Reachable Sets for Some Simple Models of the Car's Motion

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Mathematical cars

Dubins´ car

$$\dot{x} = \sin\theta$$
$$\dot{y} = \cos\theta$$
$$\dot{\theta} = u, \qquad |u| \leqslant 1$$

Reeds-Shepp´s car

$$\dot{x} = w\sin\theta$$
$$\dot{y} = w\cos\theta$$
$$\dot{\theta} = u, \quad |u| \leqslant 1, \; |w| \leqslant 1$$

L. E. Dubins (1957). On curves of minimal length with a constraint on average curvature and with prescribed initial and terminal positions and tangents. *Amer. J. Math.,* Vol. 79, 497–516.

A. A. Markov (1889). Some examples of the solution of a special kind of problem on greatest and least quantities. *Soobscenija Charkovskogo matematiceskogo obscestva*, Vol. 2, 1, N° 5, 6, 250–276.

R. Isaacs (1951). Games of pursuit. Scientific report of the RAND Corporation, Santa Monica.
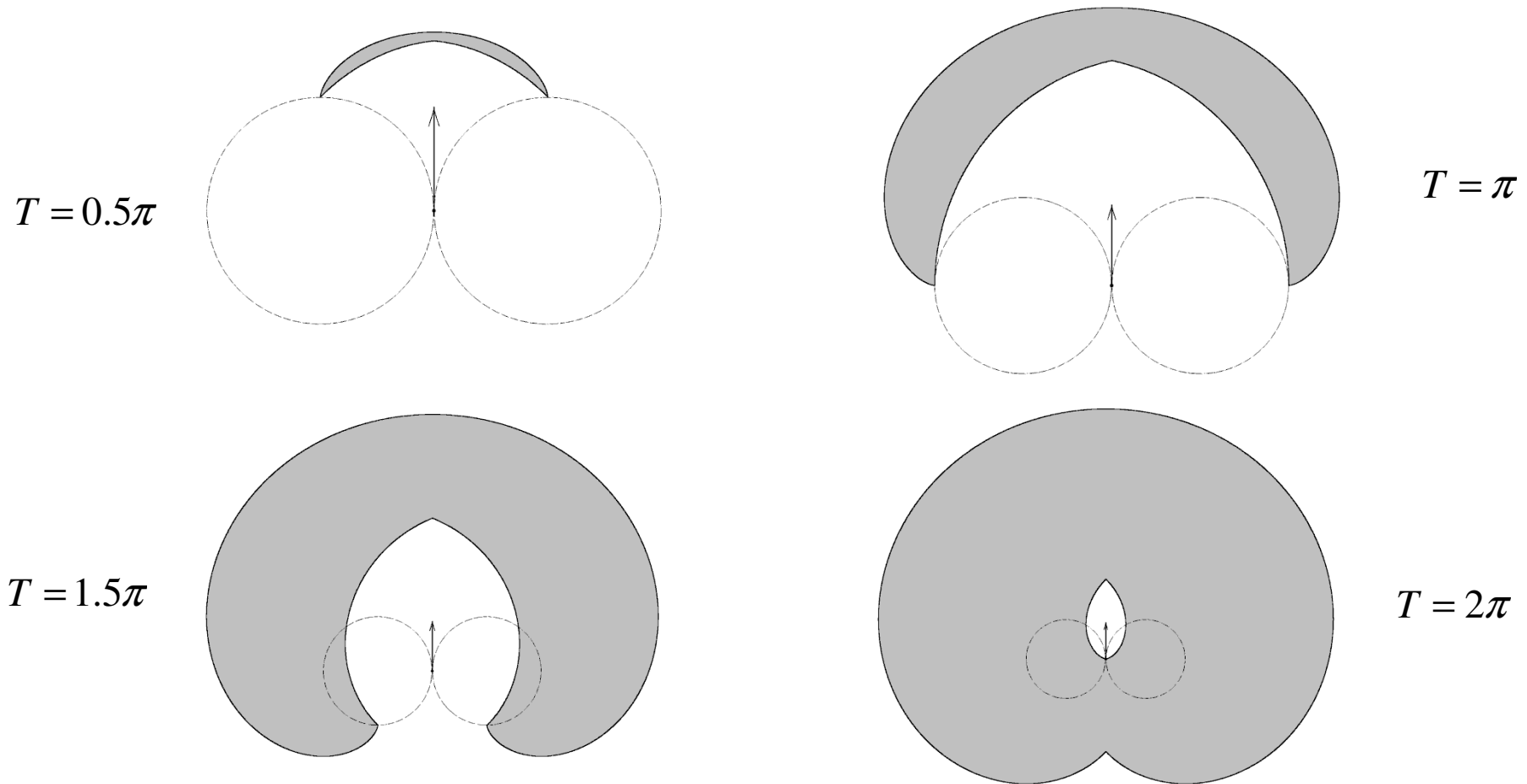
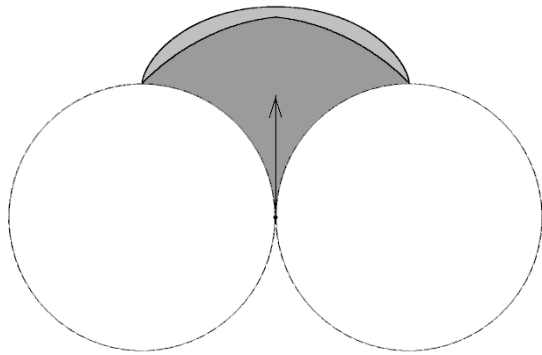**R. Isaacs (1965).** *Differential games*. NY: John Wiley.

J. A. Reeds and L. A. Shepp (1990). Optimal paths for a car that goes both forwards and backwards. *Pacific J. Math.,* Vol. 145, N° 2, 367–393.

Given-time reachable sets in the plane of geometric coordinates (Dubins´ car)

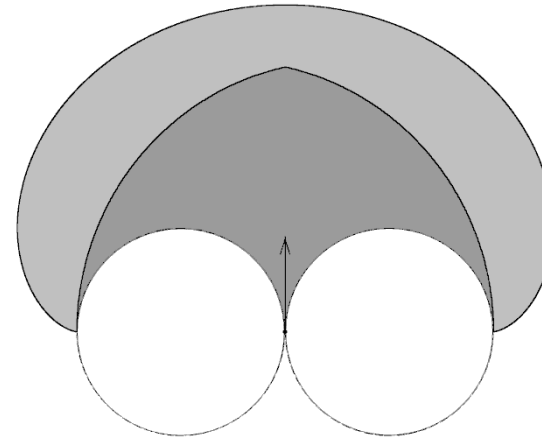
$T = 0.5\pi$

$T = \pi$

$T = 1.5\pi$

$T = 2\pi$

E. J. Cockayne, G. W. C. Hall (1975). Plane motion of a particle subject to curvature constraints. *SIAM J. Control* , Vol. 13 (1), 197–220.
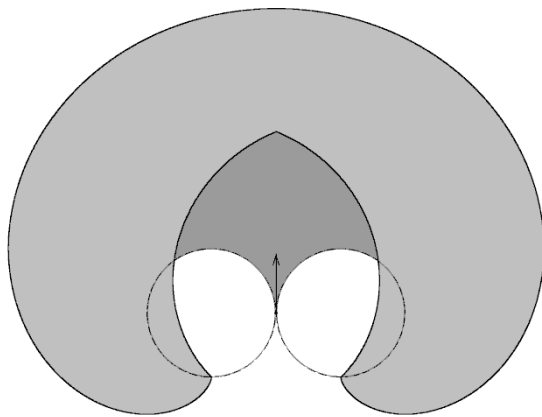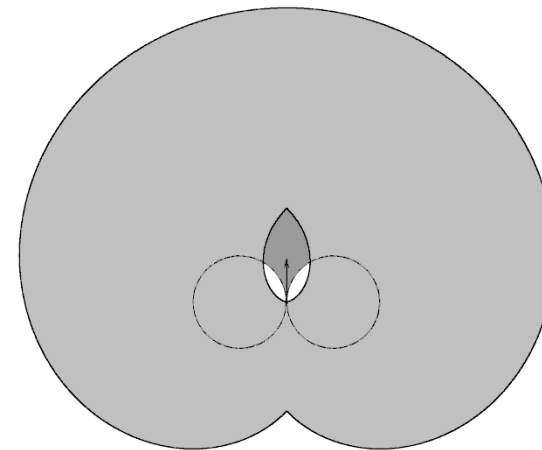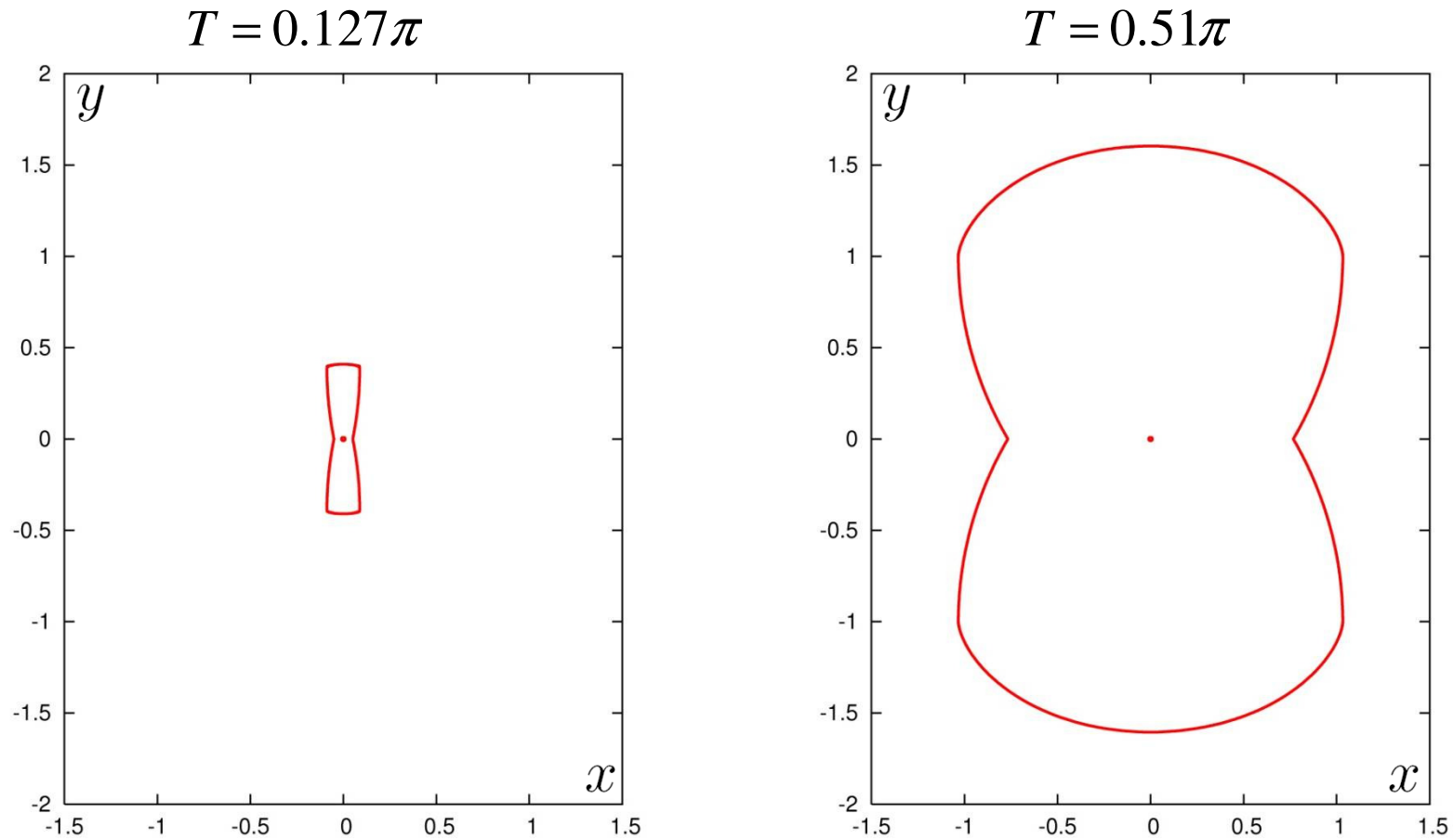
Time-limited reachable sets

$T = 0.5\pi$

$T = \pi$

$T = 1.5\pi$

$T = 2\pi$

Given-time reachable sets (Reeds-Shepp´s car)

$$T = 0.127\pi$$

$$T = 0.51\pi$$

P. Soueres, J.-Y. Fourquet, J.-P. Laumond (1994). Set of reachable positions for a car. *IEEE Trans. on Automatic Control*, Vol. 39(8), 1626−1630.

From Dubins´ car to Reeds-Shepp´s car

$$\dot{x} = w \sin \theta$$
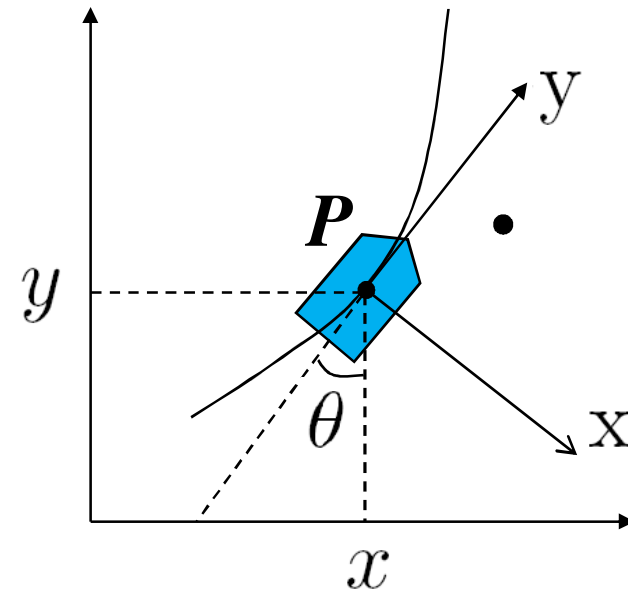$$\dot{y} = w \cos \theta$$
$$\dot{\theta} = u$$

$$|u| \leq 1, \qquad a \leq w \leq 1$$

a - parameter

Dependence of reachable sets on parameter a ?
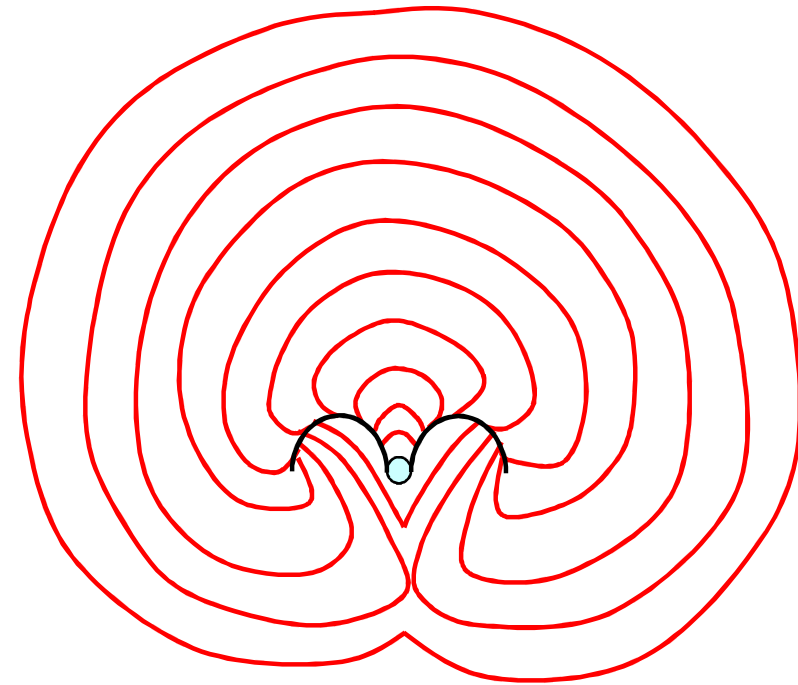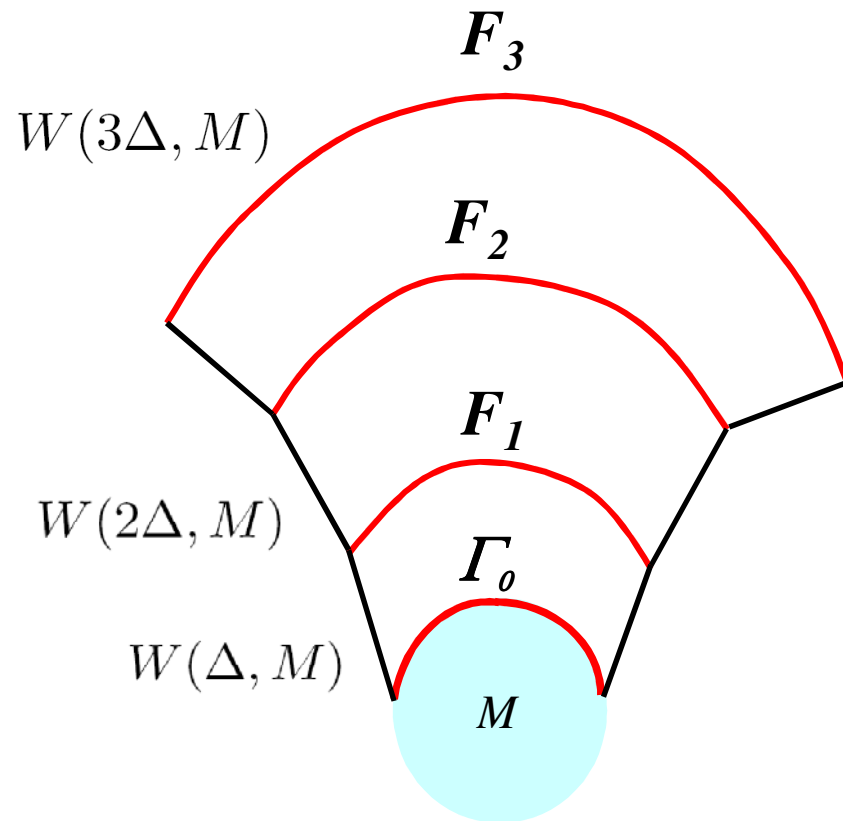
Isaacs´ transformation

$$\dot{x} = -y\,u$$
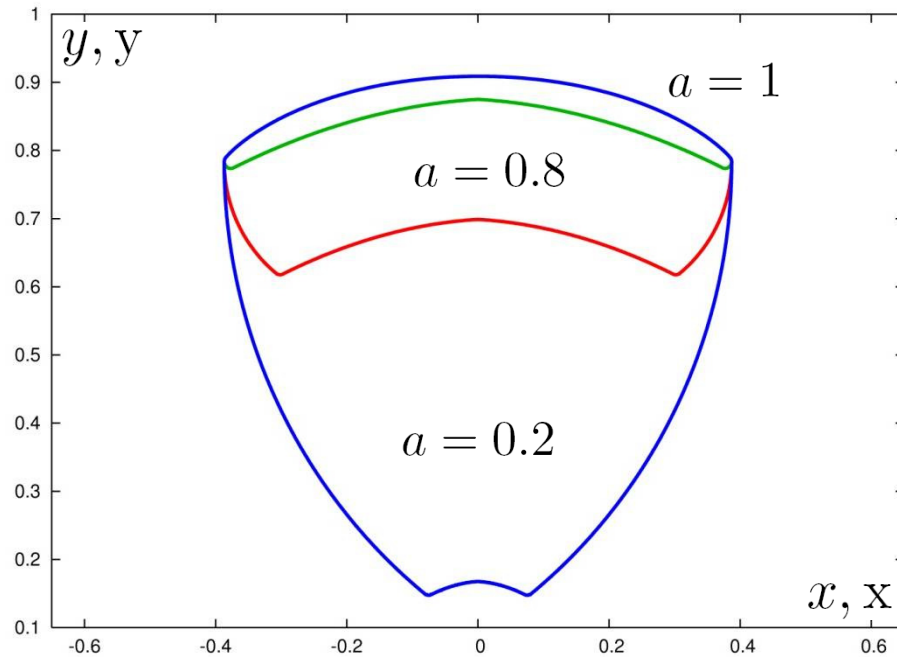$$\dot{y} = x\,u - w$$
$$|u| \le 1, \quad a \le w \le 1$$

Reachable set of the original system is the collection of points from which this system can be brought to the origin at a given time or by a given time
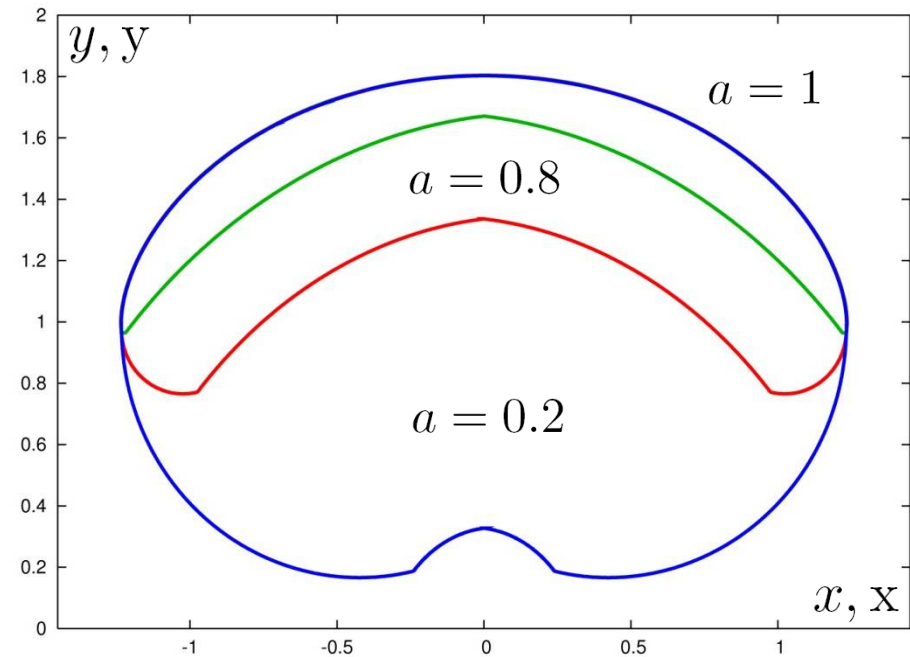
Backward construction

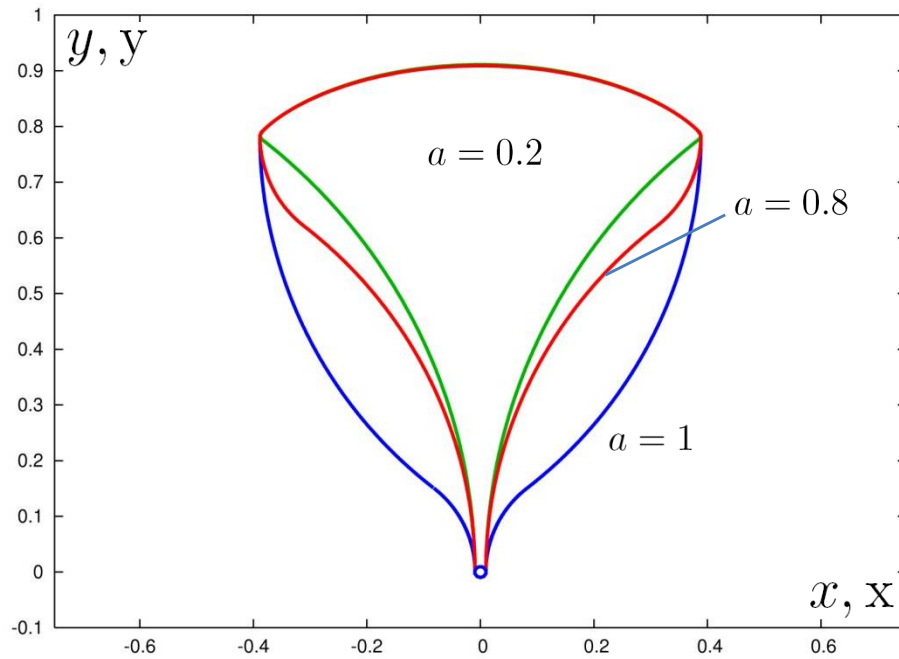# Given-time reachable sets for different values of a

$T = 0.9$

y, y

$a = 1$

$a = 0.8$

$a = 0.2$

x, x

$T = 1.8$
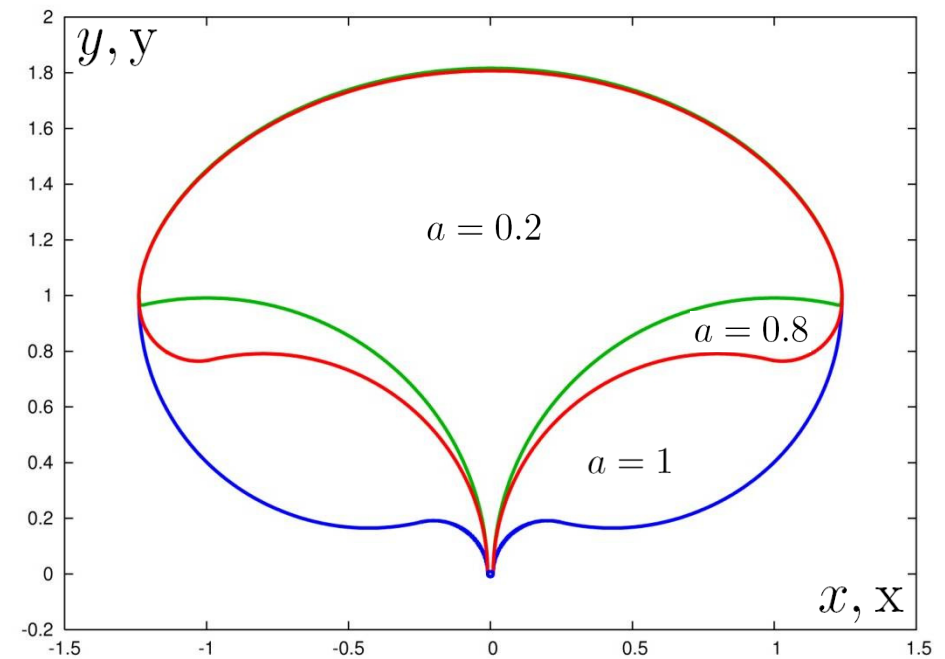
y, y

$a = 1$

$a = 0.8$

$a = 0.2$

x, x

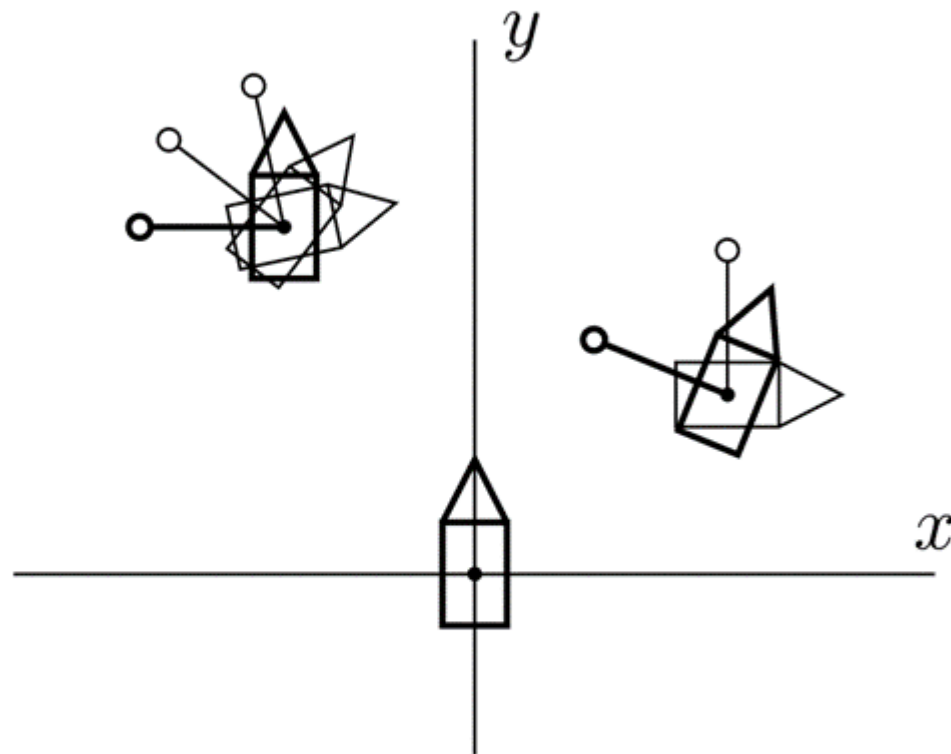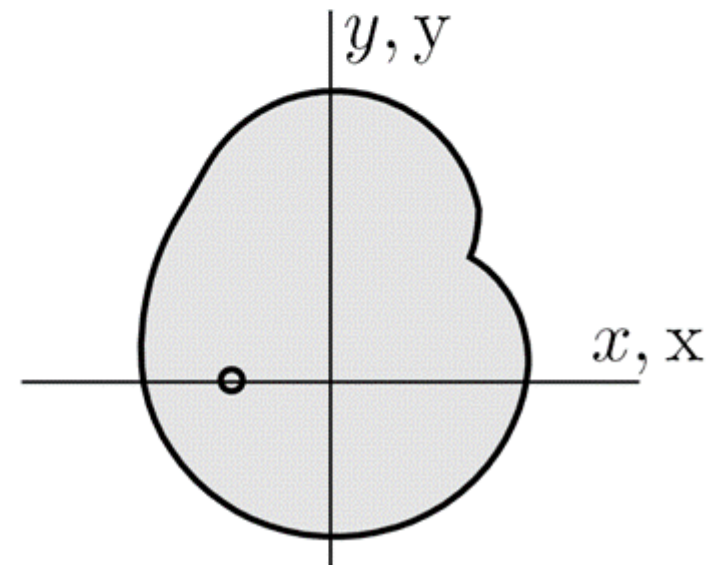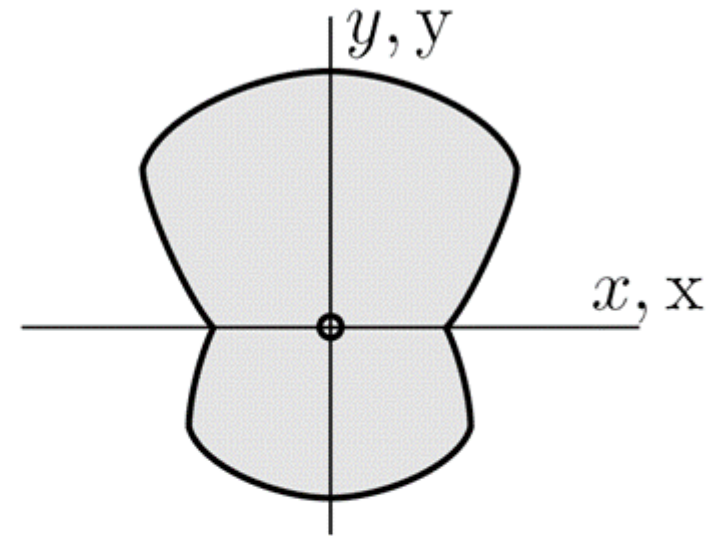# Time-limited reachable sets for different values of a

$$T = 0.9$$

$$T = 1.8$$

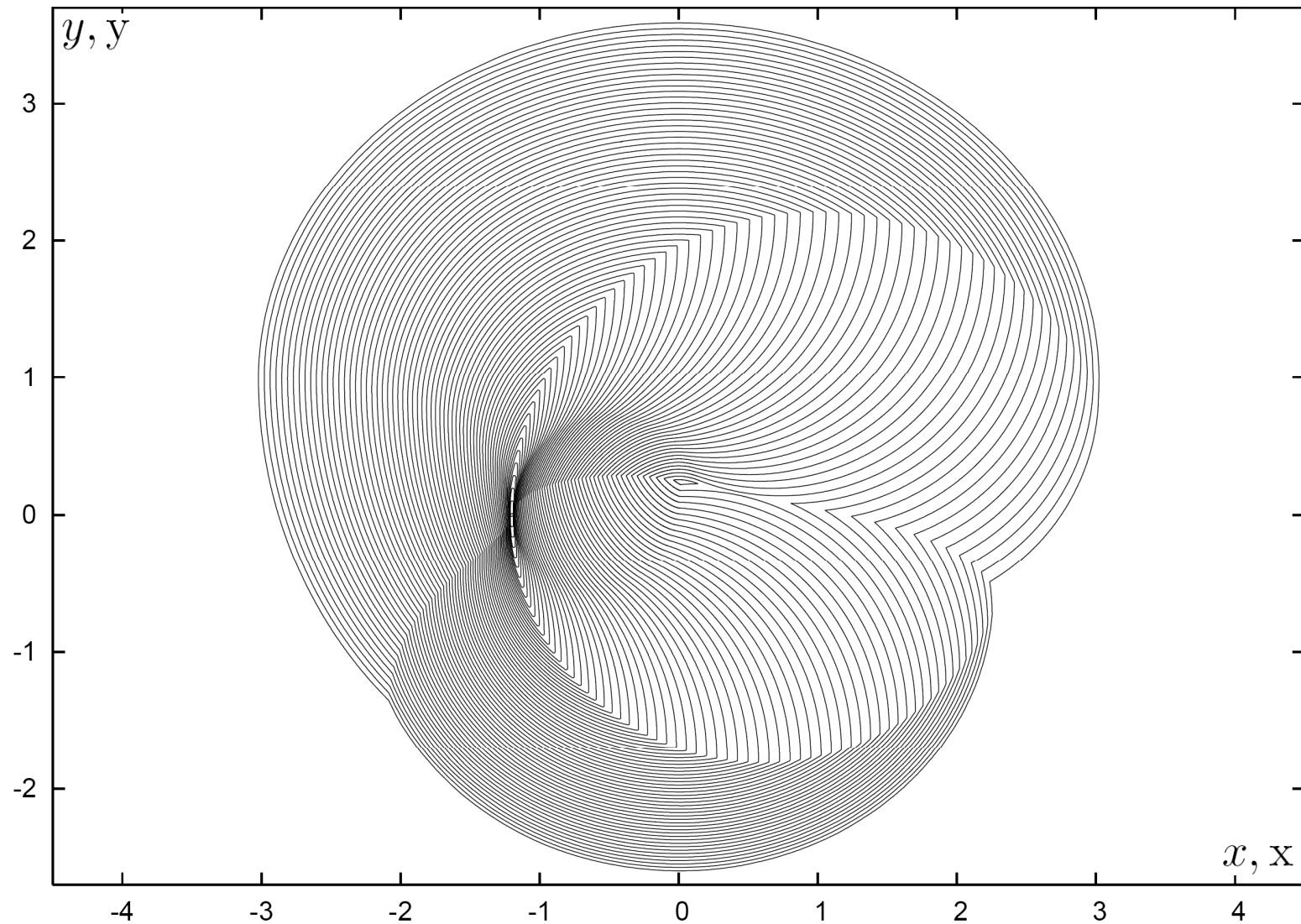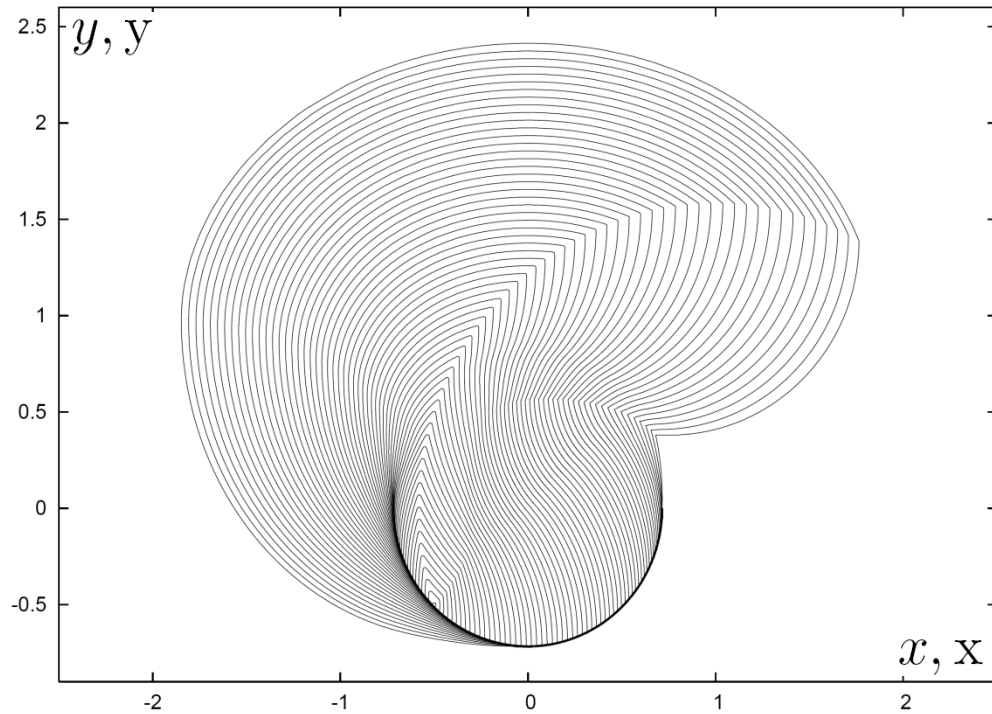# Reachable sets corrected by accounting for oriented added shift

Collection of points in the plane x, y which can be covered by given orientedly added set at time *T*
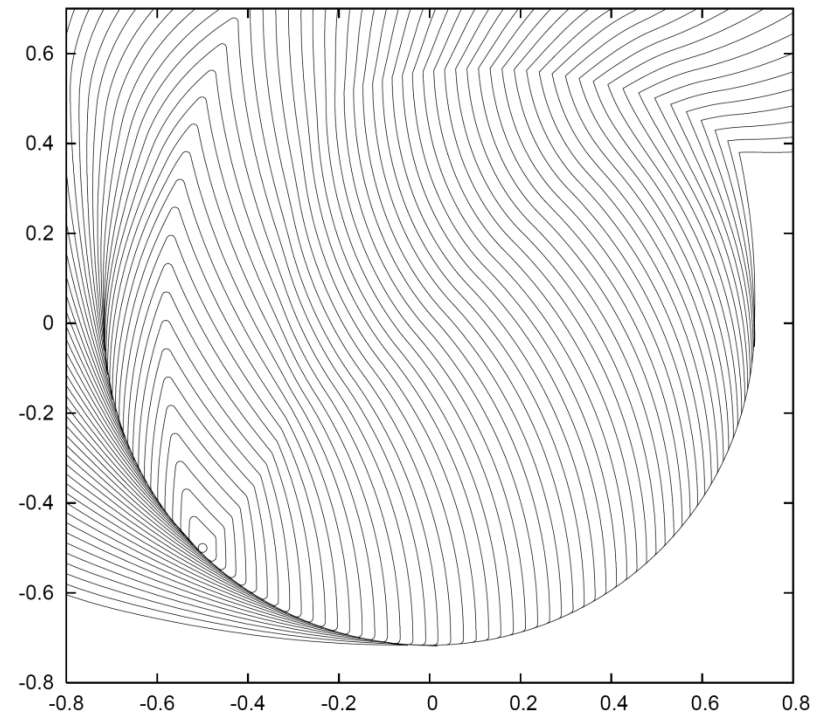
Reachable sets for oriented shift $(-1.2, 0)$

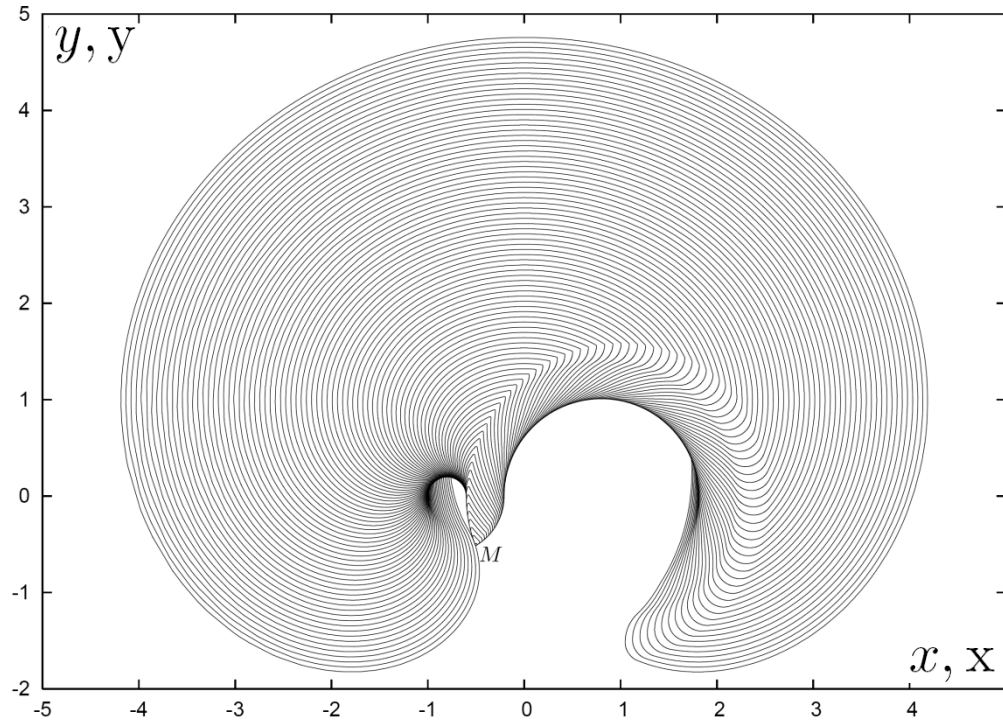$$a = -0.6, \ T = 2.6, \ \Delta = 0.002, \ \delta = 0.04$$

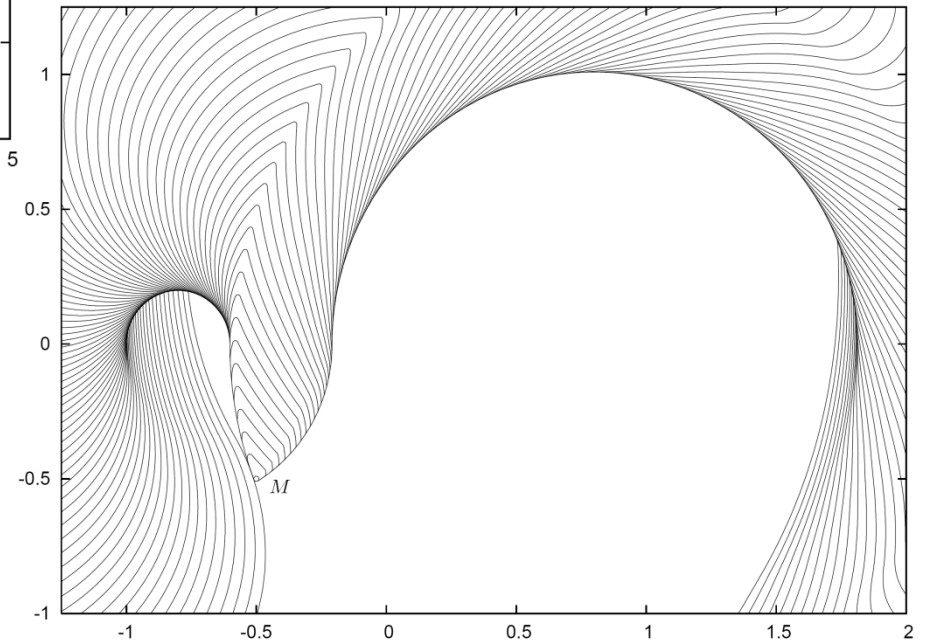# Reachable sets for oriented shift $(-0.5, -0.5)$

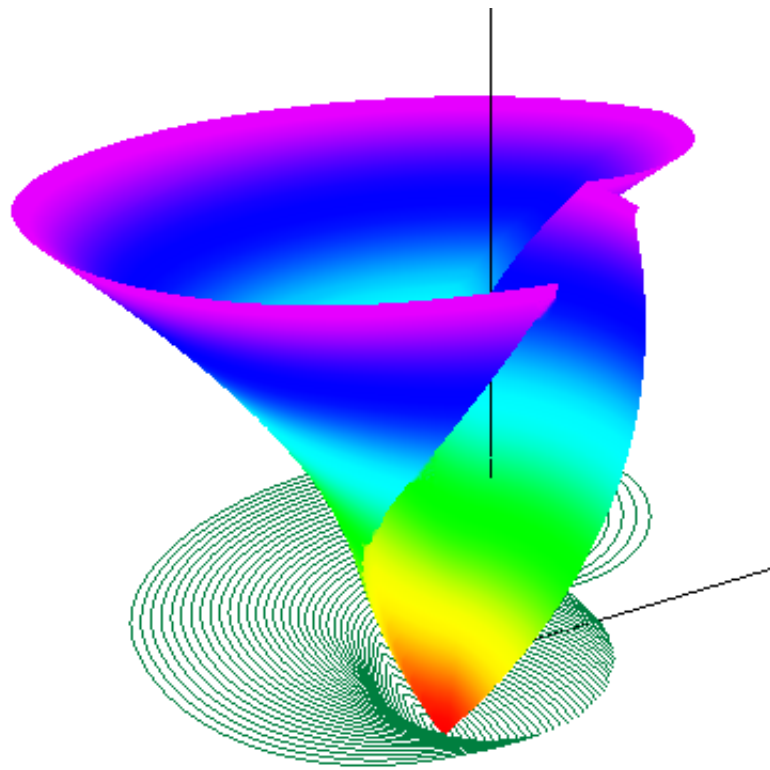$$a = 0, \ T = 2.4$$

$$\Delta = 0.002, \ \delta = 0.04$$

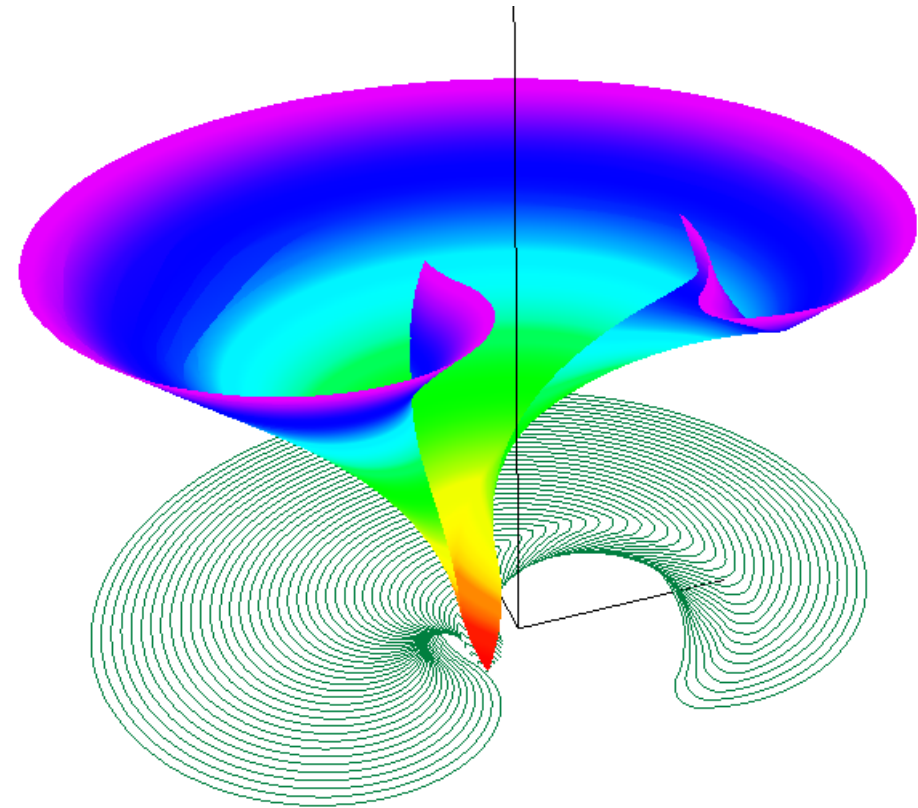# Time-limited reachable sets for oriented shift $(-0.5, -0.5)$

$$a = 0.8, \quad T = 4.75$$

$$\Delta = 0.002, \quad \delta = 0.04$$

Value function

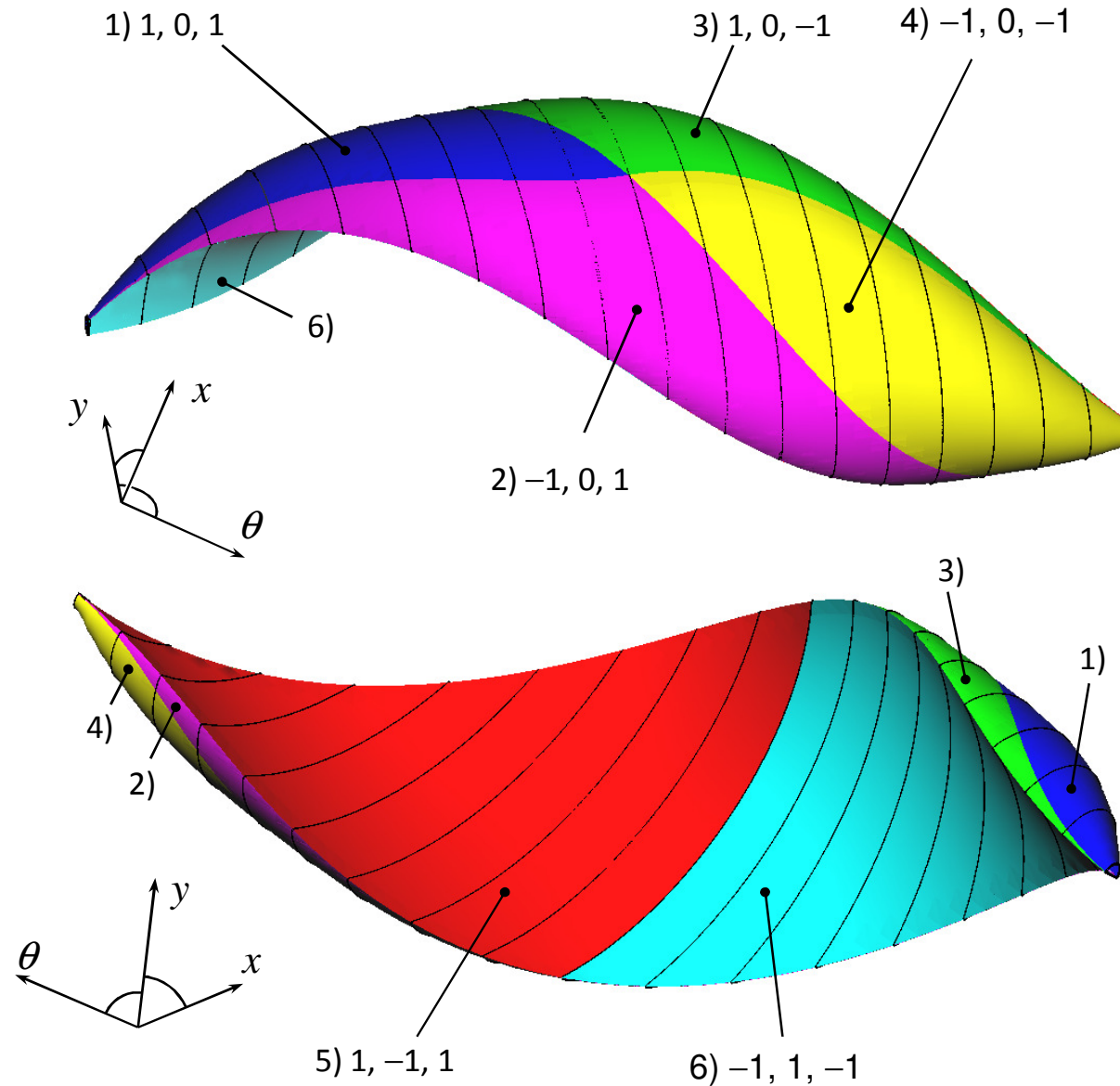
$a = 0$ $a = 0.8$

Three-dimensional given-time reachable set
for Dubins´ car

$T = \pi$

1) 1, 0, 1

3) 1, 0, −1   4) −1, 0, −1

6)

2) −1, 0, 1

x

y

θ

3)

1)

4)

2)

y

θ x

5) 1, −1, 1   6) −1, 1, −1

Time development of reachable sets for Dubins´ car

$T = \pi$

$T = 2\pi$

$T = 3\pi$

$T = 4\pi$

Violation of simple connectedness of reachable set

$T = 3.63\pi$

$T = 3\pi$

θ y x

Given-time reachable sets for angle θ modulo 2π

$T = 1.5\pi$

$T = 2\pi$

Reachable sets in cylindrical coordinates

Authors´ works

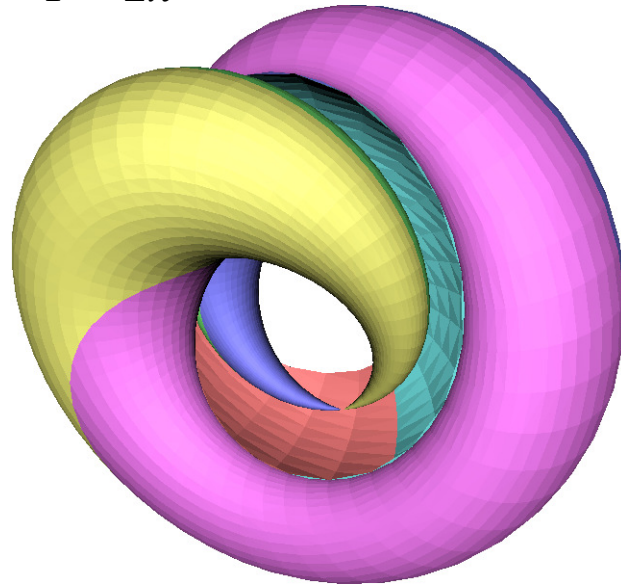V. S. Patsko, V. L. Turova (2001). Level sets of the value function in differential games with the homicidal chauffeur dynamics. *Int. Game Theory Review*, Vol. 3, N°1, 67–112.

V. S. Patsko, S. G. Pyatko, A. A. Fedotov (2003). Three-dimensional reachability set for a nonlinear control system, *J. of Computer and Systems Sciences Int.,* Vol. 42, N°3, 320–328.

V. S. Patsko, S. G. Pyatko, A. A. Fedotov, S. I. Kumkov (2005). Informational sets in model problems of aircraft tracking. *Preprints of the 16th IFAC World Congress.* Prague, Czech Republic, July 3-8, 2005.

V. S. Patsko, V. L. Turova (2008). From Dubins´ car to Reeds and Shepp´s mobile robot. *Computing and Visualization in Science* (published online: 15 April 2008).