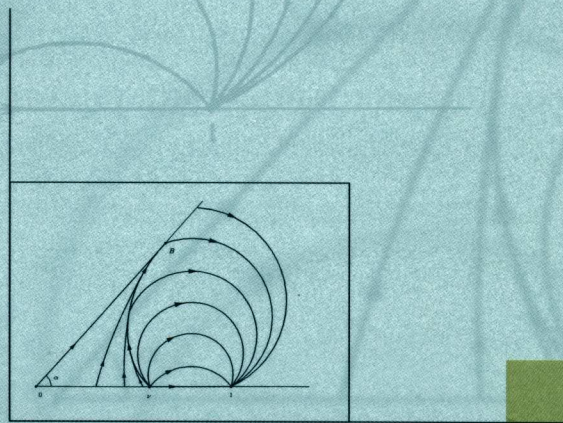


# Generalized Characteristics of First Order PDEs

Applications in Optimal  
Control & Differential Games



Arik Melikyan

To the memory  
of my parents  
Shoushanik and Artavazd

A.A. Melikyan

**Generalized Characteristics  
of First Order PDEs**

Applications in Optimal Control  
and Differential Games

1998

**Birkhäuser**

Boston • Basel • Berlin

Arik Melikyan  
Russian Academy of Science  
Institute for Problems in Mechanics  
Prospekt vernadskogo 101  
Moscow

**Library of Congress Cataloging-in-Publication Data**

Melikyan, A. A. (Arik A.), 1944-  
Generalized characteristics of first order PDEs : applications in  
optimal control and differential games / A.A. Melikyan.  
p. cm.  
Includes bibliographical references and index.  
ISBN 0-8176-3984-5  
1. Differential equations, Partial. 2. Control theory.  
3. Differential games. I. Title.  
QA374.M45 1998  
515'.353--dc21 98-4739  
CIP

Printed on acid-free paper  
© 1998 Birkhäuser Boston

*Birkhäuser* 

Copyright is not claimed for works of U.S. Government employees.  
All rights reserved. No part of this publication may be reproduced, stored in a retrieval system,  
or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording,  
or otherwise, without prior permission of the copyright owner.

Permission to photocopy for internal or personal use of specific clients is granted by  
Birkhäuser Boston for libraries and other users registered with the Copyright Clearance Center  
(CCC), provided that the base fee of \$6.00 per copy, plus \$0.20 per page is paid directly to  
CCC, 222 Rosewood Drive, Danvers, MA 01923, U.S.A. Special requests should be  
addressed directly to Birkhäuser Boston, 675 Massachusetts Avenue, Cambridge, MA 02139,  
U.S.A.

ISBN 0-8176-3984-5  
ISBN 3-7643-3984-5  
Typeset by the Author in L<sup>A</sup>T<sub>E</sub>X.  
Printed and bound by Hamilton Printing, Rensselaer, NY.  
Printed in the U.S.A.

9 8 7 6 5 4 3 2 1

# Contents

<b>Preface</b>	<b>xi</b>
<b>Introduction</b>	<b>1</b>
<b>1 Method of Characteristics in Smooth Problems</b>	<b>7</b>
1.1 Classical Cauchy problem for first order PDE . . . . .	7
1.1.1 Problem statement . . . . .	7
1.1.2 Characteristic equations . . . . .	8
1.1.3 Construction of the initial conditions . . . . .	9
1.1.4 A note on the approach in small . . . . .	11
1.1.5 Construction of twice differentiable solution . . . . .	11
1.1.6 Irregular characteristic problem . . . . .	13
1.1.7 A 2D-example; problem formulation . . . . .	17
1.1.8 Construction of the solution . . . . .	18
1.2 Cauchy problem for integral surfaces . . . . .	19
1.2.1 Geometrical formulation of the Problem 1.1 . . . . .	19
1.2.2 Generalized Cauchy problem . . . . .	21
1.2.3 Characteristic field on a manifold . . . . .	24
1.2.4 Construction of the reference solution . . . . .	27
1.2.5 Explicit expressions for $\lambda$ for small $m$ . . . . .	28
1.2.6 Sufficient conditions for the Problems 1.3, 1.4 . . . . .	29
1.2.7 The geometry of the characteristic field . . . . .	31
1.2.8 Characteristic points of the manifold $W$ . . . . .	32
1.2.9 Some particular characteristic systems for $m = 1$ . . . . .	34
1.3 Cauchy problem with movable boundary . . . . .	35
1.3.1 Regular problem with movable boundary . . . . .	35
1.3.2 Irregular problem . . . . .	38
1.3.3 Jacobi brackets of different levels . . . . .	38
1.3.4 A sufficiency condition . . . . .	39
1.3.5 Classical irregular non-characteristic problem . . . . .	41
1.3.6 Illustrative example . . . . .	52
Exercises . . . . .	54
<b>2 Generalized Solutions and Singular Characteristics of First Order PDEs</b>	<b>55</b>
2.1 Viscosity solutions and their singular manifolds . . . . .	55
2.1.1 Definition of viscosity solution . . . . .	55



2.1.2	Regular and singular points of a solution; simplest singularity . . . . .	57
2.1.3	Necessary conditions for a simplest singularity . . .	60
2.1.4	Singular characteristics, definition and classification	62
2.1.5	Some properties of IVP and TVP . . . . .	64
2.2	Dispersal surface . . . . .	64
2.2.1	General conditions . . . . .	64
2.2.2	Linear and nonlinear Hamiltonians . . . . .	66
2.3	Singular characteristics for equivocal surface . . . . .	68
2.3.1	Four types of surfaces, necessary conditions . . . . .	68
2.3.2	Equations of singular characteristics . . . . .	71
2.3.3	Some properties of characteristic system . . . . .	72
2.4	Singular characteristics for focal surface . . . . .	74
2.4.1	Six types of surfaces, necessary conditions . . . . .	74
2.4.2	Focal surface — hyperplane . . . . .	77
2.4.3	Non-symmetric surface, collinear fields . . . . .	80
2.4.4	Degenerate surfaces . . . . .	84
2.4.5	Initial conditions and identification of singular surfaces	84
2.4.6	Modifications for TVP . . . . .	85
2.5	An IVP example . . . . .	86
2.5.1	Problem formulation . . . . .	86
2.5.2	The case 1), $a < b$ . . . . .	88
2.5.3	The case 2), $a = b$ . . . . .	90
2.5.4	The case 3), $a > b$ . . . . .	91
2.5.5	Some modifications for non-symmetric case . . . . .	93
2.5.6	Concluding remarks . . . . .	96
	Exercises . . . . .	97
<b>3</b>	<b>First Order PDEs in Variation Calculus, Optimal Control and Differential Games</b>	<b>99</b>
3.1	Hamilton-Jacobi equation in Variation Calculus . . . . .	99
3.1.1	First variation formula . . . . .	99
3.1.2	The case of non-homogeneous Lagrangian . . . . .	102
3.1.3	Variational problem on geodesic line . . . . .	103
3.1.4	General homogeneous Lagrangian . . . . .	105
3.2	Bellman equation in Optimal Control . . . . .	106
3.2.1	Fixed time problem . . . . .	106
3.2.2	Time-optimal problem . . . . .	109
3.2.3	Feedback controls . . . . .	111
3.3	The Isaacs equation in Differential Games . . . . .	112
3.3.1	Fixed time game. Value function . . . . .	112
3.3.2	Pursuit-evasion games . . . . .	115
3.4	Generalized solutions of the HJBI equation . . . . .	116
3.4.1	Classical and viscosity solutions . . . . .	116
3.4.2	Generalized main equation, A.I.Subbotin's inequalities	118

3.5	Singular paths and singular characteristics . . . . .	120
3.5.1	Singular surfaces and paths: definition and classification . . . . .	120
3.5.2	Theory of the equivocal surface . . . . .	123
3.5.3	Singular paths and characteristics . . . . .	126
3.6	A linear pursuit-evasion game with elliptical vectograms . . . . .	127
3.6.1	Problem formulation . . . . .	127
3.6.2	Dispersal surface . . . . .	130
3.6.3	Focal surface . . . . .	131
3.6.4	Boundary of the indifferent zone . . . . .	132
	Exercises . . . . .	135
<b>4</b>	<b>Differential Games with Simple Motions on the Manifolds</b>	<b>137</b>
4.1	Problem statement . . . . .	137
4.1.1	Games with simple motion . . . . .	137
4.1.2	Dynamic equations . . . . .	138
4.1.3	Cost functions for two games . . . . .	139
4.2	Primary solution . . . . .	140
4.2.1	Properties of the geodesic length . . . . .	140
4.2.2	Primary and secondary domains . . . . .	141
4.3	Necessary optimality conditions . . . . .	142
4.3.1	Generalized main equation, regular paths . . . . .	142
4.3.2	Singular surface in primary domain . . . . .	145
4.3.3	Analysis of the surface $\Gamma_0$ using viscosity conditions . . . . .	148
4.4	Two branches of the equivocal surface . . . . .	153
4.4.1	Identification of the equivocal surfaces . . . . .	153
4.4.2	Main result . . . . .	155
4.4.3	Construction algorithm . . . . .	158
4.5	Game of pursuit in the presence of an obstacle . . . . .	159
4.5.1	Problem formulation . . . . .	159
4.5.2	Planar problem . . . . .	162
4.5.3	Examples . . . . .	164
	Exercises . . . . .	168
<b>5</b>	<b>Games of Simple Pursuit and Approach on Two-Dimensional Cone</b>	<b>169</b>
5.1	Game formulation in different coordinate systems . . . . .	169
5.1.1	Dynamics in Cartesian and relative variables . . . . .	169
5.1.2	Self-similar variables, complex coordinates . . . . .	173
5.1.3	Primary solutions . . . . .	175
5.2	Analysis of the primary domain . . . . .	176
5.2.1	Necessary optimality conditions . . . . .	176
5.2.2	Construction of the set $B$ , parametric analysis . . . . .	178
5.2.3	Construction of the equivocal surface . . . . .	182
5.3	Analysis of the secondary domain . . . . .	184

5.3.1	Game of pursuit . . . . .	184
5.3.2	The critical cone, $\nu = 1 - \sin \alpha$ . . . . .	188
5.3.3	Game of approach . . . . .	190
5.3.4	The case $\nu = 1$ . . . . .	194
5.3.5	On the algorithm of synthesis and computer simulation. . . . .	195
	Exercises . . . . .	198
<b>6</b>	<b>Smooth Solutions of a PDE with Nonsmooth Hamiltonian</b>	<b>199</b>
6.1	Open-loop and feedback analysis of singular paths in Optimal Control . . . . .	199
6.1.1	Introduction . . . . .	199
6.1.2	Singular arc in Optimal Control problem, open-loop approach . . . . .	201
6.1.3	Linear problem . . . . .	201
6.1.4	Two sets of variables . . . . .	203
6.1.5	Necessary conditions in invariant form . . . . .	204
6.1.6	Singular universal surface in general problem . . . . .	205
6.2	First order PDEs with nonsmooth Hamiltonian . . . . .	208
6.2.1	Necessary conditions for singular hyperplane . . . . .	208
6.2.2	An auxiliary theorem . . . . .	210
6.2.3	Necessary conditions in invariant form . . . . .	212
6.2.4	Singular characteristics for the universal surface . . . . .	213
6.2.5	Applications to the control problem . . . . .	214
6.2.6	Example . . . . .	217
6.3	Second order singularity . . . . .	220
6.3.1	Two optimal phase portraits; Kopp-Moyer condition . . . . .	220
6.3.2	Invariant form of the second order conditions . . . . .	222
6.3.3	Singular characteristics for the synthesis $S2$ . . . . .	223
	Exercises . . . . .	225
<b>7</b>	<b>Shock Waves Related to First Order PDEs</b>	<b>227</b>
7.1	Singular characteristics in two-dimensional problems . . . . .	227
7.1.1	Two-dimensional problem . . . . .	227
7.1.2	Equations for a focal line . . . . .	228
7.1.3	Equations for equivocal line . . . . .	231
7.1.4	Singular characteristics of two-dimensional Hamilton-Jacobi equation . . . . .	231
7.2	Shock waves generated by the boundary conditions . . . . .	234
7.2.1	Initial conditions . . . . .	234
7.2.2	Convexification of the function $g(p)$ . . . . .	236
7.2.3	Analysis of the second derivative . . . . .	241
7.3	Main results on the number of waves . . . . .	243
7.3.1	Simplified expressions for Jacobi brackets . . . . .	243
7.3.2	The case of simple segments . . . . .	245
7.3.3	Secondary waves . . . . .	250



7.3.4	A result concerning non-simple segment . . . . .	250
7.3.5	S.N.Kruzhkov's theorem . . . . .	252
7.3.6	Example . . . . .	253
7.3.7	Some generalizations for multidimensional case . . . . .	255
7.4	Other applications of the MSC . . . . .	257
7.4.1	Singular characteristics in conservation laws . . . . .	257
7.4.2	On a class of systems of first order PDEs . . . . .	258
	Exercises . . . . .	261
<b>8</b>	<b>Singular Surfaces of Nonsmooth Solutions to Multiple Integral Variational Problems</b>	<b>263</b>
8.1	Multiple integral Variational Problem . . . . .	263
8.1.1	Nonsmooth solution of second order PDE . . . . .	263
8.1.2	First variation formula . . . . .	264
8.1.3	Necessary conditions for singular surface . . . . .	267
8.2	Construction of singular surface . . . . .	271
8.2.1	Equations of singular characteristics . . . . .	271
8.2.2	Initial conditions . . . . .	272
8.3	Quadratic Lagrangian . . . . .	276
8.3.1	Degenerate necessary conditions . . . . .	276
8.3.2	Singular characteristics . . . . .	277
8.3.3	The perturbed problem . . . . .	280
8.3.4	Initial conditions . . . . .	281
8.4	Example . . . . .	282
8.4.1	Problem formulation . . . . .	282
8.4.2	Taylor expansions . . . . .	284
8.4.3	Particular cases . . . . .	287
	Exercises . . . . .	289
<b>9</b>	<b>Appendix</b>	<b>291</b>
9.1	Implicit function theorem . . . . .	291
9.2	Jacobi brackets . . . . .	292
9.3	Invariance of Jacobi brackets . . . . .	293
9.4	Field straightening . . . . .	296
9.5	Reduction to the simple problem . . . . .	298
	<b>Bibliography</b>	<b>301</b>
	<b>Subject Index</b>	<b>308</b>