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NUMERICAL SOLUTION OF TWO-DIMENSIONAL DIFFERENTIAL GAMES

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This paper is devoted to two-dimensional differential games. The problem of finding the set of all points for each of which the first player guarantees the achievement of a given target set within a finite time interval under any actions of the second player is considered (the game of kind). An algorithm based on operations of constructing and sewing semipermeable curves is given. One particular nonlinear game of kind is studied. An algorithm for finding level sets of the value function for linear minimum-time game problems is described. A large number of numerical examples which demonstrate specific features of solutions are presented. The paper can be useful for specialists working in the field of optimal control theory and differential games.

Editor: Corresponding member of Russian Academy of Sciences **A.I. Subbotin**

Reviewer: Professor **V.N. Ushakov**

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Introduction

This work [1] is devoted to two-dimensional differential games with the nonfixed time of termination. In such games, the first player tries to bring the state vector to a given target set, the second player tries to avoid the state vector from the meeting the target set.

According to the R.Isaacs terminology, the problem of finding the set of all states such that the first player guarantees the achievement of the target set is called the game of kind. In Chapter 1, an algorithm for solving games of kind with linear dynamics of the general form is given. The similar algorithm for one particular nonlinear system is described in Chapter 2. The main feature of these algorithms is that they are not based on the treatment of auxiliary problems where the time is considered to be the payoff function; the algorithms are based on the analysis of families of semipermeable curves. The number of such families is determined by the number of convexity-concavity cones of the Hamiltonian of a control system. The solution becomes more complicated when the number of such families increases. The complexity of problems of kind also depends on the form of the target set. The main attention is paid to the case where the target set is a single point in the plane.

An algorithm for finding level sets of value functions of linear minimum-time game problems is considered in Chapter 3. The basis of the algorithm is a backward procedure for finding fronts consisting of points where the value function is constant. Besides the front, the boundary of a level set of the value function is formed by barrier lines where the value function is discontinuous.

These algorithms were developed at the Institute of Mathematics and Mechanics of the Ural Department of the Russian Academy of Sciences in the late 1970s/ early 80s. They are related to the approach typical for the Ekaterinburg scientific school on differential games. The paper outlines only short schemes and basic ideas of the algorithms. The main points of interest are concrete examples.

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Basic notations

$\mathcal{B}$ – solvability set in the game of kind,

P – geometric restriction on the control parameter of the first player,

Q – geometric restriction on the control parameter of the second player,

m – target point in the game of kind,

H – Hamiltonian of a conflict-controlled system,

$\ell^{(1),i}$ – $+$ to $-$ zero of the function $H(\cdot, x)$ belonging to the set $\Gamma^{(1),i}$,

$\ell^{(2),i}$ – $-$ to $+$ zero of the function $H(\cdot, x)$ belonging to the set $\Gamma^{(2),i}$,

$\Phi^{(1),i}$ – family of the semipermeable curves of the first type,

$\Phi^{(2),i}$ – family of the semipermeable curves of the second type,

$p^{(n),i}$ – semipermeable curve belonging to the family $\Phi^{(n),i}$,

$\mathrm{g}^{(1)}$, $\mathrm{g}^{(2)}$ – piecewise smooth semipermeable curves of the first and the second types used in the process of construction of the set $\mathcal{B}$,

m_r – source point in the nonlinear game of kind,

M – target set in the minimum-time game problem,

Γ_0 – usable part of the target set,

Δ – step of backward procedure for the minimum-time game problem,

$W(i\Delta, M)$ – level set of the value function corresponding to the time $i\Delta$ and to the target set M in the minimum-time game problem,

F_i – front of the set $W(i\Delta, M)$.

Chapter 1

Linear two-dimensional differential games of kind

1. Statement of the problem

We consider the following system of the second order:

$$\dot{x} = Ax + u + v \tag{1.1}$$

$$x \in R^2, \ u \in P, \ v \in Q.$$

Here x is the state vector, A is a constant 2×2 matrix, u and v are control variables of the first and the second players bounded by geometric constraints P and Q, respectively. The sets P and Q are convex closed non-degenerated into points polygons. The first player strives to control system (1.1) so that the state vector arrives at a given point $m \in R^2$, the second player tries to prevent this. It is required to find the set $\mathcal{B}$ of all initial states $x_0 \in R^2$ such that a feedback strategy of the first player guarantees the transfer of system (1.1) from x_0 to m in some finite time under any actions of the second player.

In accordance with the terminology of R.Isaacs [1], problems of finding the set $\mathcal{B}$ are called games of kind. When studying the possibility for system (1.1) to be brought to m, we are only interested in reaching of m within a finite time interval but the duration of the process is not crucial. The estimate of the duration can depend on the initial point x_0 and can go to infinity when varying x_0 in the set $\mathcal{B}$. So, problems we consider are differential games with unbounded time of termination.

Let us define the set $\mathcal{B}$ more carefully. Let $\mathcal{U}$ be the set of all strategies U of the first player. Namely, this is the set of all functions defined on $R_+ \times R^2$ with values in P. Here, R_+ is the set of nonnegative reals. Let σ be an arbitrary partition of R_+ formed by points $0 < t_1 < t_2 < ...$ ($t_i \to \infty$ as $i \to \infty$), $d(\sigma)$ its diameter, and $v(\cdot)$ measurable function of time with values in Q. For fixed $\sigma, U, v(\cdot)$, we denote by $y(\cdot\,; \sigma, x_0, U, v(\cdot))$ the Euler spline [2, 3] of system (1.1) emanating from the point x_0. We denote by $\mathcal{B}$ the set of all $x_0 \in R^2$ for each of which there exist a strategy $U \in \mathcal{U}$, a time θ, and a mapping $\varepsilon \to \delta(\varepsilon)$ from R_+ into R_+ such that for any $\varepsilon > 0$, any partition σ with the diameter $d(\sigma) \leq \delta(\varepsilon)$, and any function $v(\cdot)$ with values in Q we can find a time $t \in [0, \theta]$ at which $y(t; \sigma, x_0, U, v(\cdot))$ lies in the ε-neighborhood of the point m.

By the alternative theorem [2, 3], for any point $x_0 \in R^2 \setminus \mathcal{B}$, the second player can prevent system (1.1) from attaining m within any prescribed finite time interval.

In order to simplify the algorithm of finding $\mathcal{B}$, we exclude from considerations the case of "one-type objects" where there exists a polyhedron D such that $P = -Q + D$. In this case, game (1.1) can be reduced [2, 4] to the following control problem

$$\dot{x} = Ax + w, \quad w \in D.$$

2. Smooth semipermeable curves

Our algorithm for finding the set $\mathcal{B}$ is based on the construction of special lines (semipermeable curves [1]) in the phase plane. Let us define smooth semipermeable curves.

We consider a smooth curve p in the plane (Fig. 1.1). Let $+$ and $-$ mark positive and negative sides of the curve. Denote by $\ell\ (x)$ the normal vector to p at a point x directed to the positive side of p. The curve p is called semipermeable if for any point x of this curve the following condition is fulfilled

$$\max_{u \in P} \min_{v \in Q} \langle \ell(x), Ax + u + v \rangle = 0. \tag{1.2}$$

Let us make the sense of the semipermeability property more precise. Let $u_* \in P$ be a control of the first player which gives the maximum to the left-hand side of (1.2). Then

$$\langle \ell(x), Ax + u_* + v \rangle \geq 0$$

for any control $v \in Q$ of the second player. This means that for $u = u_*$ the v-vectogram of system (1.1) lies in the positive half-space determined by the vector $\ell(x)$ (Fig. 1.1). In other words, the control u_* of the first player prevents trajectories of system (1.1) from penetrating from the positive side into the negative side of the curve p.

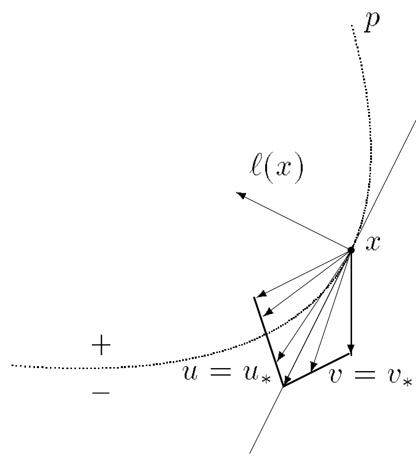


Fig. 1.1. The property of semipermeability.

Let now $v_* \in Q$ be a control of the second player giving the minimum in (1.2). Then

$$\langle \ell(x), Ax + u + v_* \rangle \leq 0$$

for any $u \in P$. This means that for $v = v_*$ the u-vectogram of system (1.1) belongs to the negative half-space determined by $\ell(x)$. So, the control v_* of the second player prevents trajectories of system (1.1) from penetrating from the negative side to the positive side of the curve p.

The solution methods for games of kind described in the book of R.Isaacs are based on constructing semipermeable curves. This methods are applicable to cases where two smooth semipermeable curves emanated from the target set and faced each to other by positive sides either have an intersection point or have infinite length. It can be often shown in this case that $\mathcal{B}$ coincides with the set contained between these two curves. However, in most interesting cases the smooth semipermeable curves emanating from the target set may not have points in common and one of them or both have finite length.

In [5–8] methods for constructing the set $\mathcal{B}$ based on a sewing the semipermeable curves are proposed. With these methods, the linear second-order differential game of kind considered in this chapter can be completely solved. A short sketch of these methods and arising from them algorithm for constructing $\mathcal{B}$ will be given in the next section. Other methods for finding $\mathcal{B}$ based on sewing semipermeable curves were studied in [9–12].

3. Families of semipermeable curves of first and second types

We consider the function

$$\varphi(\ell) = \max_{u \in P} \min_{v \in Q} \langle \ell, u + v \rangle = \max_{u \in P} \langle \ell, u \rangle + \min_{v \in Q} \langle \ell, v \rangle, \qquad \ell \in R^2.$$

Due to our assumptions about P and Q, the phase plane can be divided into even number of running in succession convex cones $K_1, ..., K_{2s}$, with the apex in the origin, the nonempty interior, and an opening less than π such that: 1) the function φ is concave for any odd $j = \overline{1, 2s}$ and is convex for any even $j = \overline{1, 2s}$; 2) the restriction of φ onto each K_j is not a linear function.

We denote by E an arbitrary consisting of $2s$ links closed polygonal line in the plane such that if E_j is its link numbered j, then $K_j = \cup_{\lambda \geq 0} \lambda E_j$. Let us agree that the cone K_{j+1} follows counterclockwise the cone K_j , $j = \overline{1, 2s-1}$ (Fig. 1.2). For any vectors ℓ_1, ℓ_2 with the ends belonging to E and not collinear one to other, the notation $\ell_1 < \ell_2$ means that the direction of the vector ℓ_1 after the counterclockwise rotation by not exceeding π angle coincides with that of the vector ℓ_2.

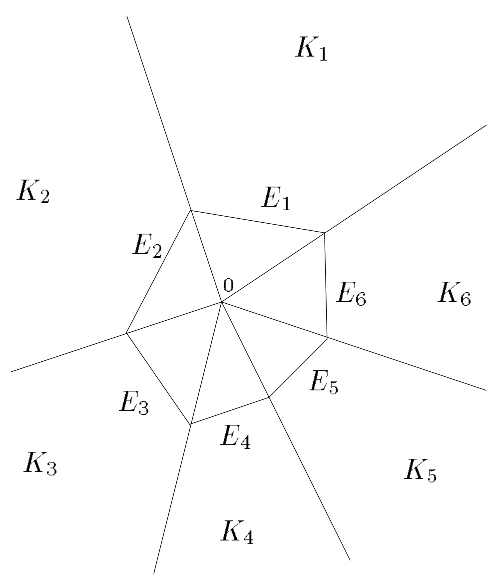


Fig. 1.2. Partition into the cones K_j.

We introduce the function

$$H(\ell, x) = \max_{u \in P} \min_{v \in Q} \langle \ell, Ax + u + v \rangle = \langle \ell, Ax \rangle + \varphi(\ell), \quad \ell \in R^2.$$

It is evident that the function $H(\cdot, x)$ inherits the convexity-concavity properties of the function φ in each of the cones K_j.

The semipermeable curves will be formed using zeros of the function $H(\cdot, x)$. We say that $\ell_* \in E$ is a $-$ to $+$ zero of the function $H(\cdot, x)$ if $H(\ell_*, x) = 0$ and

$$H(\ell, x) < 0 \quad (H(\ell, x) > 0)$$

for any $\ell < \ell_*$ $(\ell > \ell_*)$ that lies sufficiently close to ℓ_*.

We define $+$ to $-$ zeros of the function $H(\cdot, x)$ in the similar way.

We denote by $\Gamma^{(1),1}$, $\Gamma^{(1),2}$,..., $\Gamma^{(1),s}$ the pairwise unions $E_1 \cup E_2$, $E_3 \cup E_4$,..., $E_{2s-1} \cup E_{2s}$, respectively. Let $\Gamma^{(2),1}$, $\Gamma^{(2),2}$,..., $\Gamma^{(2),s}$ be the pairwise unions $E_2 \cup E_3$, $E_4 \cup E_5$,..., $E_{2s} \cup E_1$. Due to above mentioned convexity-concavity properties, the function $H(\cdot, x)$ may have at most one $+$ to $-$ zero belonging to $\Gamma^{(1),i}$, $i = \overline{1, s}$, and at most one $-$ to $+$ zero belonging to $\Gamma^{(2),i}$, $i = \overline{1, s}$. We denote by $S^{(1),i}$ $(S^{(2),i})$, $i = \overline{1, s}$, the set of all $x \in R^2$ for each of which there exists a $+$ to $-$ ($-$ to $+$) zero of the function $H(\cdot, x)$ belonging to $\Gamma^{(1),i}$ $(\Gamma^{(2),i})$. Let $\ell^{(1),i}(x) \in \Gamma^{(1),i}$ ($\ell^{(2),i}(x) \in \Gamma^{(2),i}$) be a vector which is a $+$ to $-$ ($-$ to $+$) zero. Function $\ell^{(n),i}(\cdot)$, $n = 1, 2$, $i = \overline{1, s}$, is locally Lipschitz on $S^{(n),i}$.

Let Π_+ and Π_- be matrices of the rotation by $\pi/2$ counterclockwise and clockwise, respectively.

We consider the differential equations

$$\frac{dz}{d\tau} = \Pi_+ \ell^{(1),i}(z), \quad z \in S^{(1),i}, \tag{1.3}$$

$$\frac{dz}{d\tau} = \Pi_- \ell^{(2),i}(z), \quad z \in S^{(2),i}. \tag{1.4}$$

We denote by $z^{(1),i}(\cdot, z_0)$ a solution of differential equation (1.3) satisfying the initial condition $z^{(1),i}(0, z_0) = z_0$ and maximally extended in both positive and negative directions with respect to τ. Similarly, let $z^{(2),i}(\cdot, z_0)$ be a solution of (1.4) satisfying the initial condition $z^{(2),i}(0, z_0) = z_0$. It follows from the form of equations (1.3), (1.4) that all phase trajectories of these equations are semipermeable curves. Corresponding to solutions phase trajectories form a family $\Phi^{(1),i}$ $(\Phi^{(2),i})$ filling the region $S^{(1),i}$ $(S^{(2),i})$. Curves of the first family obtained with the use of $+$ to $-$

zeros of $H(\cdot, x)$ will be called the semipermeable curves of the first type. Curves of the second family obtained with the use of $-$ to $+$ zeros will be called the semipermeable curves of the second type.

The number of families of semipermeable curves of the first and the second types depends on the number of convexity-concavity regions of the function φ. If P or Q is a segment, one can divide the plane into four running in succession convex cones K_j so that the function φ is concave in K_1, K_3 and is convex in K_2, K_4. Therefore, $s = 2$, and we have two families of semipermeable curves of the first type and two families of semipermeable curves of the second type.

4. Short scheme of algorithm for constructing solvability set $\mathcal{B}$. Computed examples for games of kind

There may occur that the set $\mathcal{B}$ consists of the single point m. This means that for any initial point $x_0 \neq m$ the second player can prevent system (1.1) from reaching the point m in any finite time. Necessary and sufficient conditions of the equality $\mathcal{B} = \{ \ m \ \}$ are given in [13–15].

The algorithm [5–8] for computing the set $\mathcal{B}$ is based on assembling some piecewise smooth curves $\mathrm{g}^{(1)}$, $\mathrm{g}^{(2)}$ from smooth semipermeable curves and on the analysis of mutual locations and intersections of $\mathrm{g}^{(1)}$, $\mathrm{g}^{(2)}$. The curve $\mathrm{g}^{(1)}$ is designed using arcs belonging to the families $\Phi^{(1),i}$, $i = \overline{1, s}$, and the curve $\mathrm{g}^{(2)}$ is composed of arcs belonging to the families $\Phi^{(2),i}$, $i = \overline{1, s}$.

In this paper we restrict ourselves by the explanation of the idea of the algorithm in the case where the set P is a segment and the set Q is a convex polygon in the plane. In this case, the curves of four families $\Phi^{(1),1}$, $\Phi^{(1),2}$, $\Phi^{(2),1}$, $\Phi^{(2),2}$ can be involved into the construction of a solution. We will also suppose that the matrix A has complex eigenvalues and the phase trajectories of the equation $\dot{x}(t) = Ax(t)$ go around the origin in the counterclockwise direction with t increasing.

We denote by $p^{(n),i}(a, b)$ a part of a semipermeable curve of the family $\Phi^{(n),i}$ that connects the points a, b (τ increases when we go from a to b). We denote by $p^{(n),i}(a)$ a part of a semipermeable curve of the family $\Phi^{(n),i}$ emanating from a point a and extended up to the boundary of the set $S^{(n),i}$.

Let $\hat{i} = 1$ when $i = 2$ and $\hat{i} = 2$ when $i = 1$.

The procedure of sewing semipermeable curves will refer to the notion of sprout points. A point b is called the sprout point of a curve $p^{(n),i}(a)$ if it is the first point (when moving from the end of this curve) such that :

1) there exists a zero ℓ_0 of the function $H(\cdot, b)$ belonging to the set $\Gamma^{(n),\widehat{i}}$;

2) the composed curve $p^{(n),i}(a,b) \cup p^{(n),\widehat{i}}(b)$ has the semipermeability property. Condition 2) can also be formulated in terms of zeros of $H(\cdot, x)$. Namely: a) the angle between the vectors $\ell^{(n),i}(b)$ and ℓ_0 is less than π; b) for any vector $\ell \in E$, $\ell^{(n),i}(b) < \ell < \ell_0$, the inequality $(-1)^n H(\ell, b) \geq 0$ holds; c) for any $\ell \in E$ which is sufficiently closed to ℓ_0 and such that $l > \ell_0$, we have $(-1)^n H(\ell, b) > 0$.

At the first step of the algorithm we check the validity of the relation $\mathcal{B} \neq \{ m \}$. Except for some subtle cases which rarely appear in practice, it is sufficient [15] to verify that the function $H(\cdot, m)$ has only one $+$ to $-$ zero, only one $-$ to $+$ zero, and the angle between these zeros taken counterclockwise from the first zero to the second one is less than π, to recognize that $\mathcal{B} \neq \{ m \}$. We suppose that this condition is fulfilled and vectors $\ell^{(1),i_1}(m)$ and $\ell^{(2),i_2}(m)$ are, respectively, $+$ to $-$ and $-$ to $+$ zeros mentioned in the condition.

In the description of the procedure for constructing $\mathrm{g}^{(1)}$, $\mathrm{g}^{(2)}$ we will use the term "branch" for indication of appearing auxiliary curves. This term seems to be meaningful because these auxiliary curves may ramify from each other like branches. We denote by $\mathrm{g}_k^{(n)}$ the kth branch of $\mathrm{g}^{(n)}$. We begin with the branch

$$\mathrm{g}_1^{(2)} = \begin{cases} p^{(2),i_2}(m), & i_1 \neq i_2 \\ p^{(2),i_2}(m, q_1^{(2)}) \cup p^{(2),\widehat{i_2}}(q_1^{(2)}), & i_1 = i_2. \end{cases}$$

Here $q_1^{(2)}$ is a sprout point of the curve $p^{(2),i_2}(m)$.

Then we construct the branches $\mathrm{g}_1^{(1)}, \mathrm{g}_2^{(2)}, \mathrm{g}_2^{(1)}, \ldots$ of the curves $\mathrm{g}^{(1)}, \mathrm{g}^{(2)}$ alternating the indices 1 and 2. Each curve in this sequence is a semipermeable curve $p^{(n),i}(q_k^{(n)})$ (with an appropriate index i) emanating from the sprout point $q_k^{(n)}$ of the previous curve. It is assumed that $q_1^{(1)} = m$. Before starting a new branch of the curve $\mathrm{g}^{(n)}$, we change the second index of the family of semipermeable curves. If, for example, we have used the family $\Phi^{(n),i}$ for constructing $\mathrm{g}_k^{(n)}$, then we are to employ the family $\Phi^{(n),\widehat{i}}$ when constructing $\mathrm{g}_{k+1}^{(n)}$. To construct the next branch $\mathrm{g}_k^{(n)}$, we are to find its starting point $q_k^{(n)}$. If the sprout point for the next branch does not exist, then the construction of $\mathrm{g}^{(n)}$ is finished. The process of developing the curve $\mathrm{g}^{(n)}$ can be represented by the sequence $\mathrm{g}_1^{(n)}(m, q_2^{(n)})$, $\mathrm{g}_2^{(n)}(q_2^{(n)}, q_3^{(n)}), \ldots$, where $\mathrm{g}_k^{(n)}(a,b)$ denotes the arc of the curve $\mathrm{g}_k^{(n)}$ drawn from the point a up to the point b.

In the case considered (the matrix A has complex eigenvalues), the curves $g^{(1)}, g^{(2)}$ can be twisting and untwisting spirals. In the course of the construction we analyse mutual locations of $g^{(1)}, g^{(2)}$.

When constructing the curve $g_k^{(1)}$, we verify the intersection with the part of the curve $g^{(2)}$ available by this stage. This type of intersection will be further called α-intersection. The typical form of the set $\mathcal{B}$ in the case of α-intersection is shown in Fig. 1.3. The part of the curve $g^{(2)}$ disposed beyond the intersection point does not give any contribution to the boundary of the set $\mathcal{B}$.

When constructing the curve $g_k^{(2)}$, we verify the intersection with the part of the curve $g^{(1)}$ obtained by this stage. This type of intersection will be called ξ-intersection. The form of the set $\mathcal{B}$ in the case of ξ-intersection is shown in Fig. 1.4. The part of the curve $g^{(1)}$ which is constructed but does not give any contribution to the boundary of $\mathcal{B}$ is drawn with a thin line.

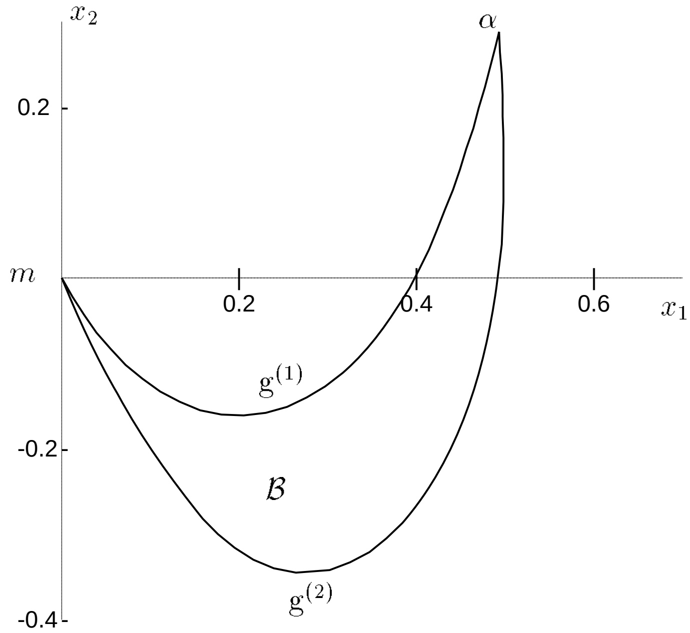
If the curve $g^{(1)}$ is a twisting spiral and has a common point with the curve $g^{(2)}$, we say that these curves have β-intersection. The structure of the algorithm in this case looks like that of one in the case of ξ-intersection. The set Ω bounded by the curves $g^{(1)}(m, \beta), g^{(2)}(m, \beta)$ does not belong to $\mathcal{B}$. The form of the set $\mathcal{B}$ in the case of β-intersection is determined by the behavior of the curve $g^{(2)}$ beyond the point of β-intersection. Namely, if $g^{(2)}$ has a limit cycle, then $\mathrm{cl}\mathcal{B} = \mathrm{cl}(\Omega^{(2)} \setminus \Omega)$ where $\Omega^{(2)}$ is the set bounded by this limit cycle ; if there is no limit cycle of $g^{(2)}$, we have $\mathrm{cl}\mathcal{B} = \mathrm{cl}(R^2 \setminus \Omega)$. An example of β-intersection is given in Fig. 1.5. The curve $g^{(2)}$ is an untwisting spiral which has not any limit cycle. The set $\mathcal{B}$ is the whole plane excluding the open set bounded by the curves $g^{(1)}(m, \beta), g^{(2)}(m, \beta)$. The part of the curve $g^{(2)}$ (the untwisting spiral) lying beyond the point of β-intersection is shown with a thin line.

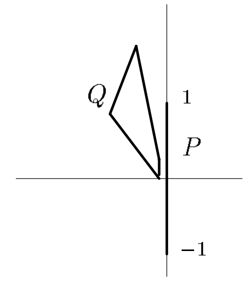
It should be noted that we verify the intersections not for the whole above considered curves but for certain fragments of the curves which are specified by the algorithm.

If in the course of constructions we obtain an intersection of the curves $g^{(1)}$ and $g^{(2)}$, then this intersection is related to one of three following types: α, ξ, or β-intersection. If the curves do not intersect each other, then the following cases may occur.

The curves $g^{(1)}, g^{(2)}$ are untwisting spirals. The set $\mathcal{B}$ is bounded by the curves $g^{(1)}$ and $g^{(2)}$; $\mathcal{B}$ is infinite but does not coincide with the whole plane (Fig. 1.6).

The curve $g^{(2)}$ is an untwisting spiral. If it has a limit cycle, we denote by $\Omega^{(2)}$ the open set bounded by this cycle. If there is not any limit cycle, we put

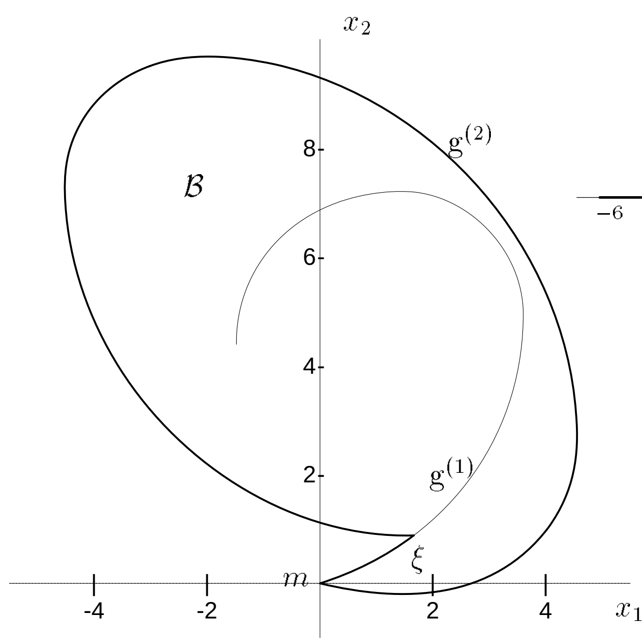


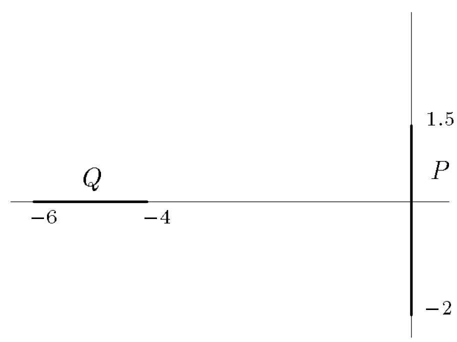


$$\dot{x}_1 = x_2 + u_1 + v_1$$

$$\dot{x}_2 = -4x_1 + u_2 + v_2$$

Fig. 1.3. The set $\mathcal{B}$ in case of α-intersection.

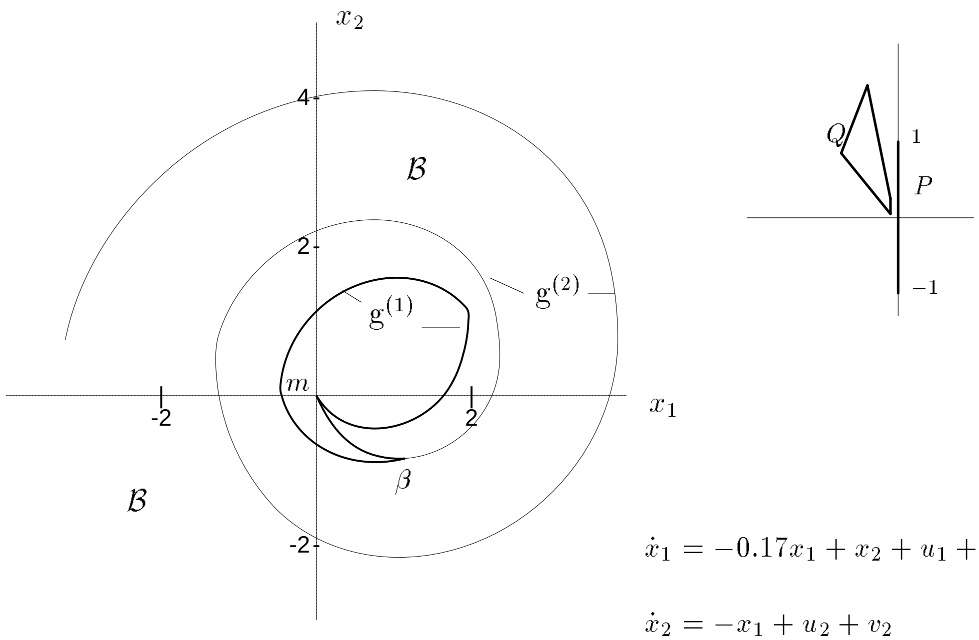




$$\dot{x}_1 = 0.3x_1 + x_2 + u_1 + v_1$$

$$\dot{x}_2 = -x_1 + u_2 + v_2$$

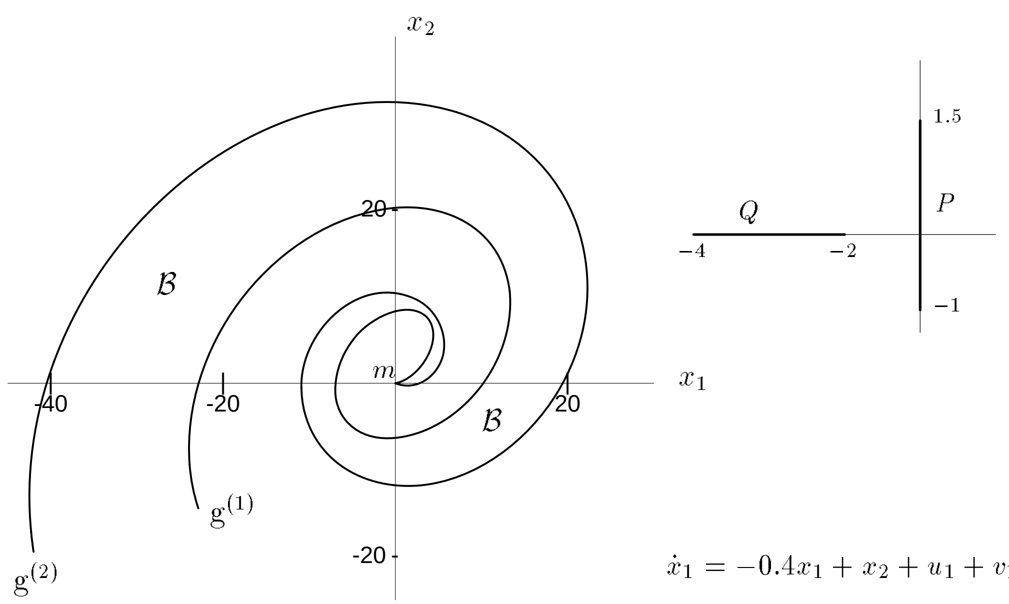
Fig. 1.4. The set $\mathcal{B}$ in case of ξ-intersection.



$$\dot{x}_1 = -0.17x_1 + x_2 + u_1 + v_1$$

$$\dot{x}_2 = -x_1 + u_2 + v_2$$

Fig. 1.5. The set $\mathcal{B}$ in case of β-intersection.



$$\dot{x}_1 = -0.4x_1 + x_2 + u_1 + v_1$$

$$\dot{x}_2 = -x_1 + u_2 + v_2$$

Fig. 1.6. The set $\mathcal{B}$ is an infinite “ribbon”.

$\Omega^{(2)} = R^2$. The curve $g^{(1)}$ is either a twisting spiral winding down onto its own limit cycle or a finite curve whose construction is finished because the corresponding sprout point does not exist. If $g^{(1)}$ has a limit cycle, we denote by $\Omega^{(1)}$ the closed set bounded by this limit cycle. If there is not any limit cycle, we put $\Omega^{(1)} = \emptyset$. In all these cases, $\mathcal{B} = \Omega^{(2)} \setminus \Omega^{(1)}$.

The set $\mathcal{B}$ may be not closed: some fragments of the boundary may not belong to $\mathcal{B}$. For example, if $\mathcal{B}$ is bounded by a limit cycle of the curve $g^{(2)}$, the limit cycle itself does not belong to $\mathcal{B}$.

The above mentioned examples were computed with the use of the program described in [8].

In the papers [5, 6], the existence of a strategy of the first player which guarantees the attainment of the point m within a finite time interval was proved in the case where the set P is an arbitrary segment. The proof is based on the dividing the set $\mathcal{B}$ into elementary cells (curvilinear polygons) $\Lambda_1, ..., \Lambda_d$. The cell Λ_1 adjoins to the point m. Each type of cells is associated with the control of the first player which has a constant value in the interior of a cell and has some switching structure on the cell's boundary. Such a control ensures transfer from the current cell to a cell with a lesser index and, finally, from Λ_1 to m.

If, for constant values of u, v, phase trajectories of system (1.1) are circles, the set $\mathcal{B}$ can be constructed with a pair of compasses. Let us explain this using the following example. Consider the control system

$$\begin{aligned} \dot{x}_1 &= \quad x_2 + u_1 + v_1 \\ \dot{x}_2 &= -x_1 + u_2 + v_2, \end{aligned} \tag{1.5}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in P, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in Q.$$

The sets P, Q are shown in Fig. 1.7. The set P is the vertical segment of the length 2μ which is symmetric with respect to the origin.

We have $u_1 \equiv 0$, which means that the control variable of the first player takes scalar values; the relevant constraint can be written as $| u_2 | \leq \mu$. We assume that μ is sufficiently large. We denote by h_1, h_2 the vertices of P, and by γ_1, γ_2 the normal vectors. The set Q is the quadrangle with the vertices r_1, r_2, r_3, r_4. We denote by $\eta_1, \eta_2, \eta_3, \eta_4$ the inward normals to Q.

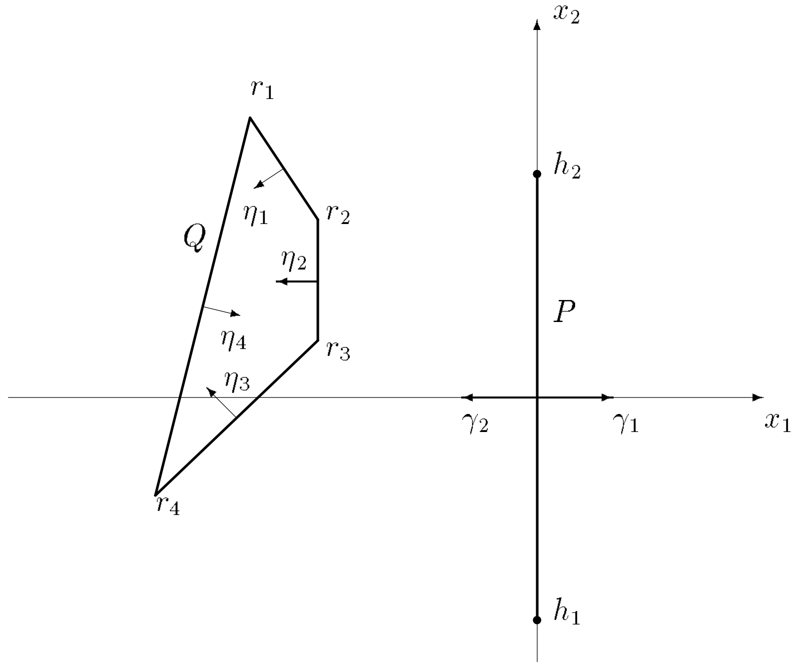


Fig. 1.7. The sets P, Q for game (1.5).

We consider the partition of the plane into the cones $K_1, ..., K_4$ (Fig. 1.8). Let E be corresponding to this partition closed polygonal line consisting of four links $E_1, ..., E_4$. The function $\varphi(\ell) = \max_{u\in P}\min_{v\in Q}\langle \ell, u+v\rangle$ is concave in the cone K_3 because the normals γ_1, γ_2 do not belong to K_3 (so, for any $\ell \in K_3$, the same element of P gives maximum to the expression $\max_{u\in P}\langle l, u\rangle$) but the normal η_3 to the set Q belongs to K_3. For similar reasons, the function φ is concave in the cone K_1. The function φ is convex in K_2, since any normal of Q does not belong to K_2 (so, the same element of Q gives minimum to the expression $\min_{v\in Q}\langle \ell, v\rangle$) but the normal γ_1 to the set P belongs to K_2. The cone K_4 containes the normals γ_2, η_2 to the sets P and Q and these normals have the same direction. The function φ is convex in K_4 because the length of the segment P is greater than the length of the parallel segment $[r_2, r_3]$.

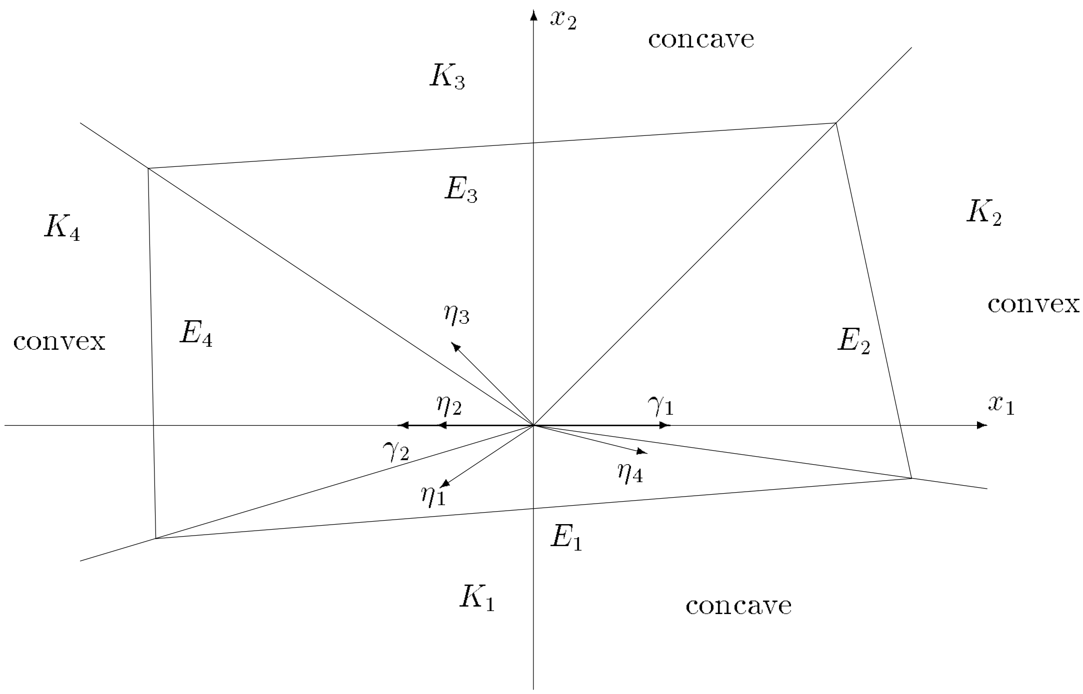


Fig. 1.8. Partition into the cones for game (1.5).

We denote $\Gamma^{(1),1} = E_1 \cup E_2,\ \Gamma^{(1),2} = E_3 \cup E_4,\ \Gamma^{(2),1} = E_2 \cup E_3,\ \Gamma^{(2),2} = E_4 \cup E_1$. We put

$$\widetilde{J} = \mathrm{cl}((P+Q) \setminus ((Q+h_1) \cup (Q+h_2))).$$

The set $\widetilde{J}$ is drawn in Fig. 1.9. Each vertex of the polygonal line bounding the set $\widetilde{J}$ has the form $u+v$, and

$$\widetilde{a} = h_1 + r_4, \quad \widetilde{b} = h_2 + r_4, \quad \widetilde{c} = h_2 + r_3, \quad \widetilde{d} = h_1 + r_2, \quad \widetilde{e} = h_1 + r_1.$$

Let $\widetilde{S}^{(n),i}$ be the set of all $y \in R^2$ for each of which there exist a + to − zero of the function $\max\limits_{u\in P} \min\limits_{v\in Q} \langle \ell, u+v-y \rangle$ in $\Gamma^{(n),i}$ if $n=1$ and, on the contrary, a − to + zero in $\Gamma^{(n),i}$ if $n=2$. It is easy to see that the sets $\widetilde{S}^{(n),i}$ have the form shown in Fig. 1.10 in the example considered. We obtain

$$S^{(n),i} = \{x \in R^2 \colon -Ax = y,\ y \in \widetilde{S}^{(n),i}\}, \quad n = 1,2,\ i = 1,2.$$

We put

$$J = \{x \in R^2 \colon -Ax = y,\ y \in \widetilde{J}\}.$$

In this example,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

So, the sets $S^{(n),i}$, J can be obtained via rotation of the sets $\widetilde{S}^{(n),i}$, $\widetilde{J}$ by $\pi/2$ clockwise.

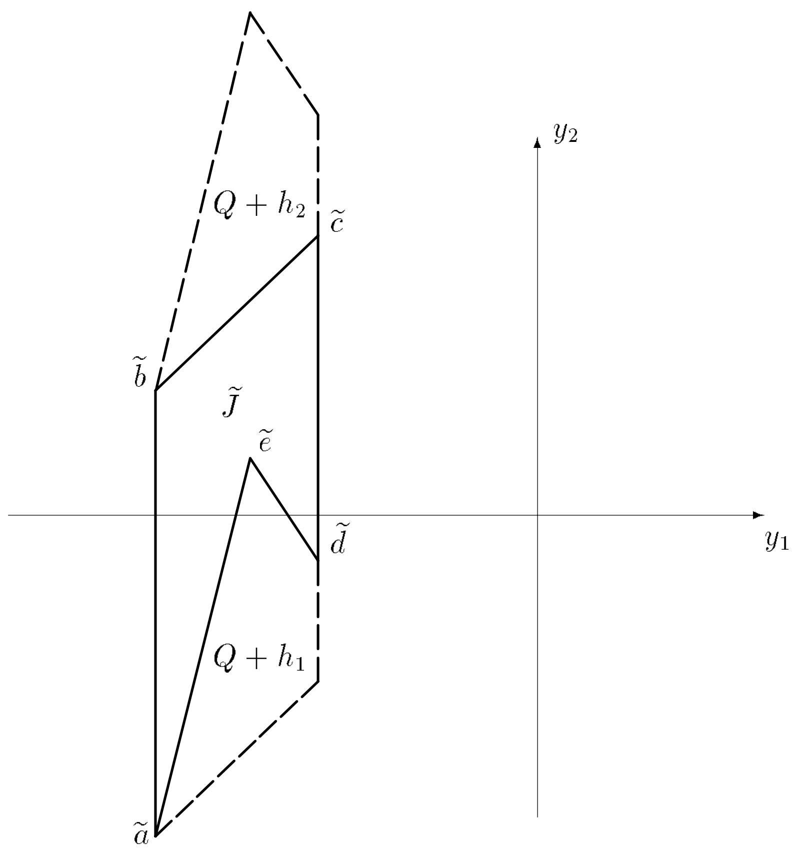


Fig. 1.9. Construction of the set $\widetilde{J}$.

For any point x which does not belong to J, there exist one + to − zero and one − to + zero of $H(\cdot, x)$. For any point from the interior of J, there exist two zeros of each type.

Semipermeable curves of families $\Phi^{(n),i}$ are composed of arcs of circles whose centers are points a, b, c, d, e. Figure 1.11 shows, for example, the semipermeable curve $p^{(1),1}$ belonging to the family $\Phi^{(1),1}$. This curve is composed of the arcs w_1w_2, w_2w_3, w_3w_4, w_4w_5 whose centers are the points d, e, a, b, respectively. Figure 1.12 shows the semipermeable curve $p^{(2),1}$ belonging to the family $\Phi^{(2),1}$. The arcs f_1f_2 and f_2f_3 have centers at b and c, respectively.

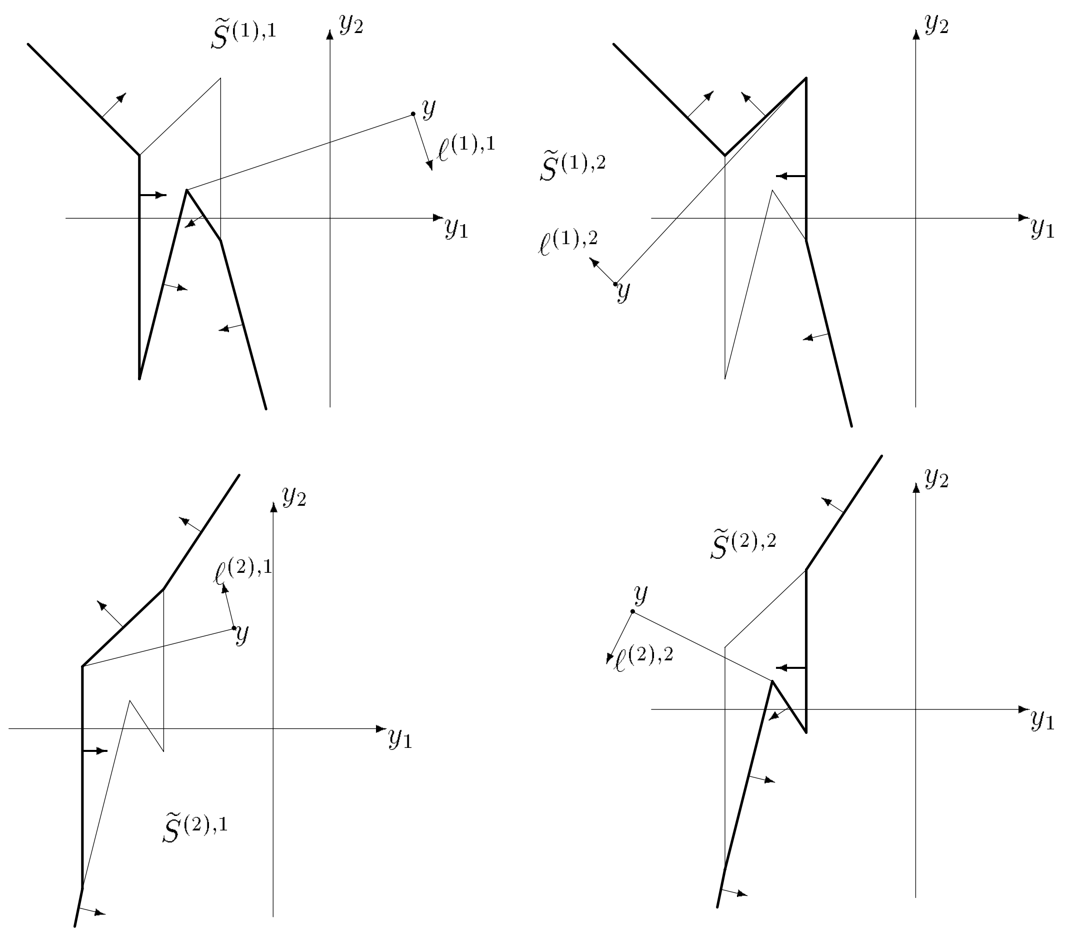


Fig. 1.10. The form of the sets $\widetilde{S}^{(n),i}$.

Let us explain how does the construction of the set $\mathcal{B}$ go, when the target point $m \in S^{(1),2} \bigcap S^{(2),2}$ (Fig. 1.13).

The semipermeable curve $p^{(2),2}(m)$ of the family $\Phi^{(2),2}$ is an arc of a circle with the center at e. We draw the arc $p^{(2),2}(m)$ up to the boundary of the set $S^{(2),2}$. The end point of this curve is denoted by ω. We seek a sprout point on the curve obtained. Starting with the end point ω, we search through points of $p^{(2),2}(m, \omega)$ and find the first point $q_1^{(2)}$ such that conditions 1),2) (see page 11) are fulfilled. It should be noted that conditions 1), 2a), 2c) hold for all points of $p^{(2),1}(q_1^{(2)}, \omega)$, while condition 2b) is valid for the point $q_1^{(2)}$ only. We draw the curve $p^{(2),1}(q_1^{(2)})$ of the family $\Phi^{(2),1}$ from the point $q_1^{(2)}$ (it is composed of arcs with the centers

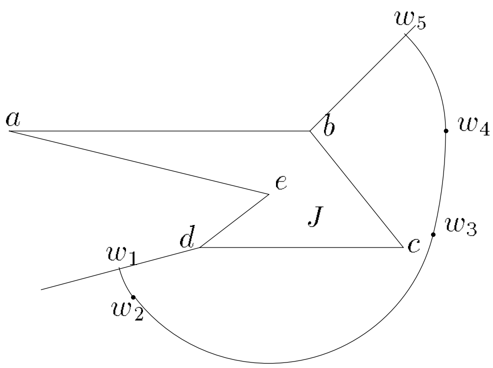


Fig. 1.11. A semipermeable curve of the family $\Phi^{(1),1}$.

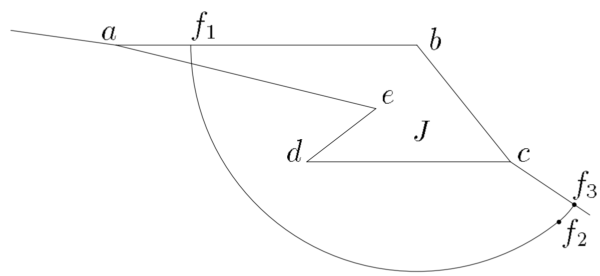


Fig. 1.12. A semipermeable curve of the family $\Phi^{(2),1}$.

at b and c). We continue this curve up to the boundary of the set $S^{(2),1}$. The curve $\mathrm{g}_1^{(2)}$ is formed by two smooth pieces

$$\mathrm{g}_1^{(2)} = p^{(2),2}(m, q_1^{(2)}) \cup p^{(2),1}(q_1^{(2)}).$$

Then we construct the first branch of $\mathrm{g}^{(1)}$. The curve $p^{(1),2}(m)$ is composed of arcs with centers at c and d. When constructing this curve, we test α-intersection with the curve $\mathrm{g}_1^{(2)}$. The α-intersection does exist. The construction is finished. The set $\mathcal{B}$ is bounded by the curves $\mathrm{g}_1^{(2)}(m, \alpha)$, $\mathrm{g}_1^{(1)}(m, \alpha)$.

In Fig. 1.13, positive and negative sides of curves are indicated with marks. A feedback control transferring the system to the point m can be designed using the following scheme: being originated from any point $x_0 \in \mathcal{B}$, any motion of system (1.1) with $u_2 = \mu$ meets the curve $p^{(2),2}(m, q_1^{(2)})$; switching the control for $u_2 = -\mu$ prevents the motion from penetrating into the negative side of this curve; if the motion comes down to the positive side of the curve $p^{(2),2}(m, q_1^{(2)})$, then after some time we put $u_2 = \mu$ again, and so on; the sliding mode along the curve $p^{(2),2}(m, q_1^{(2)})$ ensures the movement towards the point m.

The set $\mathcal{B}$ for some other target point $m \in S^{(1),1} \bigcap S^{(2),1}$ is shown in Fig. 1.14. In this specially selected case, $\mathcal{B}$ depends discontinuously on m : very small

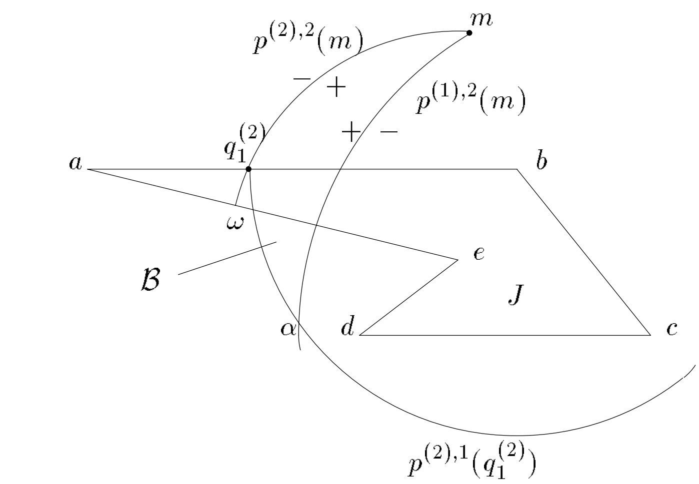


Fig. 1.13. Construction of the set $\mathcal{B}$ for $m \in S^{(1),2} \bigcap S^{(2),2}$.

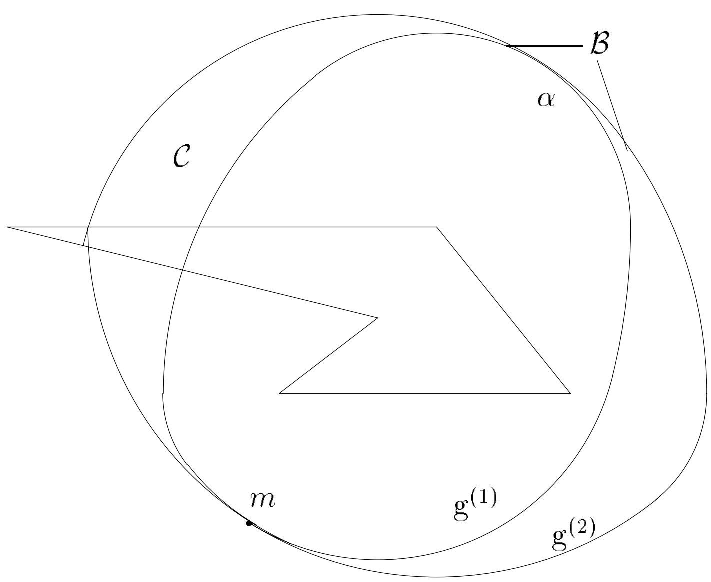


Fig. 1.14. The set $\mathcal{B}$ for $m \in S^{(1),1} \bigcap S^{(2),1}$.
The case of tangency of $\mathrm{g}^{(1)}$ and $\mathrm{g}^{(2)}$.

displacements of m cause violent changes of $\mathcal{B}$. In such a situation it is important which of formalizations is used. With the assumed by the authors formalization that corresponds to [2, 3], the set $\mathcal{B}$ contains the set $\mathcal{C}$ lying beyond the tangent point α. The set $\mathcal{C}$ may not belong to $\mathcal{B}$ for other formalizations. It should be noted that in examples shown in Fig. 1.13 and in Fig. 1.14 the set $\mathcal{B}$ is not closed: the arc $\alpha q_1^{(2)}$ in Fig. 1.13 and the similar arc in Fig. 1.14 do not belong to $\mathcal{B}$.

The sets $\Gamma^{(n),i}$ in the example examined were introduced via the polygonal line E with the links $E_1,\ E_2,\ E_3,\ E_4$. Another variant of setting the sets $\Gamma^{(n),i}$ is their definition via two parallel horizontal straight lines which we denote for convenience by $E_1,\ E_3$. The line E_1 lies below the origin, the line E_3 lies above the origin. Since P is the vertical segment, the function φ is concave on E_1 and E_3. Specification of $\Gamma^{(n),i}:\ \Gamma^{(1),1} = E_1,\ \Gamma^{(1),2} = E_3,\ \Gamma^{(2),1} = E_3,\ \Gamma^{(2),2} = E_1$. Such a definition may be more convenient in some cases and it will be used in Chapter 2. It should be taken into account that by this definition horizontal vectors do not belong to $\Gamma^{(n),i}$.

The computational algorithm for finding the set $\mathcal{B}$ does not use any explicit geometric description of the sets $\widetilde{S}^{(n),i},\ S^{(n),i},\ \widetilde{J},\ J$. These sets are useful if we construct the set $\mathcal{B}$ "by hand". The program for finding the set $\mathcal{B}$ in the case of an arbitrary matrix A with complex eigenvalues and arbitrary convex polygons $P,\ Q$ is given in [8]. This program computes associated with the point m semipermeable curves using zeros of function H, analyses mutual dispositions of these curves, and gives the boundary of $\mathcal{B}$. Several variants of the set $\mathcal{B}$ for various matrices A and polygons P, Q are demonstrated in Figs. 1.15–1.18.

A complete description of solution to games of kind in the case of scalar control of the first player (the set P is a segment) and arbitrary matrices A is given in [5, 6].

If the matrix A has real eigenvalues, the solution is, as a rule, more simple because the number of branches of the curves $\mathrm{g}^{(1)}, \mathrm{g}^{(2)}$ is less than that in case of complex eigenvalues. On the other hand, the solution may be more complicated because in some cases there may be insufficient to construct only curves associated with the point m for finding the set $\mathcal{B}$. Sometimes, it is necessary to use semipermeable curves of the first and the second types associated with one or more additional "source" points. An example of the set $\mathcal{B}$ in the case of real eigenvalues of the matrix A is shown in Fig. 1.19. The point m_1 is a source point.

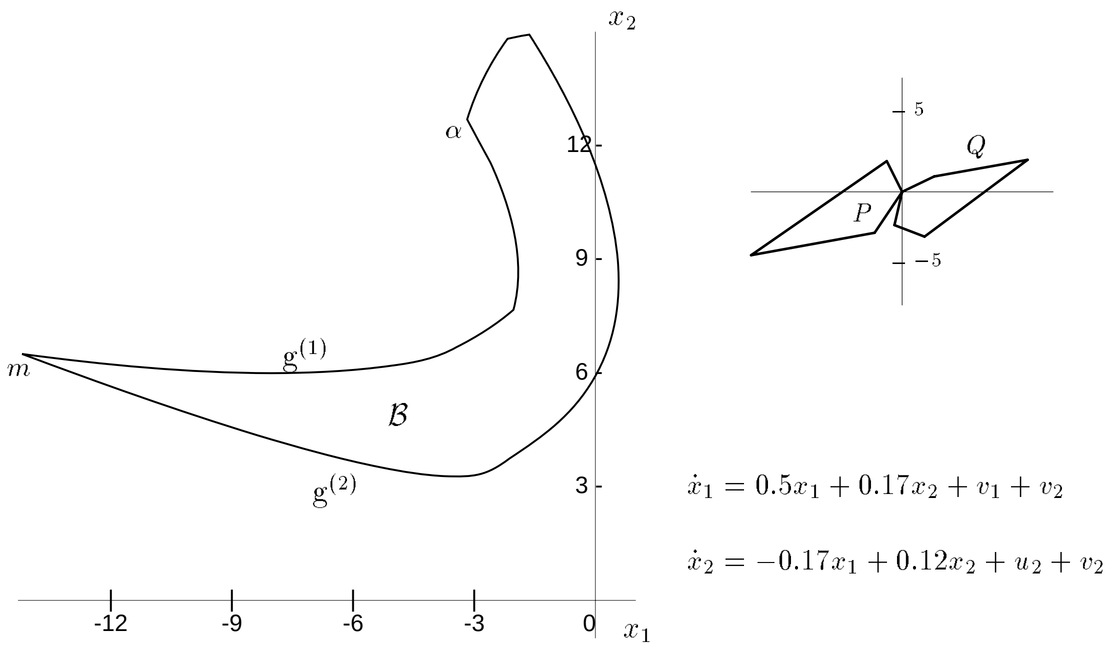


Fig. 1.15. The set $\mathcal{B}$ is determined by α-intersection point.

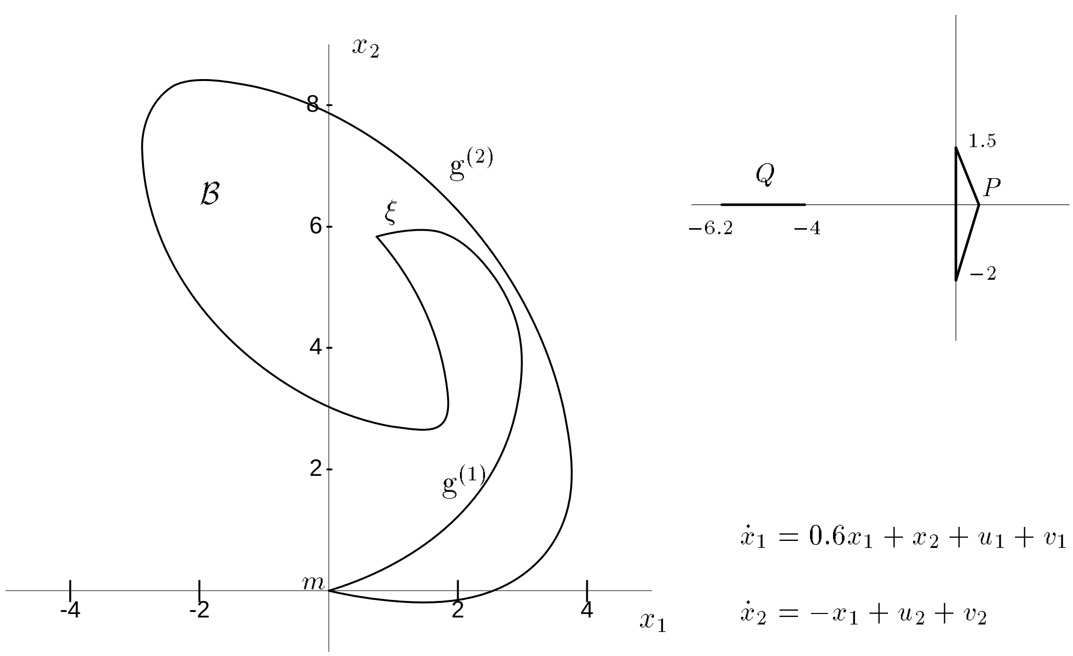


Fig. 1.16. The set $\mathcal{B}$ is determined by ξ-intersection point.

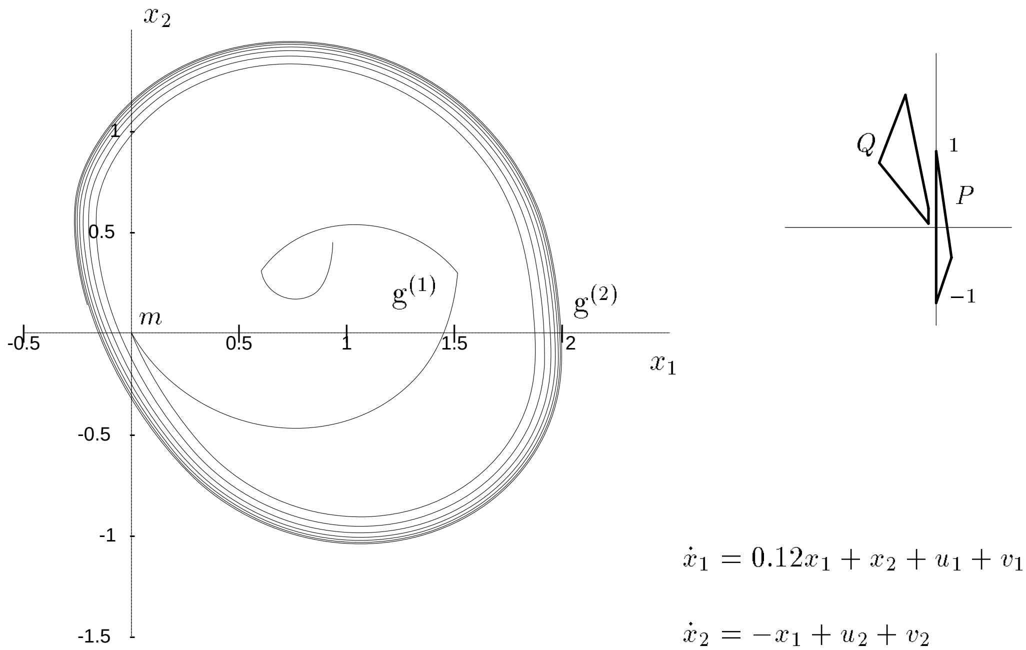


Fig. 1.17. The set $\mathcal{B}$ is bounded by the limit cycle of the curve $g^{(2)}$.

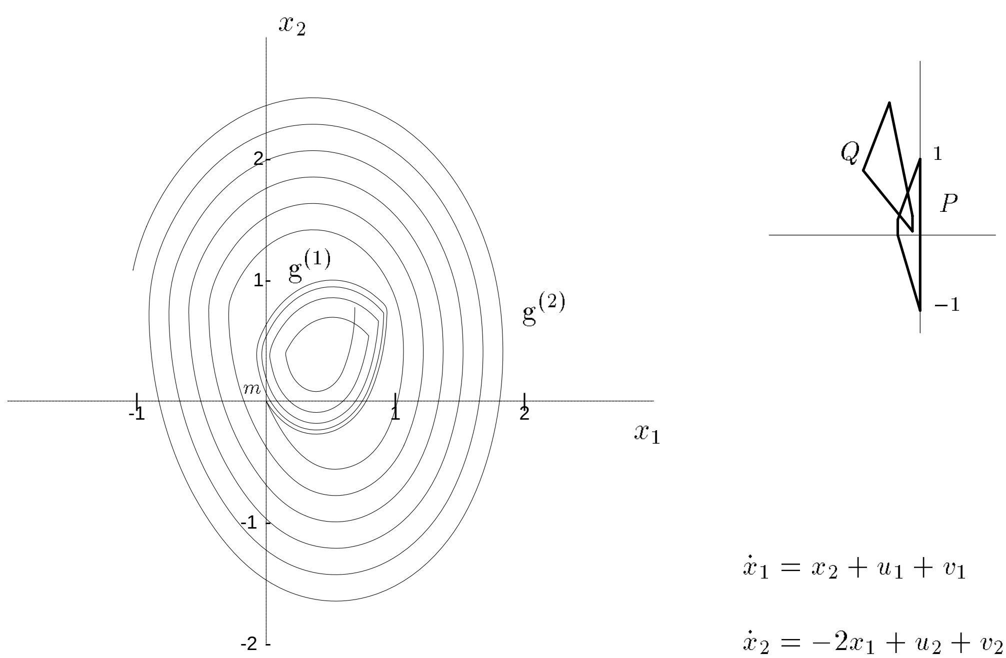


Fig. 1.18. The curve $g^{(1)}$ is the twisting spiral, the curve $g^{(2)}$ is the untwisting spiral. The set $\mathcal{B}$ is the whole plane.

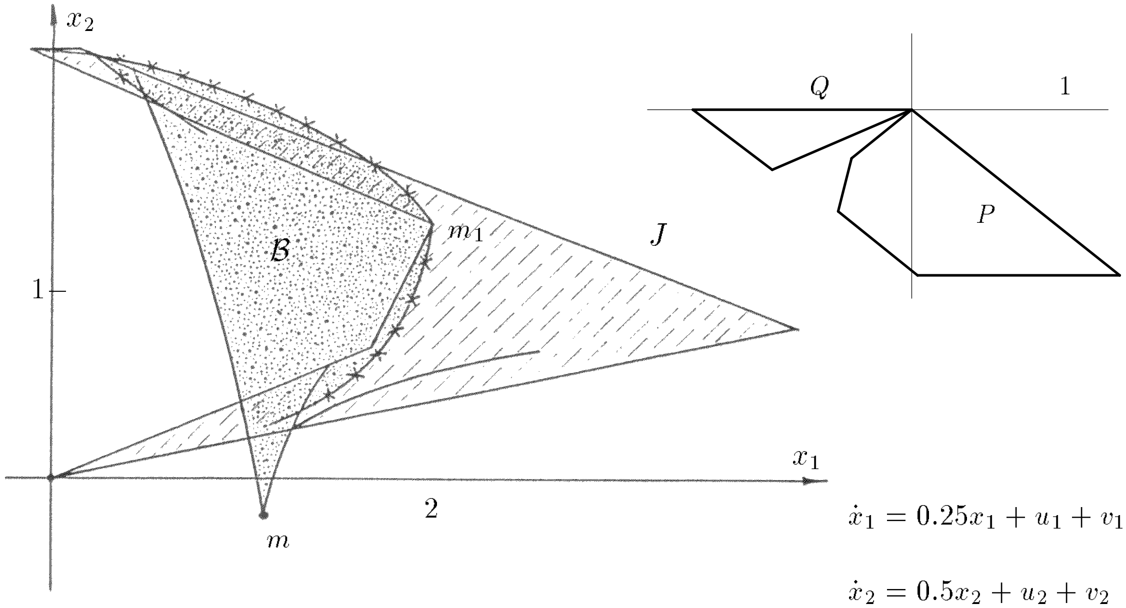


Fig. 1.19. An example of the set $\mathcal{B}$ in the case of real eigenvalues of the matrix A.

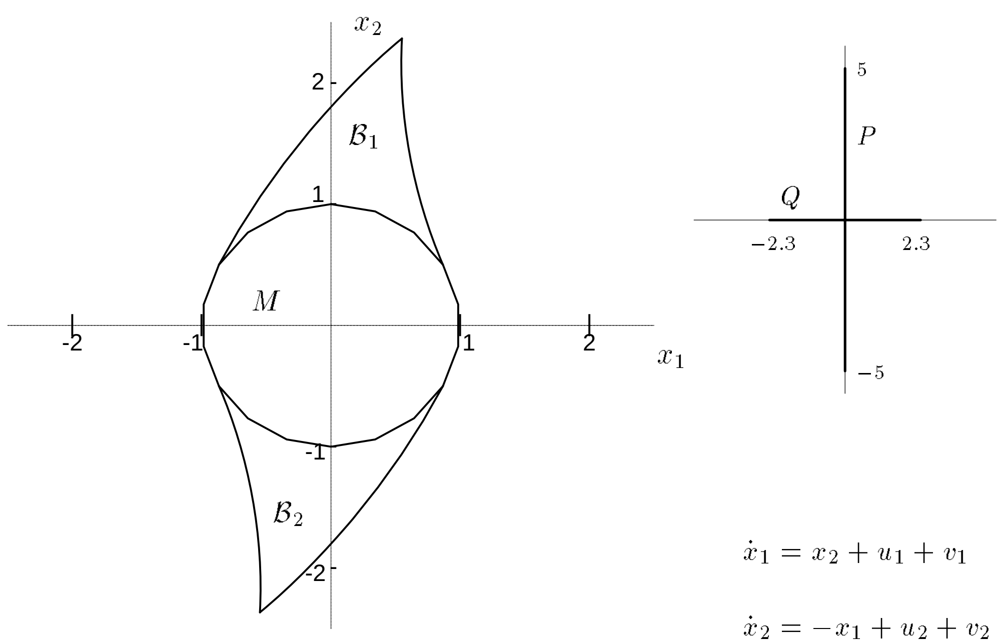


Fig. 1.20. An example of solution in the case where the target set is not a single point.

We considered the case where the target set is a single point. If the target set is an arbitrary convex compact M in the plane, the game of kind can also be solved using semipermeable curves of the first and the second types. The simplest structure of solutions in this case is the following: one or more components of the set $\mathcal{B}$ growing from the set M can be constructed independently and these components do not intersect each other. Such a simple variant is demonstrated in Fig. 1.20. The set $\mathcal{B}$ computed consists of two pieces $\mathcal{B}_1$, $\mathcal{B}_2$. Significant complications occur when the above mentioned pieces intersect each other.

Chapter 2

Nonlinear game of kind

The idea of the algorithm for finding the set $\mathcal{B}$ for two-dimensional linear differential games of kind can be extended to differential games of kind with nonlinear in phase variable dynamics. In this chapter we describe how to modify the algorithm for the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + v_1 \\ \dot{x}_2 &= -k\sin(x_1 + a) + u_2 + c \end{aligned} \tag{2.1}$$

$$v_1 \in [\nu_*, \nu^*], \quad |\, u_2 \,| \le \mu.$$

The target set is a point m in the plane. The constants c, μ, k satisfy the relation $c < \mu - k,\ k > 0$. This condition does not simplify the problem essentially, it only reduces the number of variants of the possible structure of the set $\mathcal{B}$. A more detailed description of the algorithm is given in [16, 17].

1. Description of families of semipermeable curves

By analogy with Chapter 1, we consider the function

$$\varphi(\ell) = \max_{u \in P} \min_{v \in Q} \langle \ell, u + v \rangle, \qquad \ell \in R^2,$$

where P is the segment with the vertices $(0, \mu)$, $(0, -\mu)$ and Q is the segment with the vertices (ν_*, c), (ν^*, c). As P is the vertical segment, the plane is divided by the x_1-axis into two half-planes $\{\ell_2 \ge 0\}$, $\{\ell_2 < 0\}$ so that the function φ is concave in each of these half-planes. Instead of the polygonal line E introduced in section 3 of Chapter 1, we consider the horizontal straight lines $E_1\colon\ \ell_2 = -\alpha$ (α is an arbitrary positive number) and $E_3\colon\ \ell_2 = \alpha$. We put

$$\Gamma^{(1),1} = E_1, \qquad \Gamma^{(1),2} = E_3, \qquad \Gamma^{(2),1} = E_3, \qquad \Gamma^{(2),2} = E_1.$$

The function H introduced in Chapter 1 has now the form

$$H(l, x) = \langle \ell, \begin{pmatrix} x_2 \\ -k\sin(x_1 + a) \end{pmatrix} \rangle + \varphi(\ell).$$

It inherits the concavity-convexity properties of the function φ with respect to variable ℓ. So, for any x fixed, the function H may have at most one $+$ to $-$ zero and one $-$ to $+$ zero in each set $\Gamma^{(n),i}$, $n=1,2$, $i=1,2$.

Just as in section 4 of Chapter 1, we introduce sets $\widetilde{S}^{(n),i}$ that are sets of all $y \in R^2$ for each of which there exist a $+$ to $-$ zero (if $n=1$) and a $-$ to $+$ zero (if $n=2$) of the function $\max\limits_{u\in P}\min\limits_{v\in Q}\langle l, u+v-y\rangle$ in the set $\Gamma^{(n),i}$. The sets $\widetilde{S}^{(n),i}$, $n=1,2$, $i=1,2$, are open and have the form shown in Fig. 2.1.

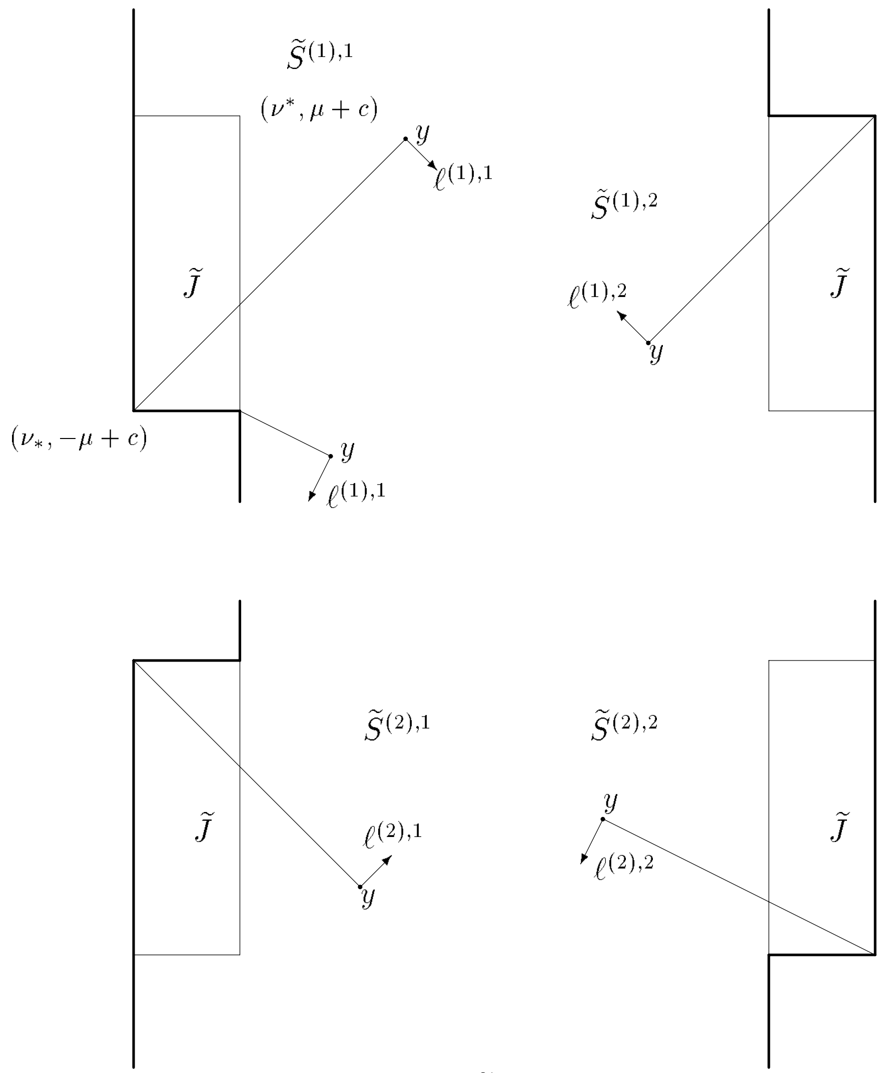


Fig. 2.1. Sets $\widetilde{S}^{(n),i}$.

We consider the mapping $\Psi\colon (x_1, x_2) \to (-x_2,\ k\sin(x_1+a))$. The preimage of the set $\widetilde{S}^{(1),i}$ ($\widetilde{S}^{(2),i}$) under the mapping Ψ is a set $S^{(1),i}$ ($S^{(2),i}$) consisting of points x such that there exists a unique zero $\ell^{(1),i}(x)$ ($\ell^{(2),i}(x)$) of $+$ to $-$ ($-$ to $+$) type of the function $H(\cdot, x)$ belonging to the set $\Gamma^{(1),i}$ ($\Gamma^{(2),i}$).

Since $c < \mu - k$, the preimage of the set $\{y\colon\ y_2 < -\mu + c\}$ under the mapping Ψ is the empty set. So, $S^{(1),1}$ ($S^{(2),2}$) is the half-plane below the line $x_2 = -\nu_*$ (above the line $x_2 = -\nu^*$).

If $\mid \mu + c \mid /k \leq 1$, then $S^{(n),i},\ n \neq i,$ is the collection of sets which are translations of the set $X^{(n),i}$ (Fig. 2.2) by $2\pi r,\ r = 0, \pm 1, \pm 2, \ldots$, along the x_1-axis. By $b,\ d,\ e$ in Fig. 2.2, we denoted the values

$$-a - \arcsin\frac{\mu + c}{k} - \pi, \qquad -a + \arcsin\frac{\mu + c}{k}, \qquad -a - \arcsin\frac{\mu + c}{k} + \pi,$$

respectively. If $\mid \mu + c \mid /k < 1$, then the sets $S^{(1),2},\ S^{(2),1}$ have the forms shown in Fig. 2.3.

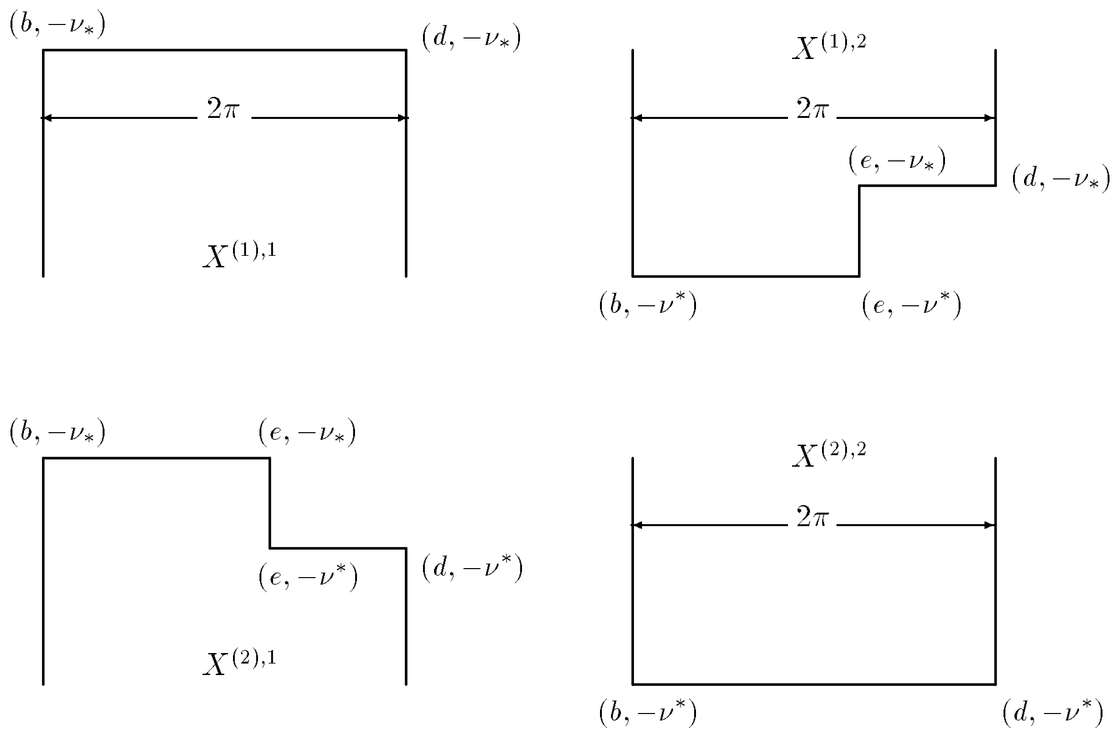


Fig. 2.2. Auxiliary sets $X^{(n),i}$.

If $\mid \mu + c \mid /k > 1$, then the preimage of the set $\{y:\ y_2 > \mu + c\}$ under the transformation Ψ is the empty set. In this case, the set $S^{(1),2}$ ($S^{(2),1}$) is the half-plane above the line $x_2 = -\nu^*$ (below the line $x_2 = -\nu_*$).

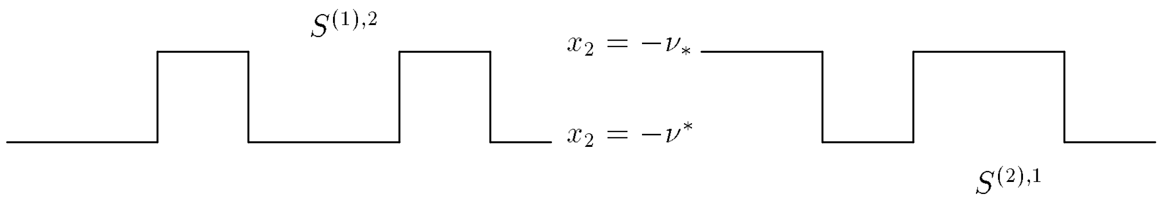


Fig. 2.3. Sets $S^{(1),2}$, $S^{(2),1}$.

The function $\ell^{(n),i}(\cdot)$ satisfies the Lipschitz condition in any closed bounded subset of $S^{(n),i}$ for any $n = 1, 2,\ i = 1, 2$. Similar to the case of linear system, we consider the following differential equations

$$\frac{dz}{d\tau} = \Pi_+ \ell^{(1),i}(z), \quad z \in S^{(1),i}, \tag{2.2}$$

$$\frac{dz}{d\tau} = \Pi_- \ell^{(2),i}(z), \quad z \in S^{(2),i}. \tag{2.3}$$

Belonging to $S^{(1),i}$ ($S^{(2),i}$) phase trajectories of maximally extended solutions of equation (2.2) ((2.3)) generate the family $\Phi^{(1),i}$ ($\Phi^{(2),i}$) of semipermeable curves.

We put

$$F = \{x \in R^2:\ k\sin(x_1 + a) < \mu + c\}, \qquad G = \{x \in R^2:\ k\sin(x_1 + a) > \mu + c\}.$$

If $\mid \mu + c \mid /k < 1$, then the set F (G) is the collection of vertical strips F_r (G_r), $r = 0, \pm 1, \pm 2, \ldots$, which are repeated periodically with the period 2π. These strips are bounded by the following vertical straight lines

$$M_r = \{x \in R^2:\ x_1 = -a + \arcsin\frac{\mu + c}{k} + 2\pi r\},$$

$$N_r = \{x \in R^2:\ x_1 = -a - \arcsin\frac{\mu + c}{k} + \pi + 2\pi r\}.$$

If $(\mu + c)/k \le -1$, then $F = \emptyset,\ G = R^2$. If $(\mu + c)/k \ge 1$, then $F = R^2,\ G = \emptyset$.

We concentrate ourselves on the most complicated case $\mid \mu + c \mid /k < 1$.

Semipermeable curves $p^{(n),i}$ of the family $\Phi^{(n),i}$ can be interpreted as phase trajectories of system (2.1). Namely, curves $p^{(1),1}$ are phase trajectories corresponding to $u_2 = -\mu$, $v_1 = \nu_*$. Curves $p^{(2),2}$ are phase trajectories with $u_2 = -\mu$, $v_1 = \nu^*$. Curves $p^{(1),2}$ are phase trajectories with $u_2 = \mu$; $v_1 = \nu_*$ in the set G and $v_1 = \nu^*$ in the set F. Curves $p^{(2),1}$ are phase trajectories with $u_2 = \mu$; $v_1 = \nu^*$ in the set G and $v_1 = \nu_*$ in the set F.

The above mentioned semipermeable curves are drawn schematically in Fig. 2.4. Arrows show the reverse time directions for motions of system (2.1) or, which is the same, directions of motion for increasing τ in equations (2.2), (2.3).

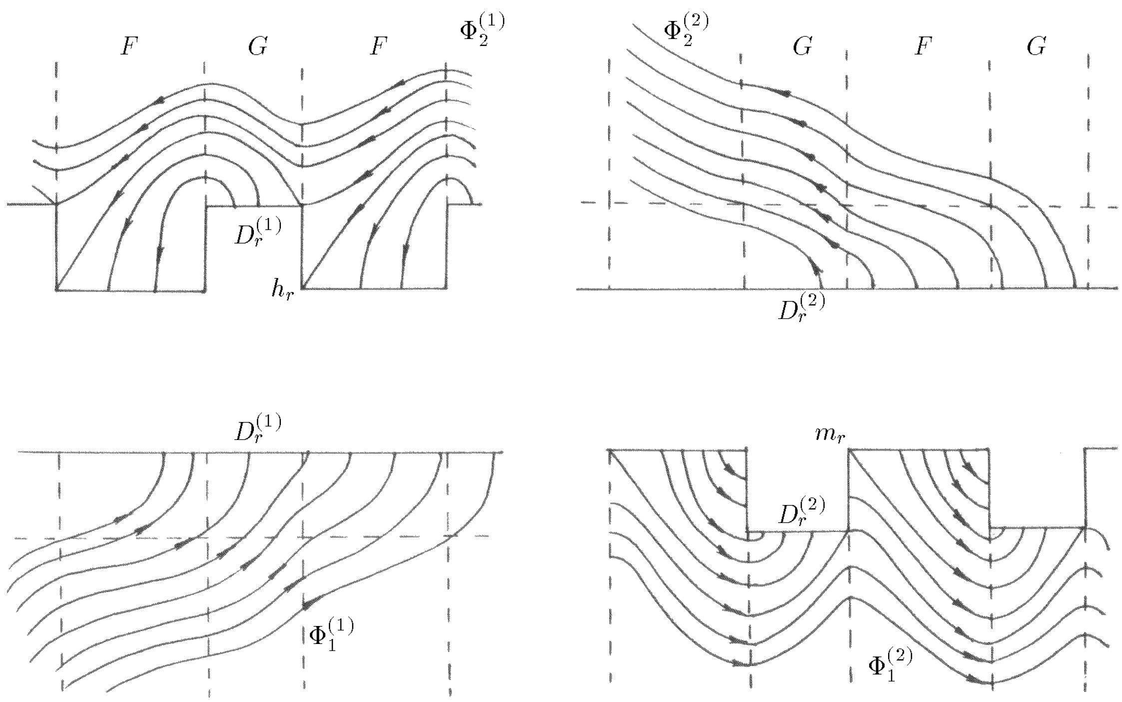


Fig. 2.4. Families of semipermeable curves.

Curves $p^{(1),1}$ and $p^{(1),2}$ have a smooth conjunction at all points of the segments $D_r^{(1)} = \{x\colon\ x_2 = -\nu_*\} \bigcap \mathrm{cl}G_r$, $r = 0, \pm 1, \pm 2, \ldots$, with the exception of the points

$$m_r = (-a - \arcsin\frac{\mu + c}{k} + \pi + 2\pi r,\ -\nu_*) \in D_r^{(1)} \bigcap N_r,$$

which are saddle points of system (2.1) for $u_2 = \mu$, $v_1 = \nu_*$. Curves $p^{(2),1}$ and

$p^{(2),2}$ have a smooth conjunction at all points of the segments $D_r^{(2)} = \{x\colon\ x_2 = -\nu^*\} \bigcap \mathrm{cl}G_r$, with the exception of the points

$$h_r = (-a - \arcsin\frac{\mu + c}{k} + \pi + 2\pi r,\ -\nu^*) \in D_r^{(2)} \bigcap N_r,$$

which are saddle points of system (2.1) for $u_2 = \mu,\ v_1 = \nu^*$.

The sets $S^{(n),i}$ are open. Nevertheless, we need the notion of semipermeable curves emanating from some points of the boundary of $S^{(n),i}$. It is clear from Fig. 2.4 that for each boundary point of $S^{(n),i}$ at most one curve of the family $\Phi^{(n),i}$ may outcome from and at most one curve of this family may arrive at such a point. So, the notation $p^{(n),i}(x_*)$ for $x_* \in \partial S^{(n),i}$ has the evident meaning.

For each point m_r, the curves $p^{(1),2}(m_r)$ and $p^{(2),1}(m_r)$ are tangent at this point to an invariant straight line associated with the positive eigenvalue of the matrix of the system obtained via linearizing system (2.1) with respect to the equilibrium point m_r, under $u_2 = \mu,\ v_1 = \nu_*$. The state vector of system (2.1) approaches asymptotically to m_r along these curves as $t \to \infty$. It should be noted that the curve $p^{(n),\widehat{n}}(m_{r_1})$ is obtained from the curve $p^{(n),\widehat{n}}(m_{r_2}),\ r_2 > r_1$, by the horizontal displacement by the value $2\pi(r_2 - r_1)$.

2. Algorithm for finding the solvability set $\mathcal{B}$. Examples of solving the game of kind

Like the idea of Chapter 1 where the dynamics of the control system is linear, the idea of the algorithm for finding the set $\mathcal{B}$ consists in constructing by turns semipermeable curves of the first and the second types. The main distinction from the case of complex eigenvalues of the matrix A is the following: for finding the set $\mathcal{B}$, besides m we may need one or several additional points called further source points. Such points arise because equilibrium states of a certain type may exist in the nonlinear case. The set $\mathcal{B}$ is determined by all semipermeable curves emanating from the point m and from source points. For the problem in question, the points m_r will be used as source points.

In the course of the construction of the set $\mathcal{B}$, curves $p^{(1),1}$ are sewn together with curves $p^{(1),2}$, and curves $p^{(2),1}$ are sewn together with curves $p^{(2),2}$. The assembling of curves with the observance of properties 1), 2) which determine the sprout point (see section 4 of Chapter 1) is realizable for curves $p^{(1),1}, p^{(1),2}$

only at points of the segments $D_r^{(1)}$, and for curves $p^{(2),1}$, $p^{(2),2}$ only at points of the segments $D_r^{(2)}$. As it follows from properties of the curves (see Fig. 2.4), curves of the first type are smoothly sewn everywhere on $D_r^{(1)}$ excluding the points m_r, and curves of the second type are smoothly sewn everywhere $D_r^{(2)}$ excluding the points h_r. The resulting line obtained after sewing the curves with the same first indices has the semipermeability property.

Like in Chapter 1, a special condition [13–15] excludes the case $\mathcal{B} = \{m\}$. In particular, $\mathcal{B} = \{m\}$ if the point m lies in the horizontal strip between the lines $x_2 = -\nu_*$, $x_2 = -\nu^*$.

Suppose $\mathcal{B} \neq \{m\}$ and describe the algorithm for constructing the set $\mathcal{B}$. Assume for definiteness that m lies below the horizontal strip specified. That is, $m \in S^{(1),1} \bigcap S^{(2),1}$.

At the first step we construct the branch

$$\mathrm{g}_1^{(2)} = \begin{cases} p^{(2),1}(m), & p^{(2),1}(m) \text{ is infinite} \\ p^{(2),1}(m, q_1^{(2)}) \cup p^{(2),2}(q_1^{(2)}), & p^{(2),1}(m) \text{ is finite.} \end{cases}$$

Here, $q_1^{(2)}$ is the sprout point of the curve $p^{(2),1}(m)$. The point $q_1^{(2)}$ coincides with the endpoint of the curve $p^{(2),1}(m)$ and belongs to the segment $D_r^{(2)}$.

The branches of the curves $\mathrm{g}^{(1)}$, $\mathrm{g}^{(2)}$ are constructed by turns : $\mathrm{g}_1^{(1)}, \mathrm{g}_2^{(2)}, \mathrm{g}_2^{(1)}, \ldots$. The branches $\mathrm{g}_1^{(1)}$, $\mathrm{g}_2^{(1)}$ are associated with the point m: $\mathrm{g}_1^{(1)} = p^{(1),1}(m, q_2^{(1)})$, $\mathrm{g}_2^{(1)} = p^{(1),2}(q_2^{(1)})$. Here, $q_2^{(1)}$ is the sprout point of the curve $p^{(1),1}(m)$. The point $q_2^{(1)}$ coincides with the endpoint of the curve $p^{(1),1}(m)$ if it belongs to $D_r^{(1)}$. If do no, we decide that the sprout point on the curve $p^{(1),1}(m)$ does not exist and put $\mathrm{g}_2^{(1)} = \emptyset$.

Let us define now the curves associated with the points m_r. We will only use those points which lie on the left from the endpoint of the curve $\mathrm{g}_1^{(1)}$. We enumerate such points from the right to the left and denote them $m_1, m_2, \ldots$. For the curves of the first type with $j \geq 3$, we put

$$\mathrm{g}_j^{(1)} = \begin{cases} p^{(1),2}(m_{\frac{j-2}{2}}), & j \text{ is even number} \\ \emptyset, & j \text{ is odd number.} \end{cases}$$

For the curves of the second type with $j \geq 2$, we let

$$\mathrm{g}_j^{(2)} = \begin{cases} p^{(2),1}(m_{\frac{j}{2}}), & j \text{ is even number} \\ p^{(2),2}(q_j^{(2)}), & j \text{ is odd number and sprout point } q_j^{(2)} \text{ exists} \\ \emptyset, & j \text{ is odd number and sprout point } q_j^{(2)} \text{ does not exist.} \end{cases}$$

So, the curves of the first type emanating from the points m_r belong to the family $\Phi^{(1),2}$, and they do not have sprout points. The curves of the second type emanating from m_r may have sprout points. In the latter case, the curves of the family $\Phi^{(2),1}$ are continued by the curves of the family $\Phi^{(2),2}$.

When constructing the current branch $g_j^{(1)}$, $j \geq 1$, we test intersections with one or two already constructed curves of the second type. If the intersection occurs (α-intersection), then the construction of $g_j^{(1)}$ beyond the point of intersection is ceased. When constructing the current branch $g_j^{(2)}$, $j \geq 2$, we test intersections with one of the already constructed curves of the first type. Namely, for even indices j, we check the intersection with $g_1^{(1)}$, for odd indices j, we analyse the intersection with $g_{j-1}^{(1)}$. If the intersection occurs (ξ-intersection), then the construction of $g_j^{(2)}$ is finished.

In the most simple cases, the set $\mathcal{B}$ is determined only by curves emanating from the point m. All these cases are listed below:

a) If in the course of the construction of $g_1^{(1)}$ or $g_2^{(1)}$ an α-intersection with $g_1^{(2)}$ occurs, then the set $\mathcal{B}$ is bounded by the curves drawn from the point m up to the point of α-intersection (Fig. 2.5);

b) If the curve $g_2^{(1)}$ is infinite and does not have points in common with the curve $g_1^{(2)}$, then the set $\mathcal{B}$ is bounded by the curves $g^{(1)}$, $g^{(2)}$ emanating from the point m (Figs. 2.6–2.8). In this case, the set $\mathcal{B}$ is infinite but does not coincide with the whole plane.

In Figs. 2.6–2.8, three variants of the set $\mathcal{B}$ are shown. For all variants, $g^{(1)}$, $g^{(2)}$ are smooth infinite curves. The pictures differ one from other because they correspond to different segments Q.

Note that Fig. 2.5 is similar to Fig. 1.3 of Chapter 1, and Figs. 2.6–2.8 are similar to Fig. 1.6.

It may happen that the curve $g_1^{(1)}$ does not have any sprout point (Fig. 2.9) or it has a sprout point but the curve $g_2^{(1)}$ is finite (Fig. 2.10). Besides, the above mentioned curves do not have any α-intersection with the curve $g_1^{(2)}$. For linear systems with complex eigenvalues of the matrix A, such a configuration would determine the set $\mathcal{B}$. (The set $\mathcal{B}$ would be either bounded by a limit cycle of the curve $g^{(2)}$ or would coincide with the whole plane if $g^{(2)}$ does not have limit cycles.) The solution of the nonlinear problem considered is more complicated: to obtain the set $\mathcal{B}$, one should use semipermeable curves of the first and the second types emanating from the points m_r. This is what the algorithm does.

In the example shown in Fig. 2.9, the following is done. We trace the curve $g_2^{(2)}$ from the point m_1 and obtain ξ-intersection with $g_1^{(1)}$. As $g_2^{(1)} = \emptyset$, $g_3^{(2)} = \emptyset$, and $g_3^{(1)} = \emptyset$, we issue the curve $g_4^{(2)}$ from the point m_2. Then we trace $g_4^{(1)}$ from

m_1. This curve is infinite. The set $\mathcal{B}$ is bounded by the curves $g_1^{(2)}(m)$, $g_1^{(1)}(m, \xi_1)$, $g_2^{(2)}(m_1, \xi_1)$, $g_4^{(1)}(m_1)$. The curve $g_4^{(2)}$ was constructed but it does not give any contribution to the boundary of $\mathcal{B}$.

Let us explain the constructions shown in Fig. 2.10. After construction of the curves $g_1^{(2)}$ and $g_1^{(1)}$, we compute the curve $g_2^{(2)}$ emanating from m_1. It has ξ-intersection with the curve $g_1^{(1)}$. Then we trace the curve $g_2^{(1)}$. It has α-intersection with $g_2^{(2)}$. Then we construct the curve $g_4^{(2)}$ from the point m_2 and the curve $g_4^{(1)}$ from the point m_1. The curve $g_4^{(1)}$ has α-intersection with $g_4^{(2)}$. The boundary of $\mathcal{B}$ is formed by the curves

$$g_1^{(2)}(m),\ g_1^{(1)}(m, q_2^{(1)}),\ g_2^{(2)}(m_1, \alpha_{1*}),\ g_2^{(1)}(q_2^{(1)}, \alpha_{1*}),\ g_4^{(2)}(m_2, \alpha_{2*}),\ g_4^{(1)}(m_1, \alpha_{2*}),$$

and also by the curves $g_{4+2K}^{(2)}(m_{2+K}, \alpha_{(2+K)*})$, $g_{4+2K}^{(1)}(m_{1+K}, \alpha_{(2+K)*})$, $K = 1, 2, ...$, that are translations of the curves $g_4^{(2)}(m_2, \alpha_{2*})$, $g_4^{(1)}(m_1, \alpha_{2*})$ in the x_1-backward direction by the value $2\pi K$.

Figures 2.5–2.10 present results obtained with the use of the computer program implementing the above described algorithm. Some of the other possible variants of the structure of the set $\mathcal{B}$ are shown in Figs. 2.11, 2.12. In the first case (Fig. 2.11), the only source point m_1 takes effect on the boundary of the set $\mathcal{B}$. In the second case (Fig. 2.12), the number of such source points is infinite (as well as in the example shown in Fig. 2.10). In [16, 17], all of the possible variants of the structure of the set $\mathcal{B}$ are given (19 variants) for the case where the point m lies below the line $x_2 = -\nu^*$. These papers describe also some variants of the structure of the set $\mathcal{B}$ when the point m lies above the line $x_2 = -\nu_*$. Two of these variants are shown in Figs. 2.13, 2.14.

The set $\mathcal{B}$ of initial states for the successful termination of the game may be not closed. A limit point $\hat{x}$ belongs to $\mathcal{B}$ if and only if one of the following conditions is satisfied: a) $\hat{x}$ is not an α or ξ- intersection point of boundary curves, and the state vector does not meet the singular points m_r, h_r when moving in direct time from $\hat{x}$ along a boundary curve passing through $\hat{x}$; b) $\hat{x}$ is an α-intersection point, and for each boundary curve passing through $\hat{x}$ the state vector does not meet the singular points m_r, h_r when moving in direct time from $\hat{x}$ along this boundary curve; c) $\hat{x}$ is an ξ-intersection point, and the state vector does not meet the singular point m_r when moving in direct time from $\hat{x}$ along the boundary curve of the first type.

In examples shown in Figs. 2.9–2.12, 2.14, the set $\mathcal{B}$ is not closed. For instance, in Fig. 2.10, the curve $\alpha_{1*} m_1 \alpha_{2*} m_2 \alpha_{3*} m_3 ... \alpha_{K*} m_K ...$ does not belong to $\mathcal{B}$.

Examples of the set $\mathcal{B}$ in the nonlinear game

1^0. Behaviour of the set $\mathcal{B}$ when resources of the second player decrease. Source points are absent.

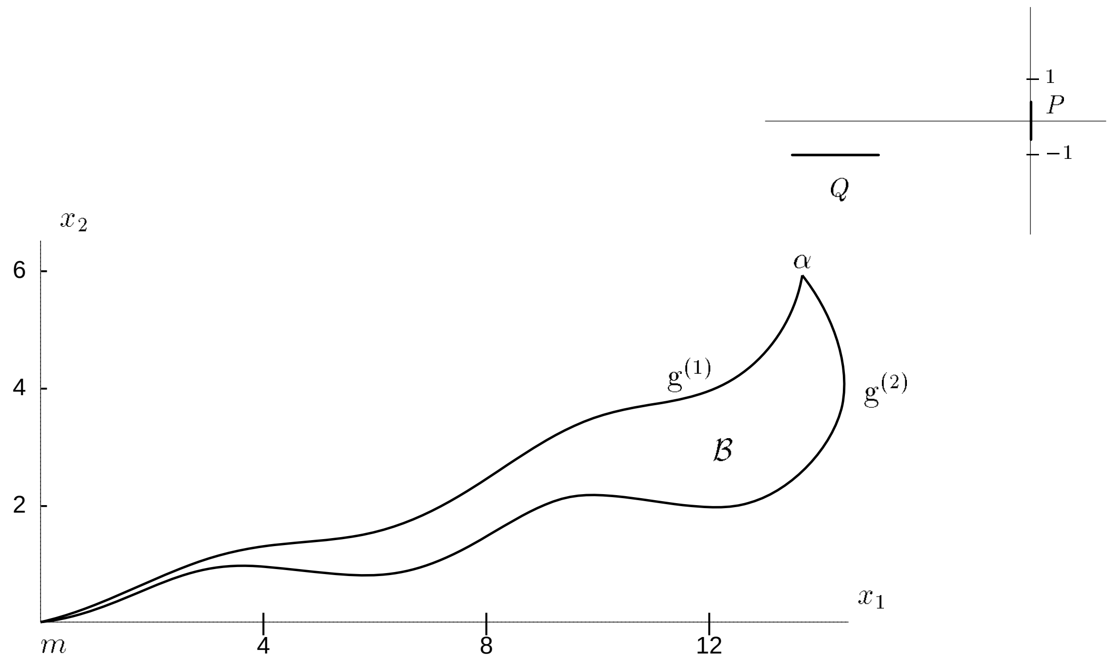


Fig. 2.5.

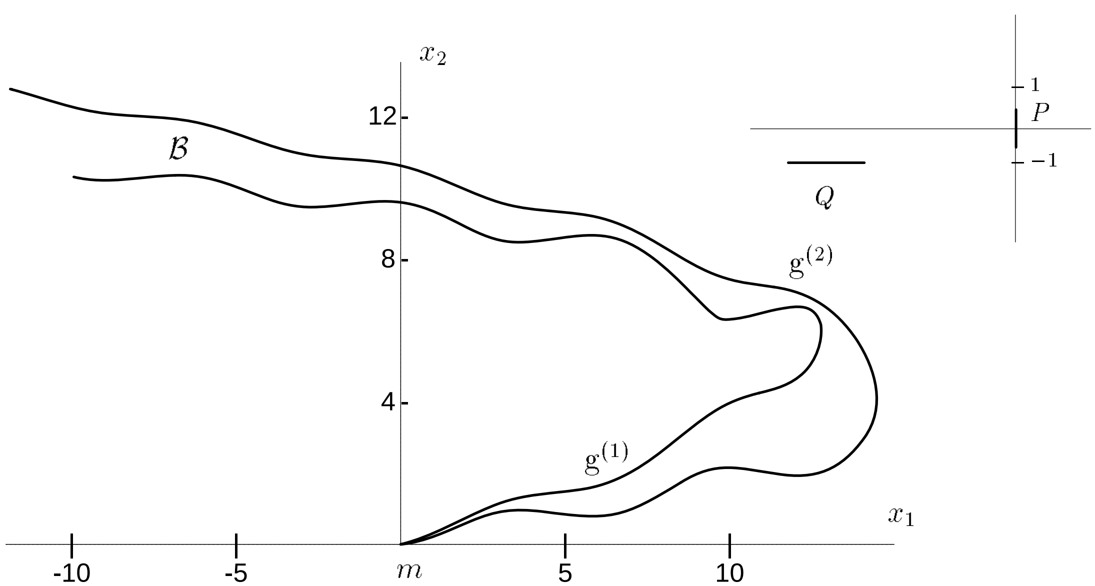


Fig. 2.6.

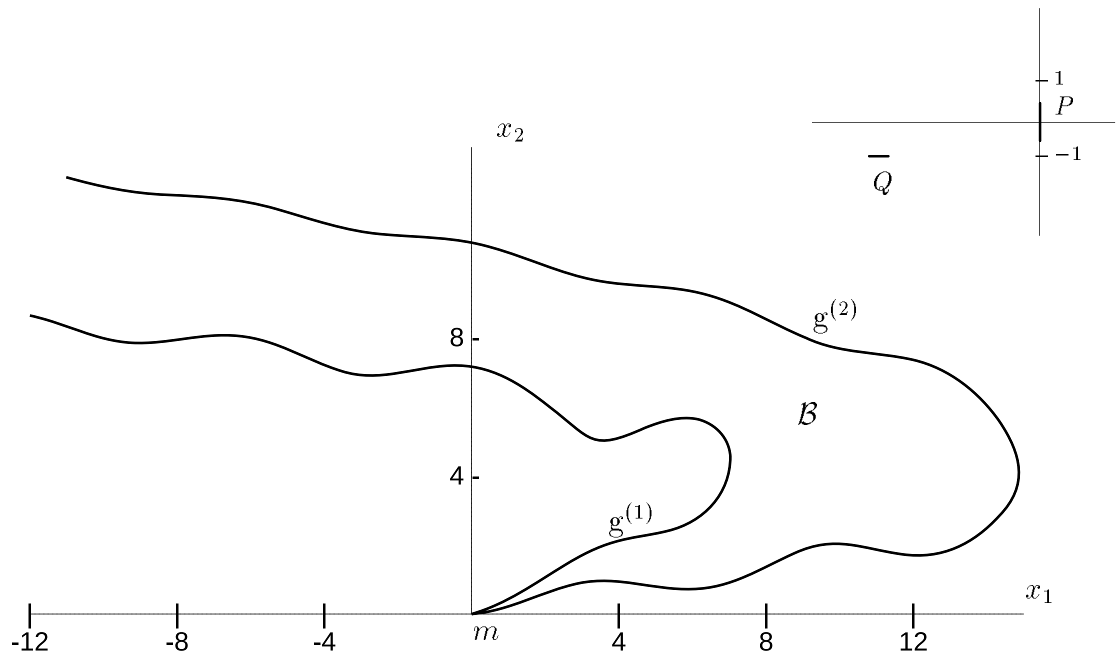


Fig. 2.7.

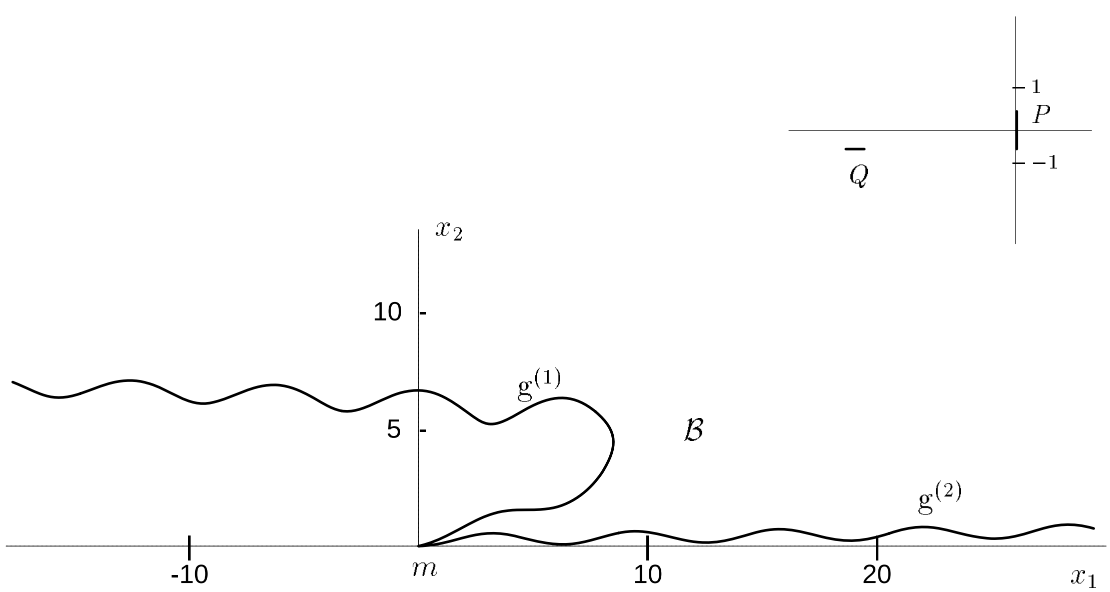


Fig. 2.8.

2^0. The form of the set $\mathcal{B}$ in cases of one and infinite number of source points.

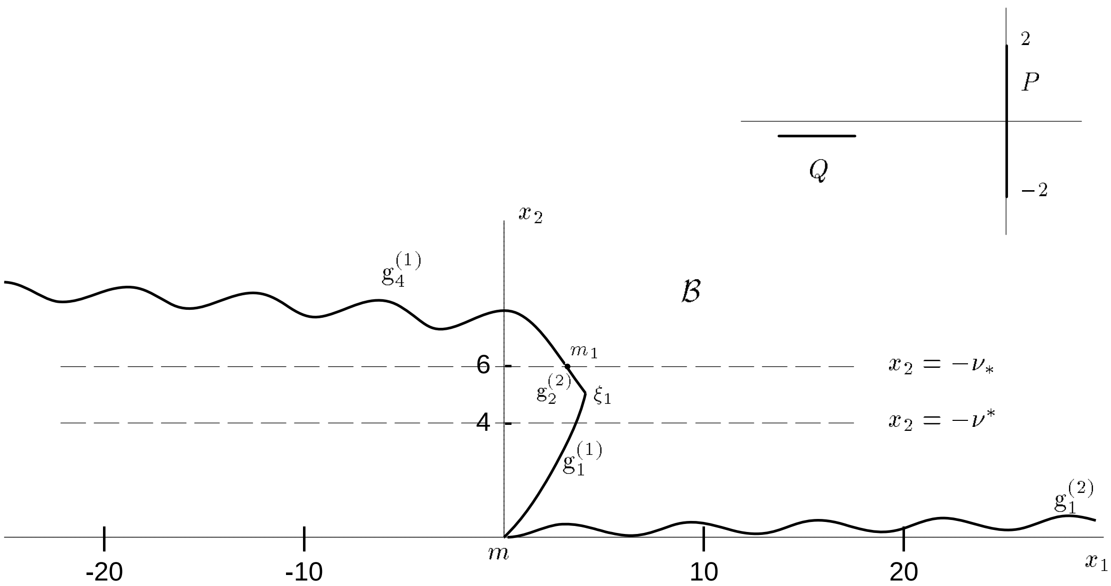


Fig. 2.9.

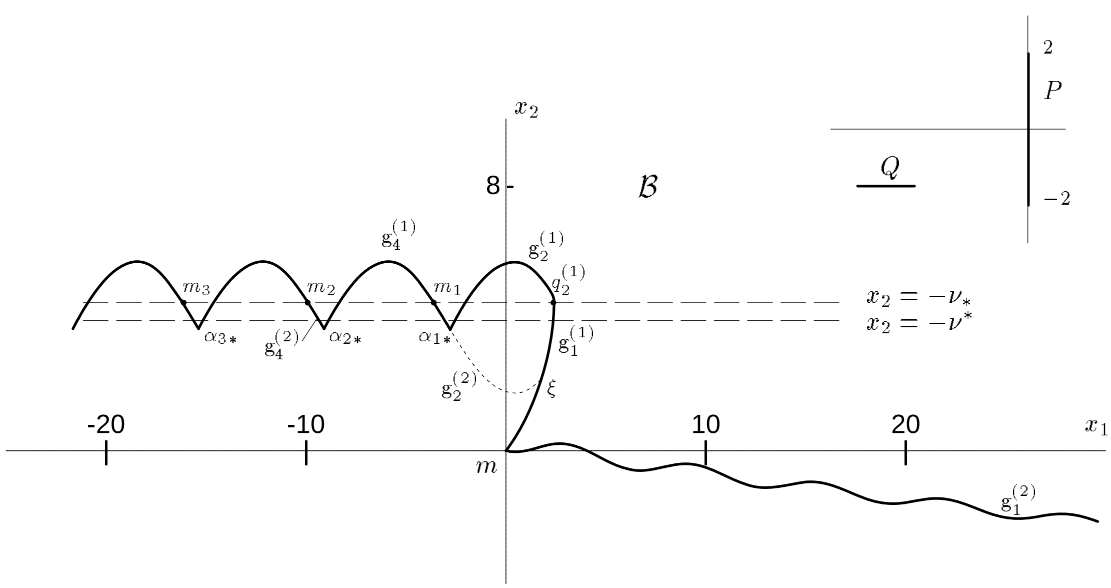


Fig. 2.10.

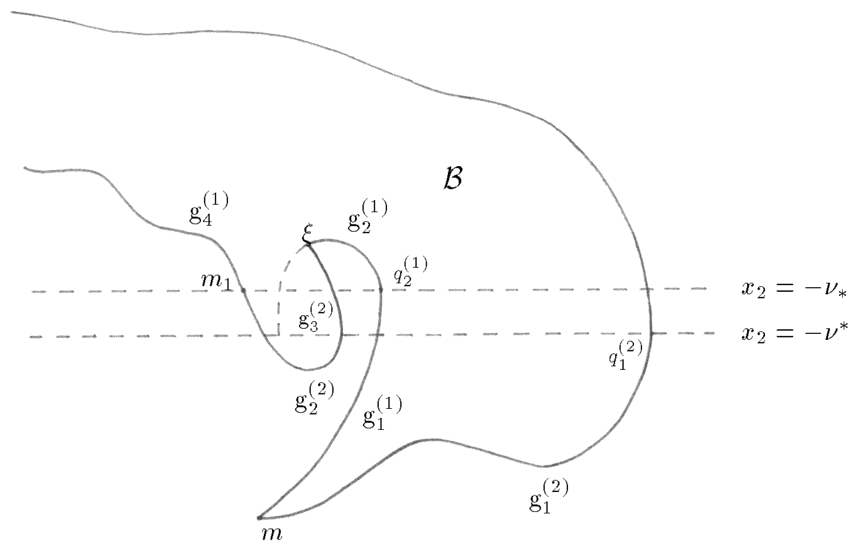


Fig. 2.11.

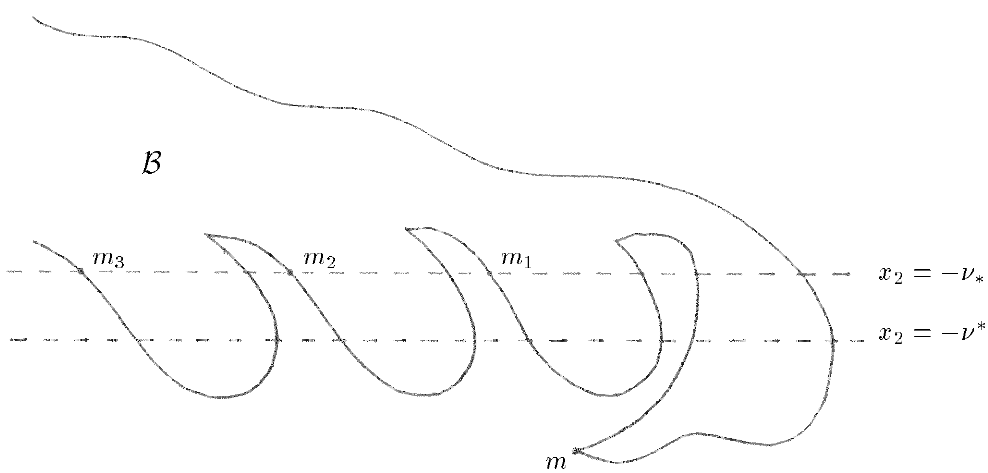


Fig. 2.12.

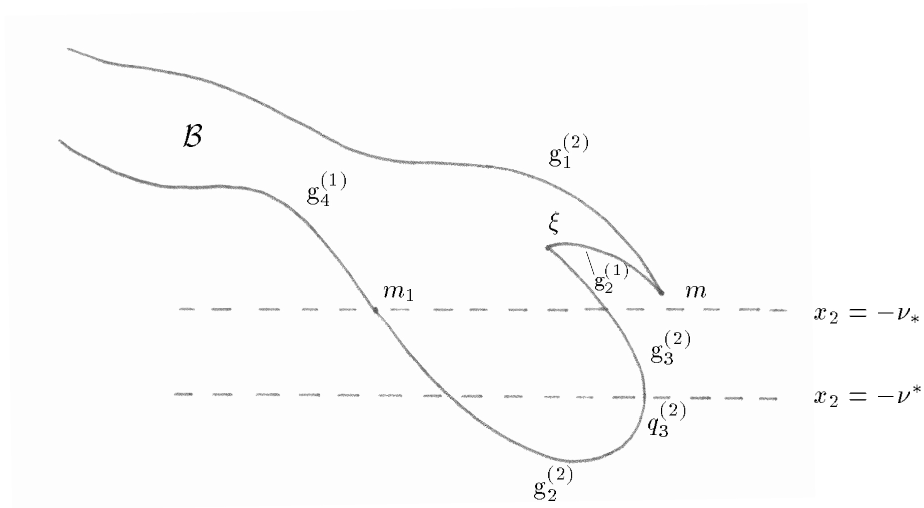


Fig. 2.13.

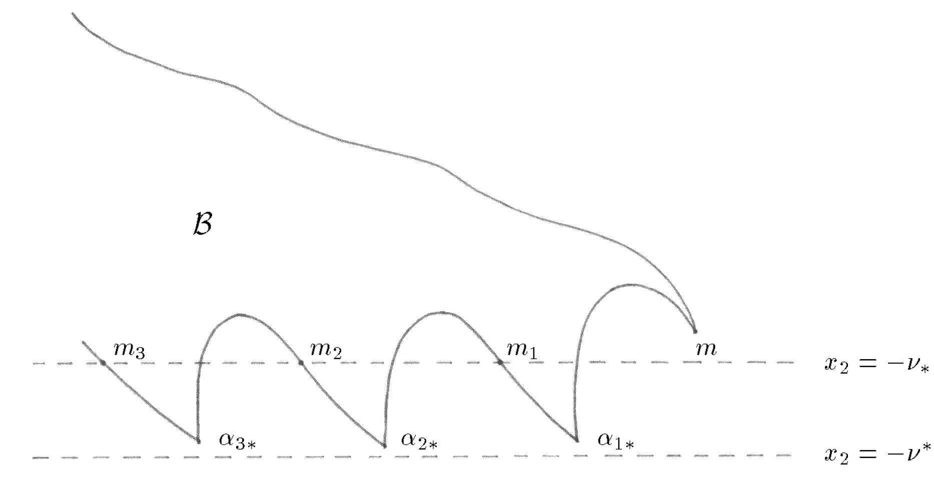


Fig. 2.14.

3. Controls of first and second players

Since we are not interested in finding time optimal controls, there are many strategies of the first player which provide the reaching of the target point m. One of such strategies can be designed using the extremal aiming procedure [2, 3] applied to the closure of a bundle $Y(x_0)$ consisting of trajectories of (2.1) starting from the point x_0 at $t_0 = 0$ and arriving at m by a time θ which can depend on x_0. The bundle $Y(x_0)$ has the u-stability property. The notions of u-stability and of extremal strategies that we use here are given in [2, 3].

Construction of the bundle with the necessary properties uses (see for details [16, 17]) a partition of the closure of $\mathcal{B}$ into "cells" Λ_κ defined via an increasing sequence of sets:

$$A_1 \subseteq A_2 \subseteq ... \subseteq A_\kappa \subseteq ..., \quad A_\kappa \subseteq \mathrm{cl}\mathcal{B}, \quad \kappa = 1, 2, ...$$

The construction of such a sequence $\{A_\kappa\}$ is explained in Figs. 2.15, 2.16. In some cases, the sets with neighbor indices may coincide. The sequence $\{A_\kappa\}$ in Fig. 2.15 corresponds to the set $\mathcal{B}$ given in Fig. 2.11. In this example, we have $A_4 = \mathrm{cl}\mathcal{B}$. In Fig. 2.16, the sets A_1, $A_2 = A_3$, $A_4 = A_5$ of an infinite sequence $\{A_\kappa\}$ corresponding to the set $\mathcal{B}$ depicted in Fig. 2.10 are shown. In general case, the boundary of A_κ is composed from semipermeable curves of the first and the second types and from some auxiliary vertical lines. If $\mathrm{cl}\mathcal{B}$ does not contain the points m_i, $i = 2, 3, ...$, then $A_\kappa = \mathrm{cl}\mathcal{B}$ for some $\kappa \leq 4$. Otherwise, the sequence $\{A_\kappa\}$ is infinite and converges to $\mathrm{cl}\mathcal{B}$ as $\kappa \to \infty$.

The cells Λ_κ are defined in the following way: $\Lambda_1 = \mathrm{cl}A_1$, $\Lambda_\kappa = \mathrm{cl}(A_\kappa \setminus A_{\kappa-1})$, $\kappa \geq 2$. If $A_\kappa = A_{\kappa-1}$, then $\Lambda_\kappa = \emptyset$. The intersection of two different cells may consist of some of their boundary curves only. In Fig. 2.17 and Fig. 2.18, partitions into cells corresponding to sets $\mathcal{B}$ from Fig. 2.15 and Fig. 2.16 are given. We put $u_2 = \mu$ for the cells with even numbers and $u_2 = -\mu$ for the cells with odd numbers. Such a control ensures the right-hand side of system (2.1) to be not equal to zero in the interior of cells for any control v_1 of the second player.

The idea of construction of the bundle $Y(x_0)$ with the above mentioned properties looks as follows. We find the smallest κ_* such that $x_0 \in \Lambda_{\kappa_*}$ and consider the cells Λ_κ with $\kappa \leq \kappa_*$. The first player uses the constant control $u_2 = \mu$ $(u_2 = -\mu)$ in the cells with even (odd) indices. There are some peculiarities of the control choice near the cells boundaries. Namely, if a motion of system (2.1) for $u_2 = \mu$ $(u_2 = -\mu)$ meets a part of the boundary of Λ_{κ_*} which does not

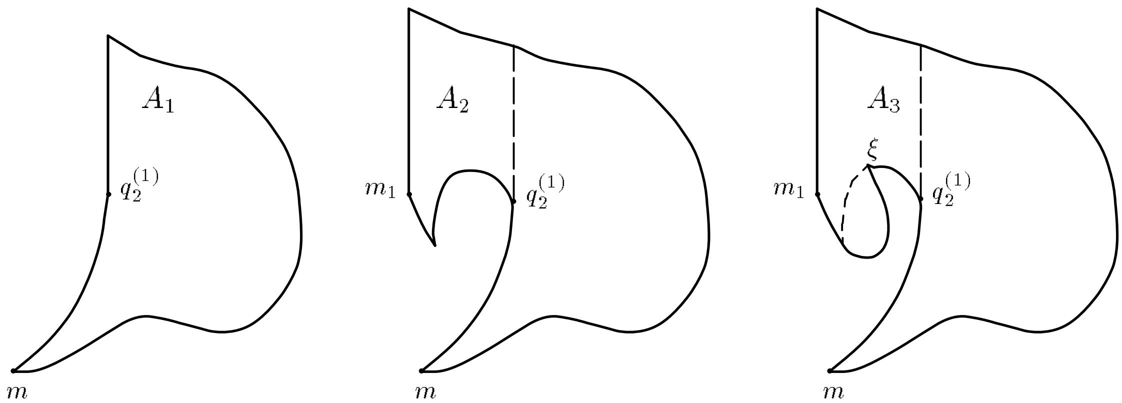


Fig. 2.15. Construction of auxiliary sets A_κ.

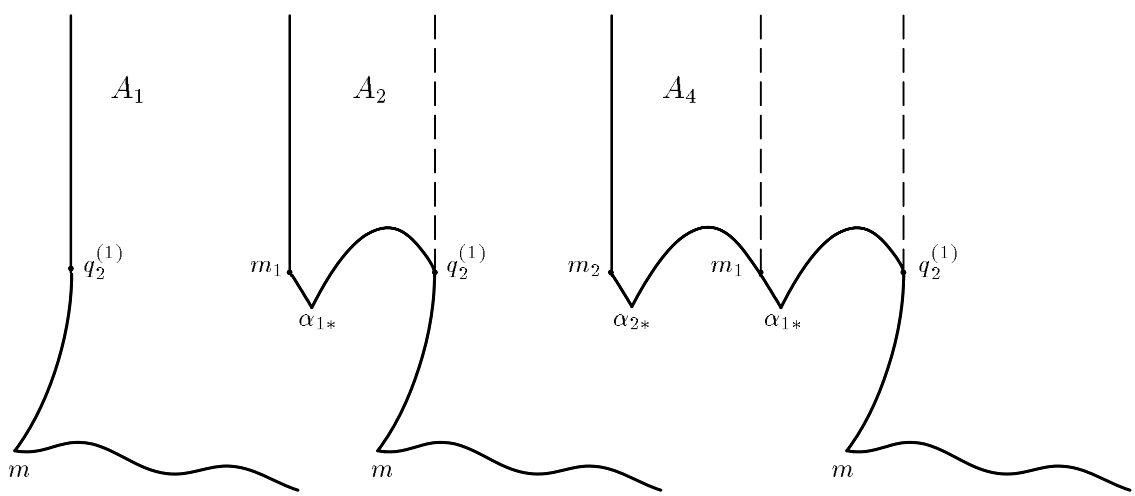


Fig. 2.16. Construction of auxiliary sets A_κ.
The case of the infinite number of source points m_i.

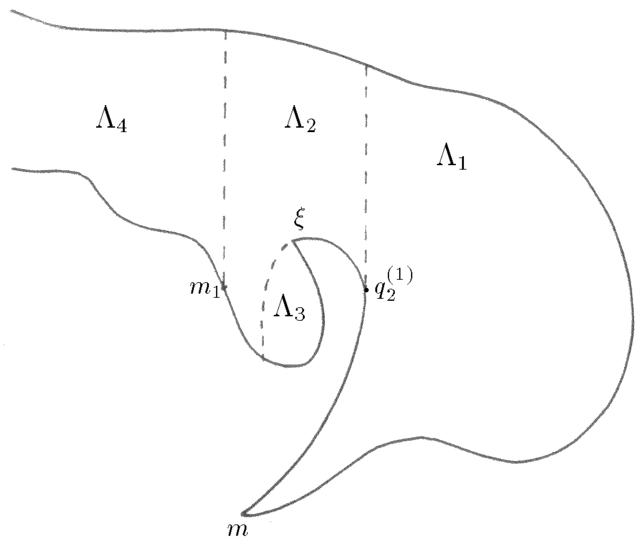


Fig. 2.17. Partition of the set $\mathcal{B}$ into cells Λ_κ.

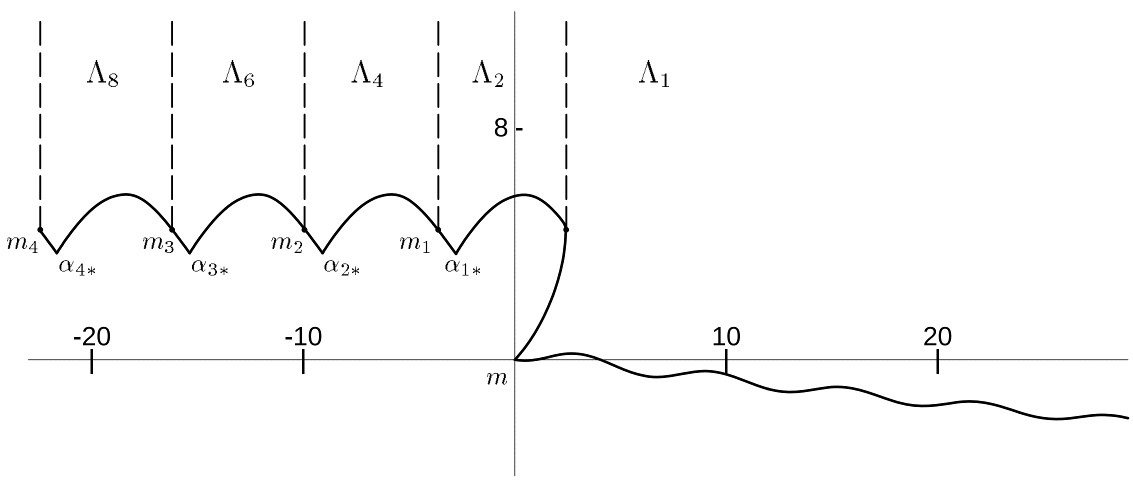


Fig. 2.18. Partition of the set $\mathcal{B}$ into cells Λ_κ.
The case of the infinite number of source points m_i.

belong to Λ_S for some $s < \kappa_*$, then we switch the control u_2 for the control with the opposite sign and keep it until the motion meets some auxiliary "tracking" line which lies close to the above mentioned boundary curve. After meeting this auxiliary curve, we put $u_2 = \mu$ $(u_2 = -\mu)$ again, and so on. Thus, the sliding mode near the cell's boundary may occur. Such a rule of the control choice provides transition of trajectories from Λ_K to Λ_S for some $s < \kappa$. For trajectories starting from the point x_0, the sequence of passable cells is not strictly determined (it depends on the actual control $v_1(\cdot)$) but the values of indices of passable cells increase necessarily. In some cases, to realize the motion of the state vector of (2.1) towards m, one should use some discrimination of the second player: the control of the first player must depend not only on the actual state position but also on the control $v_1(t)$ which is anticipated for some small time interval. The bundle $Y(x_0)$, $x_0 \in \mathcal{B}$, is defined as the set of all trajectories of (2.1) obtained using the above control rule of the first player and by union over all controls $v_1(\cdot)$ of the second player.

The way to design a strategy of the second player preventing the trajectories from meeting the point m whenever the initial states belong to the set $R^2 \setminus \mathcal{B}$ is given in [16, 17].

The considered example with the nonlinear dynamics demonstrates the possibility of finding solvability sets for games of kind using some preliminary analysis of the behavior of semipermeable curves of the first and second types.

Chapter 3

Minimum-time game problem

The previous chapters were devoted to the construction of solvability sets for games of kind. The following properties characterize points of solvability sets for games of kind: the first player guarantees the attainment of the target set within a finite time interval, but the minimum guaranteed time of attainment can not be specified.

In this chapter, we consider two-dimensional minimum-time differential game. The payoff function in such a game is the time of reaching a given target set M. The first player seeks to minimize the time of reaching M, the aim of the second player is opposite. The solution to such a problem will be obtained via construction of sets $W(\theta, M)$, $\theta > 0$. Each of them is the set of all initial states x_0 such that the first player guarantees the transition of the state vector to M by the time θ. The set $W(\theta, M)$ is the level set (the Lebesque set) of the value function of the minimum-time game problem. This set is also called the set of the positional absorption by the time θ or t-section of the maximal u-stable bridge [2, 3] corresponding to $t = \theta$. The set $W(\theta, M)$ converges to the solvability set of the corresponding game of kind as $\theta \to \infty$.

To find the sets $W(\theta, M)$, we will use backward procedures. The general ideas of backward procedures for differential games were considered in papers of R.Bellmann, R.Isaacs, W.Fleming, L.S.Pontrjagin, and B.N.Pshenichny.

The most advanced results [18–25] related to algorithmic implementations of backward constructions were obtained for linear differential games with fixed time of termination. The main peculiarities of these problems are the following: 1) the convexity of target sets implies the convexity of t-sections of maximal stable bridges; 2) if the target set is cylindrical with respect to all coordinates with exception for some k coordinates, then one can reduce the problem to an equivalent k-dimensional game. The latter enables to apply numerical methods to some important practical problems [26–30].

The above mentioned features are not inherent to differential games with nonfixed time of termination: as a rule, t-sections of maximal stable bridges are not convex, and, what is more worth, it is impossible to reduce the dimension of the problem using the standard change of variables. Numerical methods for solving nonconvex problems with fixed time of termination and for nonfixed time games are developed in papers of V.N.Ushakov and his collaborators [31–34]. The algorithm for constructing the set $W(\theta, M)$ described below is based on the ideas of the algorithm proposed in [19, 20] for linear games with fixed time of termination and uses operations on polygonal lines which are similar to parts of boundaries of convex sets.

The sets $W(\theta, M)$ can be used for finding optimal strategies of the first and second players in minimum-time differential games [2, 3]. But the problem of finding optimal strategies is rather an independent task, and we do not consider it in this paper.

The backward procedures we apply do not have immediate connections with the analysis of singular surfaces [1] of differential games. The construction of singular manifolds for solving differential games is a special field of research [1, 35–37].

1. The statement of the problem

We consider a two-dimensional linear differential game with the dynamics

$$\dot{x} = Ax + u + v \tag{3.1}$$

and with the geometric bounds on controls: $u \in P$, $v \in Q$, where P and Q are convex closed polygons in the plane.

A time $\theta > 0$ and a convex closed polygon $M \subset R^2$ are given. It is required to find the set $W(\theta, M)$ of all initial points $x_0 \in R^2$ from which a feedback control of the first player provides the reaching of M by the time θ.

We define now the set $W(\theta, M)$ more precisely [2, 3]. Let $\mathcal{U}$ be the set of all positional strategies U of the first player. Namely, this is the set of all functions defined on $[0, \theta] \times R^2$ and taking the values in P. Let σ be an arbitrary partition of the segment $[0, \theta]$ formed by the points $0 = t_1 < t_2 < ... < t_n = \theta$, $d(\sigma)$ its diameter, $v(\cdot)$ measurable function of time with values in Q, and $y(\cdot\,; \sigma, x_0, U, v(\cdot))$ the Euler spline emanating from the point x_0. We denote by $W(\theta, M)$ the set of all points $x_0 \in R^2$ for each of which there exist a strategy $U \in \mathcal{U}$ and a mapping $\varepsilon \to \delta(\varepsilon)$ from R_+ to R_+ such that for any $\varepsilon > 0$, any σ with the diameter $d(\sigma) \leq \delta(\varepsilon)$, and any function $v(\cdot)$ with values in Q there exists a time $t \in [0, \theta]$ at which $y(t; \sigma, x_0, U, v(\cdot))$ belongs to the ε-neighborhood of the set M.

In the next section we give a short sketch of the algorithm for the approximate construction of the set $W(\theta, M)$. The more detailed description of the algorithm is done in [17, 38].

2. The main idea of the algorithm

The set $W(\theta, M)$ is formed via a step-by-step backward procedure generating a sequence of embedded sets

$$W(\Delta, M) \subset W(2\Delta, M) \subset W(3\Delta, M) \subset ... \subset W(i\Delta, M) \subset ... \subset W(\theta, M). \quad (3.2)$$

Here Δ is the step of the backward procedure. Each set $W(i\Delta, M)$ consists of all initial points such that the first player brings system (3.1) into the set $W((i-1)\Delta, M)$ within the time duration Δ. We put $W(0, M) = M$.

Before doing the first step of the backward procedure, we find a usable part Γ_0 on the boundary of M. In accordance to [1], the usable part is a curve or several curves of the boundary of M attainable for trajectories of system (3.1) from points lying in the exterior of M close to the boundary of M. The usable part is defined by the following formula

$$\Gamma_0 = \mathrm{cl}\{x \in \partial M : \quad \min_{u \in P} \max_{v \in Q} \langle \ell, Ax + u + v \rangle < 0, \ \forall \ell \in K_x\}.$$

Here K_x is the cone of outward normals to the set M at x. Since the target set is convex, each curve of the usable part is locally convex in the following sense: the normal to the curve at a point x runs in only direction when x moves along the curve.

Let us introduce the term "front". We put $F_0 = \Gamma_0$. The front F_i is the set of all points on the boundary of the set $W(i\Delta, M)$ for which the minimum guaranteeing time of reaching F_{i-1} (therefore, the time of reaching the set $W((i-1)\Delta, M)$) is equal to Δ. For other points on the boundary of $W(i\Delta, M)$ the optimal time of attainment of $W((i-1)\Delta, M)$ is less than Δ. Thus, the line $\partial W(i\Delta, M) \setminus F_i$ possesses the properties of barriers [1]. The front F_i is designed using the previous front F_{i-1}. Straight lines connecting endpoints of F_i with the corresponding endpoints of F_{i-1} give the extension of the barrier lines. The boundary of the set $W(i\Delta, M)$ is formed by the front F_i, the above mentioned extentions of the barrier lines, and the line $\partial W((i-1)\Delta, M) \setminus F_{i-1}$ (Fig. 3.1).

Suppose the usable part of M consists of one curve only. Due to the linearity of system (3.1), the fronts $F_1,\ F_2,\ F_3, \ldots$ inherit the property of the local convexity of Γ_0, and this property is kept until the next front F_i does not meet the already constructed set $W((i-1)\Delta, M)$. If such a meeting happens, we say that the front collides with the set $W((i-1)\Delta, M)$. The situation of "collision" means that the current front meets the barrier part of the boundary of $W((i-1)\Delta, M)$ or the part $\partial M \setminus \Gamma_0$ of the boundary of M. In many examples, the case of collision either does not occur or it happens for sufficiently large values $i\Delta$. The property of the local convexity of fronts enable us to employ, with some small modifications,

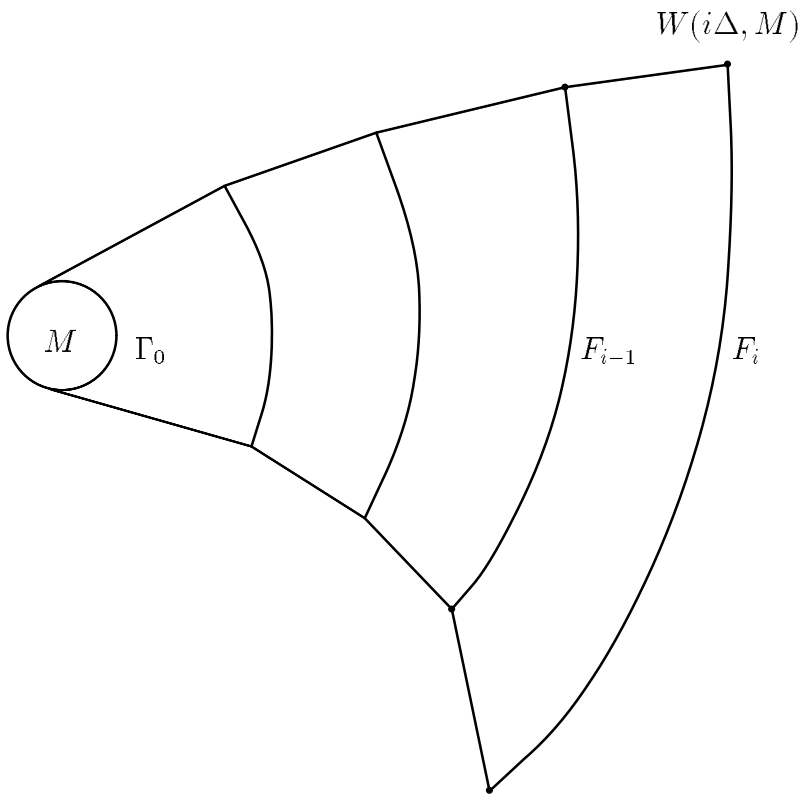


Fig. 3.1. Construction of the set $W(i\Delta, M)$.

procedures for the construction of cross-sections of maximal stable bridges which were developed for linear differential games with convex target sets and fixed time of termination. An example of constructing sequence (3.2) in the case where the collision does not happen is shown in Fig. 3.2. The lines ab, cd are barriers. Further computations are shown in Fig. 3.3 where the collision occurs.

Let the front F_i meets the set $W((i-1)\Delta, M)$. To construct the next front F_{i+1}, we should take into account that F_i and the boundary of $W((i-1)\Delta, M)$ have the nonconvex conjunction. The next front F_{i+1} may be not locally convex.

The barrier lines of the set $W(i\Delta, M)$ are stored in the corresponding computer program as ordered collections of points. Until the case of collision does not happen, updating these collections can be done very easy. The program is not applicable to very complicated cases of collision whose processing requires the exhaustion of significantly large number of variants.

If the usable part of M consists of several fragments of the boundary of M, then the construction can be carried out independently for each fragment until an intersection of the sets sprouting from these pieces does not occur.

So, the algorithm consists of the following operations:

1) Finding the usable part on the boundary of the target set.

2) Constructing the next front using the previous front.

3) Testing the intersections of the current front with the barrier part of the already constructed set. If the intersection is detected, further computations are carried out taking into account the arising nonconvex conjunction.

Let us describe now an algorithm for the construction of fronts. We assume for simplicity that the sets P and Q are segments in the plane. Suppose a current front F_i is already constructed. Let us construct F_{i+1}.

First, consider the case where the situation of collision does not occur. The front F_i is a polygonal line belonging to the boundary of $W(i\Delta, M)$ and possessing the local-convexity property which means that normals to the links of the line are rotated in the clockwise direction when we go along this line in the clockwise direction. Let us enumerate vertices of F_i and denote them z_1, z_2, ..., z_r (see Fig. 3.4). We associate the outward normals ℓ_{j-1} and ℓ_j to the links $[z_{j-1}, z_j]$ and $[z_j, z_{j+1}]$ with the vertex z_j, $j = \overline{2, r-1}$. The vertex z_1 is

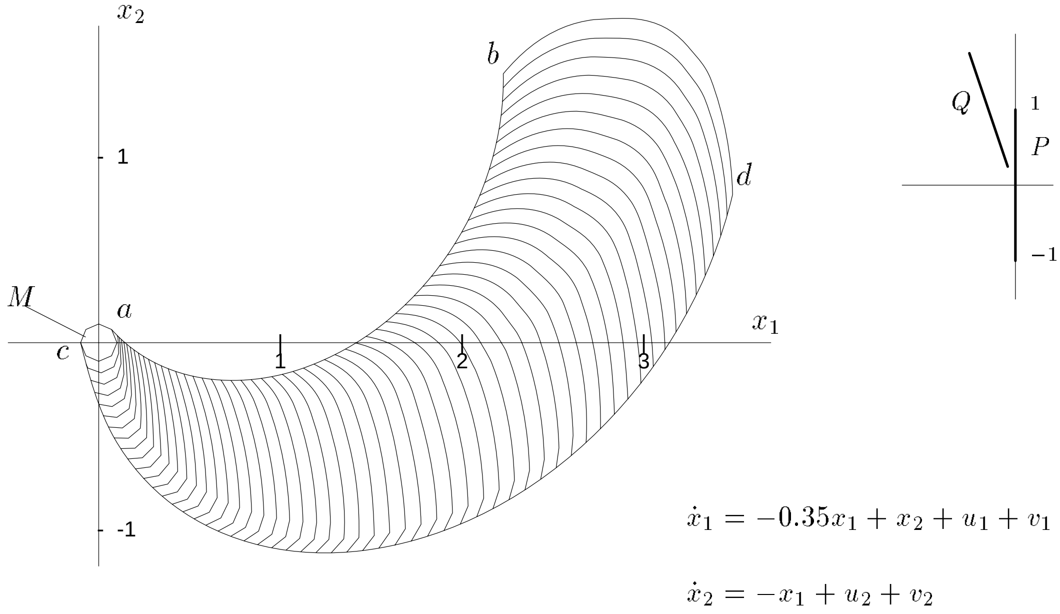


Fig. 3.2. Sequence of the sets $W(i\Delta, M)$, "collision" does not happen.

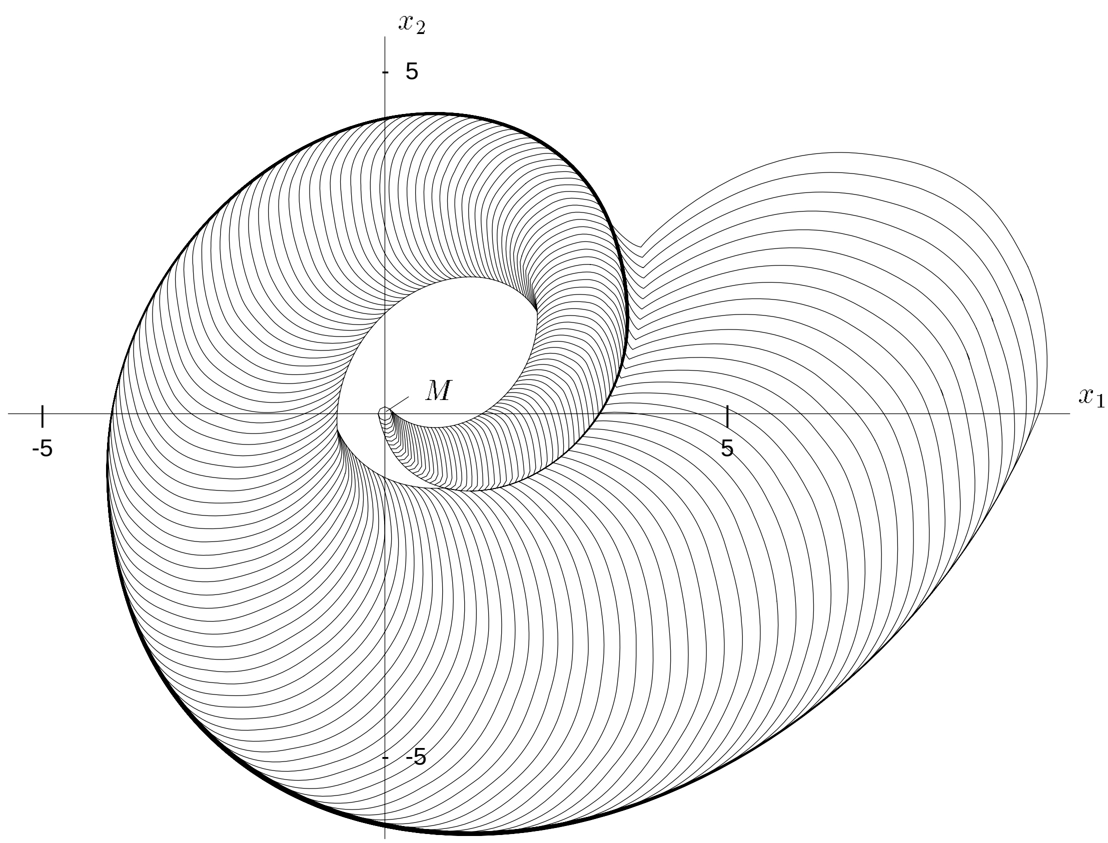


Fig. 3.3. Sequence of the sets $W(i\Delta, M)$, the case of "collision".

associated with normal ℓ_1 and a vector ℓ_0 specified by the following conditions:

$$1)\quad \min_{u\in P}\max_{v\in Q}\langle \ell_0, Az_1+u+v\rangle = 0,$$

$$2)\quad \forall \ell : \ell_1 < \ell < \ell_0 \quad \min_{u\in P}\max_{v\in Q}\langle \ell, Az_1+u+v\rangle < 0,$$

The vertex z_r is associated with the normal ℓ_{r-1} and a vector ℓ_r such that

$$1)\quad \min_{u\in P}\max_{v\in Q}\langle \ell_r, Az_r+u+v\rangle = 0,$$

$$2)\quad \forall \ell : \ell_r < \ell < \ell_{r-1} \quad \min_{u\in P}\max_{v\in Q}\langle \ell, Az_r+u+v\rangle < 0.$$

If $\min\limits_{u\in P}\max\limits_{v\in Q}\langle \ell_1, Az_1+u+v\rangle = 0$, then we put $\ell_0 = \ell_1$. Similarly, if $\min\limits_{u\in P}\max\limits_{v\in Q}\langle \ell_{r-1}, Az_r+u+v\rangle = 0$, we set $\ell_r = \ell_{r-1}$.

Let q_1, q_2 be endpoints of the segment Q. We divide the polygonal line F_i into parts $F_i^{(k)}$ so that for any normal ℓ_j to the line $F_i^{(k)}$ the same endpoint of the segment Q gives maximum over $v\in Q$ to the scalar product $\langle \ell_j, v\rangle$.

Assume for definiteness that there are two parts $F_i^{(1)} = [z_1, ..., z_\omega]$ and $F_i^{(2)} = [z_\omega, ..., z_r]$ in all, and we have

$$\arg\max_{v\in Q}\langle \ell_j, v\rangle = q_1, \quad j = \overline{0, \omega-1},$$

for the first of them and

$$\arg\max_{v\in Q}\langle \ell_j, v\rangle = q_2, \quad j = \overline{\omega, r},$$

for the second.

Such a partition means that one of the normals to the segment Q lies between the vectors $\ell_{\omega-1}$, ℓ_ω. We denote this normal by ℓ_q.

For each vertex z_j, $j = \overline{1, \omega-1}$, of $F_i^{(1)}$, we consider the following trajectories

$$z(\tau) = z_j - \tau(Az_j+u+v) \tag{3.3}$$

with $v = q_1$ and $u = u_*$ or $u = u^*$ where u_*, u^* are specified by the following conditions

$$u_* = \arg\min_{u\in P}\langle \ell_{j-1}, u\rangle, \quad u^* = \arg\min_{u\in P}\langle \ell_j, u\rangle. \tag{3.4}$$

When constructing trajectories (3.3) corresponding to the point z_ω, we replace the index j by ω and the vector ℓ_j by ℓ_q in (3.3), (3.4).

If $u_* = u^*$, we obtain a single trajectory. If $u_* \neq u^*$, we have two trajectories. If the vector ℓ_{j-1} is orthogonal to the segment P, then the vector ℓ_j is not orthogonal to P and u^* is determined uniquely. We set u_* to be that of two endpoints of P

which does not coincide with u^*. We do similarly if the vector ℓ_j is orthogonal to P. We continue the trajectories up to the time $\tau = \Delta$.

For the points $z_{\omega+1}, ..., z_r$ of $F_i^{(2)}$, the trajectories are constructed in a similar way with the replacement $v = q_1$ by $v = q_2$. When constructing the trajectories corresponding to the point z_ω, we replace the index j by ω and the vector ℓ_{j-1} by ℓ_q in (3.3), (3.4).

So, if the front F_i is divided into two parts $F_i^{(1)}$ and $F_i^{(2)}$ we deal with two families of regular extremal trajectories. Trajectories of each family can be interpreted as characteristics of the appropriate Bellmann-Isaacs equation.

Being connected consecutively, the corresponding to time $\tau = \Delta$ endpoints of the extremal trajectories emanating from the vertices of $F_i^{(1)}$ form a spline $[\xi_1, ..., \xi_S]$. The endpoints of the trajectories emanating from the vertices of $F_i^{(2)}$ form a spline $[\xi_{S+1}, ..., \xi_m]$. The method we used for constructing the trajectories ensures the type of the intersection of these splines ("the swallow-tail") which is shown in Fig. 3.4. Two trajectories emanating from the point z_ω and arriving at the points ξ_S, ξ_{S+1} are depicted in Fig. 3.5 with dash lines. If we imagine that trajectories are constructed for all points of F_i, then the part $\xi_\alpha\xi_S$ is formed by the endpoints of the trajectories (with $v = q_1$) emanating from the points of the line z^*z_ω which adjoins to z_ω from above, and the part $\xi_\alpha\xi_{S+1}$ is formed by the endpoints of the trajectories (with $v = q_2$) emanating from the points of the line $z_\omega z_*$ which adjoins to the point z_ω from below. It is clear that the trajectories which form the part $\xi_\alpha\xi_{S+1}$ intersect the trajectories which form the part $\xi_\alpha\xi_S$. The lines $\xi_\alpha\xi_S$, $\xi_\alpha\xi_{S+1}$ are eliminated. The resulting spline $[\xi_1, ..., \xi_\alpha, ..., \xi_m]$ is the next front F_{i+1}. Each of the lines $\xi_\alpha\xi_S$, $\xi_\alpha\xi_{S+1}$ may consist of several (not a single) segments. If so, we eliminate the splines $[\xi_\alpha, ..., \xi_S]$ and $[\xi_\alpha, ..., \xi_{S+1}]$.

If the polygonal line F_i is divided into three or more parts $F_i^{(k)}$, then some complicated types of intersections of the polygonal lines composed from endpoints of the extremal trajectories may occur. The algorithm eliminates the lines corresponding to the intersection of characteristics. If the extremal vector $v \in Q$ is the same for all normals ℓ_j to F_i, then we do not divide F_i into the parts $F_i^{(k)}$, and we consider the extremal trajectories corresponding to this value v. The curve formed by the endpoints of the trajectories is the next front F_{i+1}.

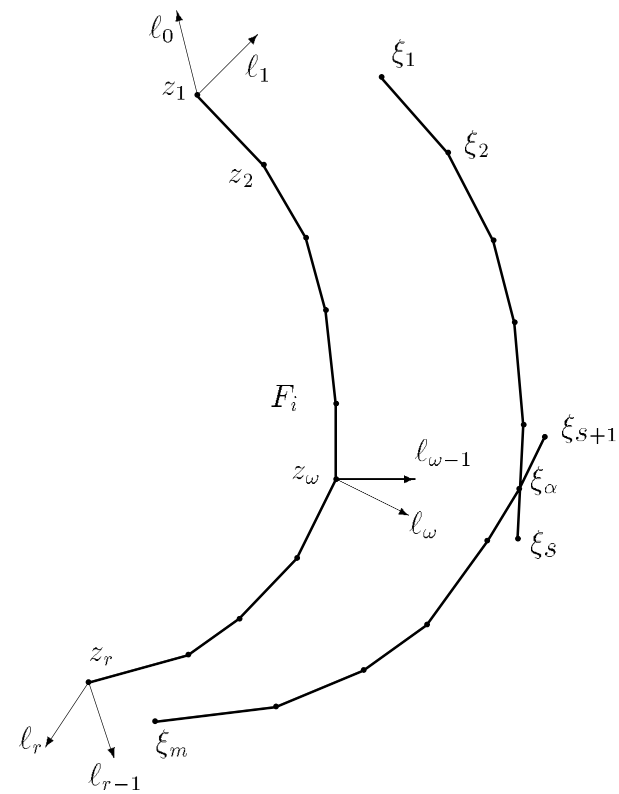


Fig. 3.4. Construction of the current front.

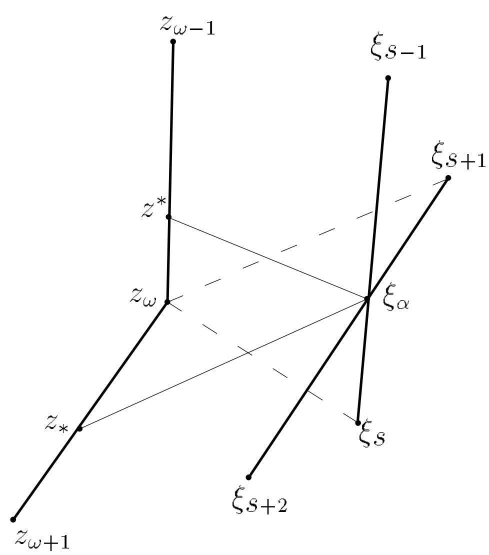


Fig. 3.5. Intersection of polygonal lines in the process of construction of the current front.

Consider now the case of collision. Let the current front meets the barrier part of the boundary of the set $W((i-1)\Delta, M)$. The part of F_i belonging to $W((i-1)\Delta, M)$ is eliminated. When constructing the front F_{i+1}, we should take into account the nonconvex conjunction of F_i with $\partial W((i-1)\Delta, M)$. We enumerate the vertices of F_i and denote them by $z_1, \ldots, z_r$. We add the small segment $z_1 z_0$ (Fig. 3.6) of the boundary of $\partial W((i-1)\Delta, M)$ to the front F_i. The point z_1 is the point of nonconvex conjunction. Consider a small neighborhood of the point z_1. When constructing trajectories started from the point z_1, we deal with the following "inverse problem ".

Assume that the aim of the second player governing the control variable v is to bring the state vector of system (3.1) to the shaded set shown in Fig. 3.6. The aim of the first player, who uses the control variable u, is to avoid the state vector from meeting the shaded set. We associate both the normal $\widetilde{\ell}_0$ to $[z_1 z_0]$ and the normal $\widetilde{\ell}_1$ to $[z_1 z_2]$ with the point z_1, and consider extremal trajectories constructed according to the above described method with the only difference that the sets P and Q change their places in the description.

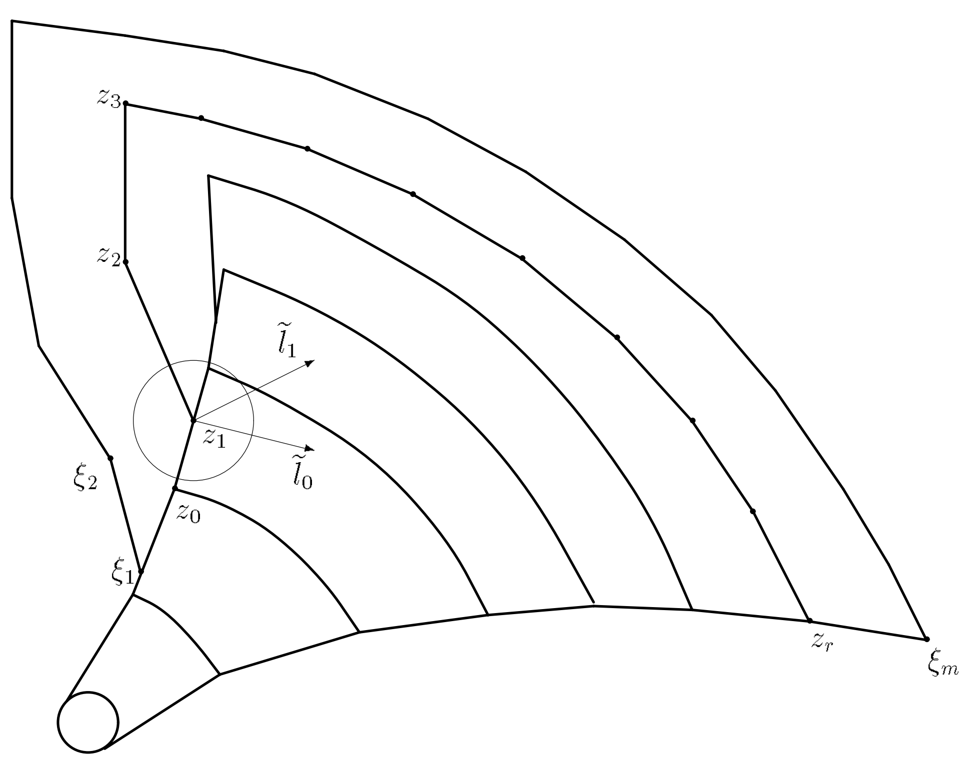


Fig. 3.6. Treatment of the case of the nonconvex conjuncture.

If the extremal control of the first player is the same for both vectors $\widetilde{\ell}_0$ and $\widetilde{\ell}_1$ (let, for example, $u = p_1$), then we consider trajectories (3.3) emanating from z_1 and corresponding to $u = p_1$ and $v = v_*$ or $v = v^*$ which are determined from the conditions

$$v_* = \arg\min_{v \in Q} \langle \widetilde{\ell}_0, v \rangle, \quad v^* = \arg\min_{v \in Q} \langle \widetilde{\ell}_1, v \rangle. \tag{3.5}$$

If $v_* = v^*$, then we obtain a single trajectory. If $v_* \neq v^*$, we have two trajectories. Suppose now that extremal controls of the first player are different for the vectors $\widetilde{\ell}_0$, $\widetilde{\ell}_1$, i.e.

$$\arg\max_{u \in P} \langle \widetilde{\ell}_0, u \rangle = p_1, \quad \arg\max_{u \in P} \langle \widetilde{\ell}_1, u \rangle = p_2.$$

This means that one of two normals to P (we denote it by ℓ_p) lies between the vectors $\widetilde{\ell}_0$, $\widetilde{\ell}_1$. First, we trace the trajectories from the point z_1 with $u = p_1$ and letting $v = v_*$ and $v = v^*$ where v_*, v^* are found from (3.5) with ℓ_p in place of $\widetilde{\ell}_1$; then, we consider trajectories with $u = p_2$ and both $v = v_*$ and $v = v^*$ where v_*, v^* are chosen from (3.5) with ℓ_p in place of $\widetilde{\ell}_0$.

From the other vertices of F_i, we trace the trajectories in the "normal" way. As the result, we obtain a collection of several curves. Now, we eliminate the parts of these curves corresponding to intersections of characteristics. The resulting polygonal line $[\xi_1, \xi_2, ..., \xi_m]$ shown in Fig. 3.6 corresponds to the case where $v_* \neq v^*$ and extremal values of u are different for the vectors ℓ_0 and ℓ_1. The vertices ξ_1 and ξ_2 are the points of the nonconvexity. We take this into account when processing these points at the next stage of the algorithm.

If P and Q are polygons, then the number of families of regular extremal trajectories appearing in the process of constructions increases. In the case of the local convexity (local concavity), the number of trajectories tracing from each vertex of the current front is determined by the number of outward normals to edges of P (Q) lying between the normals associated with this vertex.

3. Examples of solving minimum-time game problems

1. The canonical example of the minimum-time problem in the theory of optimal control has the following form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \qquad | u | \leq 1. \end{aligned}$$

We add the disturbance v to the first equation and consider the following differential game:

$$\begin{aligned} \dot{x}_1 &= x_2 + v \\ \dot{x}_2 &= u, \end{aligned} \tag{3.6}$$

$$| u | \leq 1, \quad | v | \leq 1.$$

The first player minimizes the time of reaching M, the aim of the second player is opposite.

If M is a small regular polygon with the center at the origin, then the sequence (3.2) becomes stationary starting with some index i. The solvability set for the corresponding game of kind reduces to the origin when the set M reduces to the origin. Similar situation occurs if the target set is a sufficiently small regular polygon with the center at $\bar{x}$ such that $| \bar{x}_2| < 1$. If M is a regular polygon with the center at $\bar{x}$ such that $| \bar{x}_2 | > 1$, then the solvability set of the game of kind is the whole plane; the construction of $W(i\Delta, M)$ is meaningful for any index i.

Let M be a regular octagon inscribed into the circle with the radius 0.1 and with the center at the point $(0, 2)$. We set $\Delta = 0.05$. The sets $W(\tau, M)$ ($W(\tau)$ briefly) for the time instants $\tau = k \cdot 4\Delta$, $k = \overline{0, 50}$, are shown in Fig. 3.7. The sets $W(\tau)$ for $\tau = k \cdot 20\Delta$, $k = \overline{1, 20}$, are given in Fig. 3.8. The first situation of collision happens at $\tau = 6.6$. The set $W(6.6)$ is contoured in Fig. 3.7. The front $W(9)$ is also shown. We denote by a, b the endpoints of the usable part Γ_0 of M. The curves ac and bd formed by the endpoints of the fronts are barriers. The value function is discontinuous on these curves and also on the line $\partial M \setminus \Gamma_0$. The line cf formed by the corners of the fronts is the set where the value function is not differentiable.

The differential game we consider was investigated in [39, 40] under assumptions of incomplete information about the state vector. The minimum-time game problem with dynamics (3.6) was also studied in [41]. It follows from these papers that the optimal synthesis of feedback controls is determined by the singular line $ghbacp$ (Fig. 3.9) dividing the plane into two parts. We have $u^0 = -1$, $v^0 = 1$ above this line and $u^0 = 1$, $v^0 = -1$ below it. The joining to M fragments ac, bh of the singular line are barriers. The rest of the line is formed by the equivocal curves e_1, e_2, e_3, e_4, e_5,... . The second equivocal curve e_2 is designed using the first curve e_1, the third curve e_3 is obtained using e_2, and so on. The first equivocal line e_1 is described by a differential equation. For other equivocal

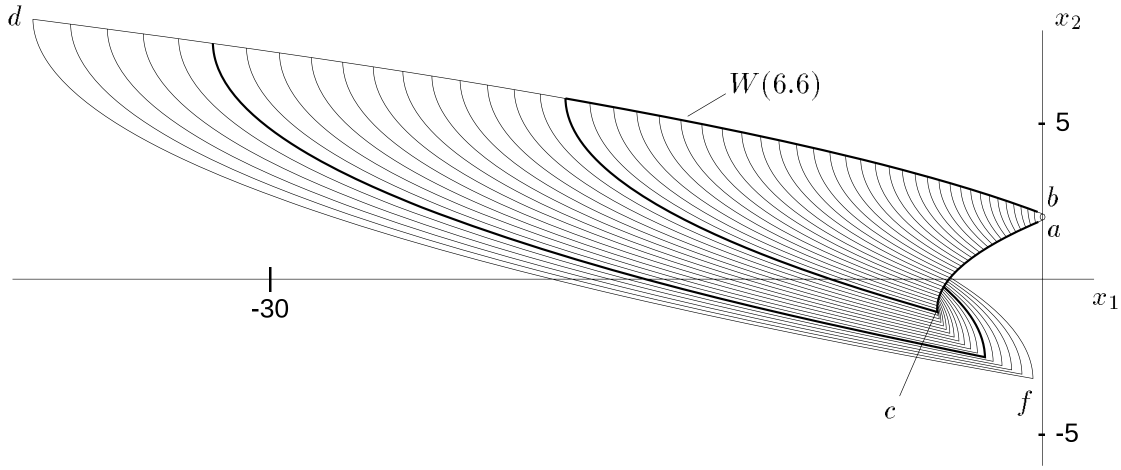


Fig. 3.7. Solution at $\tau = 10$.

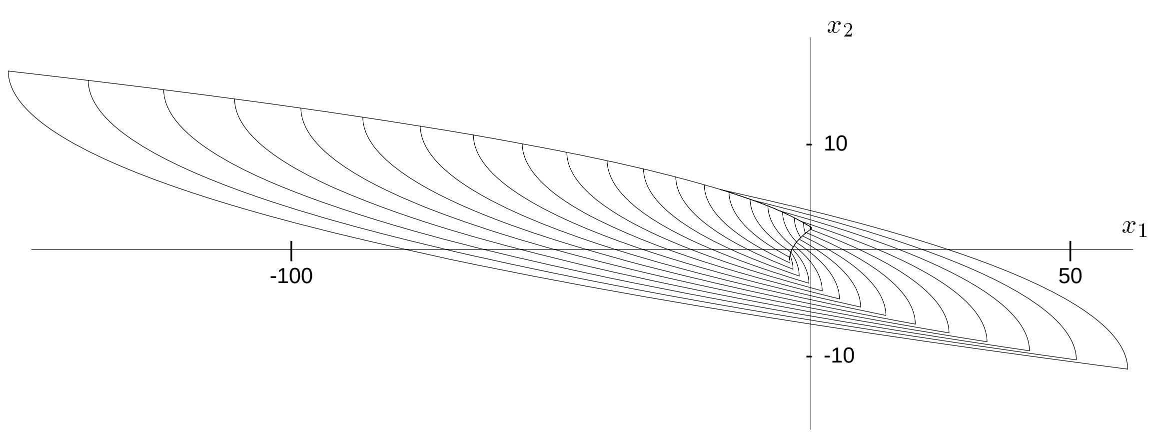


Fig. 3.8. Solution at $\tau = 20$.

lines, there are not any explicit equation, some qualitative properties can only be formulated. So, the value function does not have global analytical description.

Computations depicted in Fig. 3.7 are carried out up to the time $\tau = 10$. The line cf is an initial part of the equivocal curve e_1 depicted in Fig. 3.9. The line bd of the barrier bh shown in Fig. 3.9 ceases to grow when the endpoints of some front come together. The development of such a situation is seen in Fig. 3.8. In Fig. 3.8, the constructions are carried out up to $\tau = 20$ but the points f and d do not yet reach their limit locations (see Fig. 3.9.)

Thus, in this game with the simple dynamics, the most complicated singular lines (equivocal lines) appear. This fact makes difficult the employment of the method of characteristics [1]. When we apply the above described algorithm to this example, the only difficulty is the presence of the situation of collision but this does not lead to the loss of the local convexity of fronts.

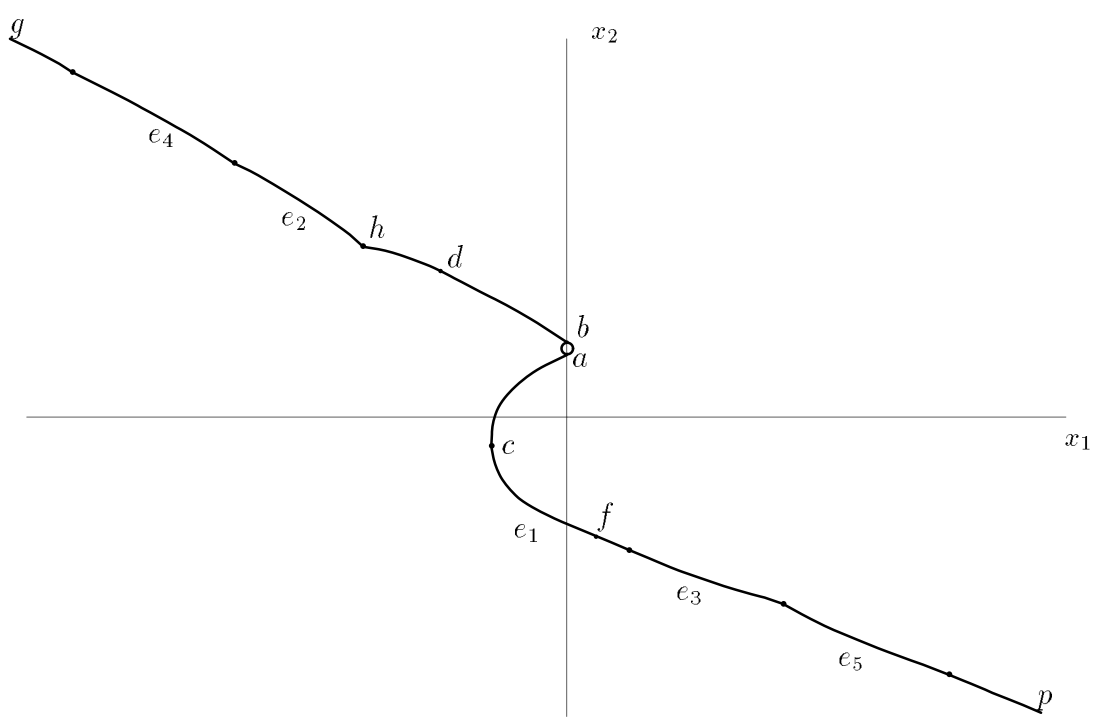


Fig. 3.9. Singular line in the problem of control of a mass point.

2. In [42], the complete analysis of the minimum-time differential game with the following dynamics

$$\begin{aligned} \dot{x}_1 &= -\lambda_1 x_1 + u + p_1 - v \\ \dot{x}_2 &= -\lambda_2 x_2 + ku + p_2 - lv, \end{aligned} \tag{3.7}$$

$$| u | \leq \mu, \quad | v | \leq \nu,$$

and the target set M consisting of a single point is done. Here, $\lambda_2 > \lambda_1 > 0$, $k > 0$, $l < 0$, $\mu > 0$, $\nu > 0$; p_1, p_2 are arbitrary numbers. Depending on parameters of the problem, classifications of types of solutions are carried out, singular lines are found, and optimal strategies of the players are designed.

In this paper, we give computational results for problem (3.7). We assume M to be a regular octagon inscribed into a circle of some small radius and with the center at the origin.

In the case depicted in Fig. 3.10, the optimal synthesis is determined by the equivocal line (formed by the corners of fronts). The equivocal line divides the domain of the value function (the region bounded by the barrier lines) into two parts. The optimal controls take the values $u^0 = \mu$, $v^0 = \nu$ above the equivocal line and the values $u^0 = -\mu$, $v^0 = -\nu$ below it.

The solution for other values of parameters is presented in Fig. 3.11. In contrast to the previous case, both barrier lines go up and should intersect each other after a sufficiently large number of steps of the backward procedure.

The simplest case is shown in Fig. 3.12. The barriers go down and intersect each other. Singular lines do not appear inside the domain of the value function. The optimal controls in the interior of the domain of the value function take the values $u^0 = -\mu$, $v^0 = -\nu$.

For the values of the parameters corresponding to Fig. 3.13, the attainment of the origin can be guaranteed for any initial state. The computations in Fig. 3.13 are carried up to $\tau = 4.4$. The curves ac, bd are barriers, cf is the line with the equivocal property.

Remark. In [42], there are misprints in formulas (1.1), (2.1), (2.2), and in the formula for $h(u, v)$: the signs of the terms which contain the control variable v should be replaced by the opposite ones.

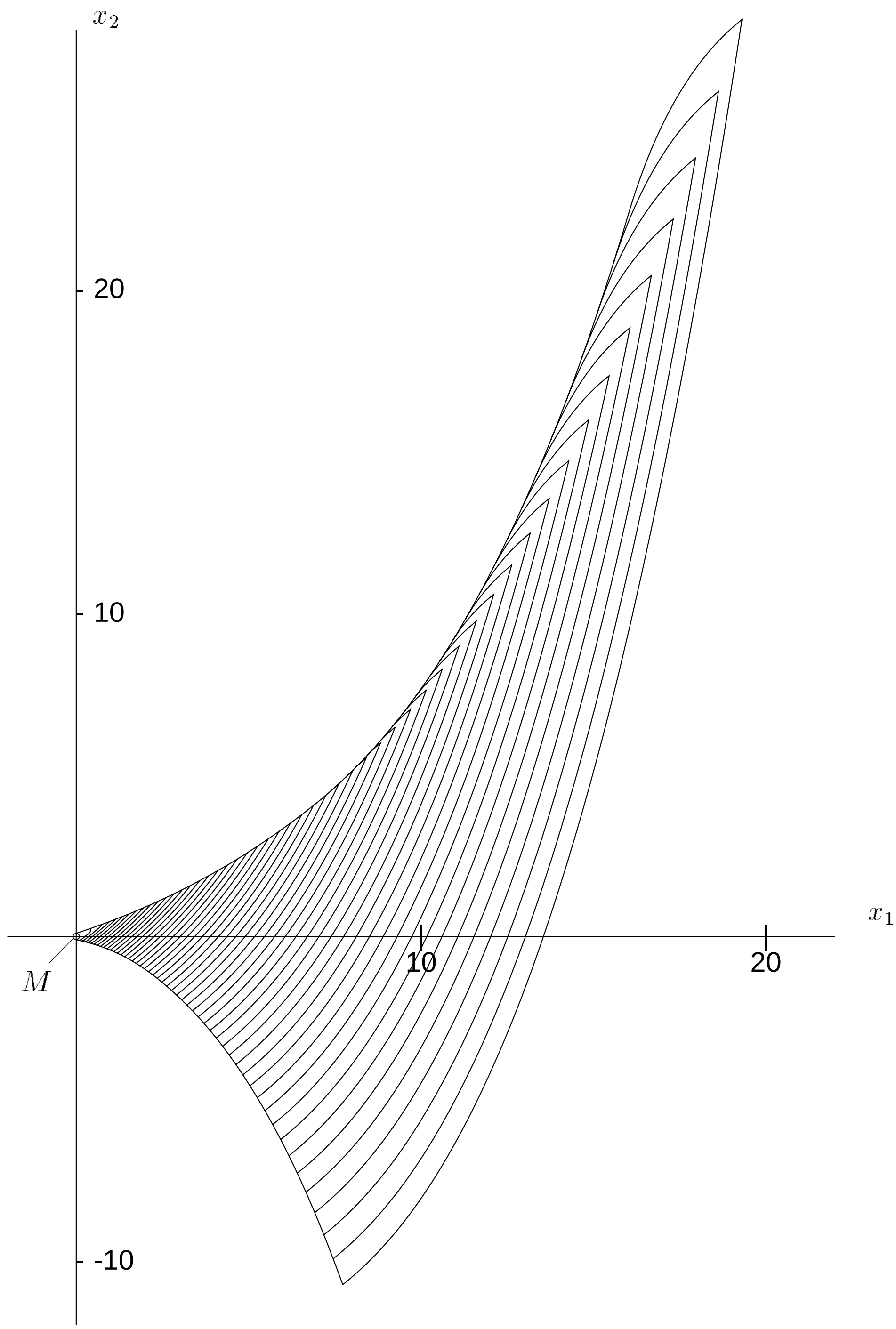


Fig. 3.10. Solution of stable system: $\lambda_1 = 0.1,\ \lambda_2 = 0.4,$

$k = 5/6,\ l = -1,\ \mu = 0.6,\ \nu = 0.15,\ p_1 = -1.2,\ p_2 = -0.3.$

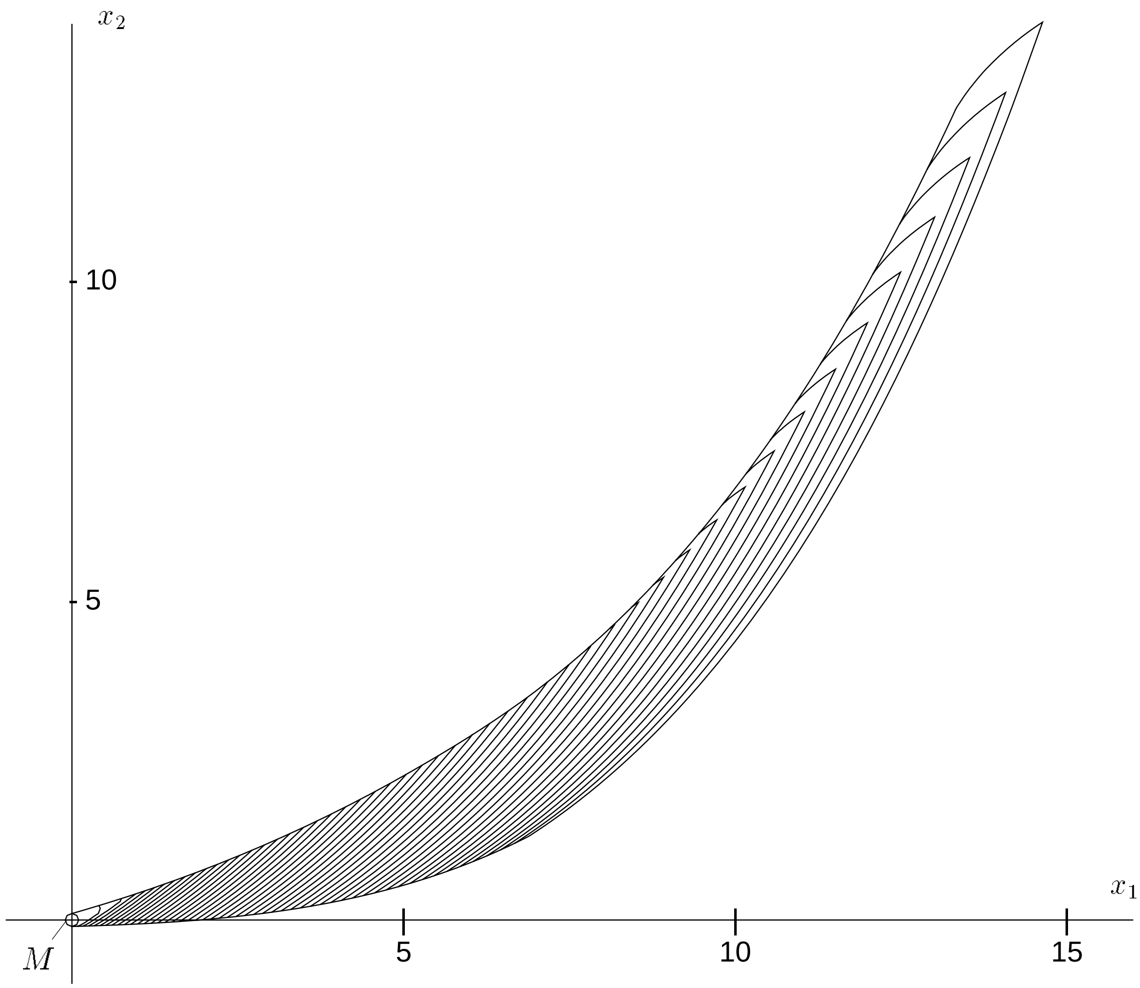


Fig. 3.11. Solution of stable system: $\lambda_1 = 0.1,\ \lambda_2 = 0.4,$

$k = 2/3,\ l = -1,\ \mu = 0.6,\ \nu = 0.15,\ p_1 = -1.2,\ p_2 = -0.3.$

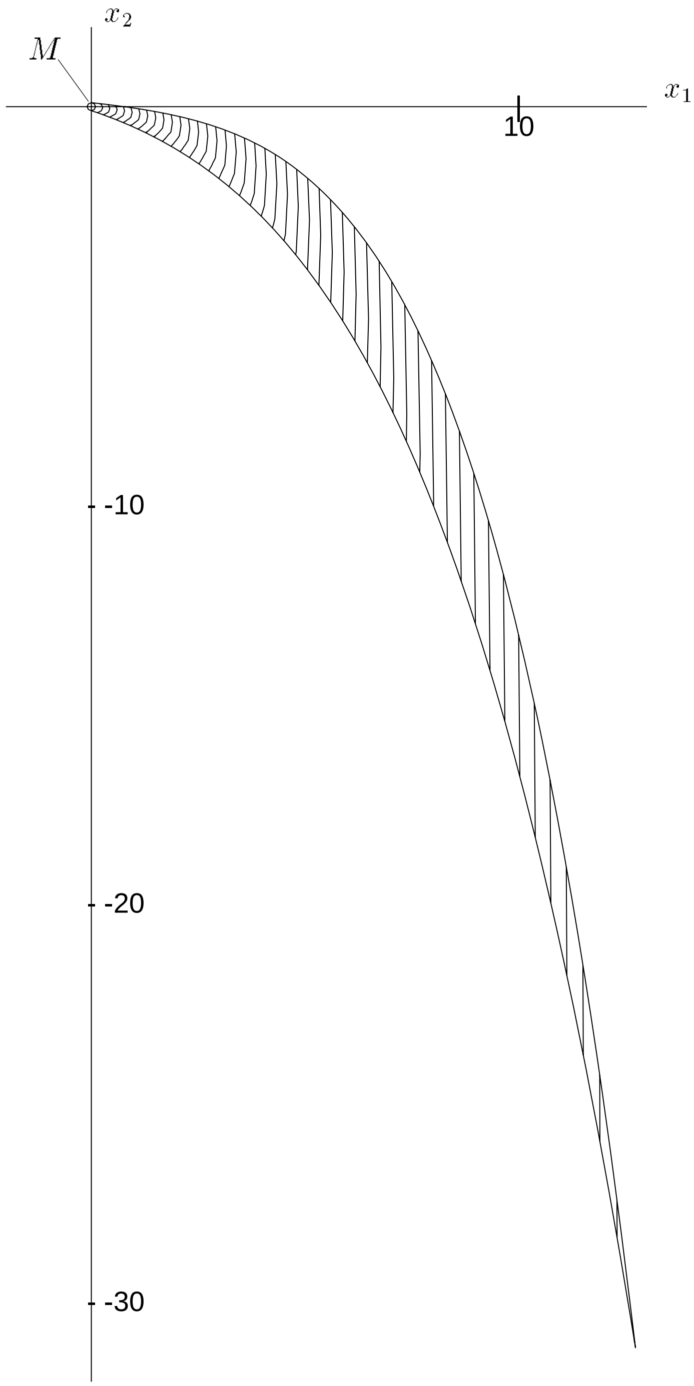


Fig. 3.12. Solution of stable system: $\lambda_1 = 0.1,\ \lambda_2 = 0.6,$

$k = 1,\ l = -1/3,\ \mu = 0.1,\ \nu = 0.3,\ p_1 = -0.6,\ p_2 = 0.3.$

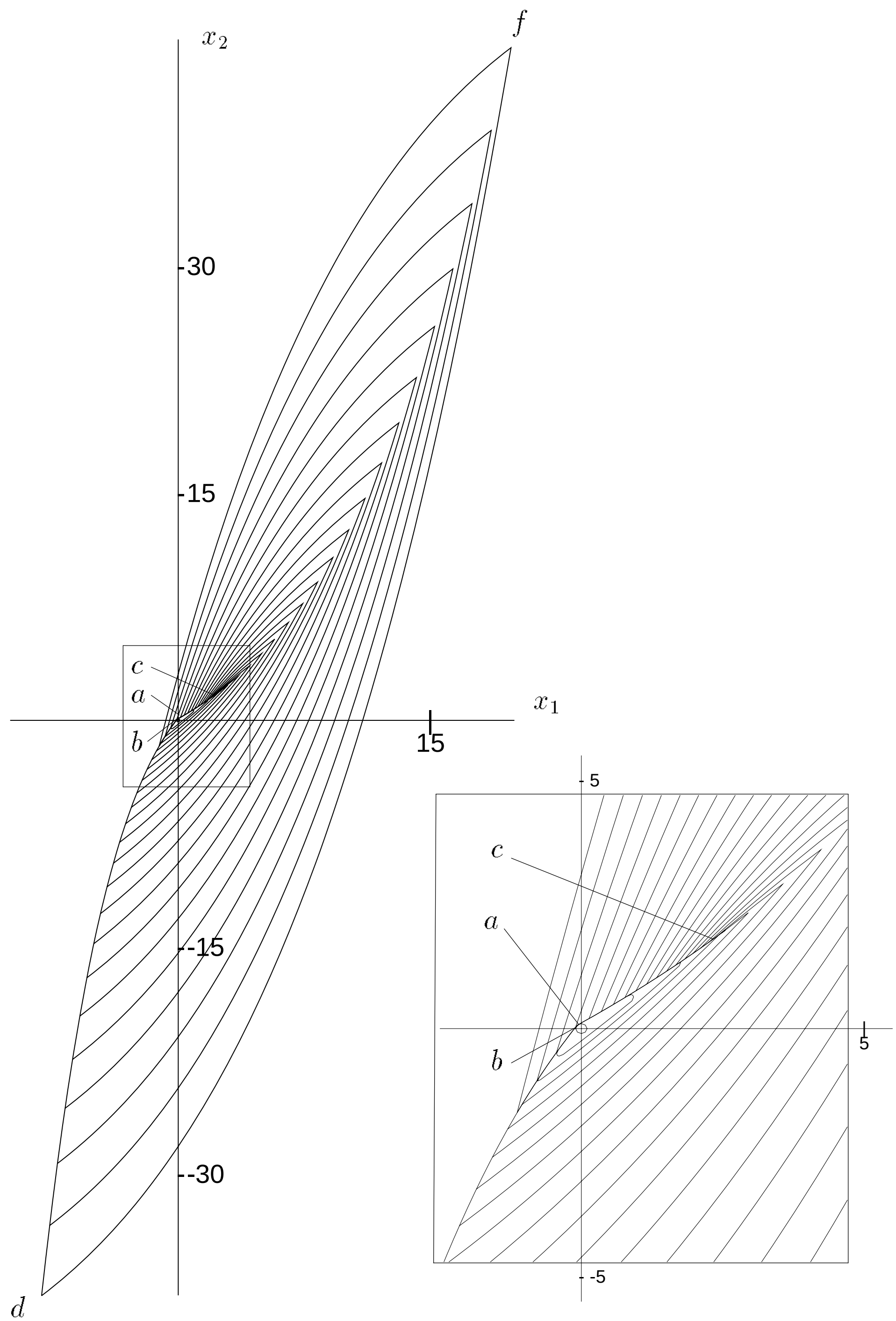
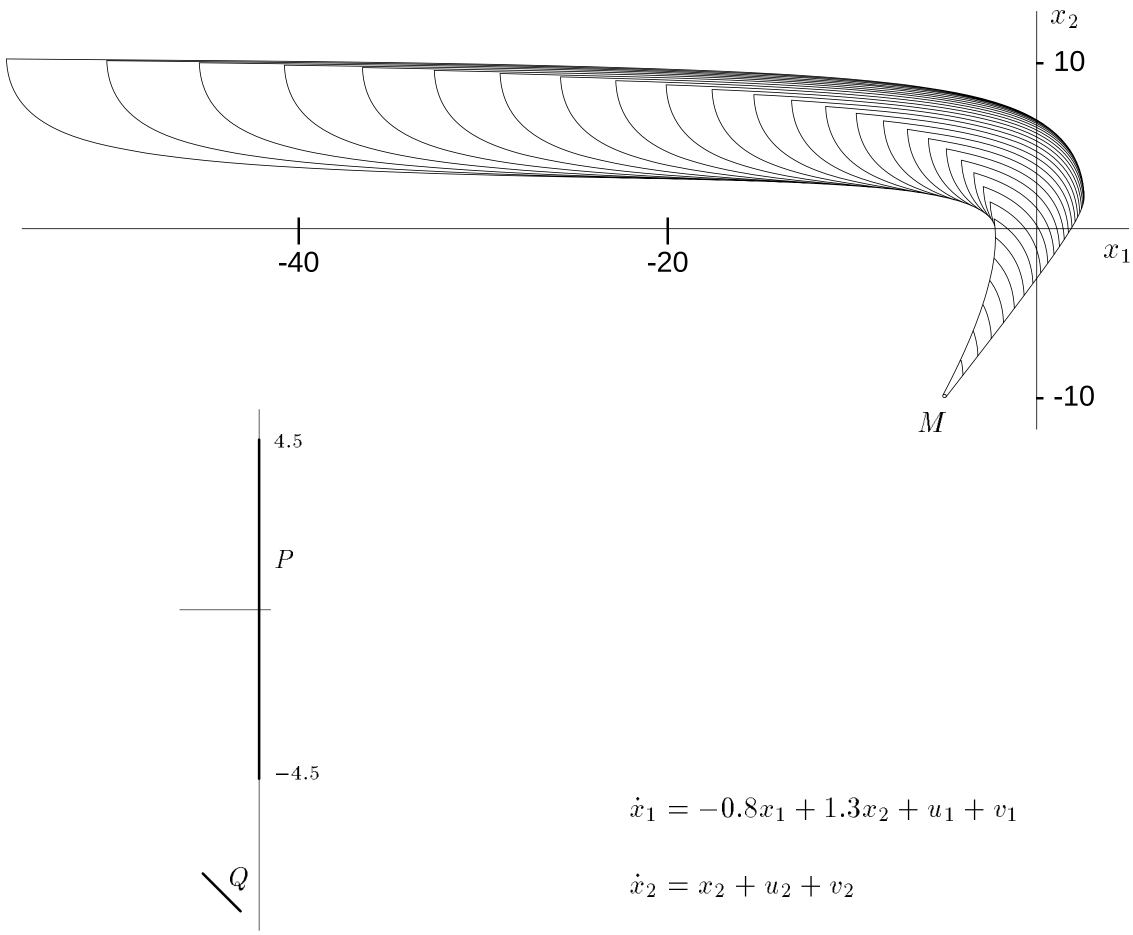


Fig. 3.13. Solution of stable system: $\lambda_1 = 0.1,\ \lambda_2 = 0.6,$

$k = 5/6,\ l = -1,\ \mu = 3,\ \nu = 0.15,\ p_1 = -1.2,\ p_2 = -0.3.$

3. In Figs. 3.14–3.16, results of computing the sets $W(i\Delta, M)$ for a system with a matrix A which has real eigenvalues of the opposite sign are presented. The set M is a regular octagon inscribed into the circle with the radius 0.1 and the center at $(-5, -10)$. The matrix A and the segment P are the same for all Figs. 3.14–3.16, the pictures differ one from other because they correspond to different segments Q.



$$\dot{x}_1 = -0.8x_1 + 1.3x_2 + u_1 + v_1$$

$$\dot{x}_2 = x_2 + u_2 + v_2$$

Fig. 3.14.

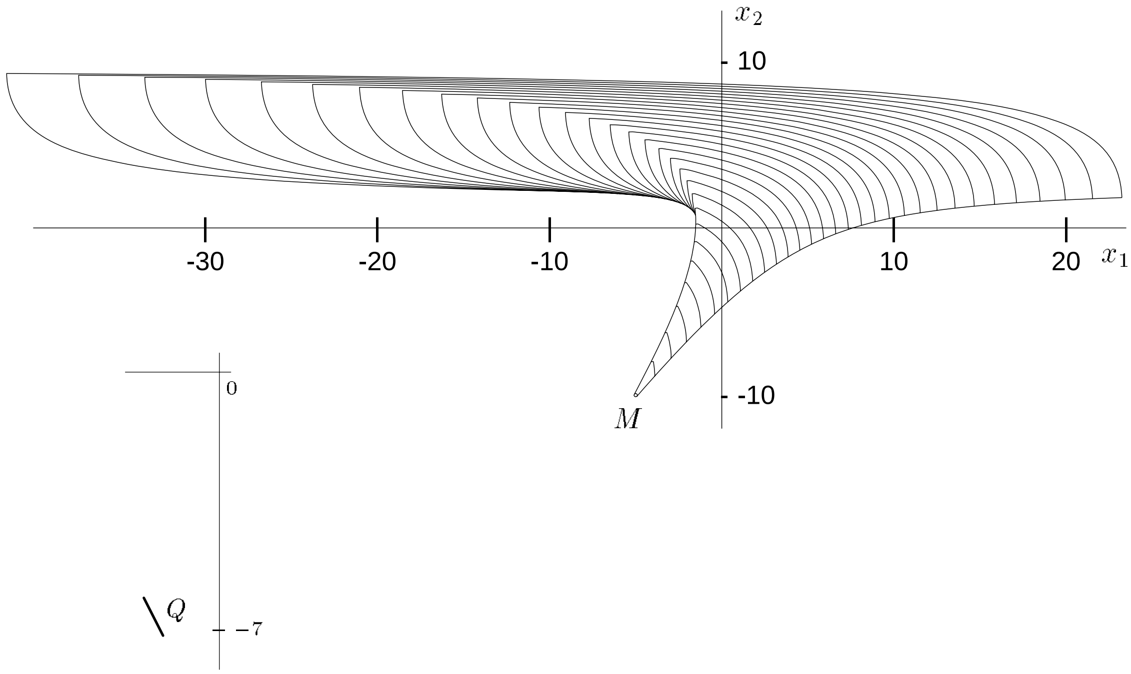


Fig. 3.15.

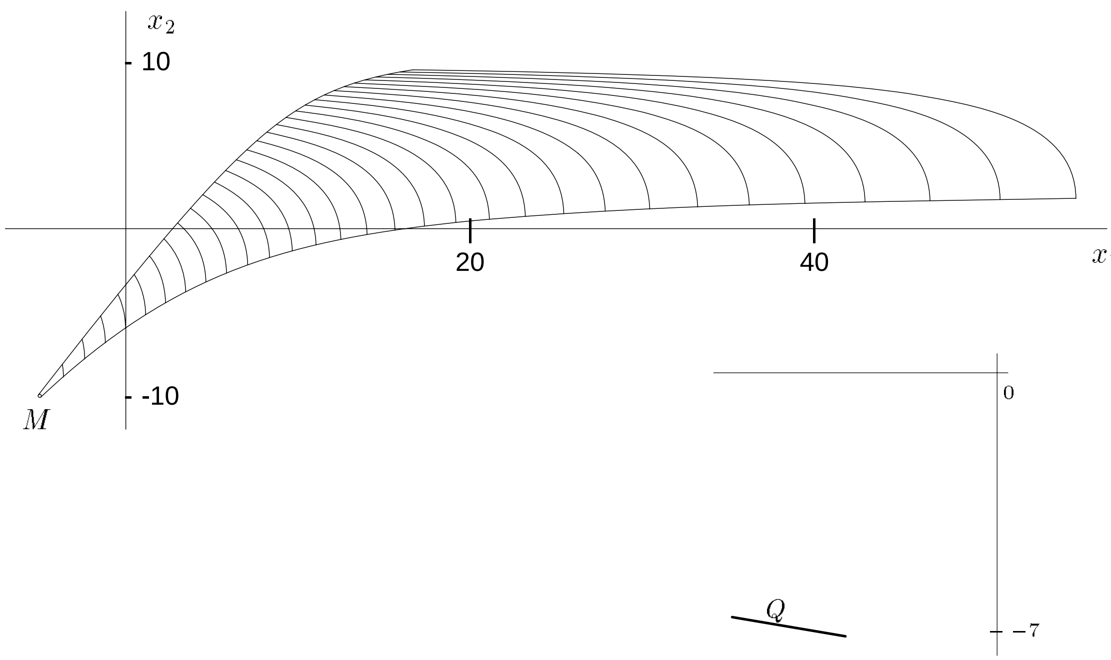


Fig. 3.16.

4. Consider the following oscillating system

$$\begin{aligned} \dot{x}_1 &= \quad x_2 + u_1 + v_1 \\ \dot{x}_2 &= -x_1 + u_2 + v_2, \end{aligned} \tag{3.8}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in P, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in Q.$$

Figs. 3.17–3.19 demonstrate the development of the set $W(\tau, M)$ when τ increases. The target set M is a regular octagon with the center at the origin. The sets P, Q are drawn in Fig. 3.17. The step of the backward procedure is $\Delta = 0.05$. The calculations are carried out up to $\tau = 5$ in Fig. 3.13 and up to $\tau = 6.45$ in Fig. 3.18. At $\tau = 6.45$, the front collides with the set M. For $\tau > 6.45$, the front is divided into two parts. Each of these parts loses the property of local convexity at some instant of time. In Fig. 3.19, filling "the lune" that finishes by the time $\tau = 8.15$ is shown. The fronts in the lower part of the picture are computed up to $\tau = 6.95$.

Changes of the solution caused by the reduction of resources of the second player in differential game (3.8) are demonstrated in Fig. 3.20. The left barrier line terminates at some instant of time, then the front begins to go around this barrier line, one of the endpoints of the front slides along the outward side of the barrier. As the result, the lune appearing in the previous example does not arise. The calculations are carried out up to $\tau = 4.9$.

5. Fig. 3.21 corresponds to an example with the oscillating dynamics given in Fig. 1.6 of Chapter 1. The target set is a regular octagon inscribed into a circle of some small radius and with the center at the origin. The process of filling the solvability set $\mathcal{B}$ of the game of kind by level sets of the value function of the corresponding minimum-time game is demonstrated. The calculations are carried out up to $\tau = 12$. Fronts are nonsmooth for relatively small interval of the parameter τ.

6. For all examples we have considered, the usable part of M consists of a single arc. Fig. 3.22 demonstrates an example where the usable part consists of two parts $\Gamma_0^{(1)}$, $\Gamma_0^{(2)}$. The dynamics of the control system is defined by equations (3.6). The set M is a regular 12-gon inscribed into the unit circle with the center at the origin. The step of the backward procedure is $\Delta = 0.05$. The problem is symmetric with respect to the origin. So, the set $W(\tau, M)$ is formed by two symmetric parts. The calculations are carried out up to $\tau = 4$.

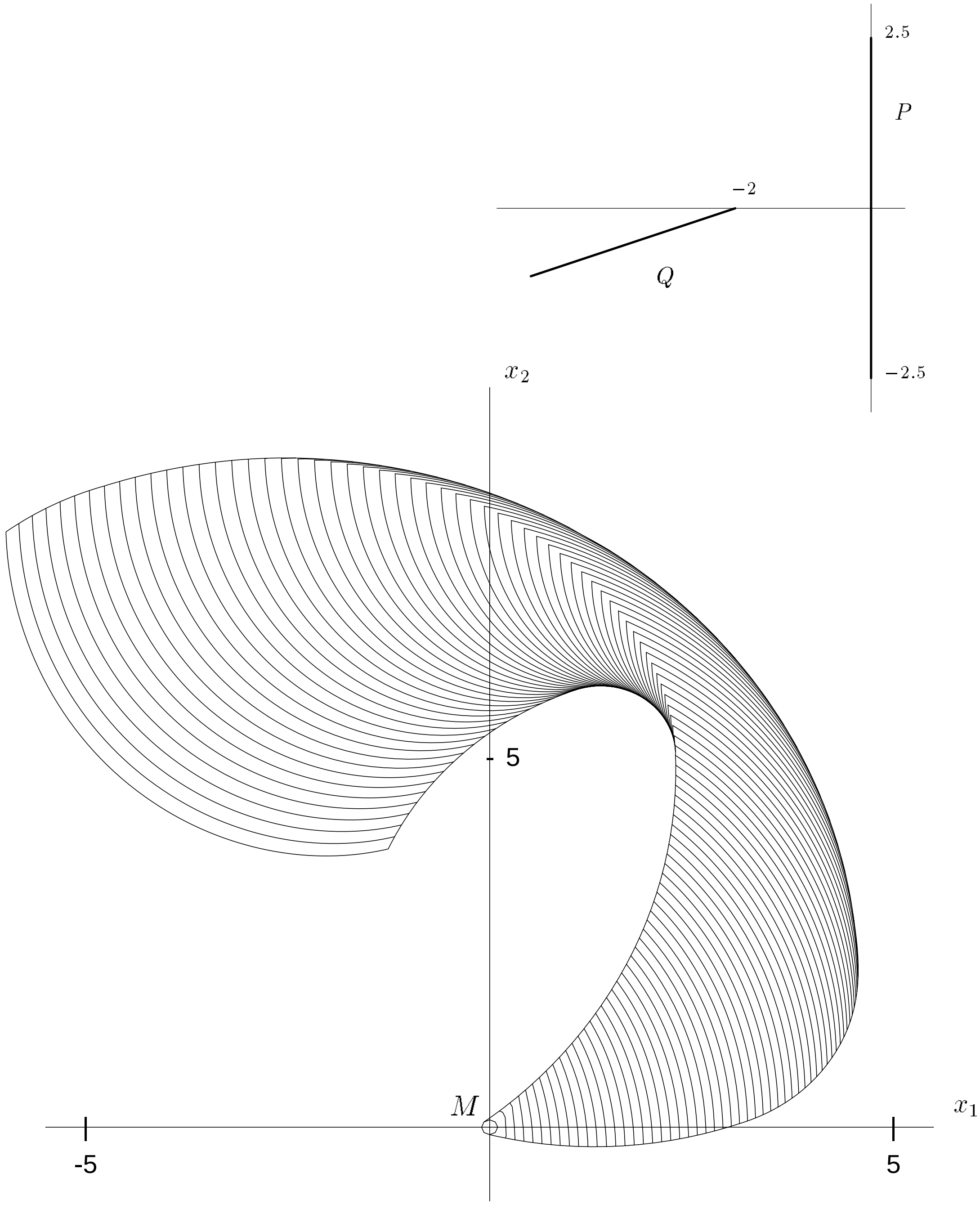


Fig. 3.17. Initial stage of constructions.

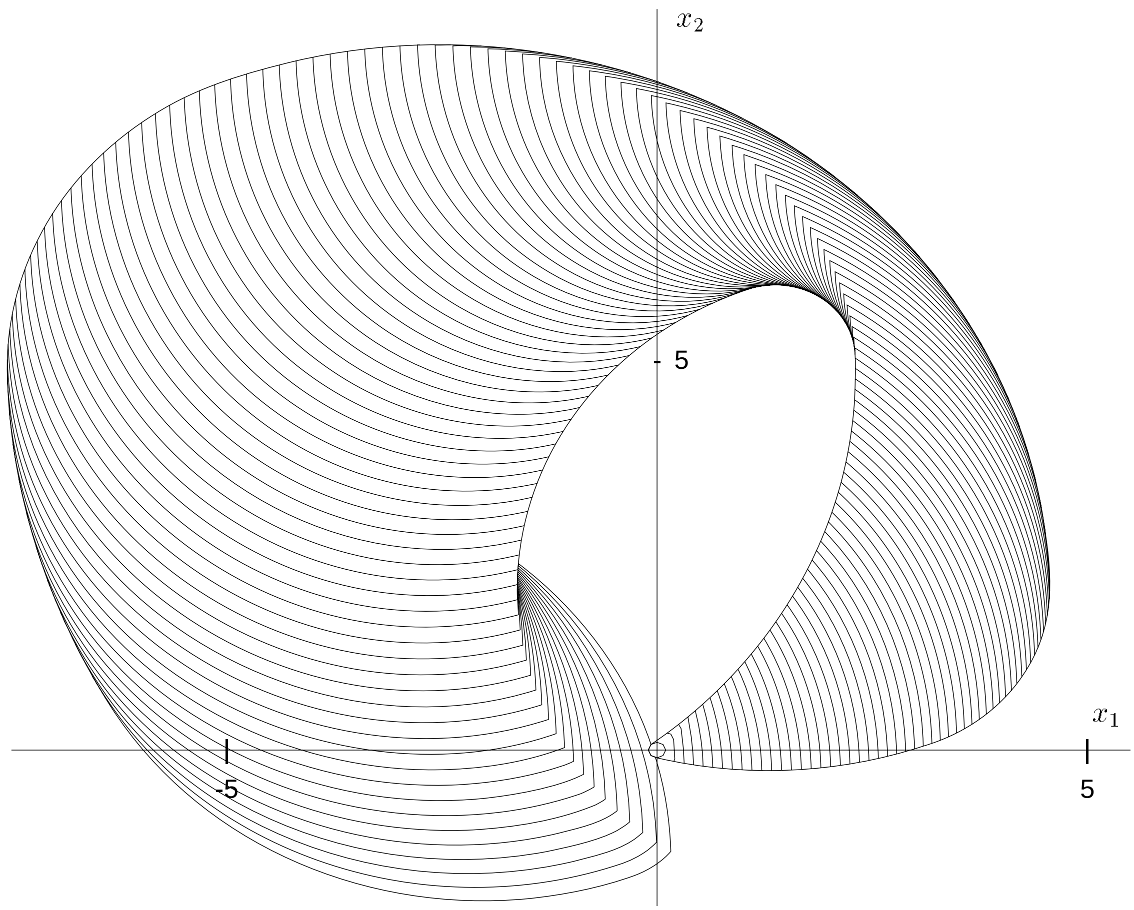


Fig. 3.18. The case of "collision".

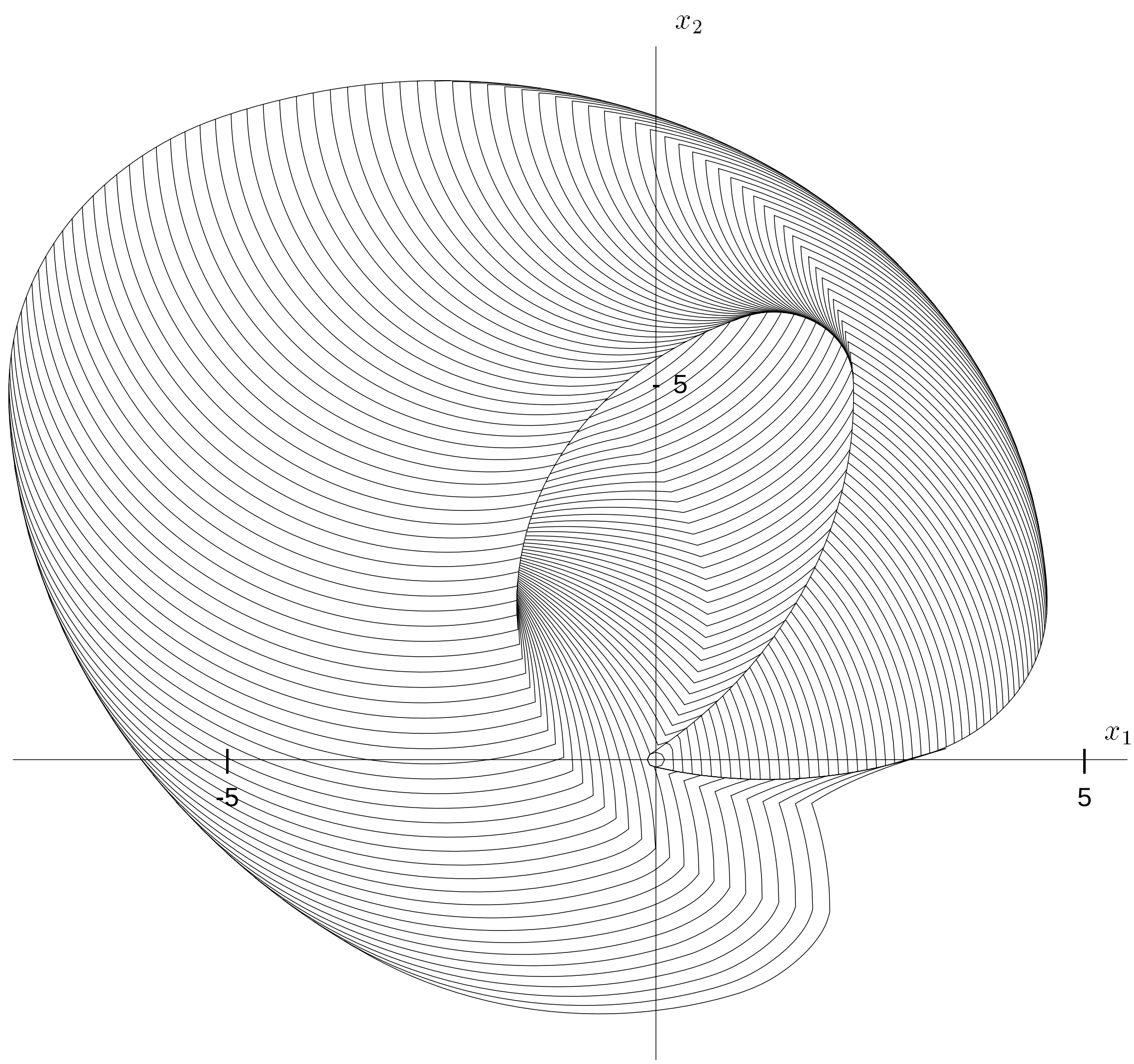


Fig. 3.19. Filling out the “lune”.

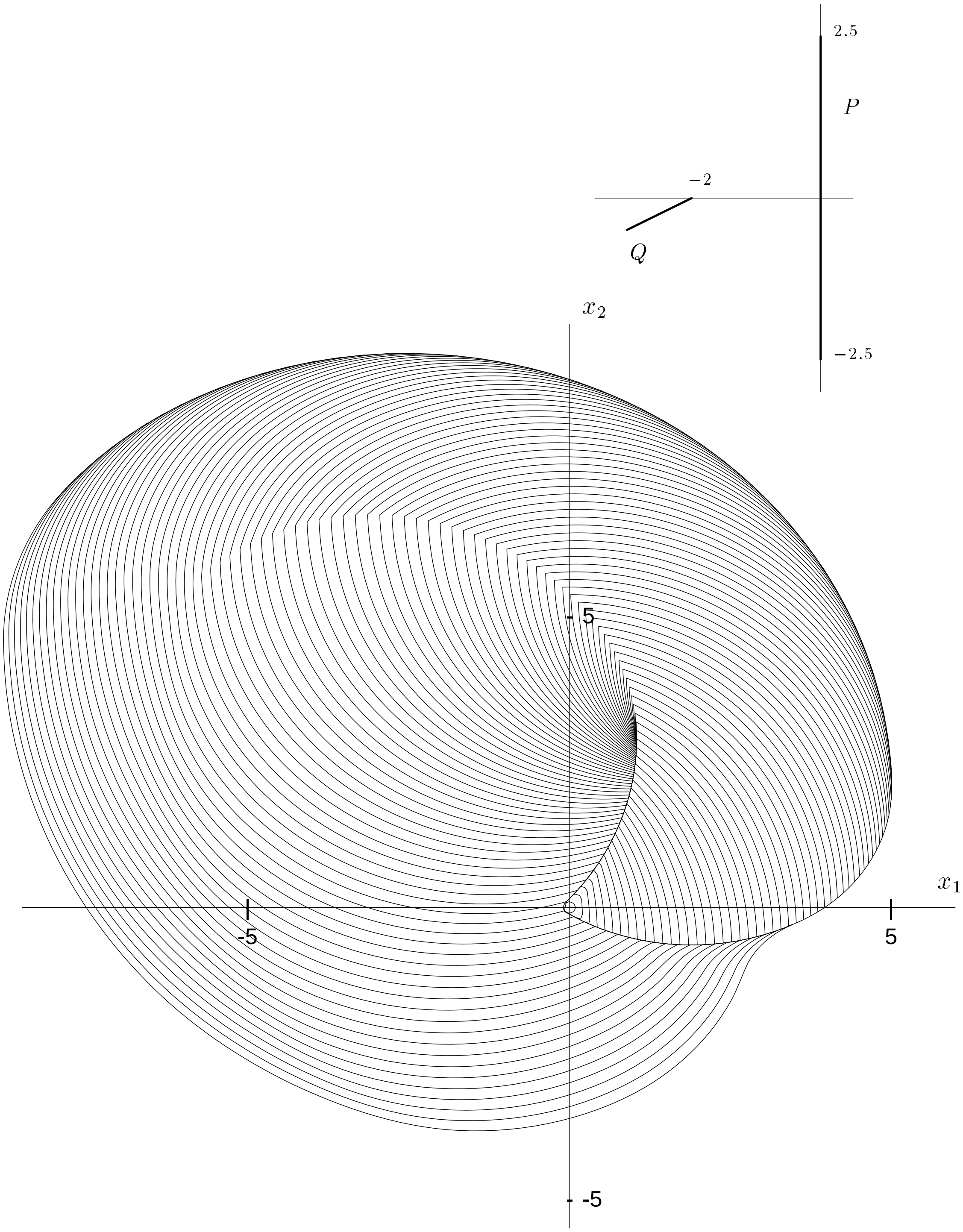


Fig. 3.20. Disappearing the "lune" by the reduction of resources of the second player.

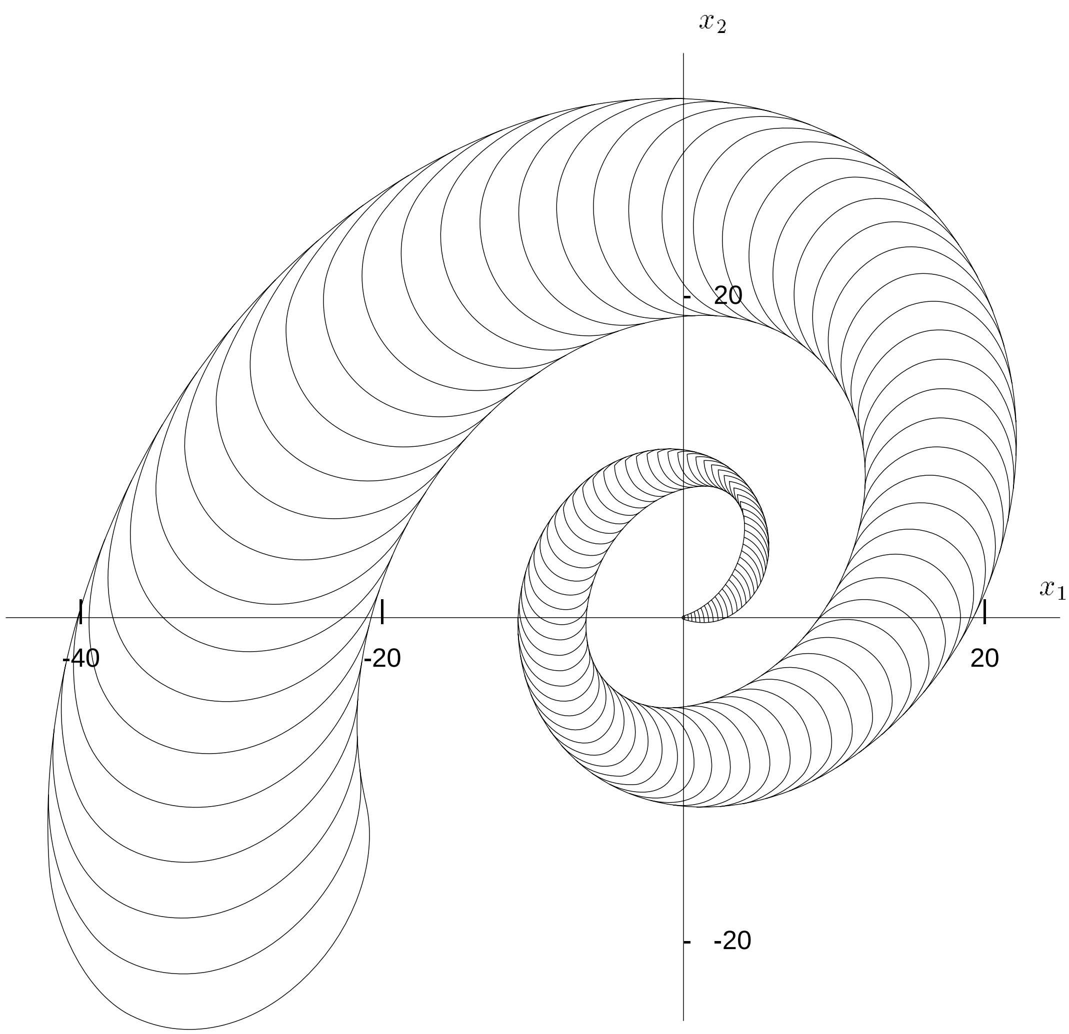


Fig. 3.21. Oscillating system (3.8). The process of filling out the solvability set of the game of kind (compare with Fig. 1.6).

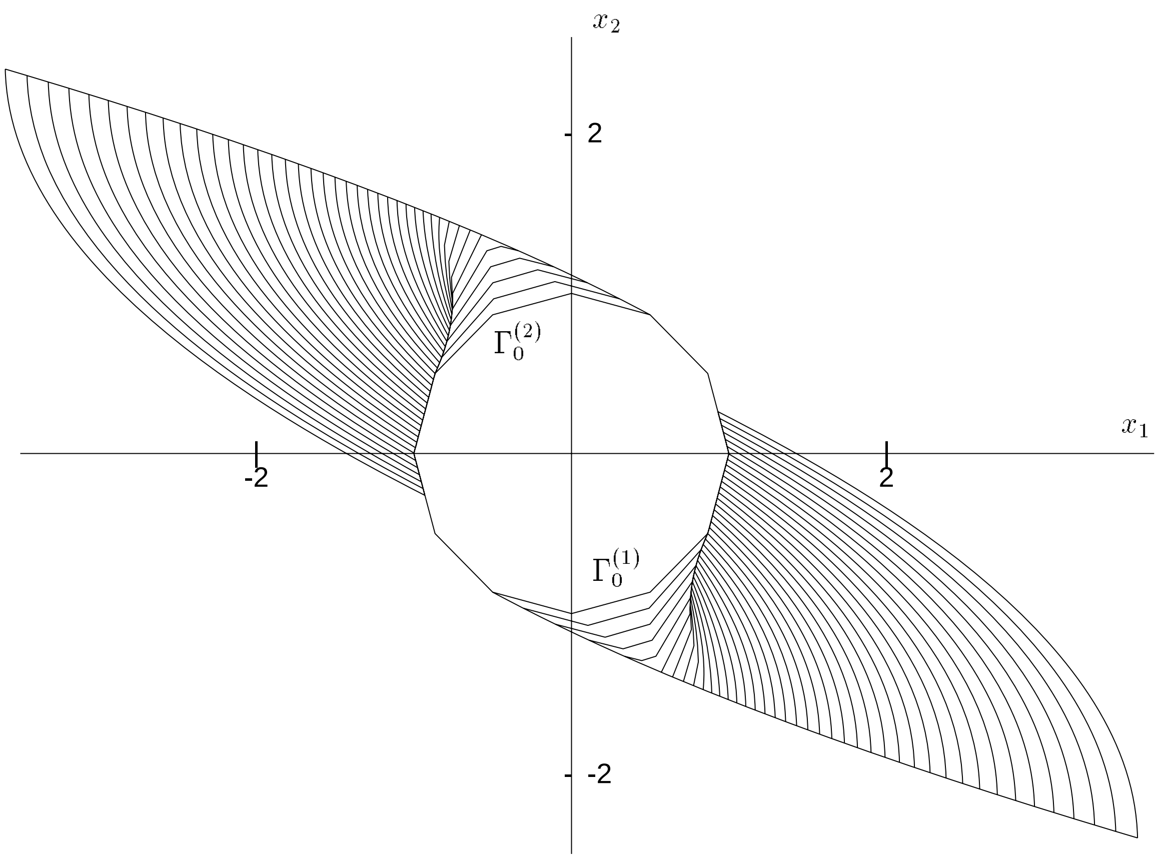


Fig. 3.22. System (3.6). Usable part of M
consists of two arcs $\Gamma_0^{(1)}$, $\Gamma_0^{(2)}$.

Conclusions

In this paper, two-person differential games in the plane are considered. The controls of the players are restricted via geometric bounds.

In the first chapter, an algorithm for constructing solvability sets of a linear control system is described that is an agorithm for finding the set of all initial states from which the first player provides the state vector to be brought to a given target set under any actions of the second player (game of kind). The algorithm is based not on the embedding the game of kind within corresponding game of degree, but on operations of constructing and sewing semipermeable curves. The number of families of semipermeable curves is determined by the number of convexity-concavity cones of the Hamiltonian of the control system. The ideas of the first chapter can be extended to control systems with nonlinear dynamics. This is demonstrated in the second chapter, where the game of kind for the nonlinear pendulum is considered. In the third chapter, an algorithm for finding level sets of the value function for linear minimum-time game problems (games of degree) is described.

A number of numerical examples which demonstrate specific features of solutions are presented. Definite number of examples shows the connection between games of kind and games of degree.

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