Families of Semipermeable Curves and Their Application to Some Complicated Variants of the Homicidal Chauffeur Problem

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Time-limited reachable sets for the simplest (Dubins´) car

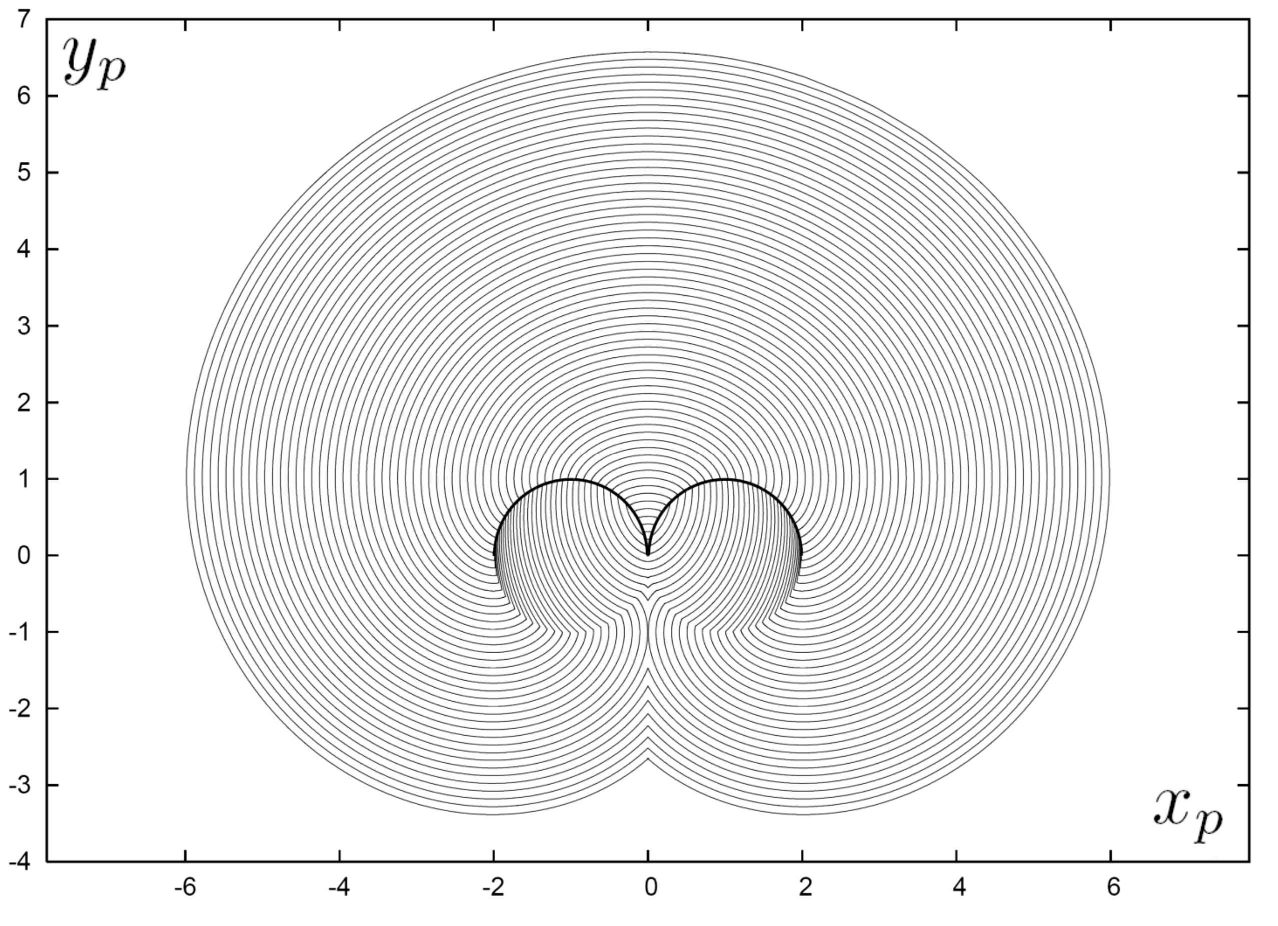
Dynamics in normalized coordinates

$$\dot{x}_p = \sin\theta$$

$$\dot{y}_p = \cos\theta$$

$$\dot{\theta} = u$$

$$|u| \leqslant 1$$



Isaacs´ homicidal chauffeur game

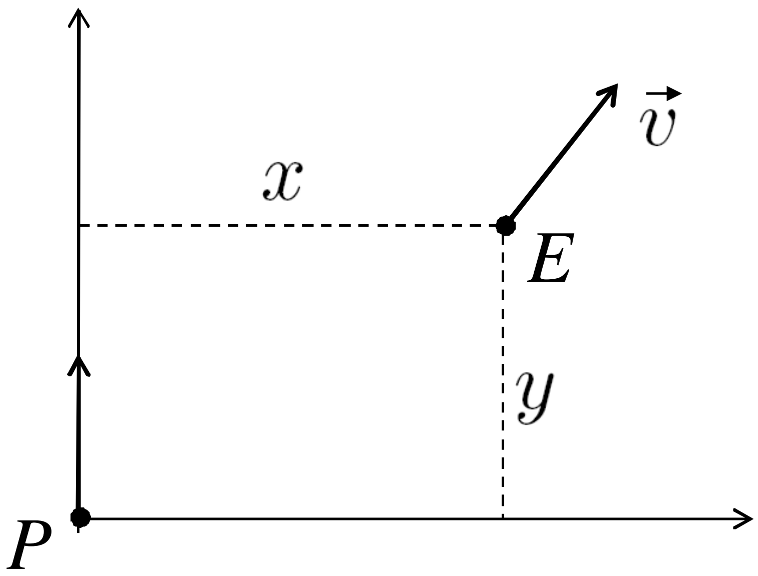
$$P: \quad \dot{x}_p = \sin\theta$$
$$\dot{y}_p = \cos\theta$$
$$\dot{\theta} = u, \quad |u| \leqslant 1$$

$$E: \quad \dot{x}_e = v_1$$
$$\dot{y}_e = v_2$$
$$v = (v_1, v_2)' \in Q$$

Q - circle of radius ν

In reduced coordinates

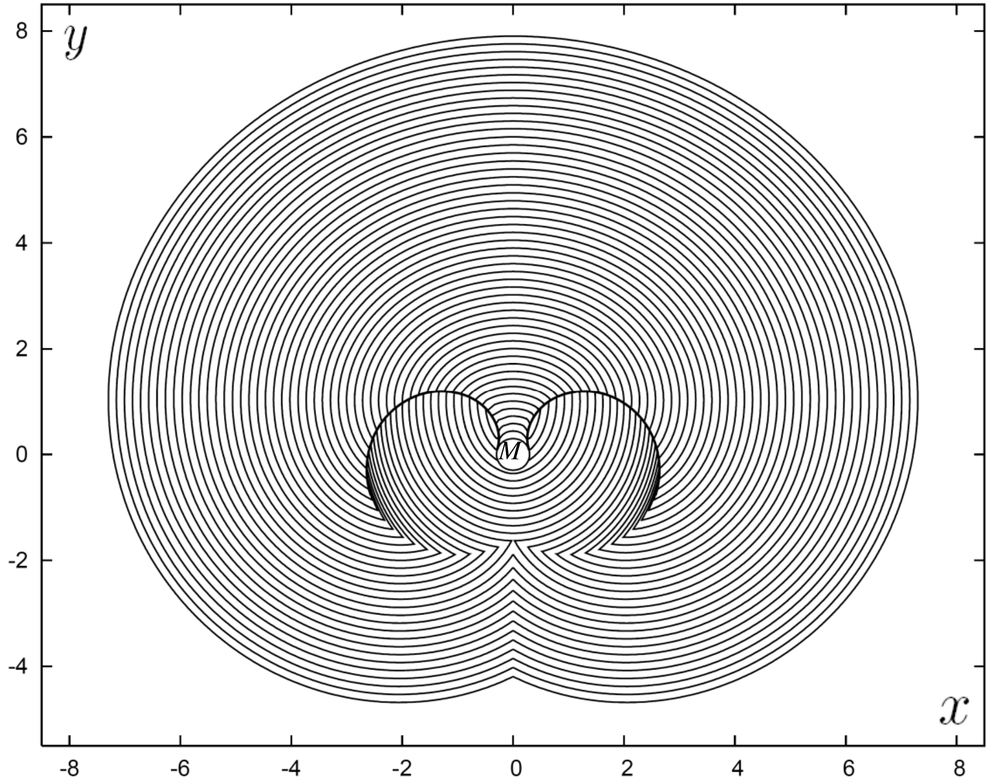
$$\dot{x} = -yu + v_x$$
$$\dot{y} = \quad xu + v_y - 1$$
$$|u| \leqslant 1, \qquad v \in Q$$

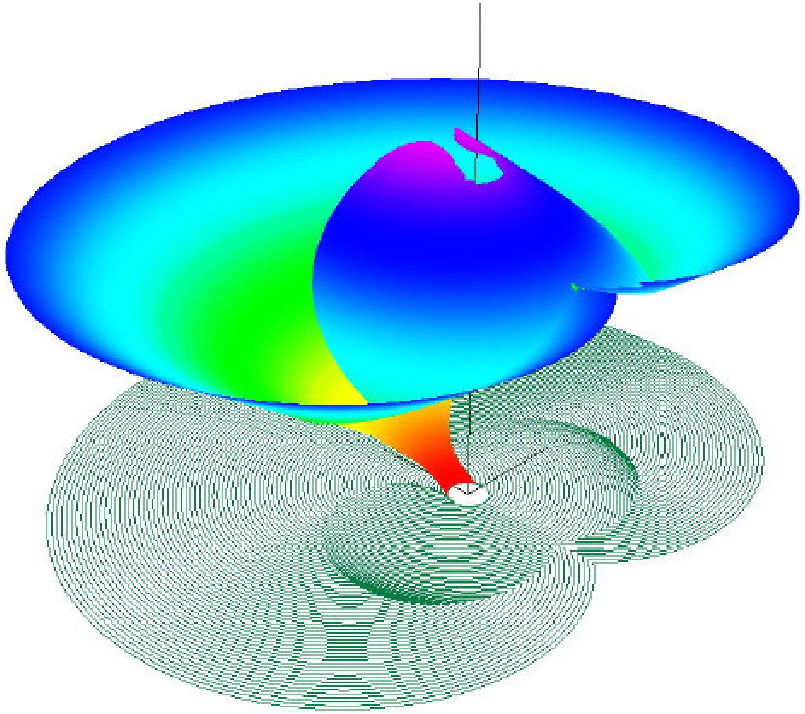


In the classical setting, target set is a circle of radius r centred at the origin

Level sets and graph of the value function

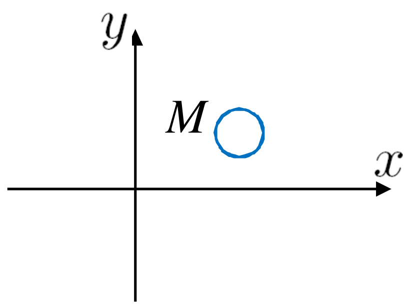
$$\nu = 0.3, \quad r = 0.3$$



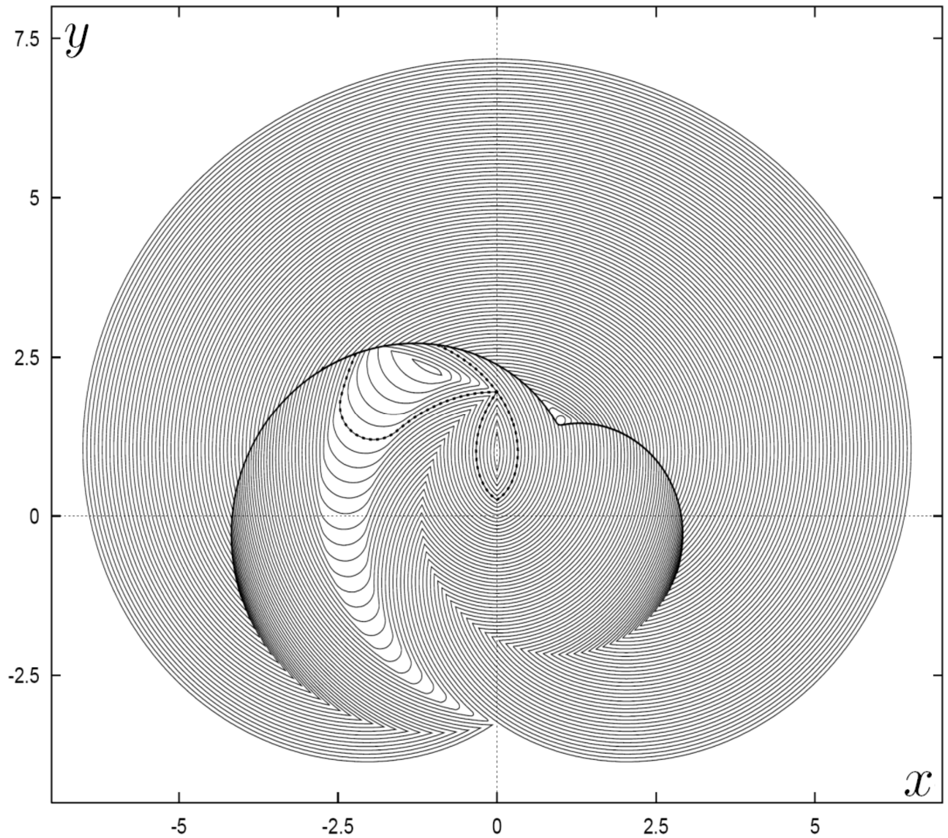


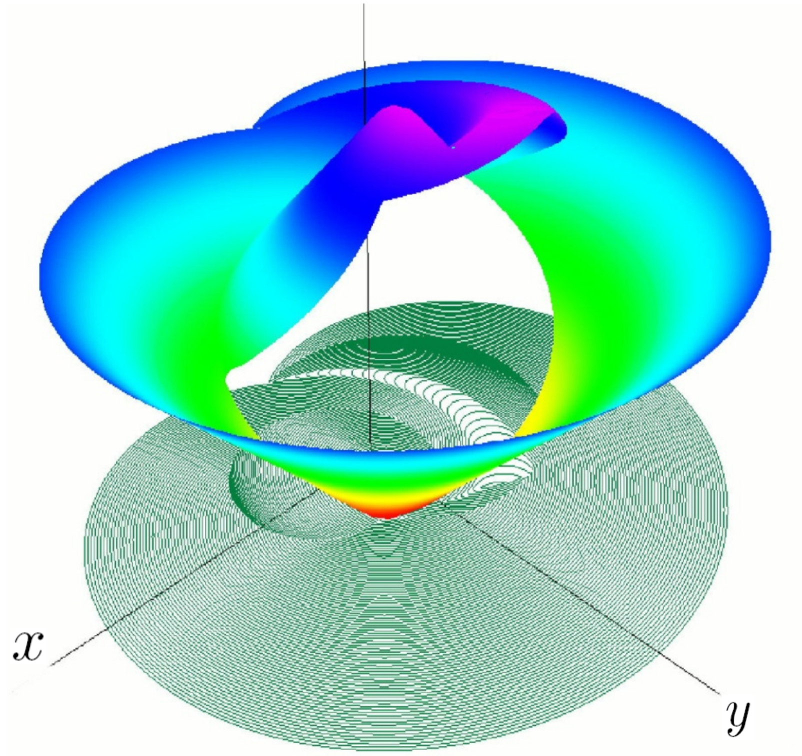
$$\Delta = 0.01, \;\; \tau_f = 10.3, \;\; \delta = 0.2$$

Example with shifted target set



$$\nu = 0.3,\ r = 0.075,\ m_x = 1,\ m_y = 1.5$$

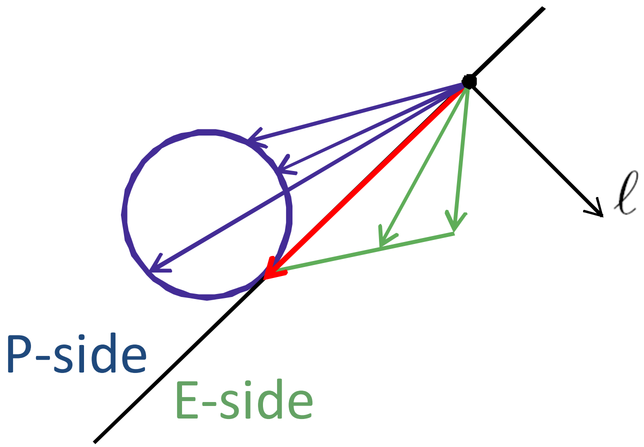


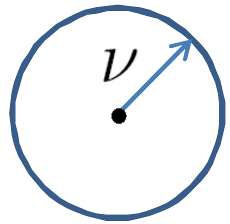


$$\Delta = 0.01,\ \tau_f = 9.5,\ \delta = 0.08$$

Semipermeable directions and semipermeable curves

$$H(\ell, z) = \min_{|u|\le 1} \max_{v\in Q} \ell' f(z,u,v) = \max_{v\in Q} \min_{|u|\le 1} \ell' f(z,u,v), \quad z = \begin{pmatrix} x \\ y \end{pmatrix}$$



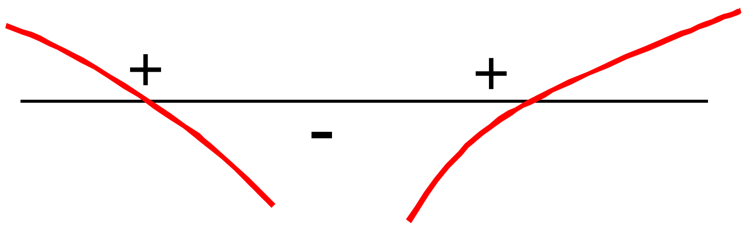
$Q =$ 

Fix $z \in R^2$ and find $\ell \in R^2$

such that $H(\ell, z) = 0$

$\ell^{(1)}$ is root from + to -

$\ell^{(2)}$ is root from - to +



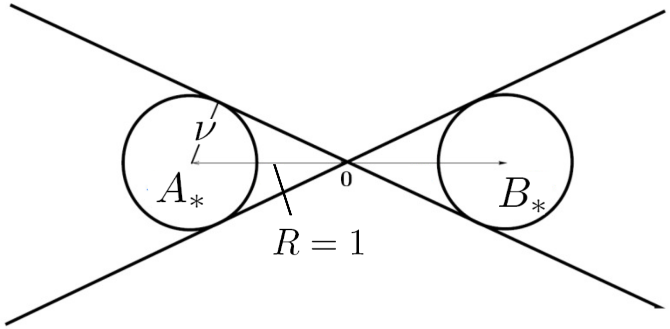
For any point $z \in R^2$ there exist two roots of each type at most

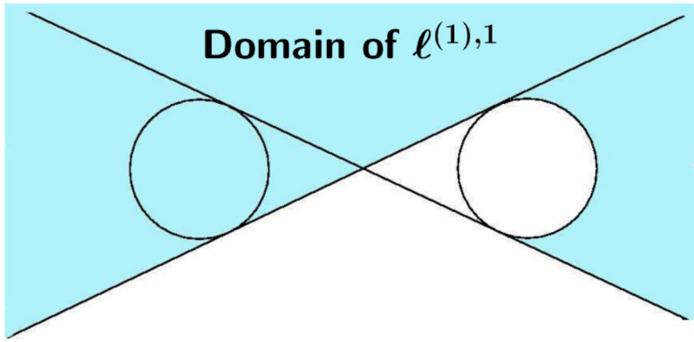
$\ell^{(1),1}(z),\ \ell^{(1),2}(z)$ and $\ell^{(2),1}(z),\ \ell^{(2),2}(z)$

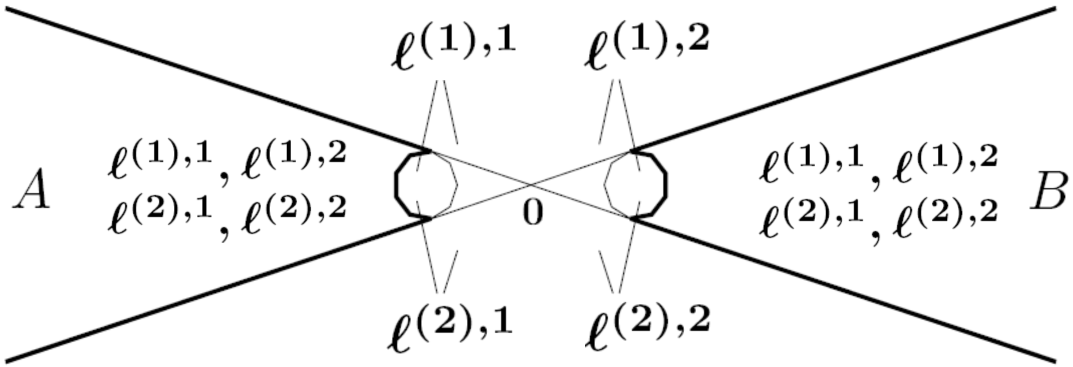
Differential equations for semipermeable curves. Domains of functions $\ell^{(j),i}$

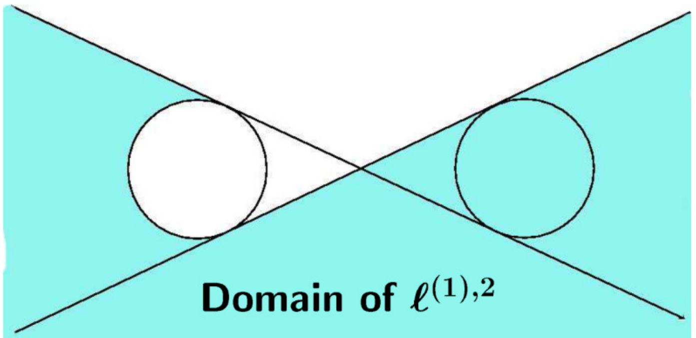
$$\frac{dz}{dt} = \Pi\,\ell^{(j),i}(z), \quad z = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$i = 1, 2,\ \ j = 1, 2, \quad z \in \text{domain of } \ell^{(j),i}$$

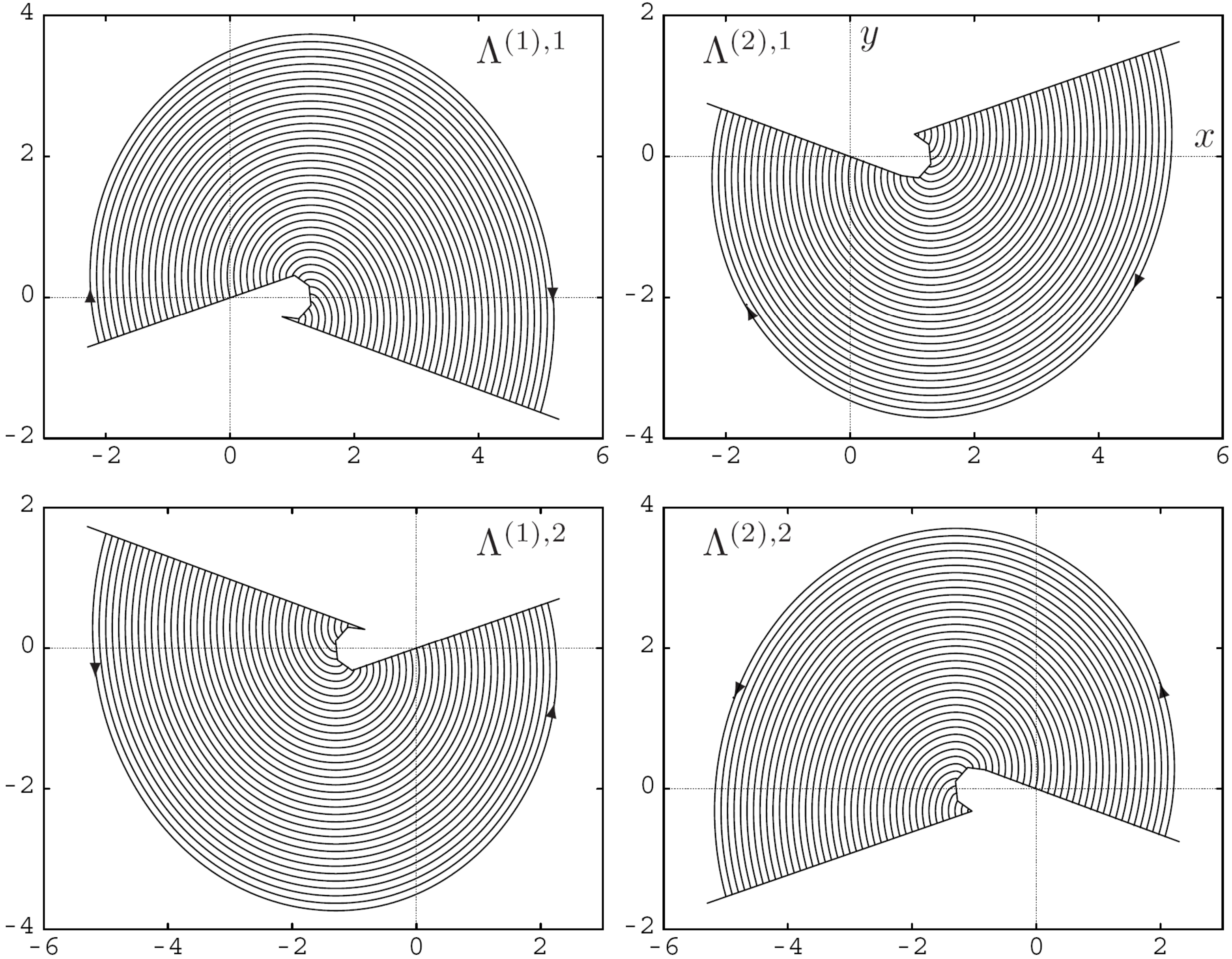




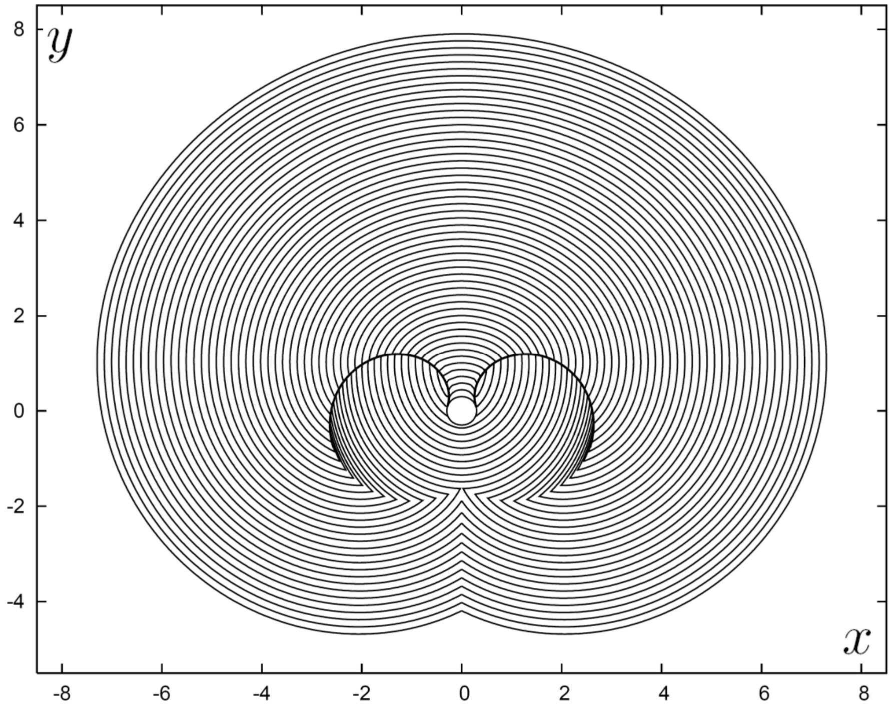


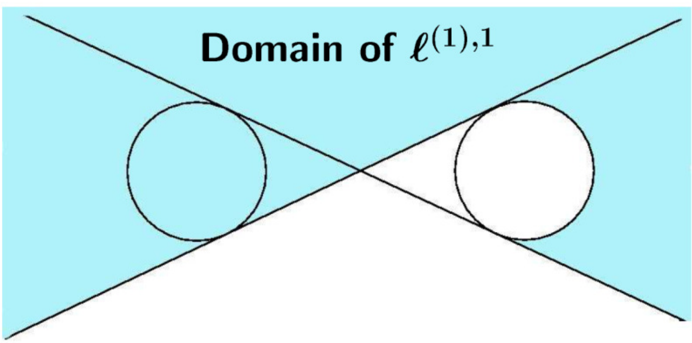


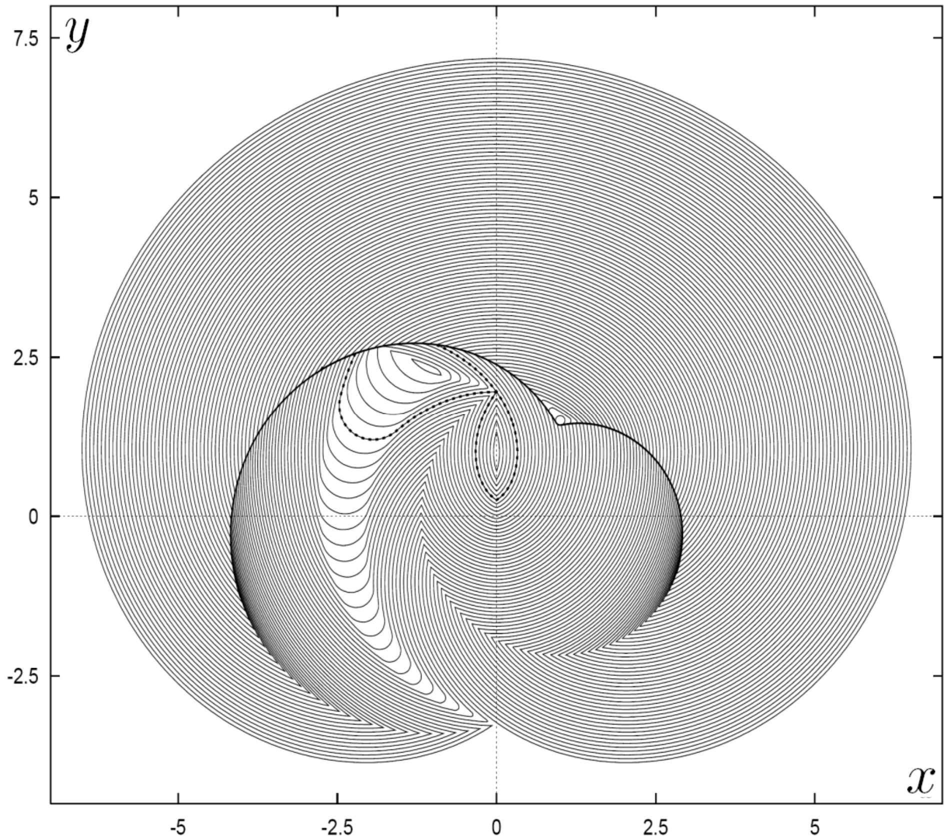
Families of semipermeable curves



Discontinuity lines are semipermeable curves of families $\Lambda^{(1),1}$, $\Lambda^{(2),2}$

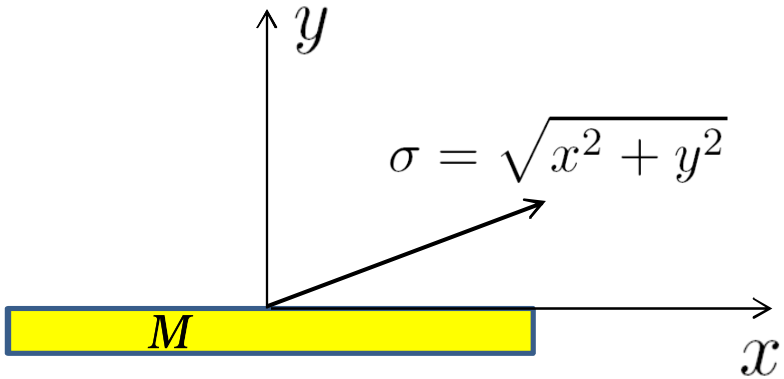






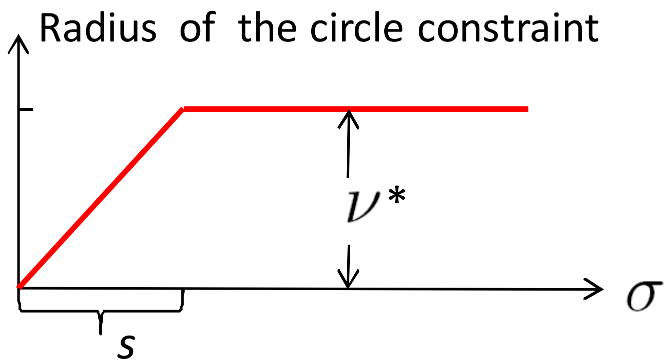
Acoustic game

P. Cardaliaguet, M. Quincampoix, P. Saint-Pierre (1999). Set-valued numerical analysis for optimal control and differential games. In: M. Bardi, T.E.S. Raghavan, T. Parhasarathy (eds.), *Stochastic and Differential Games: Theory and Numerical Methods, Annals of the International Society of Dynamic Games.* Boston: Birkhäuser, Vol. 4, 177–247.

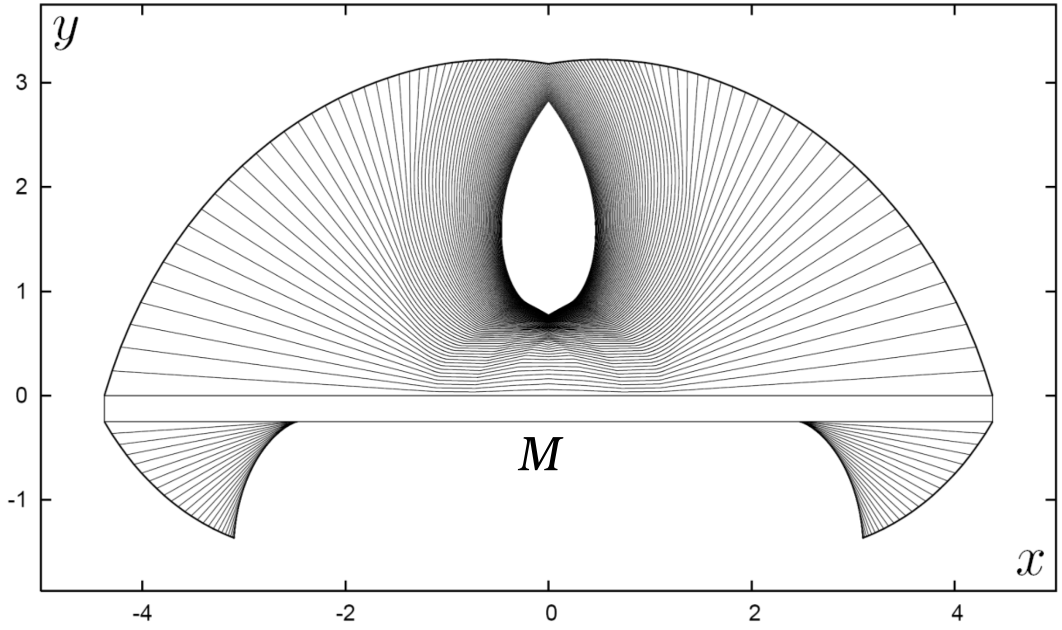


$$\dot{x} = -yu + v_x$$
$$\dot{y} = \quad xu + v_y - 1$$

$$|u| \le 1, \quad v \in$$ radius depends on σ

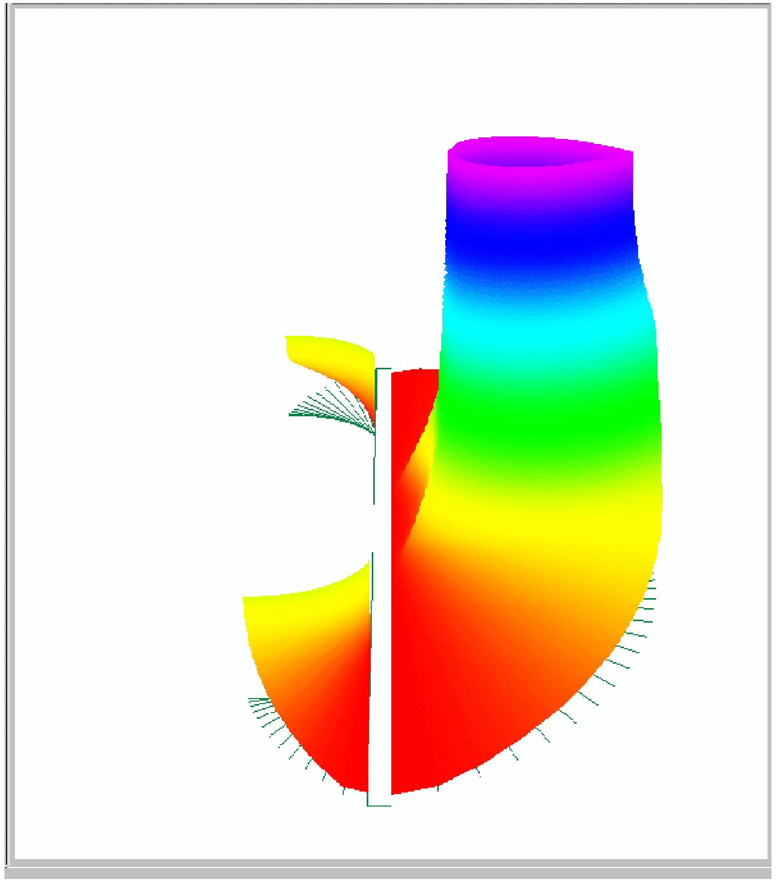


Regions with infinite values of the game



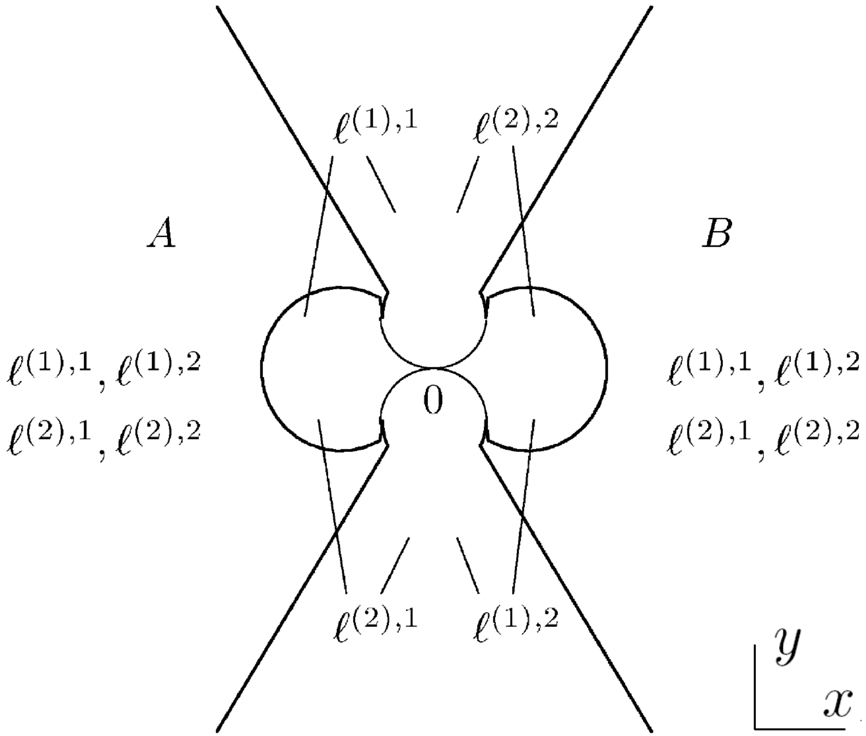
$$\nu^* = 1.5, \quad s = 0.9375$$

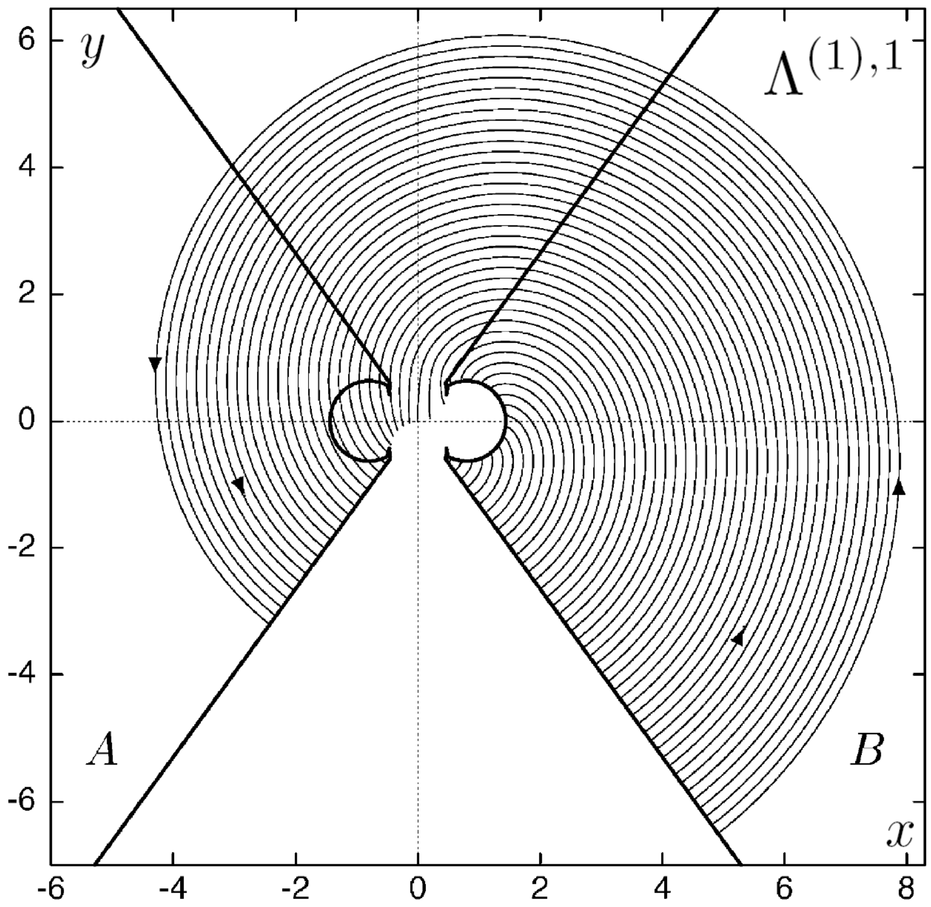
$$\Delta = 0.00625, \; \delta = 0.0625$$



Semipermeable curves in acoustic game

$$\nu^* = 0.8,\ s = 0.9375$$

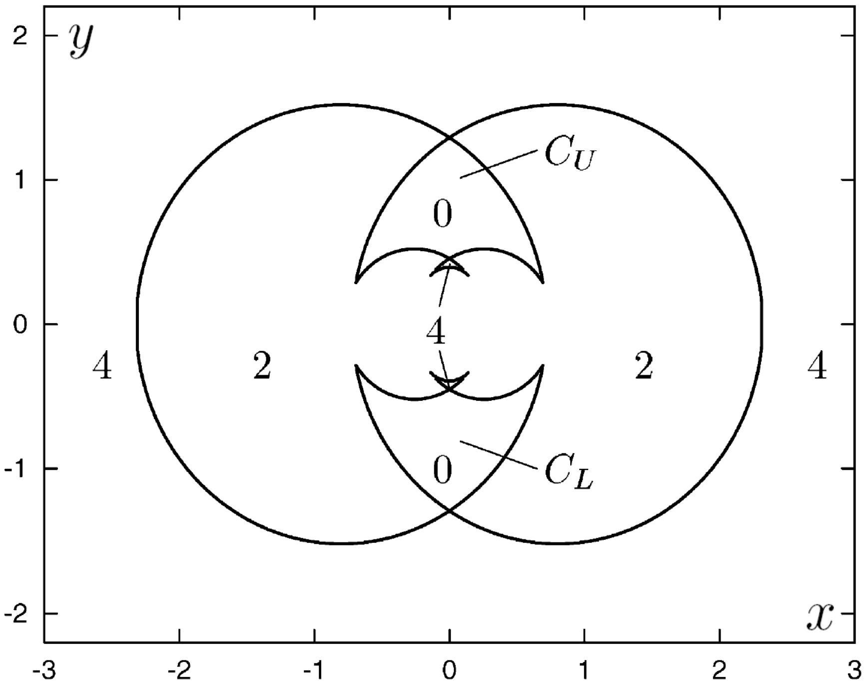




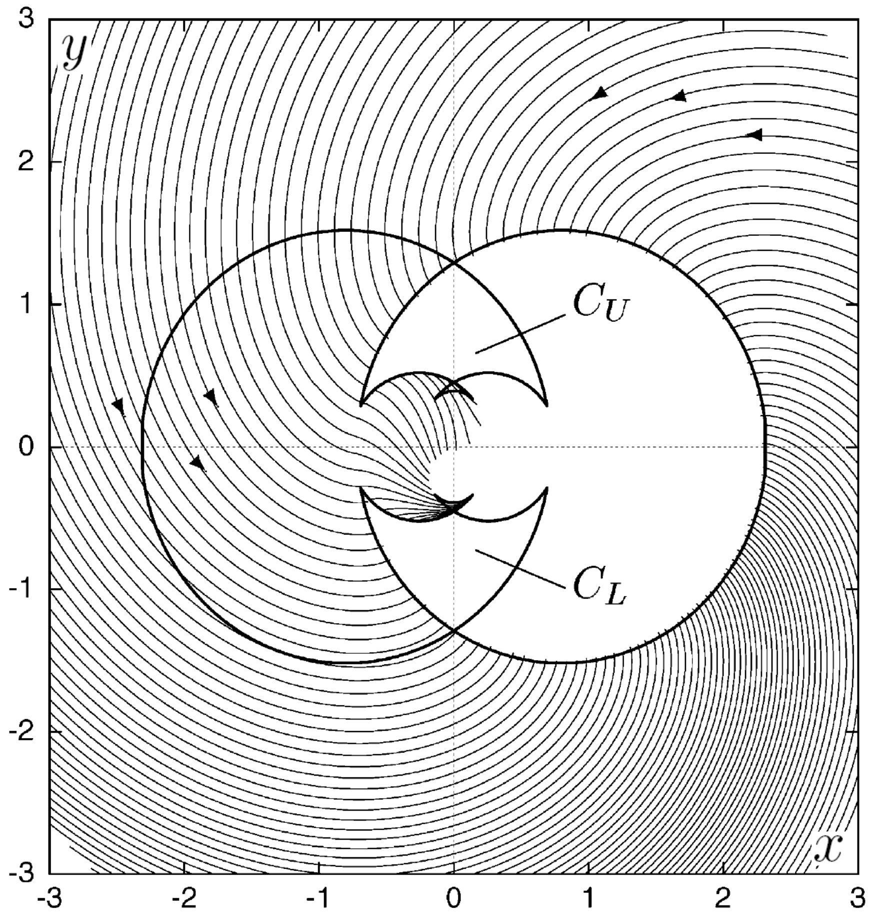
Semipermeable curves in acoustic game

$$\nu^* = 1.8, \;\; s = 0.9375$$

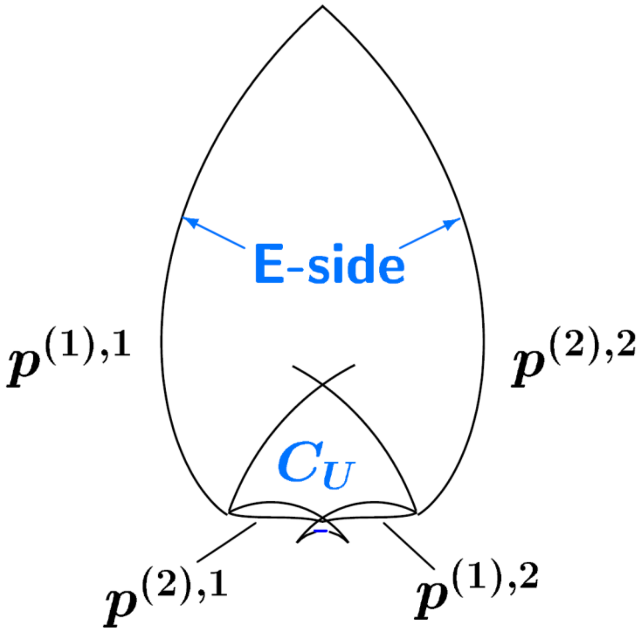
C_U, C_L - superiority sets of player E

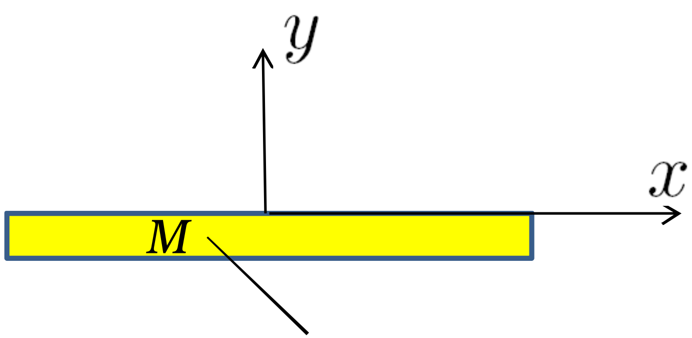


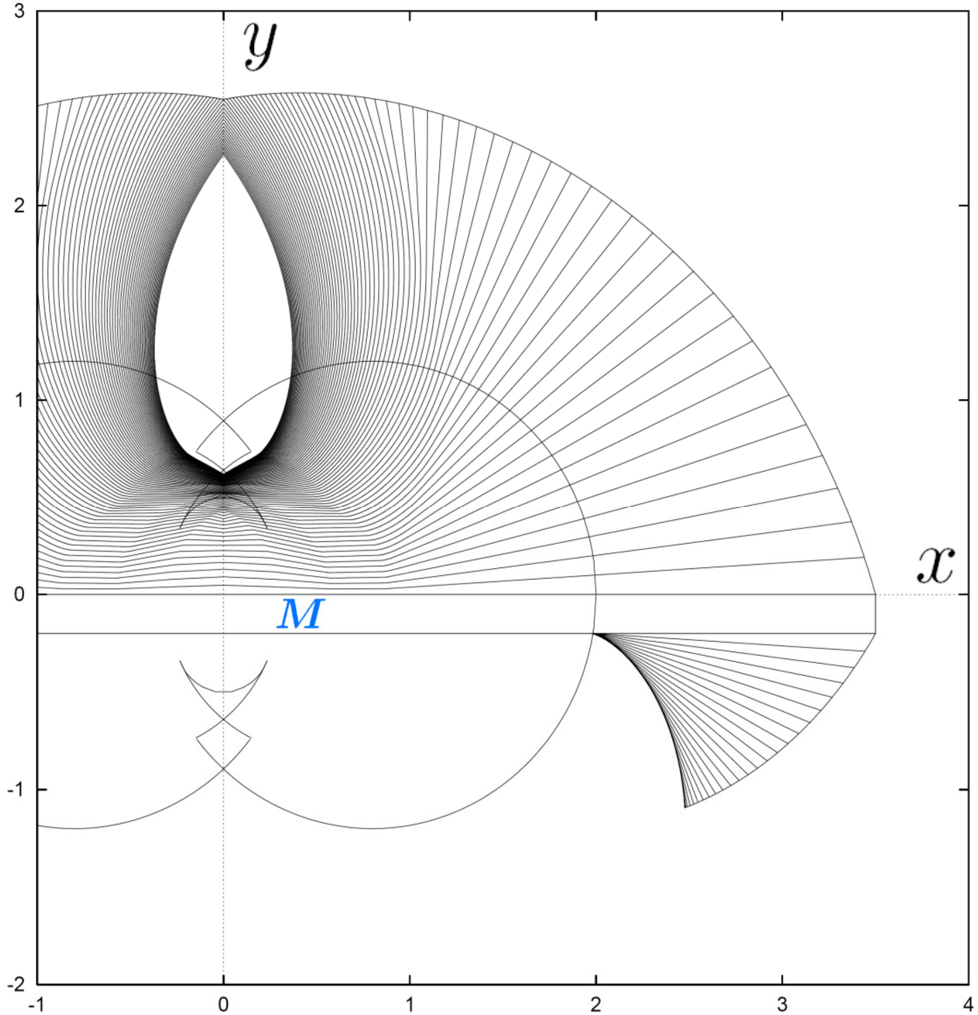
$\Lambda^{(1),1}$



Semipermeable curves and superiority sets of player E



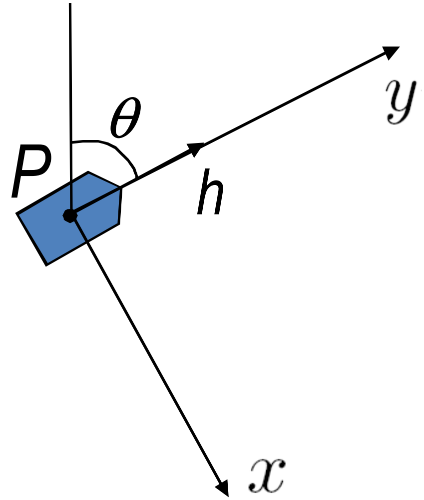




Reeds-Shepp´s car as a pursuer

J. A. Reeds and L. A. Shepp (1990). Optimal paths for a car that goes both forwards and backwards. Pacific J. Math., Vol. 145, N° 2, 367–393.

$$\dot{x}_p = w \sin\theta$$
$$\dot{y}_p = w \cos\theta$$
$$\dot{\theta} = u, \qquad |u| \leqslant 1, \ |w| \leqslant 1$$



p. 373: "…for slowly moving vehicles, such as carts, this seems like a reasonable compromise to achieve tractability".

$$\dot{x} = -yu + v_x$$
$$\dot{y} = \ xu - w + v_y$$
$$|u| \leqslant 1, \ w \in [-1, 1], \ v = (v_x, v_y)', \ |v| \leqslant \nu$$

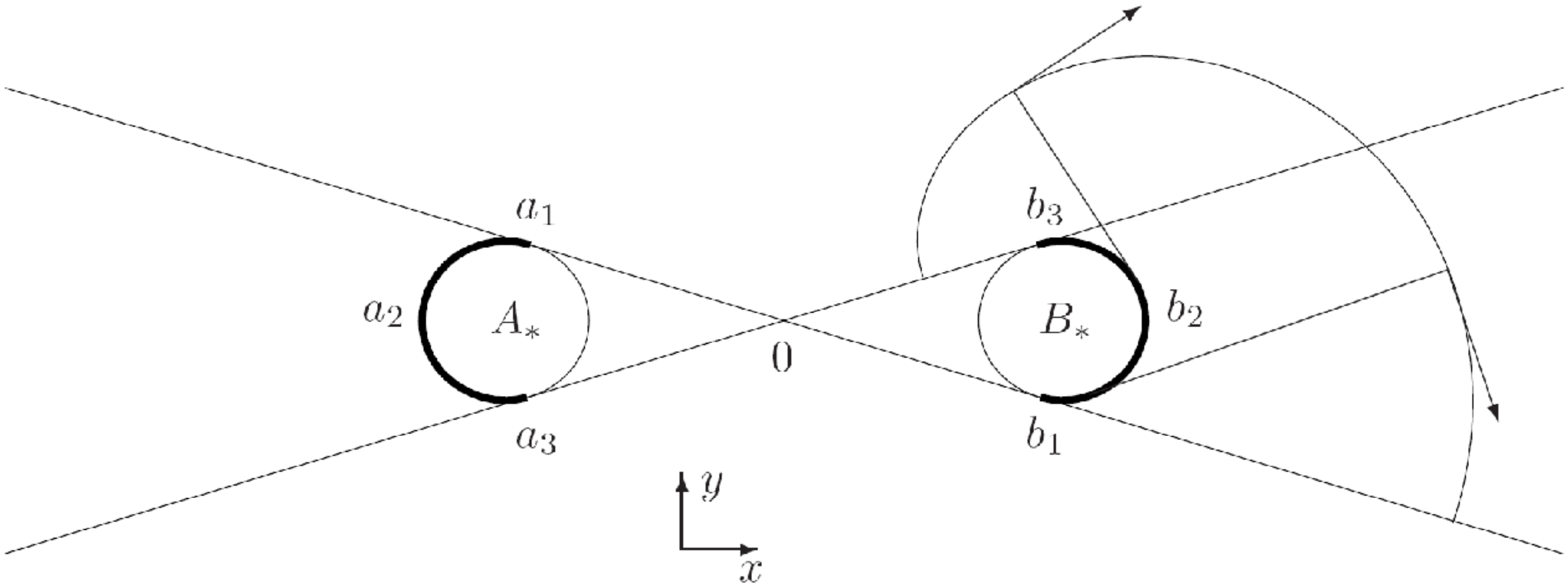
$$\boxed{w \in [a, 1]}$$

Families of semipermeable curves in the classical homicidal chauffeur problem

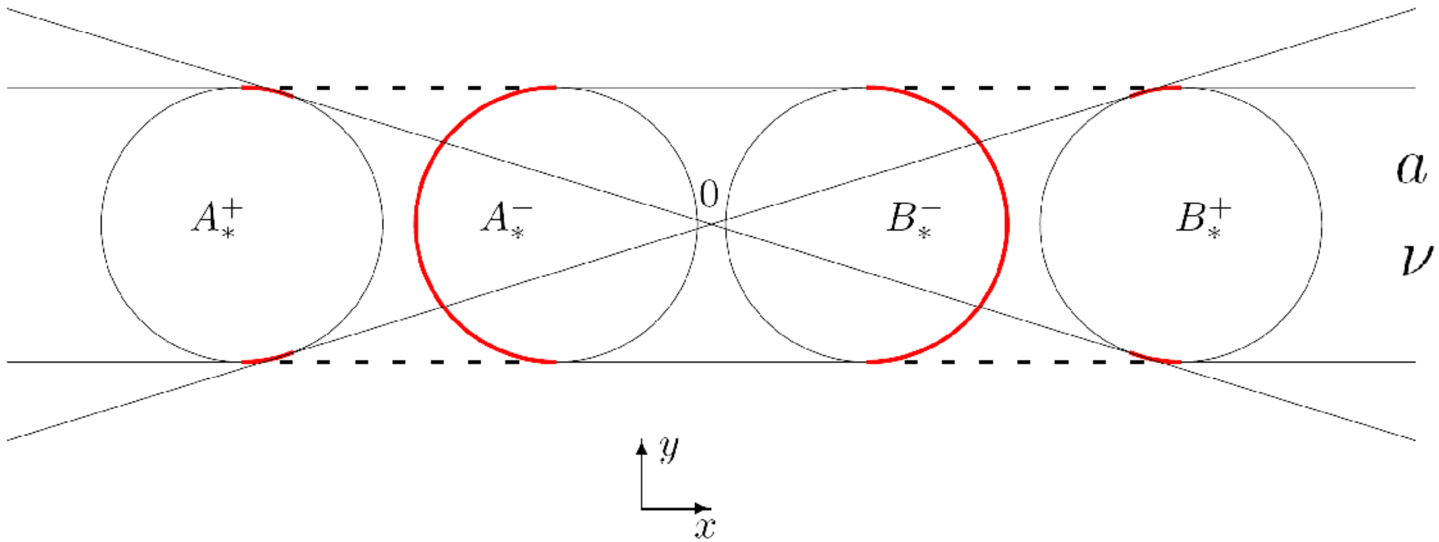
$$a = 1, \quad |v| \le \nu, \quad \nu \in (0,1)$$

$$A_* = \{(x,y):\ y + v_x = 0,\ -x - 1 + v_y = 0,\ |v| \le \nu\}$$

$$B_* = \{(x,y):\ -y + v_x = 0,\ x - 1 + v_y = 0,\ |v| \le \nu\}$$



Basis for families of semipermeable curves in reinforced homicidal chauffeur dynamics



$$a = 0.33$$

$$\nu = 0.3$$

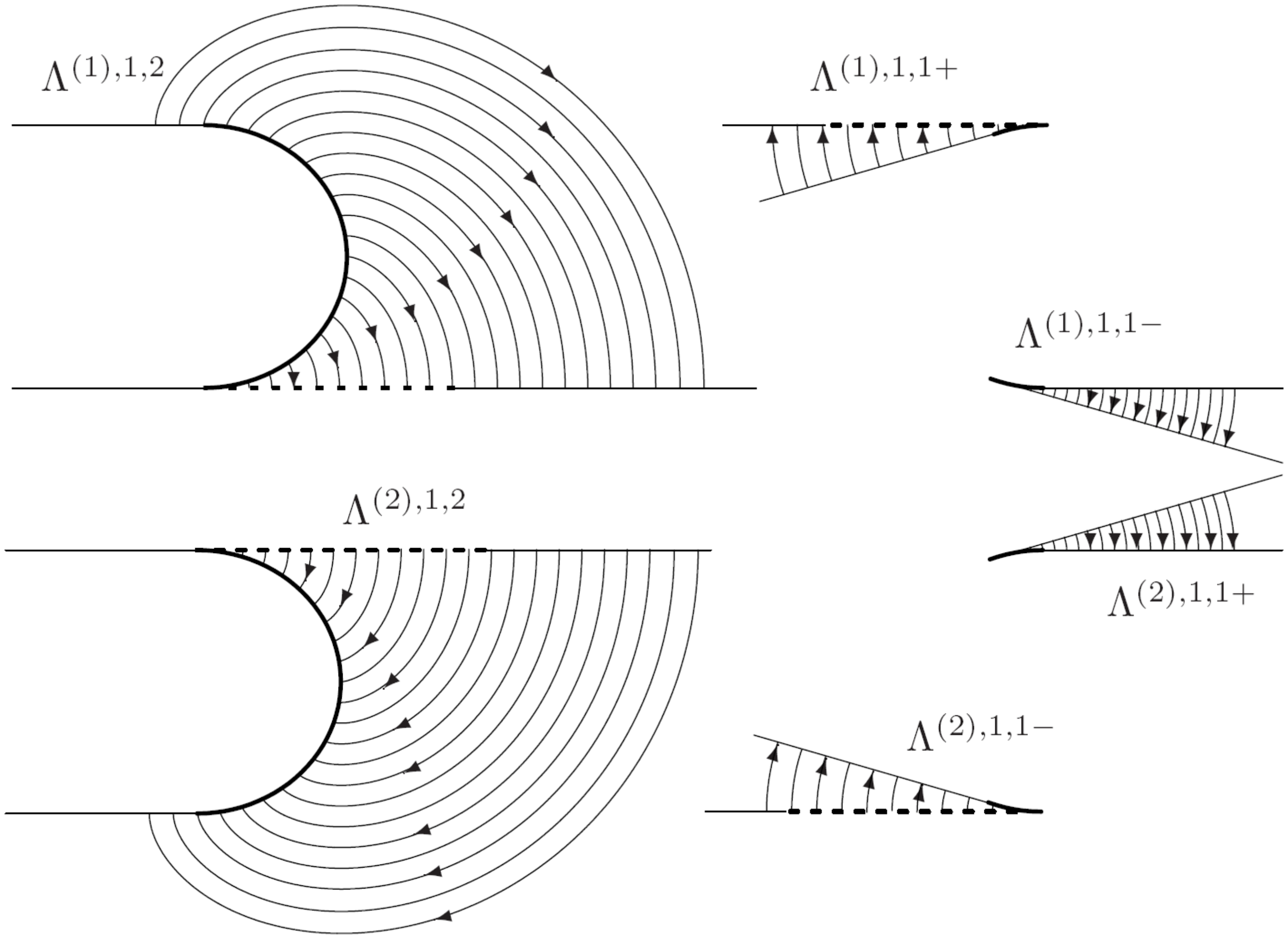
$$B_*^+ = \{(x, y) : \ -y + v_x = 0, \ x - 1 + v_y = 0, \ |v| \leq \nu\}$$

$$A_*^+ = -B_*^+$$

$$B_*^- = \{(x, y) : \ -y + v_x = 0, \ x - a + v_y = 0, \ |v| \leq \nu\}$$

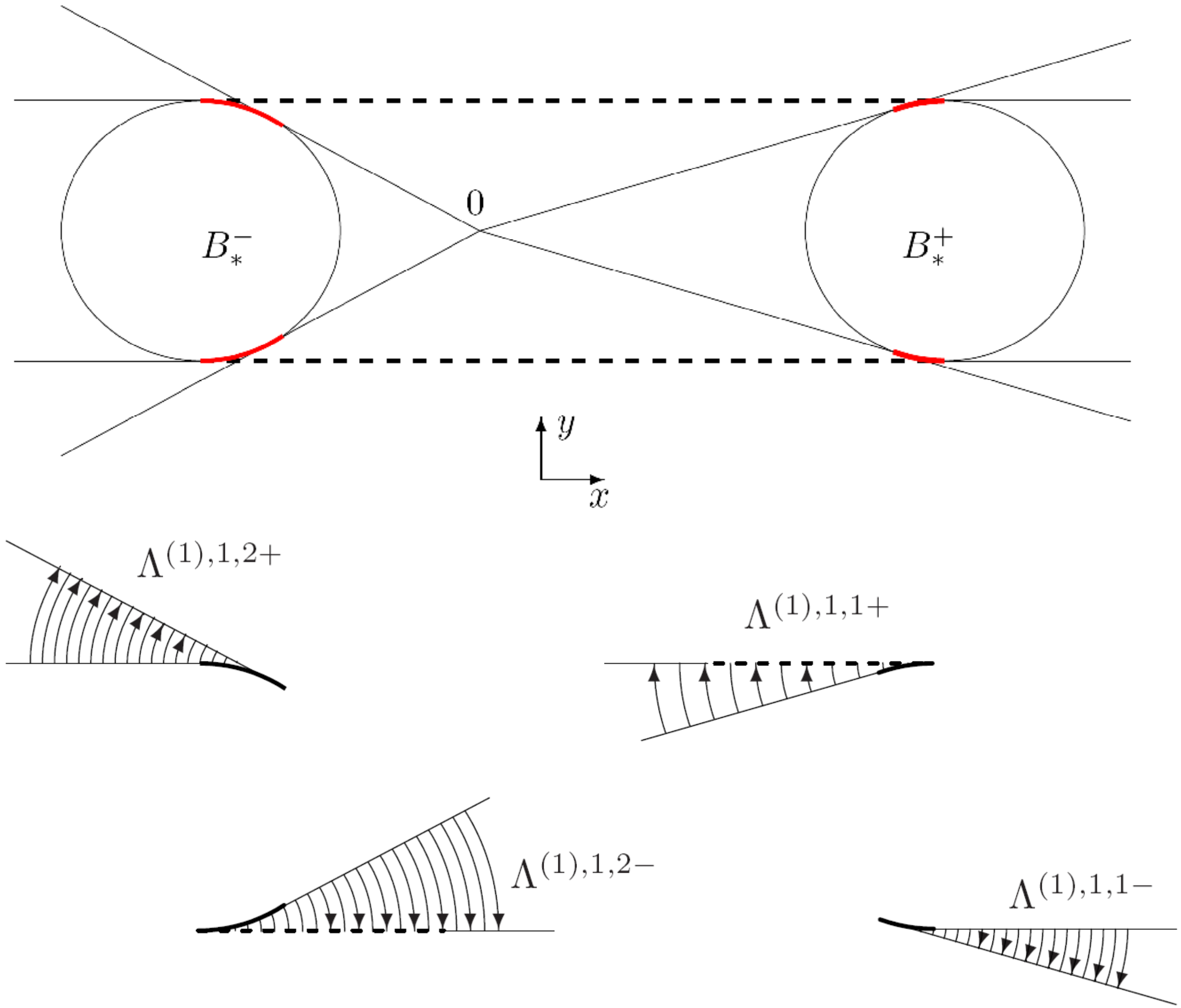
$$A_*^- = -B_*^-$$

Families of semipermeable curves for reinforced homicidal chauffeur dynamics ($a \geq -\nu$)



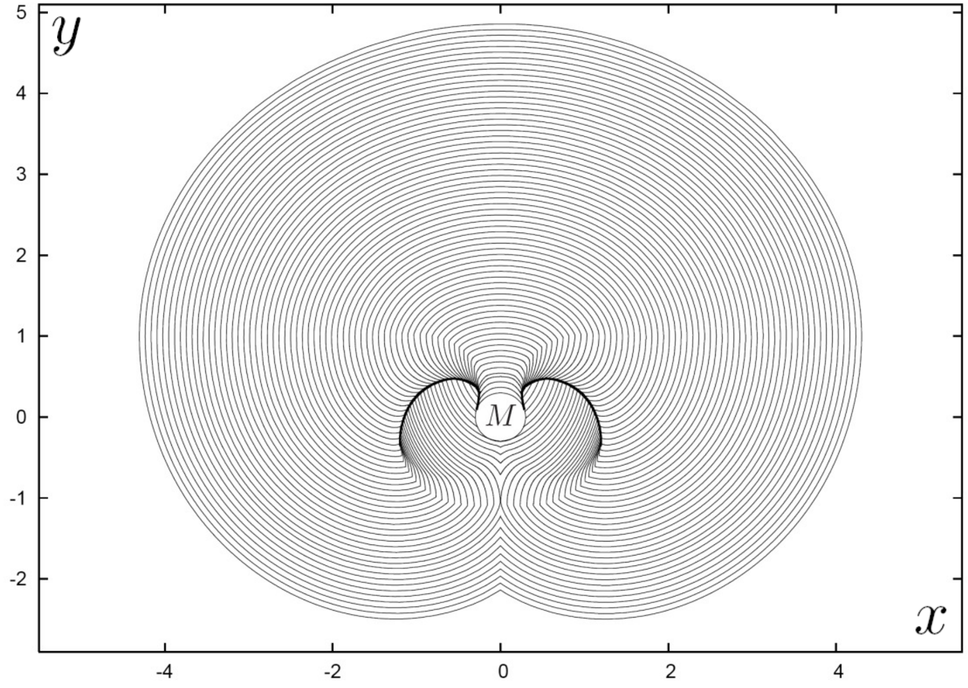
$\Lambda^{(2),2,1+}$, $\Lambda^{(2),2,1-}$, $\Lambda^{(2),2,2}$, $\Lambda^{(1),2,1+}$, $\Lambda^{(1),2,1-}$, $\Lambda^{(1),2,2}$

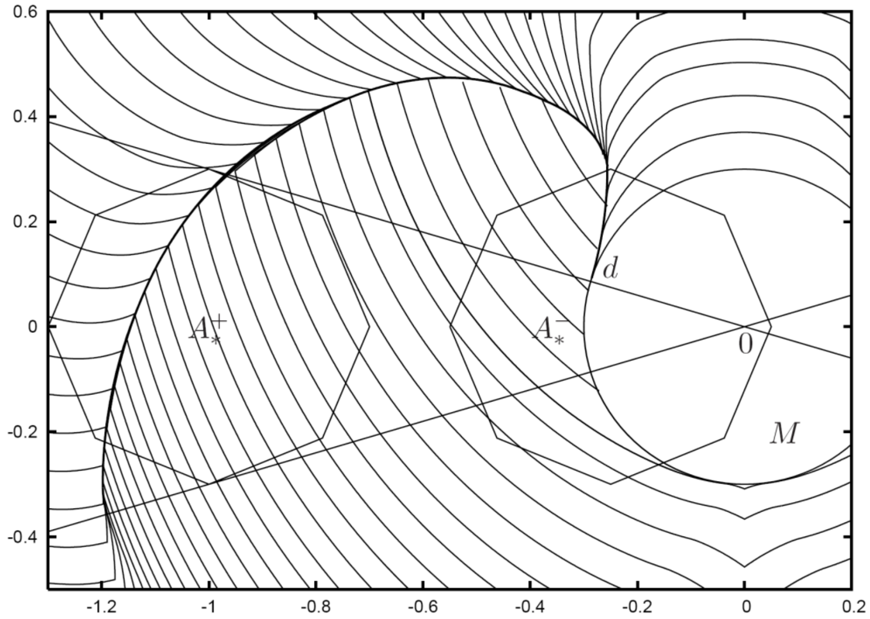
Families of semipermeable curves for reinforced homicidal chauffeur dynamics ($a \leq -\nu$)



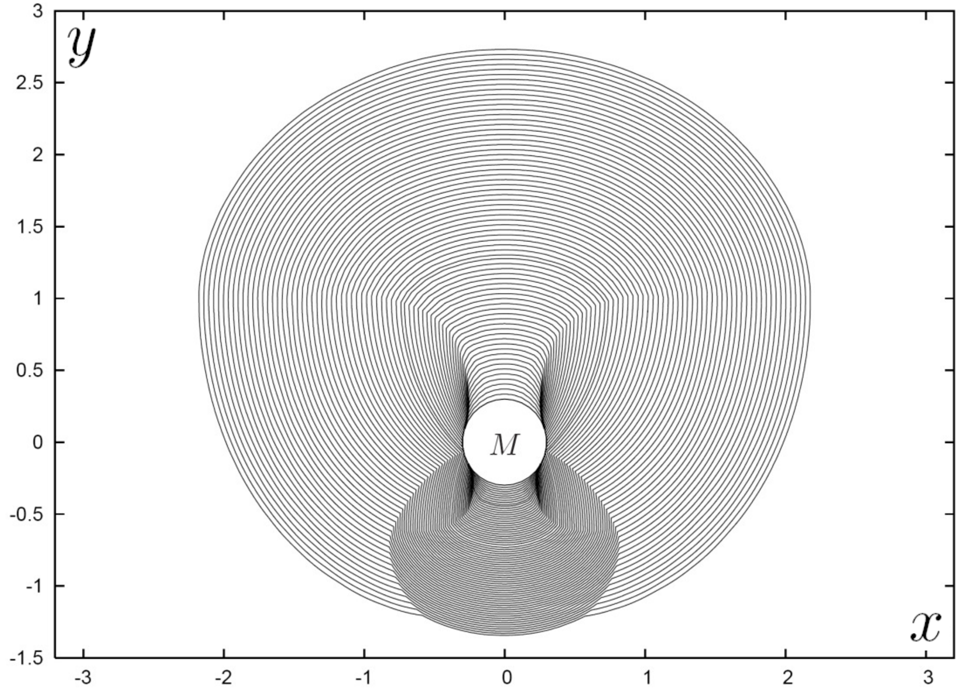
Level sets of the value function. Discontinuity lines

$$a = 0.25, \ \ \nu = 0.3\,, \ \ r = 0.3$$

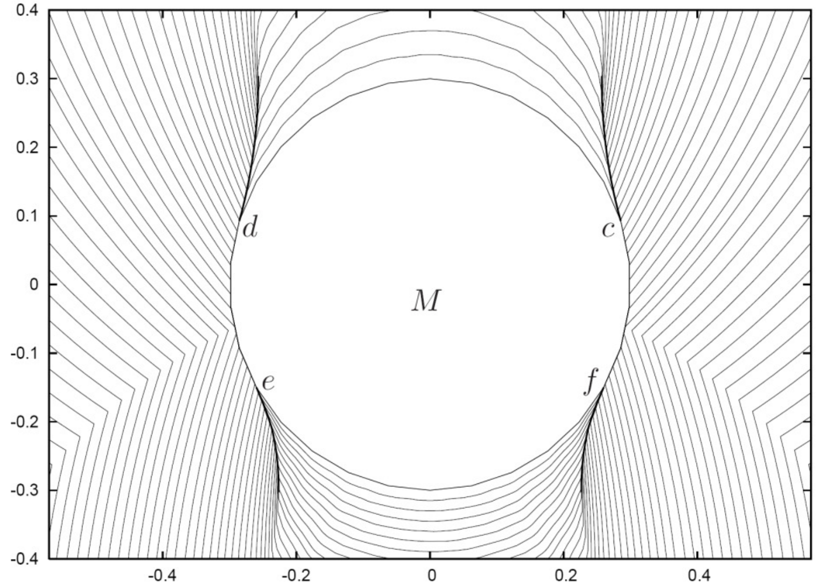




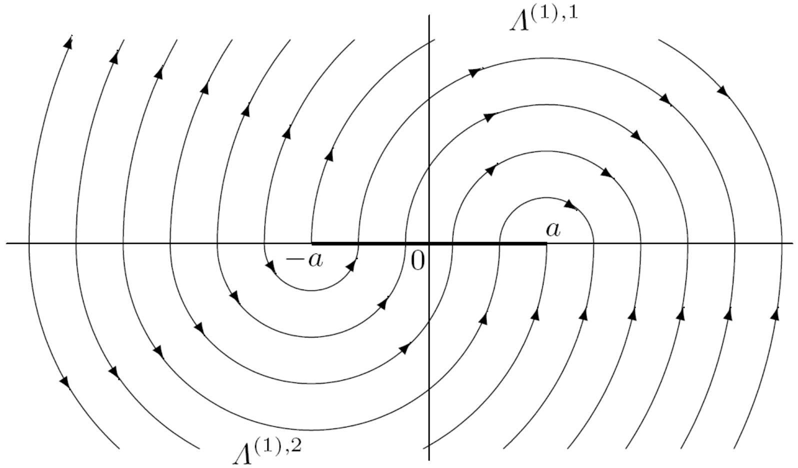
Level sets of the value function. Discontinuity lines

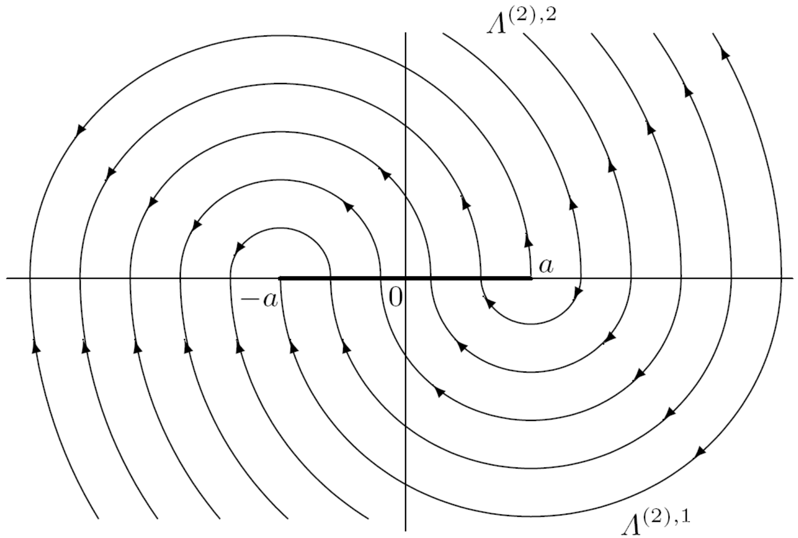


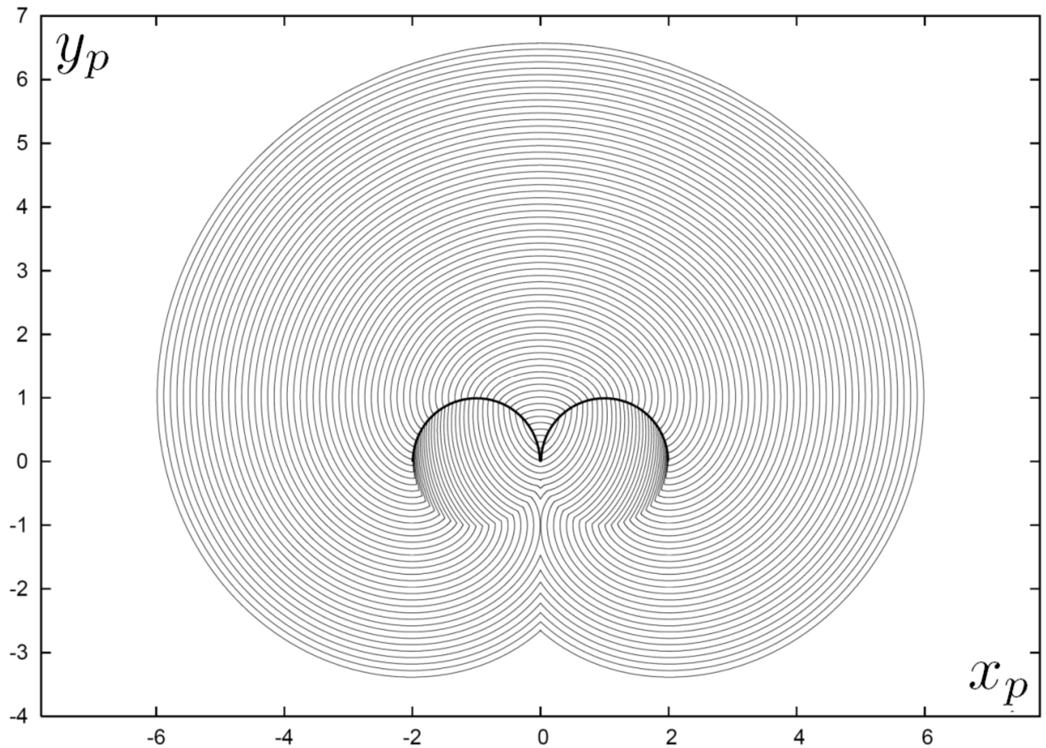
$$a = -0.6,\ \nu = 0.3,\ r = 0.3$$



Families of semipermeable curves for $a > 0,\ \nu = 0$







Related authors´works

V. S. Patsko, V. L. Turova (2001). Level sets of the value function in differential games with the homicidal chauffeur dynamics. *Int. Game Theory Review,* Vol.3, N°1, 67–112.

V. S. Patsko, V. L. Turova (2004). Families of semipermeable curves in differential games with the homicidal chauffeur dynamics. *Automatica,* Vol.40, N°12, 2059–2068.

V. S. Patsko, V. L. Turova (2007). Numerical study of the homicidal chauffeur differential game with the reinforced pursuer. *Game Theory and Applications,* Vol. 12, Chapter 8, 123–152.