u_3 + g) = 0.$$

Here v is the velocity vector, and $n = (n_1, n_2, n_3)$ is a normal vector to the surface, the gradient of the function R:

$$v = (\dot{x}, \dot{y}, \dot{z}), \quad n = \nabla R = (k_1 x, k_2 y, k_3 z) \quad \left(k_1 = \frac{2}{a^2}, k_2 = \frac{2}{b^2}, k_3 = \frac{2}{c^2}\right).$$

Generally, the dynamical system (1) has six phase coordinates $(x, \dot{x}, y, \dot{y}, z, \dot{z})$ and three control variables, $u = (u_1, u_2, u_3)$. Using the first two equations in (3), one can reduce the number of coordinates to four, and using the last equation one can get a two-dimensional control vector. If, in addition, the control force of maximal absolute value is used (which is often recommended by the maximum principle), i.e., $| u | = \mu$, then one more control parameter is eliminated, and one gets a scalar control. Thus, the dynamic system on an ellipsoid can be characterized by four coordinates and one control parameter.

3 Dynamics of a Point Mass on a Solid Ellipsoid

When a point mass moves along a solid (stiff) surface then, generally, three forces act on a point: the control force, the external force (gravity), and the reaction force of the surface. In case of frictionless motion, the reaction force is orthogonal to the surface, and the dynamic equations take the form:

$$\ddot{x} = N_1 + u_1, \quad \ddot{y} = N_2 + u_2, \quad \ddot{z} = N_3 + u_3 + g, \tag{4}$$

where $N = \lambda n$. The control force is assumed to be tangent to the surface, which gives the restriction:

$$\langle u, n \rangle = u_1 n_1 + u_2 n_2 + u_3 n_3 = 0. \tag{5}$$

The parameters N_i can be excluded by writing the Lagrange equation of the second kind for these systems with two degrees of freedom. The resulting dynamic equations include four phase variables and two control parameters. For motion with maximal force, one has only one control variable.

4 Isaacs' Model of a Car

The control force here is orthogonal to the point's velocity:

$$\langle u, v\rangle = u_1\dot{x} + u_2\dot{y} + u_3\dot{z} = 0 \tag{6}$$

so that the vector v does not change its length during the motion. Here, the dynamic equation can be reduced to four phase variables and one control variable. A motion with maximal value of the force does not contain any control freedom. For the plane model of a car, when $R = 0$ determines a plane and there is no gravity, it is known that time-optimal trajectories consist of segments of straight lines (the force vanishes and the point performs a free motion) and parts of circles (the force has maximal absolute value). The straight lines are the shortest lines (geodesics) in a plane. This property also takes place for the motion on a general surface, as shown in the next section. However, the motion with maximal force, generally, is a spiral on a surface rather than a closed trajectory.

5 Free Motion on a Surface (Ellipsoid)

Consider the motion of a mass point on a (stiff) surface in the absence of friction, gravity, and other external forces. The dynamic equations have the form:

$$\ddot{x} = N_1, \quad \ddot{y} = N_2, \quad \ddot{z} = N_3, \tag{7}$$

where $N = (N_1, N_2, N_3)$ is the force acting from the surface to the mass point. In the absence of friction, it is a normal force, so that

$$N = \lambda n$$

for some scalar, and the equations take the form:

$$\ddot{x} = \lambda n_1, \ \ddot{y} = \lambda n_2, \ \ddot{z} = \lambda n_3. \tag{8}$$

One can show that these equations of free motion have the solutions which are the geodesic lines of the surface – the shortest lines connecting two points of the

surface. Geodesic lines on the surface are the solutions of the following variational problem, with one restriction:

$$J = \int_{\theta_0}^{\theta_1} \sqrt{(\mathrm{d}x/\mathrm{d}\theta)^2 + (\mathrm{d}y/\mathrm{d}\theta)^2 + (\mathrm{d}z/\mathrm{d}\theta)^2}\mathrm{d}\theta \to \min \tag{9}$$

$$R(x, y, z) = 0,$$

where $(x(\theta), y(\theta), z(\theta)), \theta_0 \leq \theta \leq \theta_1$, is a parametrization of the curve searched for. The classical variation a calculus says that the solution of the above problem is the solution of an unconditioned problem with the integrand:

$$F(x, y, z, \acute{x}, \acute{y}, \acute{z}) = \sqrt{(\mathrm{d}x/\mathrm{d}\theta)^2 + (\mathrm{d}y/\mathrm{d}\theta)^2 + (\mathrm{d}z/\mathrm{d}\theta)^2} + \bar{\mu}R(x, y, z),$$

with some scalar $\bar{\mu} = \bar{\mu}(x, y, z)$. Here the prime sign means differentiation with respect to θ. The Euler system of equations

$$F_x - \frac{\mathrm{d}}{\mathrm{d}\theta}F_{\acute{x}} = 0, \;\; F_y - \frac{\mathrm{d}}{\mathrm{d}\theta}F_{\acute{y}} = 0, \;\; F_z - \frac{\mathrm{d}}{\mathrm{d}\theta}F_{\acute{z}} = 0$$

for such a problem for the surface $R(x, y, z) = 0$ takes the form

$$x'' = \bar{\mu}n_1, \;\; y'' = \bar{\mu}n_2, \;\; z'' = \bar{\mu}n_3. \tag{10}$$

One can see that the solutions of (8) and (10) coincide, the only difference being the parametrization of the curves. In the case of an ellipsoid, these curves are appropriate ellipses.

References

1. Isaacs, R. Differential Games. *Wiley, New York*, 1965.
2. Bryson, A.E., Ho, Yu-Chi "Larry". Applied Optimal Control. *Washington, DC: Hemisphere*, 1975.
3. Imado, F. Optimal Control Problem on an Ellipsoid, JSPS report, 2006.
4. Melikyan A.A. Generalized Characteristics of First Order PDEs: Applications in Optimal Control and Differential Games. *Birkhauser, Boston*, 1998.