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ROBUST CONTROL IN GAME PROBLEMS WITH LINEAR DYNAMICS

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Antagonistic linear differential games of two persons with fixed terminal instant are considered. The objective of the first player is to lead the system to a prescribed terminal set or closely to it at the fixed terminal time. The second player (the disturbance) hinders this. The first player's control is scalar and bounded. The peculiarity of the formulation is that there is no any prescribed constraint for the second player's control. The work studies a method for constructing a first player's feedback control, which should work acceptably for wide-range disturbances and to parry a disturbance of a "low" level by a "low" level realization. Such a control is called robust in this work.

A method for constructing a robust control is suggested and justified. For the case of the phase variable of low dimension, a numerical algorithm and a corresponding computer programs has been worked out. Two model examples are numerically investigated, one of them is a problem of aircraft landing control under wind disturbance. The work can be interesting for specialists in area of optimal control and its applications.

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Contents

Introduction

In the theory of antagonistic differential games (see, for example, [Isaacs, 1965; Krasovskii and Subbotin, 1974; Krasovskii and Subbotin, 1988; Bardi and Capuzzo-Dolcetta, 1997]), formulations are typical, where geometric constraints both for the useful control and disturbance are given a priori. But in many practical situations, introducing the exact constraint for disturbance is unnatural. For example, when a problem of aircraft landing is formulated, it is difficult to explain, why the possible deviation of the wind velocity from some average value is taken in advance not greater than 10 m/sec, but not than 12 m/sec. With that, the optimal strategy obtained from the solution of a corresponding differential game depends on the taken disturbance level.

Let us agree that a feedback control is called *robust*, if in the case of a "low" disturbance (which is unknown in advance), it provides good quality of the process by some "low" level useful control. With increasing the disturbance level, the level of the useful control guaranteeing good quality of the process grows too. This sense of the concept of "robust control" coheres with that used in mathematical literature (for example, see [Fleming, 1988]).

Constructing a linear robust control for an H^{∞}-problem on the basis of theory of differential games with linear-quadratic cost functional is described in [Basar and Bernhard, 1991]. Linear robust regulators in the sense of L^1-optmization were investigated in [Barabanov, 1996; Dahleh and Pearson, 1987; Sokolov, 2003].

This work suggests an approach to constructing a nonlinear robust control. The method is oriented to problems with linear dynamics, where a geometric constraint for the useful control is prescribed a priori. The scheme worked out is based on results of the theory of differential games with fixed terminal time and geometric constraints for both players' controls.

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Notations

$T = [\vartheta_0, \vartheta]$ – the interval of the game;

$\mathbf{x}$ – a point in the phase space R^m of original controlled system (1);

x – a point in the phase space R^n of controlled systems (2), (3);

u – a scalar control of the first player (the useful control);

v – a control of the second player (the disturbance);

$B(t)$ – a vector multiplier (column vector) of the control u of the first player in the right-hand side of systems (2), (3);

$C(t)$ – a matrix multiplier of the control v of the second player in the right-hand side of systems (2), (3);

σ – maximum of the value $|B(t)|$ in the interval T;

β – the Lipschitz constant of the function $t \mapsto B(t)$.

$X(\vartheta, t)$ – the fundamental Cauchy matrix of system (1);

$P = \big\{u \in R\colon |u| \le \mu\big\}$ – the constraint for the first player's control in systems (1), (2);

M – the terminal set for systems (1), (2);

$\mathcal{P}$ – a constraint for the first player's control in differential game (3);

$\mathcal{Q}$ – a constraint for the second player's control in differential game (3);

$\mathcal{M}$ – a terminal set in differential game (3);

$\mathbb{W}(\mathcal{P}, \mathcal{Q}, \mathcal{M})$ – the maximal stable bridge in game (3) corresponding to the parameters $(\mathcal{P}, \mathcal{Q}, \mathcal{M})$;

$Q_{\max}$ – the critical constraint for the second player's control, which should be chosen for construction of a feedback robust control;

$W = \mathbb{W}(P, Q_{\max}, M)$ – the maximal stable bridge in game (3) corresponding to the parameters $(P, Q_{\max}, M)$;

ρ – an auxiliary positive parameter, which should be defined when constructing the feedback robust control;

$\mathbb{B}(\varepsilon)$ – the closed ball in R^n with the radius ε and the center at the origin;

$\widehat{W} = \mathbb{W}\big(\{0\}, Q_{\max}, \rho M\big)$ – the maximal stable bridge in game (3) corresponding to the parameters $\mathcal{P} = \{0\}$ (i.e., the first player is absent), $\mathcal{Q} = Q_{\max}$, and $\mathcal{M} = \rho M$;

W_k – a stable bridge in game (3) defined by means of bridges W and $\widehat{W}$ with the help of the real-valued numerical parameter $k \geq 0$;

V – a scalar function defined in the space $T \times R^n$, which has the sets W_k, $k \geq 0$, as level sets (Lebesgue sets);

λ – the Lipschitz constant of the function V on x;

$\overline{V}$ – the cut-off of the function V by the level 1;

$\Pi(t)$ – the switching surface for the control u corresponding to an instant t;

$\Pi_+(t)$, $\Pi_-(t)$ – the parts of the space R^n located on the sides of the surface $\Pi(t)$;

$\mathbf{U}^0$ – a multivalued function defining the robust control of the first player;

$\Pi^r(t)$ – the geometric r-neighborhood of the surface $\Pi(t)$;

int – the symbol of interrior.

1 Construction of robust control

1.1 Formulation of problem

Let us consider a linear differential game with fixed terminal time:

$$\begin{gathered}\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)u + \mathbf{C}(t)v, \\ \mathbf{x} \in R^m, \quad t \in T, \quad u \in P = \{u \in R\colon |u| \leqslant \mu\}, \quad v \in R^q.\end{gathered} \tag{1}$$

Here, P is the constraint for the first player's scalar control u, $T = [\vartheta_0, \vartheta]$ is the time interval of the game. The matrix-functions $\mathbf{A}$ and $\mathbf{C}$ are continuous. The vector-function $\mathbf{B}$ is Lipschitzian in the time interval T.

The first player tries to lead system (1) to a set M at the terminal instant ϑ. The second one hinders this. The set M is supposed to be a convex compactum in a subspace $R^n \subset R^m$ of some n chosen components of the vector $\mathbf{x}$. Let us assume that the set M includes a neighborhood of the origin of this subspace.

Unlike the standard formulation [Isaacs, 1965; Krasovskii and Subbotin, 1974; Krasovskii and Subbotin, 1988] of a differential game, system (1) does not include any constraint for the second player's control v.

Let the initial point of system (1) is close to the origin. In this case, a first player's robust control can be informally understood as a feedback control obeying the following conditions:

- if the second player applies a "low level" control, the first player should lead the system to the terminal set closely to the origin of the subpace R^n. Moreover, the realization of the first player's control should be also of a "low level";
- if the second player's control is "stronger", the first player should still lead the system to the terminal set, maybe, by a "stronger" or even maximal control;
- in the case, when the second player involves "very strong" control and the first player (acting within the framework of his constraint) cannot guarantee reaching the terminal set, he may allow some terminal miss, but tries to minimize it.

A concept of robustness, close to the described above, was used in [Turetsky and Glizer, 2004] for the case, when the aim of conrol is reachable in a wide-range disturbance with the level, which is not fixed in advance.

It is necessary to elaborate a method for constructing a robust feedback control for system (1).

1.2 Differential game without phase variable in the right-hand side

By means of the standard change of variables [Krasovskii and Subbotin, 1974, p. 160; Krasovskii and Subbotin, 1988, pp. 89–91], let us pass to a system, which right-hand side does not include the phase vector:

$$\begin{gathered} \dot{x} = B(t)u + C(t)v, \\ x \in R^n, \quad t \in T, \quad u \in P, \quad v \in R^q. \end{gathered} \tag{2}$$

The passage is provided by the following relations:

$$x(t) = X_{n,m}(\vartheta, t)\mathbf{x}(t), \qquad B(t) = X_{n,m}(\vartheta, t)\mathbf{B}(t), \qquad C(t) = X_{n,m}(\vartheta, t)\mathbf{C}(t),$$

where $X_{n,m}(\vartheta, t)$ is the matrix combined of n corresponding to the subspace R^n rows of the fundamental Cauchy matrix of the system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$.

Here again, the first player tries to lead system (2) to the set M at the terminal instant ϑ, and the second one hinders this. The set M is a convex compact set in R^n including a neighborhood of the origin.

All the following considerations will be given for system (2). The robust control obtained for system (2) will be adopted for system (1).

1.3 Stable bridges. Addition and multiplication by a coefficient

Let us consider a standard differential game with fixed terminal time ϑ, a terminal set $\mathcal{M}$, and geometric constraints $\mathcal{P}$ and $\mathcal{Q}$ for the players' controls:

$$\begin{gathered} \dot{x} = B(t)u + C(t)v, \\ x \in R^n, \quad t \in T, \quad \mathcal{M}, \quad u \in \mathcal{P}, \quad v \in \mathcal{Q}. \end{gathered} \tag{3}$$

Convex compact sets $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{M}$ will be counted as parameters of the game.

The authors use formalization of game (3) according to [Krasovskii and Subbotin, 1974; Krasovskii and Subbotin, 1988].

For an arbitrary set $E \subset T \times R^n$, let us define its section at an instant t by the formula

$$E(t) = \{x \in R^n \colon (t, x) \in E\}.$$

Below, operations of addition and multiplication by a non-negative scalar will be introduced for sets in the space $T \times R^n$ having non-empty sections at any time instant $t \in T$. These set operations will be based on common concepts of algebraic sum (Minkowski sum) and multiplication by a scalar.

Definition 1. *Sum of two sets E_1, $E_2 \subset T \times R^n$ is a set*

$$E_1 + E_2 = \big\{(t, x) \in T \times R^n : x \in E_1(t) + E_2(t)\big\}.$$

Definition 2. *Multiplication of a set $E \subset T \times R^n$ by a real number $k \geqslant 0$ is a set*

$$kE = \big\{(t, x) \in T \times R^n : x \in kE(t)\big\}.$$

On the basis of [Krasovskii and Subbotin, 1974; Krasovskii and Subbotin, 1988], let us give definitions of stable and maximal stable bridges.

Below, $u(\cdot)$ and $v(\cdot)$ will denote some measured functions of time with their values in the sets $\mathcal{P}$ and $\mathcal{Q}$, respectively. The symbol $x\big(\cdot; t_*, x_*, u(\cdot), v(\cdot)\big)$ will denote a motion of system (3) (and, consequently, (2)) emanating from the point x_* at the instant t_* under controls $u(\cdot)$ and $v(\cdot)$.

Definition 3. *A set $W \subset T \times R^n$ is called a stable bridge for system* (3) *with some fixed $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{M}$ if $W(\vartheta) = \mathcal{M}$ and it possesses the following property. For any position $(t_*, x_*) \in W$ and any second player's control $v(\cdot)$, the first player can choose his control $u(\cdot)$ such that the pair $\big(t, x(t)\big) = \big(t, x(t; t_*, x_*, u(\cdot), v(\cdot))\big)$ stays in W at any instant $t \in (t_*, \vartheta]$, and, consequently, the motion $x(\cdot)$ reaches the set $\mathcal{M}$ at the terminal instant: $x(\vartheta) \in \mathcal{M}$.*

Definition 4. *A set W, $W(\vartheta) = \mathcal{M}$, possessing the stability property and maximal by inclusion in the space $T \times R^n$, is called the maximal stable bridge for system* (3)*.*

We introduce the following notations:
$\mathcal{W}(\mathcal{P}, \mathcal{Q}, \mathcal{M})$ is the collection of all stable bridges for system (3) with parameters $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{M}$;
$\mathbb{W}(\mathcal{P}, \mathcal{Q}, \mathcal{M}) \in \mathcal{W}(\mathcal{P}, \mathcal{Q}, \mathcal{M})$ is the maximal stable bridge for game (3) with parameters $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{M}$.

Proposition 1. *Let $F \in \mathcal{W}(\mathcal{P}, \mathcal{Q}, \mathcal{M})$ with some $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{M}$. Then $kF \in \mathcal{W}(k\mathcal{P}, k\mathcal{Q}, k\mathcal{M})$ for any $k \geqslant 0$.*

Proof. When $k = 0$, the proposition is evident. Below, it is supposed that $k > 0$.

Fix an arbitrary point $(t_*, x_*) \in kF$ and an instant $t^* \in [t_*, \vartheta]$. Let the second player choose some control $v(t) \in k\mathcal{Q}$ in the interval $[t_*, t^*]$.

We shall show how a first player's control $u(t) \in k\mathcal{P}$, $t \in [t_*, t^*]$, can be constructed so that the inclusion $\bigl(t^*, x(t^*)\bigr) \in kF$ holds for the motion $x(\cdot) = x\bigl(\cdot; t_*, x_*, u(\cdot), v(\cdot)\bigr)$.

Denote $z_* = {}^1\!/_k \cdot x_*$, $\bar{v}(t) = {}^1\!/_k \cdot v(t)$. One has $(t_*, z_*) \in F$. Since F is a stable set, for any second player's control $\bar{v}(t) \in \mathcal{Q}$, $t \in [t_*, t^*]$, it is possible to find a control $\bar{u}(t) \in \mathcal{P}$, $t \in [t_*, t^*]$, such that the motion $z(\cdot) = x\bigl(\cdot; t_*, z_*, \bar{u}(\cdot), \bar{v}(\cdot)\bigr)$ gives the inclusion $\bigl(t^*, z(t^*)\bigr) \in F$.

Let $u(t) = k\bar{u}(t)$, $t \in [t_*, t^*]$. Taking into account the type of system (2), one gets that $x(t) = kz(t)$ for any $t \in [t_*, t^*]$. Thus, $\bigl(t^*, x(t^*)\bigr) \in kF$, that means stability of the set kF. $\square$

Proposition 2. *Multiplication of the maximal stable bridge, corresponding to parameters $(\mathcal{P}, \mathcal{Q}, \mathcal{M})$, by a number $k \geqslant 0$ gives the maximal stable bridge, corresponding to parameters $(k\mathcal{P}, k\mathcal{Q}, k\mathcal{M})$:*

$$k\mathbb{W}(\mathcal{P}, \mathcal{Q}, \mathcal{M}) = \mathbb{W}(k\mathcal{P}, k\mathcal{Q}, k\mathcal{M}).$$

Proof. When $k = 0$, the proposition is evident. Below, we suppose that $k > 0$.

Let $F = \mathbb{W}(\mathcal{P}, \mathcal{Q}, \mathcal{M})$. Denote $\widetilde{F} = kF$. From Proposition 1, one gets $\widetilde{F} \in \mathcal{W}(k\mathcal{P}, k\mathcal{Q}, k\mathcal{M})$.

Assume that $\widetilde{F} \neq \mathbb{W}(k\mathcal{P}, k\mathcal{Q}, k\mathcal{M})$. Let $\widetilde{\mathbf{F}} = \mathbb{W}(k\mathcal{P}, k\mathcal{Q}, k\mathcal{M})$. One has $\widetilde{\mathbf{F}} \supset \widetilde{F}$, $\widetilde{\mathbf{F}} \neq \widetilde{F}$.

Consider the set $\mathbf{F} = {}^1\!/_k \cdot \widetilde{\mathbf{F}}$. Then $\mathbf{F} \supset F$, $\mathbf{F} \neq F$. Proposition 1 gives that the inclusion $\mathbf{F} \in \mathcal{W}(\mathcal{P}, \mathcal{Q}, \mathcal{M})$ is true. But $F = \mathbb{W}(\mathcal{P}, \mathcal{Q}, \mathcal{M})$. So, we get a contradiction. $\square$

Proposition 3. *Sum of two stable bridges F_1 and F_2, corresponding to parameters $(\mathcal{P}_1, \mathcal{Q}_1, \mathcal{M}_1)$ and $(\mathcal{P}_2, \mathcal{Q}_2, \mathcal{M}_2)$ respectively, is a stable bridge corresponding to the parameters $(\mathcal{P}_1 + \mathcal{P}_2, \mathcal{Q}_1 + \mathcal{Q}_2, \mathcal{M}_1 + \mathcal{M}_2)$:*

$$\begin{aligned} &F_1 \in \mathcal{W}(\mathcal{P}_1, \mathcal{Q}_1, \mathcal{M}_1), \quad F_2 \in \mathcal{W}(\mathcal{P}_2, \mathcal{Q}_2, \mathcal{M}_2) \;\Rightarrow \\ &\qquad \Rightarrow\; F_1 + F_2 \in \mathcal{W}(\mathcal{P}_1 + \mathcal{P}_2, \mathcal{Q}_1 + \mathcal{Q}_2, \mathcal{M}_1 + \mathcal{M}_2). \end{aligned} \tag{4}$$

Proof. Let $\widetilde{F} = F_1 + F_2$. Take an arbitrary point $(t_*, x_*) \in \widetilde{F}$ and an instant $t^* \in [t_*, \vartheta]$. Let the second player chooses a control $v(t) \in \mathcal{Q}_1 + \mathcal{Q}_2$, $t \in [t_*, t^*]$. Below, a first player's control $u(t) \in \mathcal{P}_1 + \mathcal{P}_2$, $t \in [t_*, t^*]$, will be constructed such that for the motion $x(\cdot) = x\bigl(\cdot; t_*, x_*, u(\cdot), v(\cdot)\bigr)$ the inclusion $\bigl(t^*, x(t^*)\bigr) \in \widetilde{F}$ is held.

Let us choose points z_*^1 and z_*^2 such that $(t_*, z_*^1) \in F_1$, $(t_*, z_*^2) \in F_2$, and $z_*^1 + z_*^2 = x_*$.

Take some controls $v_1(\cdot)$ and $v_2(\cdot)$ such that $v_1(t) \in \mathcal{Q}_1$, $v_2(t) \in \mathcal{Q}_2$, and $v_1(t) + v_2(t) = v(t)$, $t \in [t_*, t^*]$.

Let $i = 1, 2$. Using stability of the set F_i, one can find a control $u_i(t) \in \mathcal{P}_i$, $t \in [t_*, t^*]$, so that the motion $z^i(\cdot) = x\big(\cdot; t_*, z^i_*, u_i(\cdot), v_i(\cdot)\big)$ gives the inclusion $\big(t^*, z^i(t^*)\big) \in F_i$.

Denote $u(t) = u_1(t) + u_2(t)$, $t \in [t_*, t^*]$. Then taking into account the type of system (2), one gets that $x(t) = z^1(t) + z^2(t)$. So, $\big(t^*, x(t^*)\big) \in \widetilde{F}$, that implies the inclusion (4). □

Remark 1. Let $F_1 = \mathbb{W}(\mathcal{P}_1, \mathcal{Q}_1, \mathcal{M}_1)$ and $F_2 = \mathbb{W}(\mathcal{P}_2, \mathcal{Q}_2, \mathcal{M}_2)$, i.e., the sets F_1 and F_2 are maximal stable bridges. Then the inclusion (4) is true. An example can be easily constructed when

$$F_1 + F_2 \neq \mathbb{W}(\mathcal{P}_1 + \mathcal{P}_2, \mathcal{Q}_1 + \mathcal{Q}_2, \mathcal{M}_1 + \mathcal{M}_2).$$

So, under addition of maximal stable bridges, the result is a stable bridge, but generally speaking, not the maximal stable one.

1.4 Robust feedback control. Theorem about guarantee

Below, a description of constructing robust control for systems (2) and (1) is given. Sometimes, instead of "feedback control" we shall use "strategy".

1) Let us choose a set $Q_{\max} \subset R^q$ that will represent the "maximal" constraint on control of the second player, which the first player "agrees" to consider reasonable for the problem of guiding system (2) to the set M. The set $Q_{\max}$ should include the origin of its space. Denote by W the maximal stable bridge for system (3) corresponding to the parameters P, $Q_{\max}$, and M:

$$W = \mathbb{W}(P, Q_{\max}, M).$$

Let $\mathbb{B}(\varepsilon) = \{x \in R^n : \|x\| \leqslant \varepsilon\}$ be a ball in R^n with the radius ε and the center at the origin.

Assume the set $Q_{\max}$ is such that

$$\exists\, \varepsilon > 0 : \forall\, t \in T \quad \mathbb{B}(\varepsilon) \subset W(t). \tag{5}$$

This demands inclusion of the origin with some neighborhood into each time section $W(t)$ of the bridge W.

Increasing the size of the set $Q_{\max}$ enlarges the disturbance level, which can be successfully parried by the first player if the initial position is inside the corresponding stable bridge, but decreases the size of this bridge.

2) To construct the robust control, one needs additionally a maximal stable bridge for system (3) satisfying the following conditions: the first player's control is absent ($\mathcal{P} = \{0\}$), the second player's control is constrained by $\mathcal{Q} = Q_{\max}$, the terminal set is $\mathcal{M} = \rho M$, where $\rho > 0$. Denote the bridge by

$$\widehat{W} = \mathbb{W}\big(\{0\}, Q_{\max}, \rho M\big).$$

Let the chosen multiplier ρ be small enough, but such that

$$\exists\, \varepsilon > 0 : \forall\, t \in T \;\; \mathbb{B}(\varepsilon) \subset \widehat{W}(t). \tag{6}$$

3) We construct a family of sets

$$W_k = \begin{cases} kW & \text{if } 0 \leqslant k \leqslant 1, \\ W + (k-1)\widehat{W} & \text{if } k > 1. \end{cases}$$

Due to definition of addition operation and operation of multiplication by a scalar and relations (5) and (6), one has inclusions

$$W_{k_1} \subset W_{k_2} \subset W \subset W_{k_3} \subset W_{k_4}$$

for any $0 < k_1 < k_2 < 1 < k_3 < k_4$.

Proposition 2 gives that the sets W_k, when $0 \leqslant k \leqslant 1$, are maximal stable bridges corresponding to parameters $(kP, kQ_{\max}, kM)$. Propositions 1 and 3 imply that the sets W_k, when $k > 1$, are stable bridges, constructed with parameters $\big(P, kQ_{\max}, M + (k-1)\rho M\big)$.

So, with increasing the coefficient k, one obtains a growing collection of stable bridges, where any larger bridge corresponds to a larger constraint for the second player's control.

The main idea of the suggested method for constructing a robust feedback control is the following. Let $k^* \geqslant 0$. If the second player's control $v(t)$ for any t belongs to some constraint $k^* Q_{\max}$ and the initial position (t_0, x_0) is in the bridge W_{k^*} corresponding to this number k^*, then the system will not leave the bridge W_{k^*}. With that, the realization of the first player's control belongs to the set $\min(k^*, 1)P$. This means that if $k^* < 1$ then the system can be brought to the terminal set by means of a control, which level is less than the maximal possible one.

4) Define a function $V : T \times R^n \to R$ in the following way:

$$V(t, x) = \min\big\{k \geqslant 0 \colon (t, x) \in W_k\big\}.$$

The level sets (Lebesgue sets) of this function coincide with the stable bridges.

Because the set $Q_{\max}$ and the number ρ are chosen such that for some $\varepsilon > 0$ relations (5) and (6) are held, then the function $x \mapsto V(t,x)$ for any $t \in T$ satisfies the Lipschitz condition with the constant $\lambda = 1/\varepsilon$.

Denote by $\mathcal{A}(t,x)$ a line in the space R^n parallel to the vector $B(t)$ and passing through the point x:

$$\mathcal{A}(t,x) = \{z \in R^n \colon z = x + \alpha B(t),\ \alpha \in R\}.$$

Let

$$\mathcal{V}(t,x) = \min_{z \in \mathcal{A}(t,x)} V(t,z).$$

The minimum is reached, since the function $x \mapsto V(t,x)$ is continuous and tends to infinity as $|x| \to \infty$. Because the function is quasiconvex (i.e., all its Lebesgue sets are convex), the set of points, where the minimum is reached, is either a point or a segment.

If $B(t) = 0$, it is assumed $\mathcal{V}(t,x) \equiv V(t,x)$.

5) For any $t \in T$, let

$$\begin{aligned}\Pi(t) &= \bigl\{x \in R^n \colon V(t,x) = \mathcal{V}(t,x)\bigr\},\\ \Pi_-(t) &= \bigl\{x \in R^n \colon x + \alpha B(t) \notin \Pi(t),\ \forall \alpha \geqslant 0\bigr\},\\ \Pi_+(t) &= \bigl\{x \in R^n \colon x + \alpha B(t) \notin \Pi(t),\ \forall \alpha \leqslant 0\bigr\}.\end{aligned}$$

The set $\Pi(t)$ is closed, the sets $\Pi_-(t)$ and $\Pi_+(t)$ are on different sides of $\Pi(t)$. These three sets divide the space R^n into three parts.

6) We define a function

$$\overline{V}(t,x) = \min\bigl\{V(t,x), 1\bigr\}$$

and a multifunction

$$\mathbf{U}^0(t,x) = \begin{cases} -\overline{V}(t,x)\mu & \text{if } x \in \Pi_-(t),\\ \overline{V}(t,x)\mu & \text{if } x \in \Pi_+(t),\\ \bigl[-\overline{V}(t,x)\mu, \overline{V}(t,x)\mu\bigr] & \text{if } x \in \Pi(t).\end{cases}$$

As the strategy U of the first player, any one-valued selection from the multifunction $\mathbf{U}^0$ can be taken:

$$U(t,x) \in \mathbf{U}^0(t,x), \qquad (t,x) \in T \times R^n.$$

Thus, the control $U(t,x)$ "switches" at the set $\Pi(t)$. For simplicity, the set $\Pi(t)$ is called a *switching surface* corresponding to the instant t.

7) Below, a theorem about guarantee, provided by an arbitrary one-valued strategy $U \in \mathbf{U}^0$ of the first player, will be formulated. To characterize influence of small inaccuracies in construction of the switching surfaces $\Pi(t)$, we shall consider sets $\Pi^r(t) \supset \Pi(t)$, $r > 0$, and introduce a multivalued function $\mathbf{U}^r$ such that $\mathbf{U}^0(t,x) \subset \mathbf{U}^r(t,x)$.

When $B(t) \neq 0$, let

$$\Pi^r(t) = \left\{ x \in R^n \colon x = z + \alpha \frac{B(t)}{\bigl|B(t)\bigr|}, z \in \Pi(t), |\alpha| \leqslant r \right\}.$$

The set $\Pi^r(t)$ is a geometric r-extension of the set $\Pi(t)$ in the direction of the vector $B(t)$. When $B(t) = 0$, it is assumed that $\Pi^r(t) = \Pi(t) = R^n$.

Let us introduce the sets

$$\Pi^r_-(t) = \bigl\{ x \in R^n \colon x + \alpha B(t) \notin \Pi^r(t), \forall \alpha \geqslant 0 \bigr\},$$
$$\Pi^r_+(t) = \bigl\{ x \in R^n \colon x + \alpha B(t) \notin \Pi^r(t), \forall \alpha \leqslant 0 \bigr\}.$$

Now define a multivalued function

$$\mathbf{U}^r(t,x) = \begin{cases} -\overline{V}(t,x)\mu & \text{if } x \in \Pi^r_-(t), \\ \overline{V}(t,x)\mu & \text{if } x \in \Pi^r_+(t), \\ \bigl[-\overline{V}(t,x)\mu, \overline{V}(t,x)\mu\bigr] & \text{if } x \in \Pi^r(t). \end{cases}$$

Suppose that the first player applies a one-valued strategy $U \in \mathbf{U}^r$ in a discrete scheme of control [Krasovskii and Subbotin, 1988] with a time step Δ. In any interval of the discrete scheme, the generated control is constant. Taking an open-loop control $v(\cdot)$ of the second player and an initial position (t_0, x_0), one gets a motion $t \mapsto x(t)$ of system (2).

Let β be the Lipschitz constant of the function $B(t)$ and $\sigma = \max\limits_{t \in T} \bigl|B(t)\bigr|$.

The following theorem about guarantee is true.

Theorem. *Let $r \geqslant 0$ and U be some strategy of the first player such that $U(t,x) \in \mathbf{U}^r(t,x)$ for all $(t,x) \in T \times R^n$. Choose arbitrary $t_0 \in T$, $x_0 \in R^n$, and $\Delta > 0$. Suppose that in the interval $[t_0, \vartheta]$ the second player's control is bounded by a set $k^* Q_{\max}$, $k^* \geqslant 0$. Denote*

$$c^* = V(t_0, x_0), \qquad s^* = \max(k^*, c^*).$$

Let $x^*(\cdot)$ be the motion of system (2) emanating from the point x_0 at the instant t_0 under the control U in a discrete scheme with the time step Δ and some control $v(\cdot)$ of the second player. Then the realization $u(t) = U\bigl(t, x^*(t)\bigr)$ of the first player's control obeys to the inclusion

$$u(t) \in \min\bigl(s^* + \Lambda(t, t_0, \Delta, r), 1\bigr)P, \qquad t \in [t_0, \vartheta]. \tag{7}$$

With that, the value $V\bigl(t, x^*(t)\bigr)$ of the function V satisfies the inequality

$$V\bigl(t, x^*(t)\bigr) \leqslant s^* + \Lambda(t, t_0, \Delta, r), \qquad t \in [t_0, \vartheta]. \tag{8}$$

Here,

$$\Lambda(t, t_0, \Delta, r) = 2\lambda\sqrt{(2\sigma\mu\Delta + r)\beta\mu}(t - t_0) + 4\lambda\sigma\mu\Delta + \lambda r.$$

8) Returning to system (1), let us introduce a multifunction

$$\widetilde{\mathbf{U}}^0(t, \mathbf{x}) = \mathbf{U}^0\bigl(t, X_{n,m}(\vartheta, t)\mathbf{x}\bigr).$$

Any its one-valued selection $\widetilde{U}(t, \mathbf{x})$ gives a robust control for system (1). Herewith, according to the Theorem, some small inaccuracies are allowed in constructing the switching surface $\Pi(t)$.

By the above explanations, the construction of robust feedback control has been described. It essentially uses the ordering of the stable sets W_k and is based on the construction of the switching surface $\Pi(t)$, which changes in time. The proof of the Theorem is given in Section 2. It repeats significantly the scheme of reasoning from works [Botkin and Patsko, 1983; Patsko, 2004a; Patsko, 2004b]. There, the switching surfaces were used to build an optimal feedback control of the minimizing player in linear antagonistic differential games with fixed terminal time and geometric constraints on the players' controls.

For numerical constructing robust control, one should keep sections $W(t)$ of the bridge W and switching surfaces $\Pi(t)$ for some grid $\{t_i\}$ of time instants. Having at the instant t a position $\mathbf{x}(t)$ of system (1), one transforms it to the coordinates of system (2) by the mapping $x(t) = X_{n,m}(\vartheta, t)\mathbf{x}(t)$. The sign of the control $\widetilde{U}\bigl(t, \mathbf{x}(t)\bigr) = U\bigl(t, x(t)\bigr)$ is defined by the relative position of the point $x(t)$ with respect to the switching surface $\Pi(t)$. Analyzing the position of the point $x(t)$ with respect to the boundary of the section $W(t)$ of the bridge W, one computes the absolute value $\bigl|\widetilde{U}\bigl(t, \mathbf{x}(t)\bigr)\bigr|$. Here, homothety of sets $W_k(t)$ for $k \leqslant 1$ is used.

Remark 2. Authors do not state that the suggested method for robust control realizes any optimality criterion. Note also that when constructing the robust control, one has some freedom in the choice of the set $Q_{\max}$ and the number ρ.

Remark 3. Fixing the set $Q_{\max}$, we deals with constructing a feedback control for the case of disturbance, bounded by a constraint of known shape, but unknown level.

1.5 Construction of robust control for the case of two-dimensional system (2)

If the original controlled system (1) has the set M defined only by two coordinates of the phase vector $\mathbf{x}$ (i.e., $n = 2$), then after passing to system (2) one gets a new two-dimensional phase vector x. In this case, the sets $W(t)$ and $\widehat{W}(t)$ are in the plane.

Construction of the robust control consists of two steps: choosing sign of the control and its absolute value. At any instant t, one has a family of inserted sets $W_k(t)$ (see Fig. 1, where $k_1 < 1 < k_2$). Let us find points at the boundary of these sets, where the support line is parallel to the vector $B(t)$. Joining the points obtained in this way, one gets the *switching line* $\Pi(t)$ defining the sign of the control. The absolute value of the control can be computed by the formula

$$|U(t,x)| = \begin{cases} \dfrac{l}{L}\,\mu & \text{if } x \in W(t), \\ \mu & \text{if } x \notin W(t). \end{cases}$$

In this formula, l is the length of the vector x, L is the length of the segment connecting the origin with the boundary of the set $W(t)$ and containing the point x.

For the case $n = 2$, effective algorithms and programs for constructing maximal stable bridges in linear antagonistic differential games have been carried out [Isakova et al., 1984; Kumkov and Patsko, 2001]. These programs can be used for computing sections $W(t)$ and $\widehat{W}(t)$ of maximal stable bridges corresponding to parameters $(P, Q_{\max}, M)$ and $\big(\{0\}, Q_{\max}, \rho M\big)$.

To construct the robust feedback control, one has to store sections $W(t_i)$ of the set W and switching lines $\Pi(t_i)$ in some grid $\{t_i\}$ of instants. The sections $W(t_i)$ are convex polygons and can be kept in some appropriate form. Each switching line $\Pi(t_i)$ is a polygonal line having four linear parts.

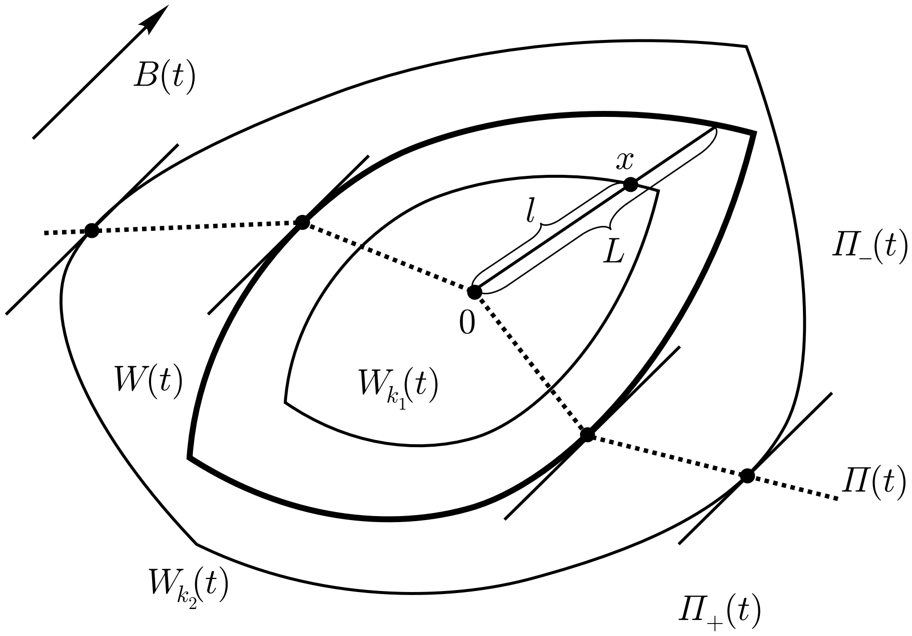


Figure 1: Constructing the robust control in the case $n = 2$

To keep it in computer memory, one has to store five its vertices (one of them is the origin).

2 Proof of the Theorem

2.1 Auxiliary statements

For compact sets X, Y in R^n, let

$$d(X,Y) = \max_{x\in X}\min_{y\in Y}|x-y|$$

be the Hausdorff deviation of the set X from the set Y.

Let us denote by $G_{k^*}(t;\bar t,\bar x)$ the attainability set of system (2) at the instant $t \geqslant \bar t$ from the initial point $\bar x$ and the initial instant $\bar t$ under all measurable open-loop controls $u(t) \in P$, $v(t) \in k^*Q_{\max}$ in the time interval $[\bar t, t]$. Let

$$\mathbf{G}_{k^*}(t;\bar t,\bar x) = G_{k^*}(t;\bar t,\bar x) + \mathbb{B}\big(2(t-\bar t)\sigma\mu\big).$$

Here, $\mathbb{B}(r)$ is a ball in R^n with the radius r and the center at the origin.

Let a second player's control satisfies the inclusion $v(t) \in k^*Q_{\max}$. In some initial position $(\bar t,\bar x)$, let the inequality $c_* = V(\bar t,\bar x) \geqslant k^*$ holds. Assume $\bar c = \min(c_*, 1) = \overline{V}(\bar t,\bar x)$. Then the definition of the function V implies that there is a control $u(\cdot)$ of the first player such that $u(t) \in \bar cP$ and for any instant $t \geqslant \bar t$ the following inequality is true:

$$V\big(t, x(t;\bar t,\bar x,u(\cdot),v(\cdot))\big) \leqslant V(\bar t,\bar x).$$

Lemma 1. *Let $k^* \geqslant 0$, $\bar t \in T$, $\bar x \notin \operatorname{int} W_{k^*}(\bar t)$, $\delta > 0$, $\bar t + \delta \leqslant \vartheta$. Let $x^*(\cdot)$ be the motion of system* (2) *under open-loop controls $u(t) \in P$ and $v(t) \in k^*Q_{\max}$ emanating from the point $\bar x$ at the instant $\bar t$. Then the following estimation holds:*

$$\mathcal{V}\big(\bar t+\delta, x^*(\bar t+\delta)\big) \leqslant V(\bar t,\bar x) + \lambda\beta\mu\delta^2. \tag{9}$$

Proof. Denote $\hat t = \bar t + \delta$. Since $\bar x \notin \operatorname{int} W_{k^*}(\bar t)$, one has $c_* = V(\bar t,\bar x) \geqslant k^*$. Thus, on the basis of the control $v(\cdot)$, one can find a control $u^\diamond(\cdot)$ such that $u^\diamond(t) \in \bar cP \subset P$, where $\bar c = \overline{V}(\bar t,\bar x)$, and the motion $x^\diamond(\cdot)$, starting from the point $\bar x$ at the instant $\bar t$ and generated by the controls $u^\diamond(\cdot)$ and $v(\cdot)$, obeys the inclusion

$$x^\diamond(t) \in W_{c_*}(t), \qquad t \in [\bar t, \hat t]. \tag{10}$$

One has

$$
\begin{gathered}
x^*(\hat{t}) - x^\diamond(\hat{t}) = \int\limits_{\bar{t}}^{\hat{t}} B(t)\bigl(u(t) - u^\diamond(t)\bigr)dt = \\
\int\limits_{\bar{t}}^{\hat{t}} \bigl(B(t) - B(\hat{t})\bigr)\bigl(u(t) - u^\diamond(t)\bigr)dt + B(\hat{t}) \int\limits_{\bar{t}}^{\hat{t}} \bigl(u(t) - u^\diamond(t)\bigr)dt.
\end{gathered} \tag{11}
$$

Denote by π the operator of orthogonal projecting of the space R^n to the subspace orthogonal to the vector $B(\hat{t})$.

Because the absolute values of the controls $u(t)$ and $u^\diamond(t)$ are bounded by the number μ, the function $B(t)$ is Lipschitzian with the constant β, and $\pi B(\hat{t}) = 0$, relation (11) gives

$$
\bigl|\pi x^*(\hat{t}) - \pi x^\diamond(\hat{t})\bigr| \leqslant \beta\mu\delta^2. \tag{12}
$$

Let $\tilde{x}$ be the point in the line $\mathcal{A}\bigl(\hat{t}, x^*(\hat{t})\bigr)$ closest to the set $W_{c_*}(\hat{t})$. Taking into account the inclusion

$$
x^\diamond(\hat{t}) \in W_{c_*}(\hat{t}),
$$

which follows from (10), and the definition of the operator π, one gets

$$
d\bigl(\{\tilde{x}\}, W_{c_*}(\hat{t})\bigr) \leqslant \bigl|\pi\tilde{x} - \pi x^\diamond(\hat{t})\bigr| = \bigl|\pi x^*(\hat{t}) - \pi x^\diamond(\hat{t})\bigr|.
$$

Thus,

$$
V(\hat{t}, \tilde{x}) \leqslant c_* + \lambda\bigl|\pi x^*(\hat{t}) - \pi x^\diamond(\hat{t})\bigr| = V(\bar{t}, \bar{x}) + \lambda\bigl|\pi x^*(\hat{t}) - \pi x^\diamond(\hat{t})\bigr|.
$$

Considering inequality (12), we have that demanded inequality (9) follows from

$$
\mathcal{V}\bigl(\hat{t}, x^*(\hat{t})\bigr) \leqslant V(\hat{t}, \tilde{x}).
$$

$\square$

Lemma 2. *Let $k^* \geqslant 0$, $\bar{t} \in T$, $\bar{x} \notin \operatorname{int} W_{k^*}(\bar{t})$, $\delta > 0$, $\bar{t} + \delta \leqslant \vartheta$. Suppose that*

$$
\mathbf{G}_{k^*}(\bar{t} + \delta; \bar{t}, \bar{x}) \subset \Pi_+(\bar{t} + \delta) \qquad \bigl(\mathbf{G}_{k^*}(\bar{t} + \delta; \bar{t}, \bar{x}) \subset \Pi_-(\bar{t} + \delta)\bigr).
$$

Let $x^*(\cdot)$ be the motion of system (2) emanating from the point $\bar{x}$ at the instant $\bar{x}$ under a constant control

$$
u(t) \equiv \overline{V}(\bar{t}, \bar{x})\mu \qquad \bigl(u(t) \equiv -\overline{V}(\bar{t}, \bar{x})\mu\bigr)
$$

and some open-loop control $v(t) \in k^* Q_{\max}$. Then the following estimation holds:

$$V\big(\bar{t}+\delta, x^*(\bar{t}+\delta)\big) \leqslant V(\bar{t}, \bar{x}) + \lambda\beta\mu\delta^2. \tag{13}$$

Proof. Denote $\hat{t} = \bar{t} + \delta$, $c_* = V(\bar{t}, \bar{x})$. Let $V(t) = V\big(t, x^*(t)\big)$ be the value of the function V along the motion $x^*(\cdot)$.

Case 1. Suppose that

$$V(t) \geqslant c_*, \qquad t \in [\bar{t}, \hat{t}]. \tag{14}$$

In the same way, as in the initial part of Lemma 1 proof, on the basis of the control $v(\cdot)$, one can construct a control $u^\diamond(t) \in \bar{c}P$, where $\bar{c} = \overline{V}(\bar{t}, \bar{x})$, such that the appearing motion $x^\diamond(\cdot)$ obeys the relation

$$x^\diamond(t) \in W_{c_*}(t), \qquad t \in [\bar{t}, \hat{t}]. \tag{15}$$

Condition (14) and the definition of the function $\overline{V}$ imply that $\overline{V}\big(t, x^*(t)\big) \geqslant \overline{V}(\bar{t}, \bar{x})$. Together with (15), this gives the inequality

$$\big|u(t)\big| \geqslant \big|u^\diamond(t)\big|, \qquad t \in [\bar{t}, \hat{t}].$$

Set

$$\tilde{z} = x^\diamond(\hat{t}) + B(\hat{t}) \int_{\bar{t}}^{\hat{t}} \big(u(t) - u^\diamond(t)\big) dt.$$

Let us show that

$$V(\hat{t}, \tilde{z}) \leqslant V\big(\hat{t}, x^\diamond(\hat{t})\big). \tag{16}$$

Consider the case

$$\mathbf{G}_{k^*}(\hat{t}; \bar{t}, \bar{x}) \subset \Pi_+(\hat{t}), \qquad u(t) \equiv \overline{V}(\bar{t}, \bar{x})\mu.$$

Due to the first inclusion, one obtains

$$x^\diamond(\hat{t}) \in \Pi_+(\hat{t}), \qquad \tilde{z} \in \Pi_+(\hat{t}).$$

Because

$$\tilde{z} \in \mathcal{A}\big(\hat{t}, x^\diamond(\hat{t})\big), \qquad u(t) \geqslant u^\diamond(t), \qquad t \in [\bar{t}, \hat{t}],$$

the vectors $\tilde{z} - x^\diamond(\hat{t})$ and $B(\hat{t})$ are co-directed. Together with quasiconvexity of the function $x \mapsto V(t, x)$, this gives inequality (16).

In the case

$$\mathbf{G}_{k^*}(\hat{t}; \bar{t}, \bar{x}) \subset \Pi_-(\hat{t}), \qquad u(t) \equiv -\overline{V}(\bar{t}, \bar{x})\mu,$$

inequality (16) can be proved in the same way.

Since the right-hand side of inequality (16) is less or equal to c_*, we have the inclusion $\tilde{z} \in W_{c_*}(\hat{t})$. Therefore,

$$d\big(\{x^*(\hat{t})\}, W_{c_*}(\hat{t})\big) \leqslant \big|x^*(\hat{t}) - \tilde{z}\big|.$$

Using the definition of the vector $\tilde{z}$, one has

$$x^*(\hat{t}) - \tilde{z} = x^*(t) - x^\diamond(\hat{t}) - B(\hat{t}) \int\limits_{\bar{t}}^{\hat{t}} \big(u(t) - u^\diamond(t)\big) dt =$$

$$\int\limits_{\bar{t}}^{\hat{t}} \big(B(t) - B(\hat{t})\big)\big(u(t) - u^\diamond(t)\big) dt.$$

Thus,

$$\big|x^*(\hat{t}) - \tilde{z}\big| \leqslant \beta\mu\delta^2.$$

Demanded inequality (13) follows from

$$V\big(\hat{t}, x^*(\hat{t})\big) \leqslant V(\hat{t}, \tilde{z}) + \lambda\big|x^*(\hat{t}) - \tilde{z}\big|, \qquad V(\hat{t}, \tilde{z}) \leqslant V(\bar{t}, \bar{x}).$$

Case 2. Let at some instant, the inequality $V(t) < c_*$ takes place.

If $V(\hat{t}) \leqslant c_*$, then demanded inequality (13) holds.

Assume that $V(\hat{t}) > c_*$. One can find the maximal instant $\tilde{t}$ such that $V(\tilde{t}) = c_*$ and $V(t) > c_*$, $t \in [\tilde{t}, \hat{t}]$. For the interval $[\tilde{t}, \hat{t}]$, the condition of Case 1 is held with change of $\bar{t}$ by $\tilde{t}$. Hence,

$$V(\hat{t}) \leqslant V(\tilde{t}) + \lambda\beta\mu\delta_1^2 \qquad (\delta_1 = \hat{t} - \tilde{t}).$$

Inequality (13) follows from $V(\tilde{t}) = c_*$ and $\delta_1 < \delta$. □

Lemma 3. *Let $k^* \geqslant 0$, $\bar{t} \in T$, $\hat{t} \in (\bar{t}, \vartheta]$. Let $x^*(\cdot)$ be the motion of system (2) emanating from the point $\bar{x}$ at the instant $\bar{t}$ under a constant control*

$$u(t) \equiv \overline{V}(\bar{t}, \bar{x})\mu \qquad \big(u(t) \equiv -\overline{V}(\bar{t}, \bar{x})\mu\big)$$

and some open-loop control $v(t) \in k^* Q_{\max}$. Assume

$$x^*(t) \in \Pi_+(t) \setminus \operatorname{int} W_{k^*}(t) \qquad \big(x^*(t) \in \Pi_-(t) \setminus \operatorname{int} W_{k^*}(t)\big)$$

for all $t \in [\bar{t}, \hat{t}]$. Then the following estimation holds:

$$V\big(\hat{t}, x^*(\hat{t})\big) \leqslant V(\bar{t}, \bar{x}). \tag{17}$$

Proof. Without loss of generality, we consider the case when

$$u(t) \equiv \overline{V}(\bar{t}, \bar{x})\mu, \quad x^*(t) \in \Pi_+(t) \setminus \operatorname{int} W_{k^*}(t), \qquad t \in [\bar{t}, \hat{t}].$$

Assume that

$$V\big(\hat{t}, x^*(\hat{t})\big) > V(\bar{t}, \bar{x}). \tag{18}$$

Let $\tilde{t}$ be the maximal instant in $[\bar{t}, \hat{t}]$ when $V\big(t, x^*(t)\big) = V(\bar{t}, \bar{x})$.

Separate the interval $[\tilde{t}, \hat{t}]$ by instants $t_1, t_2, \ldots, t_s$ ($t_1 = \tilde{t}$, $t_s = \hat{t}$) with the time step δ in such a way that for any $k = 1, 2, \ldots, s-1$, the following relation holds:

$$\mathbf{G}_{k^*}\big(t_{k+1}; t_k, x^*(t_k)\big) \subset \Pi_+(t_{k+1}).$$

It can be done by means of the assumption about location of $x^*(t)$ with respect to $\Pi(t)$.

Consider an interval $[t_k, t_{k+1}]$. The symbol $\tilde{x}_k(\cdot)$ denotes a motion of system (2) emanating from the point $x^*(t_k)$ at the instant t_k under the first player's constant control $\tilde{u}_k(t) \equiv \overline{V}\big(t_k, x^*(t_k)\big)\mu$ and the second player's control $v(\cdot)$ assumed in the Lemma formulation.

Due to Lemma 2, one has inequality

$$V\big(t_{k+1}, \tilde{x}(t_{k+1})\big) \leqslant V\big(t_k, x^*(t_k)\big) + \lambda\beta\mu\delta^2. \tag{19}$$

Estimate $V\big(t_{k+1}, x^*(t_{k+1})\big)$ on the basis of $V\big(t_{k+1}, \tilde{x}(t_{k+1})\big)$:

$$V\big(t_{k+1}, x^*(t_{k+1})\big) \leqslant V\big(t_{k+1}, \tilde{x}(t_{k+1})\big) + \lambda\big|\tilde{x}(t_{k+1}) - x^*(t_{k+1})\big|. \tag{20}$$

Since

$$\tilde{x}(t_{k+1}) - x^*(t_{k+1}) = \int\limits_{t_k}^{t_{k+1}} B(t)\big(\tilde{u}_k(t) - u(t)\big)dt =$$

$$\int\limits_{t_k}^{t_{k+1}} B(t)\Big(\overline{V}\big(t_k, x^*(t_k)\big) - \overline{V}(\bar{t}, \bar{x})\Big)\mu\, dt,$$

the inequality

$$\big|\tilde{x}(t_{k+1}) - x^*(t_{k+1})\big| \leqslant \sigma\mu\Big(\overline{V}\big(t_k, x^*(t_k)\big) - \overline{V}(\bar{t}, \bar{x})\Big)\delta. \tag{21}$$

holds. Here, we use the inequality $\overline{V}\big(t_k, x^*(t_k)\big) \geqslant \overline{V}(\bar{t}, \bar{x})$, which follows from the inequality $V\big(t_k, x^*(t_k)\big) \geqslant V(\bar{t}, \bar{x})$.

Due to inequalities (20) and (21), one obtains

$$V\big(t_{k+1}, x^*(t_{k+1})\big) \leqslant V\big(t_{k+1}, \tilde{x}(t_{k+1})\big) + \lambda\beta\mu\delta\big(\overline{V}\big(t_k, x^*(t_k)\big) - \overline{V}(\bar{t}, \bar{x})\big). \quad (22)$$

Because $\overline{V}\big(t_k, x^*(t_k)\big) \geqslant \overline{V}(\bar{t}, \bar{x})$, it implies

$$\overline{V}\big(t_k, x^*(t_k)\big) - \overline{V}(\bar{t}, \bar{x}) \leqslant V\big(t_k, x^*(t_k)\big) - V(\bar{t}, \bar{x}). \quad (23)$$

Inequalities (19), (22), and (23) imply

$$V\big(t_{k+1}, x^*(t_{k+1})\big) \leqslant V\big(t_k, x^*(t_k)\big) + \lambda\beta\mu\delta^2 + \lambda\sigma\mu\delta\big(V\big(t_k, x^*(t_k)\big) - V(\bar{t}, \bar{x})\big). \quad (24)$$

Let

$$\operatorname{Var}\big(V, [t^\flat, t^\sharp]\big) = V\big(t^\sharp, x^*(t^\sharp)\big) - V\big(t^\flat, x^*(t^\flat)\big)$$

be the variation of the function V along the motion $x^*(\cdot)$ in the interval $[t^\flat, t^\sharp]$.

Rewrite inequality (24) using the new notation:

$$\operatorname{Var}\big(V, [t_k, t_{k+1}]\big) \leqslant \lambda\beta\mu\delta^2 + \lambda\sigma\mu\delta \operatorname{Var}\big(V, [\bar{t}, t_k]\big). \quad (25)$$

With that,

$$\begin{gathered} \operatorname{Var}\big(V, [\bar{t}, t_k]\big) = \operatorname{Var}\big(V, [t_1, t_k]\big) = \sum_{p=1}^{k-1} \operatorname{Var}\big(V, [t_p, t_{p+1}]\big), \\ \operatorname{Var}\big(V, [t_1, t_2]\big) \leqslant \lambda\beta\mu\delta^2. \end{gathered} \quad (26)$$

We are interested to get an estimation of the value

$$V\big(\hat{t}, x^*(\hat{t})\big) - V(\bar{t}, \bar{x}) = \operatorname{Var}\big(V, [t_1, t_s]\big) = \sum_{k=1}^{s-1} \operatorname{Var}\big(V, [t_k, t_{k+1}]\big). \quad (27)$$

Consider the geometric progression

$$a_k = a_1 q^{k-1}, \quad k = 1, \ldots, s-1, \quad (28)$$

where

$$a_1 = \lambda\beta\mu\delta^2, \quad q = 1 + \lambda\sigma\mu\delta. \quad (29)$$

From relations (25), (26), and (28), it follows

$$\operatorname{Var}\big(V, [t_k, t_{k+1}]\big) \leqslant a_k, \quad k = 1, \ldots, s-1. \quad (30)$$

One has

$$\sum_{k=1}^{s-1} a_k = a_1 \frac{q^{s-1}-1}{q-1} = \frac{\beta\delta}{\sigma}\big((1+\lambda\sigma\mu\delta)^{s-1}-1\big).$$

Since $s-1 = (\hat{t}-\tilde{t})/\delta$, we can see that

$$\sum_{k=1}^{s-1} a_k = \frac{\beta\delta}{\sigma}\big((1+\lambda\sigma\mu\delta)^{(\hat{t}-\tilde{t})/\delta}-1\big) \leqslant \frac{\beta\delta}{\sigma}\big(e^{\lambda\sigma\mu(\hat{t}-\tilde{t})}-1\big) \leqslant \frac{\beta\delta}{\sigma}\big(e^{\lambda\sigma\mu T}-1\big). \tag{31}$$

Relations (27), (30), and (31) give the inequality

$$V\big(\hat{t}, x^*(\hat{t})\big) \leqslant V(\bar{t},\bar{x}) + \frac{\beta\delta}{\sigma}\big(e^{\lambda\sigma\mu T}-1\big),$$

which is true for any sufficiently small partition of the interval $[\tilde{t},\hat{t}]$. This gives a contradiction with relation (18).

Thus,

$$V\big(\hat{t}, x^*(\hat{t})\big) \leqslant V(\bar{t},\bar{x}).$$

□

The following Lemma gives a trivial estimation of the variation of the function V along a motion of system (2) under an admissible open-loop control of the first player and some bounded open-loop control of the second one.

Lemma 4. *Let $k^* \geqslant 0$, $\bar{t} \in T$, $\bar{x} \notin \operatorname{int} W_{k^*}(\bar{t})$, $\hat{t} \in (\bar{t}, \vartheta]$. Let $x^*(\cdot)$ be the motion of system* (2) *emanating from the point $\bar{x}$ at the instant $\bar{t}$ under open-loop controls $u(t) \in P$ and $v(t) \in k^* Q_{\max}$. Then the following estimation holds:*

$$V\big(\hat{t}, x^*(\hat{t})\big) \leqslant V(\bar{t},\bar{x}) + 2\lambda\sigma\mu(\hat{t}-\bar{t}). \tag{32}$$

Proof. Let $c_* = V(\bar{t},\bar{x})$. Using the properties of the function V, on the basis of the control $v(\cdot)$, one can find a control $u^\diamond(\cdot)$ such that $u^\diamond(t) \in P$, and the motion $x^\diamond(\cdot)$, starting from the point $\bar{x}$ at the instant $\bar{t}$ and generated by the controls $u^\diamond(\cdot)$ and $v(\cdot)$, obeys the inclusion

$$x^\diamond(t) \in W_{c_*}(t), \qquad t \in [\bar{t},\hat{t}].$$

Consequently,

$$V\big(\hat{t}, x^\diamond(\hat{t})\big) \leqslant V(\bar{t},\bar{x}). \tag{33}$$

Since

$$x^*(\hat{t}) - x^\diamond(\hat{t}) = \int\limits_{\bar{t}}^{\hat{t}} B(t)\big(u(t) - u^\diamond(t)\big)dt,$$

one can see

$$\big|x^*(\hat{t}) - x^\diamond(\hat{t})\big| \leqslant 2\sigma\mu(\hat{t} - \bar{t}). \tag{34}$$

Let $V\big(\hat{t}, x^*(\hat{t})\big) \geqslant V\big(\hat{t}, x^\diamond(\hat{t})\big)$. We have

$$V\big(\hat{t}, x^*(\hat{t})\big) - V\big(\hat{t}, x^\diamond(\hat{t})\big) \leqslant \lambda\big|x^*(\hat{t}) - x^\diamond(\hat{t})\big|.$$

Together with (33) and (34), this implies inequality (32). □

2.2 Finalizing proof of the Theorem

Let us prove inequality (8). If that inequality is true, then inclusion (7) holds, because

$$\big|U(t,x)\big| \leqslant \overline{V}(t,x)\mu = \min\big(V(t,x), 1\big)\mu.$$

In the interval $[t_0, \vartheta]$, we select closed subintervals, where $x(t) \notin \operatorname{int} W_{k^*}(t)$. Outside them, $V\big(t, x^*(t)\big) \leqslant k^* \leqslant s^*$, and inequality (8) is true.

Let $[\xi, \zeta]$ be one of these intervals. Assume that it can not be extended to the left with keeping the condition $x^*(t) \notin \operatorname{int} W_{k^*}(t)$. Then either $V\big(\xi, x^*(\xi)\big) = k^*$ or $V\big(\xi, x^*(\xi)\big) = c^* > k^*$. The latter is possible only if $\xi = t_0$.

Let us prove relation (8) in the interval $[\xi, \zeta]$.

To write the variation of the function V along the motion $x^*(\cdot)$ in some time interval $[t^\flat, t^\sharp]$, we use the following notation:

$$\operatorname{Var}\big(V, [t^\flat, t^\sharp]\big) = V\big(t^\sharp, x^*(t^\sharp)\big) - V\big(t^\flat, x^*(t^\flat)\big).$$

1. Let $\beta > 0$, $\sigma > 0$. Assume

$$h = \sqrt{(2\sigma\mu\Delta + r)/\beta\mu}. \tag{35}$$

A. Select along the motion $x^*(\cdot)$ "loops", connected to visiting the set $\Pi^r(t)$. Define also free intervals.

Going from ξ to ζ, let us find the first instant t when $x^*(t) \in \Pi^r(t)$. This instant will be called the beginning of the first loop and denoted by t_1. Further, an instant $\tilde{t}_1$ should be found when the loop finishes. It can be find as the last instant t in the interval $[t_1, t_1+h] \cap [\xi, \zeta]$ when $x^*(t) \in \Pi^r(t)$. In particular, the instant $\tilde{t}_1$ can coincide with t_1.

As the beginning of the second loop t_2, we take the first instant $t \in [t_1 + h, \zeta]$ when $x^*(t) \in \Pi^r(t)$. Then we mark the instant $\tilde{t}_2$ when the second loop terminates. It is found as the last instant t in the interval $[t_2, t_2+h] \cap [\xi, \zeta]$ when $x^*(t) \in \Pi^r(t)$.

Continuing this process, one obtains a collection of loops in the interval $[\xi, \zeta]$.

Remove from the closed interval $[\xi, \zeta]$ interior of the intervals of the loops. One gets an ordered set of intervals. Each of them will be called free. They can be degenerated, that is, consisting of one point only.

If there is no loops in the interval $[\xi, \zeta]$, then the whole interval is called free.

B. Let $[\tau, \eta]$ be a free interval. Let us show that the variation of the function V in the interval is described by the inequality

$$\mathrm{Var}_f\big(V, [\tau, \eta]\big) \leqslant 2\lambda\sigma\mu\Delta. \tag{36}$$

The subscript f emphasizes that the variation of the function V is computed in a free interval.

In the interior of a free interval, the motion $x^*(\cdot)$ goes on a certain side from the set

$$\Pi^r = \big\{(t, x) \colon t \in T,\ x \in \Pi^r(t)\big\},$$

and, consequently, on a certain side from the set

$$\Pi = \big\{(t, x) \colon t \in T,\ x \in \Pi(t)\big\}.$$

At the beginning t_Δ of a time step, a control

$$u(t_\Delta) = \overline{V}\big(t_\Delta, x^*(t_\Delta)\big)\mu \qquad \big(u(t_\Delta) = -\overline{V}\big(t_\Delta, x^*(t_\Delta)\big)\mu\,\big)$$

is chosen if

$$x^*(t_\Delta) \in \Pi^r_+(t_\Delta) \qquad \big(x^*(t_\Delta) \in \Pi^r_-(t_\Delta)\big).$$

And further, this control acts until the beginning of the next time step. If $t_\Delta + \Delta \leqslant \eta$, then due to Lemma 3 one gets

$$\mathrm{Var}\big(V, [t_\Delta, t_\Delta + \Delta]\big) \leqslant 0.$$

If $t_\Delta + \Delta > \eta$, one has

$$\mathrm{Var}\big(V, [t_\Delta, \eta]\big) \leqslant 0.$$

Summing this inequalities for all time steps, starting at $t_\Delta > \tau$, we get

$$\operatorname{Var}\bigl(V, [\tau + \Delta, \eta]\bigr) \leqslant 0.$$

For the interval $[\tau, \tau + \Delta]$, according to Lemma 4, one has

$$\operatorname{Var}\bigl(V, [\tau, \tau + \Delta]\bigr) \leqslant 2\lambda\sigma\mu\Delta.$$

Summing two latter inequalities, one comes to estimation (36).

C. Let us say that $[\tau, \eta]$ is an interval of type E_1 if it consists of some loop $[t_i, \tilde{t}_i]$ and a free interval adjacent to the loop from the right. The interval $[\tau, \eta]$ of type E_1 with additional condition $\tau + h \leqslant \eta$ will be called an interval of type E_2.

Estimate the variation of the function V along the motion $x^*(\cdot)$ in an interval of type E_1.

Consider the interval of loop $[t_i, \tilde{t}_i]$. Applying Lemma 1 with $\delta = \tilde{t}_i - t_i$, one has

$$\mathcal{V}\bigl(\tilde{t}_i, x^*(\tilde{t}_i)\bigr) \leqslant V\bigl(t_i, x^*(t_i)\bigr) + \lambda\beta\mu(\tilde{t}_i - t_i)^2.$$

Since $\tilde{t}_i - t_i \leqslant h$, the second term in the right-hand side can be changed by $\lambda\beta\mu h(\tilde{t}_i - t_i)$. Taking into account the inequality

$$V\bigl(\tilde{t}_i, x^*(\tilde{t}_i)\bigr) \leqslant \mathcal{V}\bigl(\tilde{t}_i, x^*(\tilde{t}_i)\bigr) + \lambda r,$$

one passes to the relation

$$\operatorname{Var}\bigl(V, [t_i, \tilde{t}_i]\bigr) \leqslant \lambda\beta\mu h(\tilde{t}_i - t_i) + \lambda r. \tag{37}$$

One has inequality (36) in a free interval $[\tilde{t}_i, \eta]$. Consider it together with inequality (37) for $\tau = t_i$ and the inequality $\tilde{t}_i - t_i \leqslant \eta - \tau$. This implies

$$\operatorname{Var}_1\bigl(V, [\tau, \eta]\bigr) \leqslant \lambda\beta\mu h(\eta - \tau) + 2\lambda\sigma\mu\Delta + \lambda r. \tag{38}$$

The subscript emphasizes that the variation of the function V is computed in an interval of type E_1.

Let us pass to the estimation of the variation Var_2 of the function V along the motion $x^*(\cdot)$ in an interval of type E_2. We have $\eta - \tau \geqslant h$ in this case. Therefore, relation (35) gives the inequality

$$2\lambda\sigma\mu\Delta + \lambda r \leqslant \lambda\beta\mu h(\eta - \tau).$$

Applying inequality (38), one gets

$$\operatorname{Var}_2\bigl(V, [\tau, \eta]\bigr) \leqslant 2\lambda\beta\mu h(\eta - \tau). \tag{39}$$

D. Consider an interval $[\xi, t]$ $(t \leqslant \zeta)$. It can be presented in the following way. At first, a free interval $[\xi, \bar{t}]$ appears. Then there are some finite number of intervals of type E_2 located after each other from the instant $\bar{t}$ until some instant $\hat{t}$. (Their summary interval is $[\bar{t}, \hat{t}]$.) And, finally, there is a residual interval $[\hat{t}, t]$ of type E_1. Applying sequentially estimates (36), (39), and (38), one has

$$\mathrm{Var}\big(V, [\xi, t]\big) = \mathrm{Var}_f\big(V, [\xi, \bar{t}]\big) + \mathrm{Var}\big(V, [\bar{t}, \hat{t}]\big) + \mathrm{Var}_1\big(V, [\hat{t}, t]\big) \leqslant$$
$$2\lambda\sigma\mu\Delta + 2\lambda\beta\mu h(\hat{t} - \bar{t}) + \lambda\beta\mu h(t - \hat{t}) + 2\lambda\sigma\mu\Delta + \lambda r =$$
$$2\lambda\beta\mu h(t - \bar{t}) + 4\lambda\sigma\mu\Delta + \lambda r.$$

After substitution of h according to formula (35), we get

$$\mathrm{Var}\big(V, [\xi, t]\big) \leqslant \Lambda(t, \xi, \Delta, r). \tag{40}$$

2. Let $\beta = 0$, $\sigma \geqslant 0$. Going from ξ to t $(t \leqslant \zeta)$, find the first instant when $x^*(t) \in \Pi^r(t)$. Denote it by $\bar{t}$. Let $\hat{t}$ be the last instant in $[\xi, t]$ such that $x^*(t) \in \Pi^r(t)$. One has

$$x^*(t) \notin \Pi^r(t), \qquad t \in [\xi, \bar{t}) \cup (\hat{t}, t].$$

For the intervals $[\xi, \bar{t}]$ and $[\hat{t}, t]$ on the basis of Lemmas 3, 4 (similarly to proving inequality (36)), one gets

$$\mathrm{Var}\big(V, [\xi, \bar{t}]\big) \leqslant 2\lambda\sigma\mu\Delta, \tag{41}$$

$$\mathrm{Var}\big(V, [\hat{t}, t]\big) \leqslant 2\lambda\sigma\mu\Delta. \tag{42}$$

For the interval $[\bar{t}, \hat{t}]$ due to Lemma 1 with $\beta = 0$, we have

$$\mathcal{V}\big(\hat{t}, x^*(\hat{t})\big) \leqslant V\big(\bar{t}, x^*(\bar{t})\big),$$

and, therefore, taking into account the inequality

$$V\big(\hat{t}, x^*(\hat{t})\big) \leqslant \mathcal{V}\big(\hat{t}, x^*(\hat{t})\big) + \lambda r,$$

one gets the estimation

$$\mathrm{Var}\big(V, [\bar{t}, \hat{t}]\big) \leqslant \lambda r. \tag{43}$$

Considering inequalities (41)–(43), we have

$$\mathrm{Var}\big(V, [\xi, t]\big) \leqslant 4\lambda\sigma\mu\Delta + \lambda r. \tag{44}$$

3. On the basis of inequality (40) in the case $\beta > 0,\ \sigma > 0$ and of inequality (44) in the case $\beta = 0,\ \sigma \geqslant 0$, one has the estimation

$$V\big(t, x^*(t)\big) \leqslant V\big(\xi, x^*(\xi)\big) + \Lambda(t, \xi, \Delta, r). \tag{45}$$

At the instant ξ, the value $V(\xi, x^*(\xi))$ of the function V equals to either k^* or c^*. Thus,

$$V\big(\xi, x^*(\xi)\big) \leqslant \max(k^*, c^*) = s^*.$$

Substitute this inequality to (45) and take into account that $\Lambda(t, t_0, \Delta, r) \geqslant \Lambda(t, \xi, \Delta, r)$. This gives inequality (8). □

3 Example 1. Conflict-controlled pendulum

3.1 Formulation of problem

Let a system describing a linearized pendulum be the following:

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{x}_2 + v, \\ \dot{\mathbf{x}}_2 &= -\mathbf{x}_1 + u. \end{aligned} \tag{46}$$

Here, u and v are scalar controls of the first and second players (the useful control and the disturbance). The absolute value of the first player's control is bounded:

$$|u| \leqslant \mu = 1.$$

This inequality defines the set P.

The system is studied in the time interval $T = [0, 10]$. In the plane of the phase coordinates $\mathbf{x}_1$, $\mathbf{x}_2$, the terminal set M is defined as a circle with the radius 2 and the center at the origin. The first player tries to lead system (46) to the set M at the terminal instant $\vartheta = 10$.

The constraint $Q_{\max}$ for the second player's control is taken as

$$|v| \leqslant \nu = 1.$$

In Fig. 2, the sections $W(t)$ of the maximal stable bridge W, which corresponds to the parameters P, $Q_{\max}$, and M, are shown for the instants $t = 0, 2, 4, 6, 8, 10$. The switching lines $\Pi(t)$ are given also. The symbols "+" and "−" denotes the sign of the control in the corresponding domains. These objects are drawn in the coordinates x_1, x_2 of game (2).

3.2 Simulation of motions

We suppose that the control u in system (46) is generated by the robust feedback law $\widetilde{U}(t, \mathbf{x})$ in a discrete scheme of control with the time step $\Delta = 0.05$.

To model a motion of the system, one should define the initial position and the second player's control.

Choose the initial point

$$\mathbf{x}(0) = (0, 0.5).$$

One has

$$x(0) = X_{2,2}(10, 0)\mathbf{x}(0) \in W(0).$$

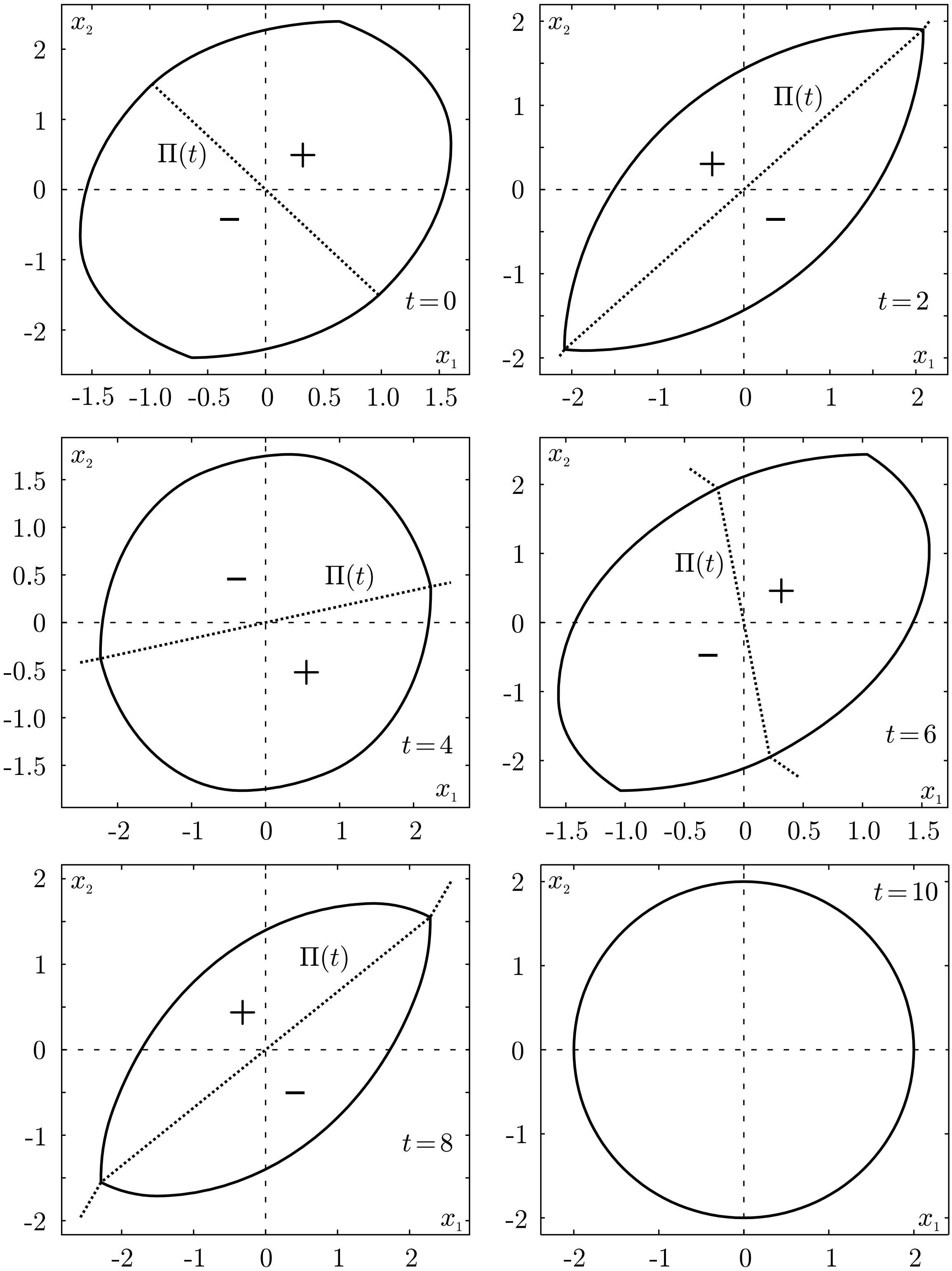


Figure 2: Example 1. Sections $W(t)$ of the maximal stable bridge W shown together with switching lines $\Pi(t)$

There were two variants of the second player's control. One of them is sinusoids of some fixed frequency and different amplitudes. The second variant is a feedback control formed on the basis of some auxiliary differential game. This game has fixed terminal time, given constraints for controls of both players, and the terminal payoff function of Minkowski type generated by the set M. The algorithm of constructing the second player's optimal strategy for such a game by means of switching surfaces is described in [Botkin et al., 1983; Botkin et al., 1984; Zarkh and Patsko, 1987; Zarkh, 1990].

Three levels for the second player's control have been considered:

- $|v| \leqslant 0.5$, a level less then the chosen maximal one (which is equal to 1);
- $|v| \leqslant 1$, the level equal to the chosen maximal one;
- $|v| \leqslant 1.5$, a level greater than the chosen maximal one (there is no guarantee of reaching the terminal set in this case).

So, totally there are 6 variants for the disturbance (two ways to construct the control and three levels for each of them). For all these variants in Figs. 3–8, trajectories in the plane $\mathbf{x}_1$, $\mathbf{x}_2$, the terminal position at the terminal instant $\vartheta = 10$, and the realizations of the players' controls are shown.

Figures 3–5 correspond to the case of sinusoidal disturbance, Figs. 6–8 show the case of the extremal (i.e., optimal for the correspondent disturbance level) one generated by the second player's feedback control from an auxiliary differential game.

3.3 Discussion of simulation results

On the basis of Figs. 3–8, one can see that the suggested robust control successfully parries any disturbance if it obeys the chosen constraint $Q_{\max}$.

Also, it can be seen that with passing from the sinusoidal disturbance to the extremal one, the results become sufficiently worse. With that, the maximal level of the control realization and the terminal position of the system correspond to the theoretic results.

Type of disturbance:

sinusoidal

Maximum of disturbance:

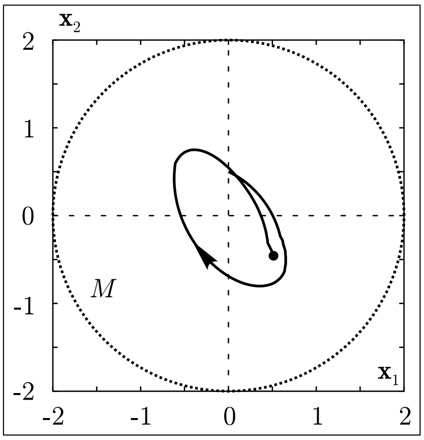
0.5 (0.5ν)

Termination:

successful

Maximum of control:

0.39 (0.39μ)



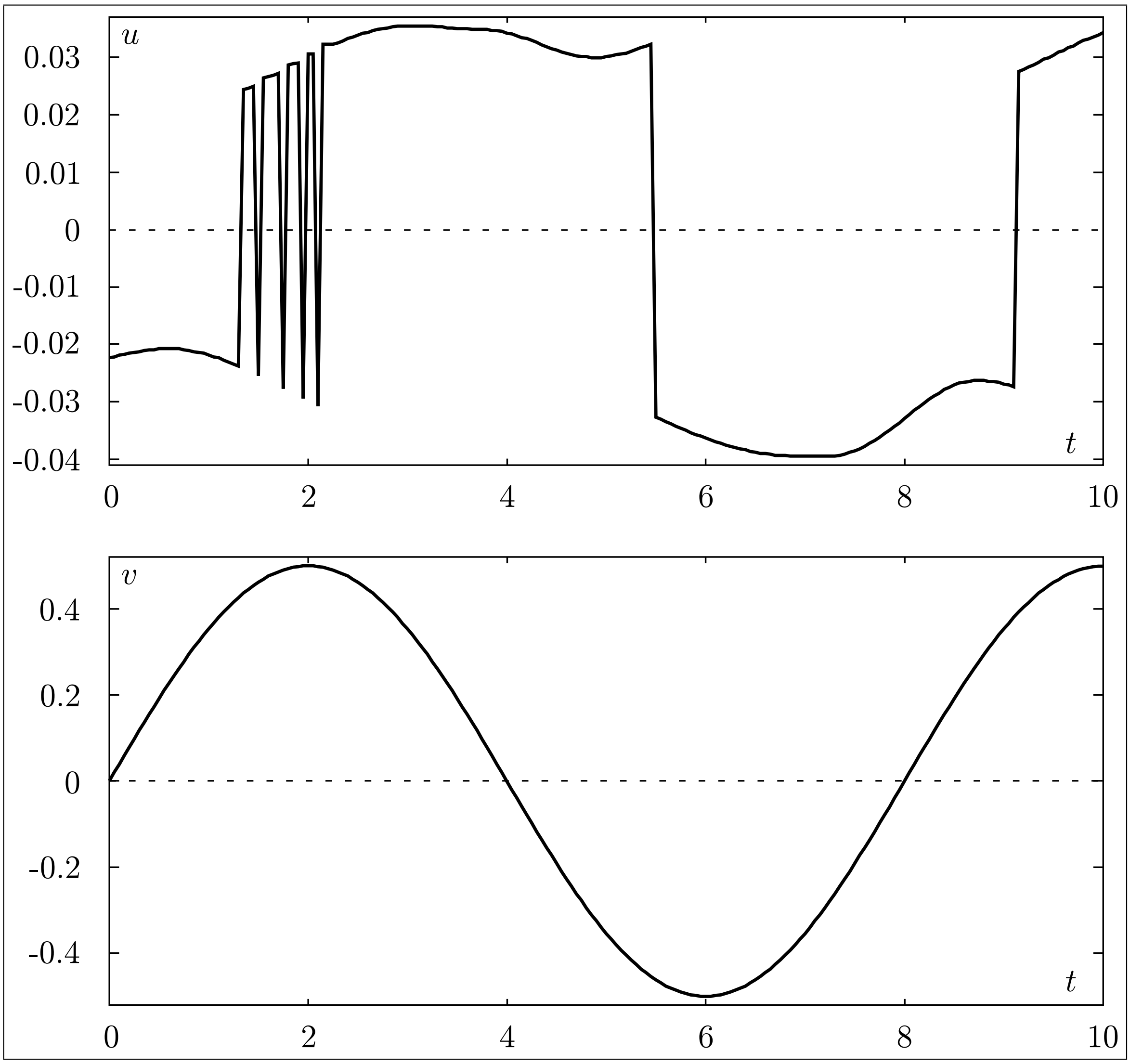
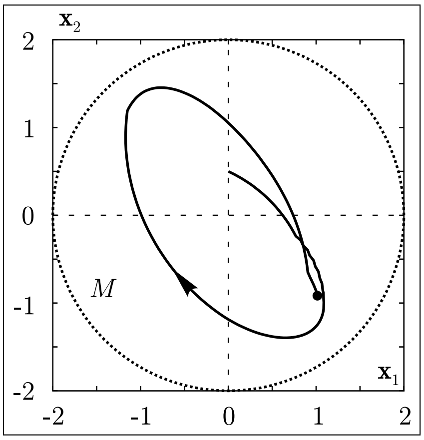


Figure 3: Example 1. Trajectory of the system (in the original coordinates $\mathbf{x}_1$, $\mathbf{x}_2$) and its state at the terminal instant (the circle in the upper figure); realizations of the players' controls: of the first player (the middle figure) and second one (the lower figure)

Type of disturbance:
sinusoidal
Maximum of disturbance:
1.0 (1.0ν)
Termination:
successful
Maximum of control:
0.78 (0.78μ)



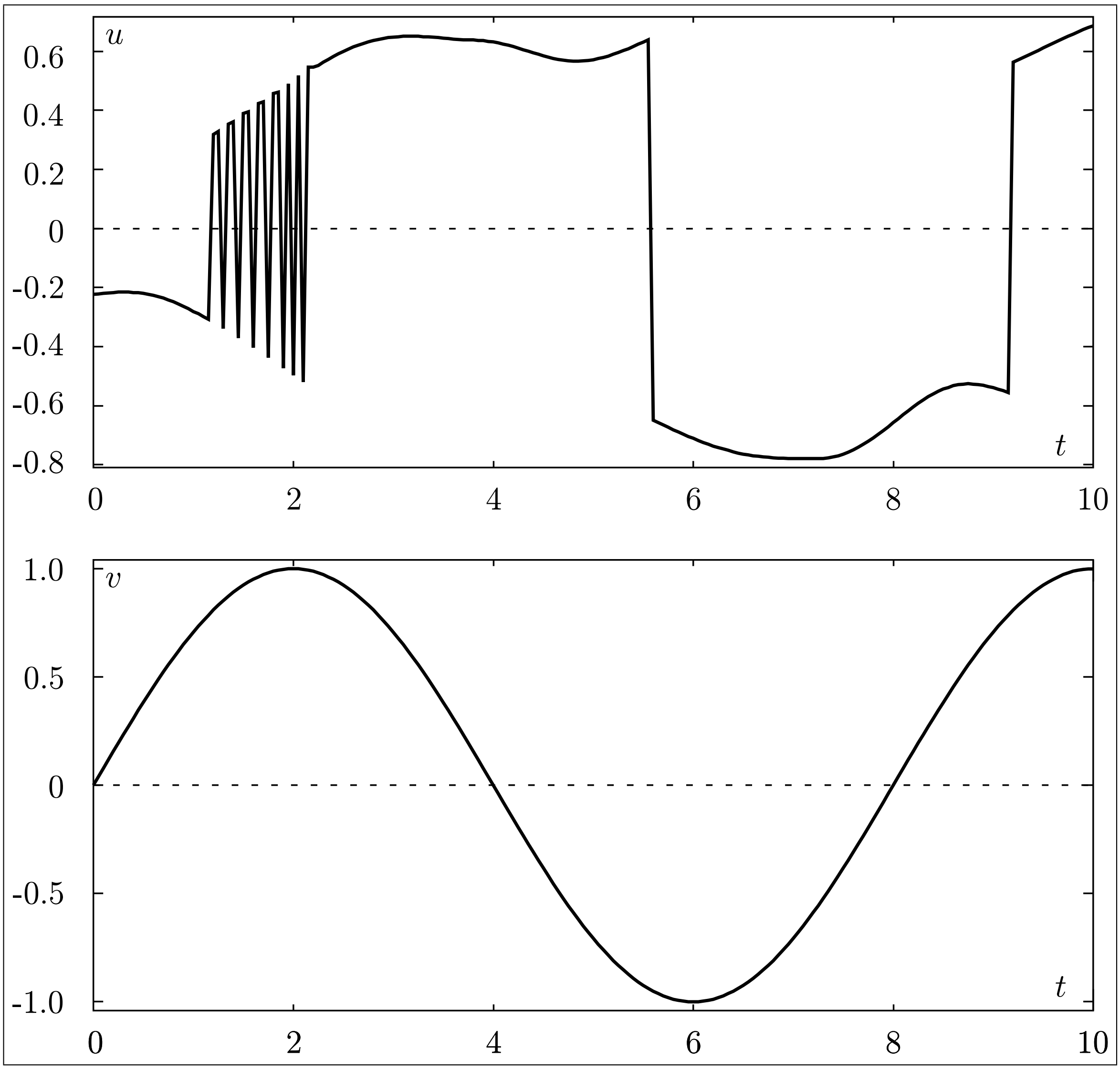
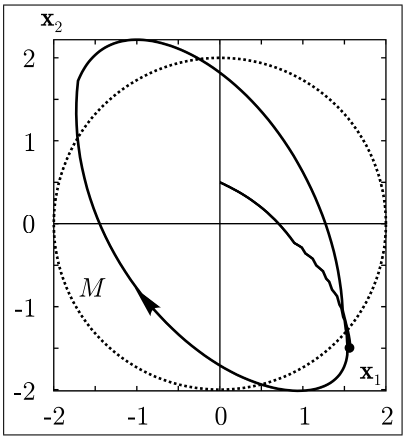


Figure 4: Example 1. Trajectory of the system (in the original coordinates $\mathbf{x}_1$, $\mathbf{x}_2$) and its state at the terminal instant (the circle in the upper figure); realizations of the players' controls: of the first player (the middle figure) and second one (the lower figure)

Type of disturbance:
sinusoidal
Maximum of disturbance:
1.5 (1.5ν)
Termination:
unsuccessful
Maximum of control:
1.0 (1.0μ)



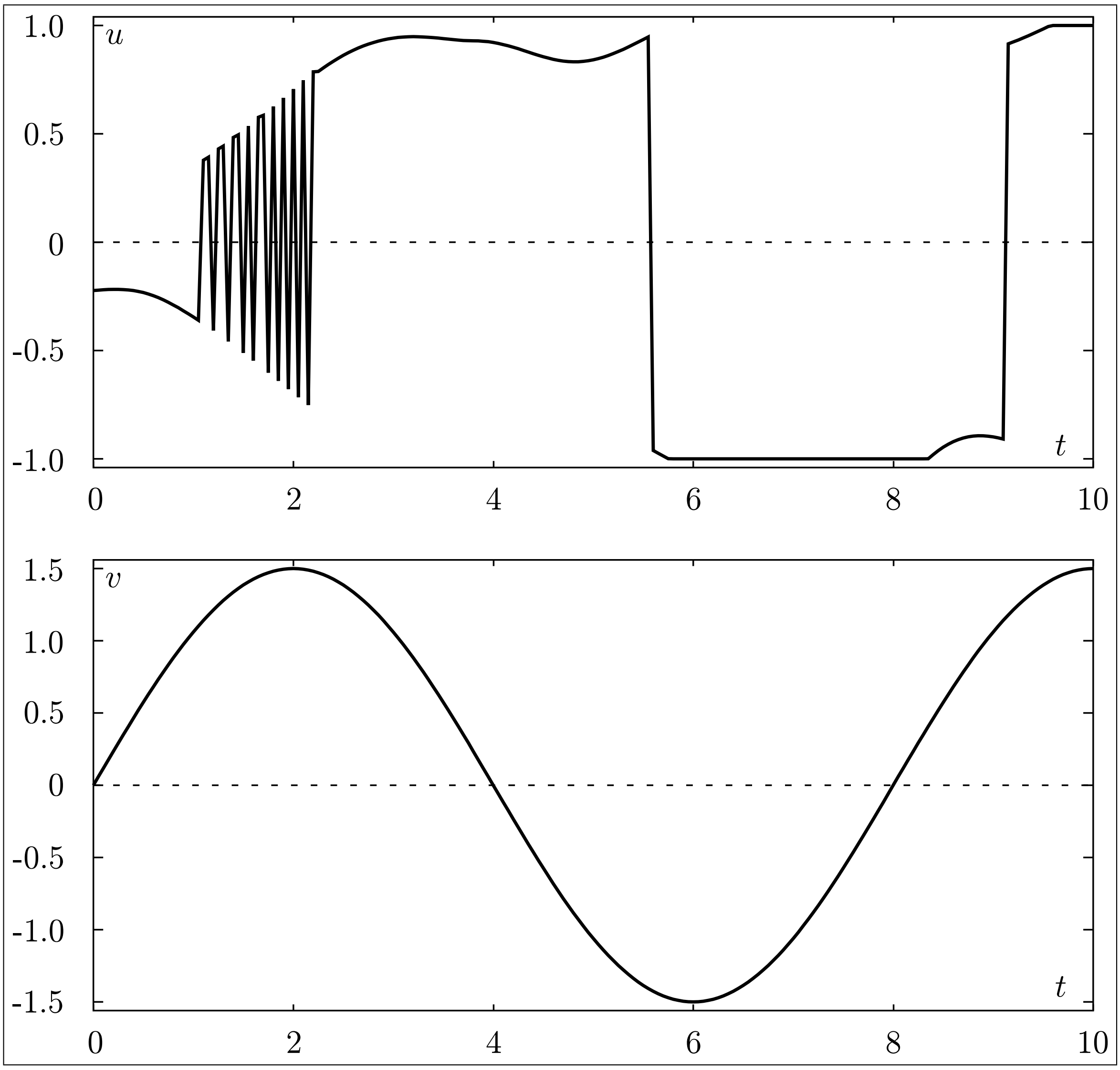


Figure 5: Example 1. Trajectory of the system (in the original coordinates $\mathbf{x}_1$, $\mathbf{x}_2$) and its state at the terminal instant (the circle in the upper figure); realizations of the players' controls: of the first player (the middle figure) and second one (the lower figure)

Type of disturbance:
optimal
Maximum of disturbance:
0.5 (0.5ν)
Termination:
successful
Maximum of control:
0.48 (0.48μ)

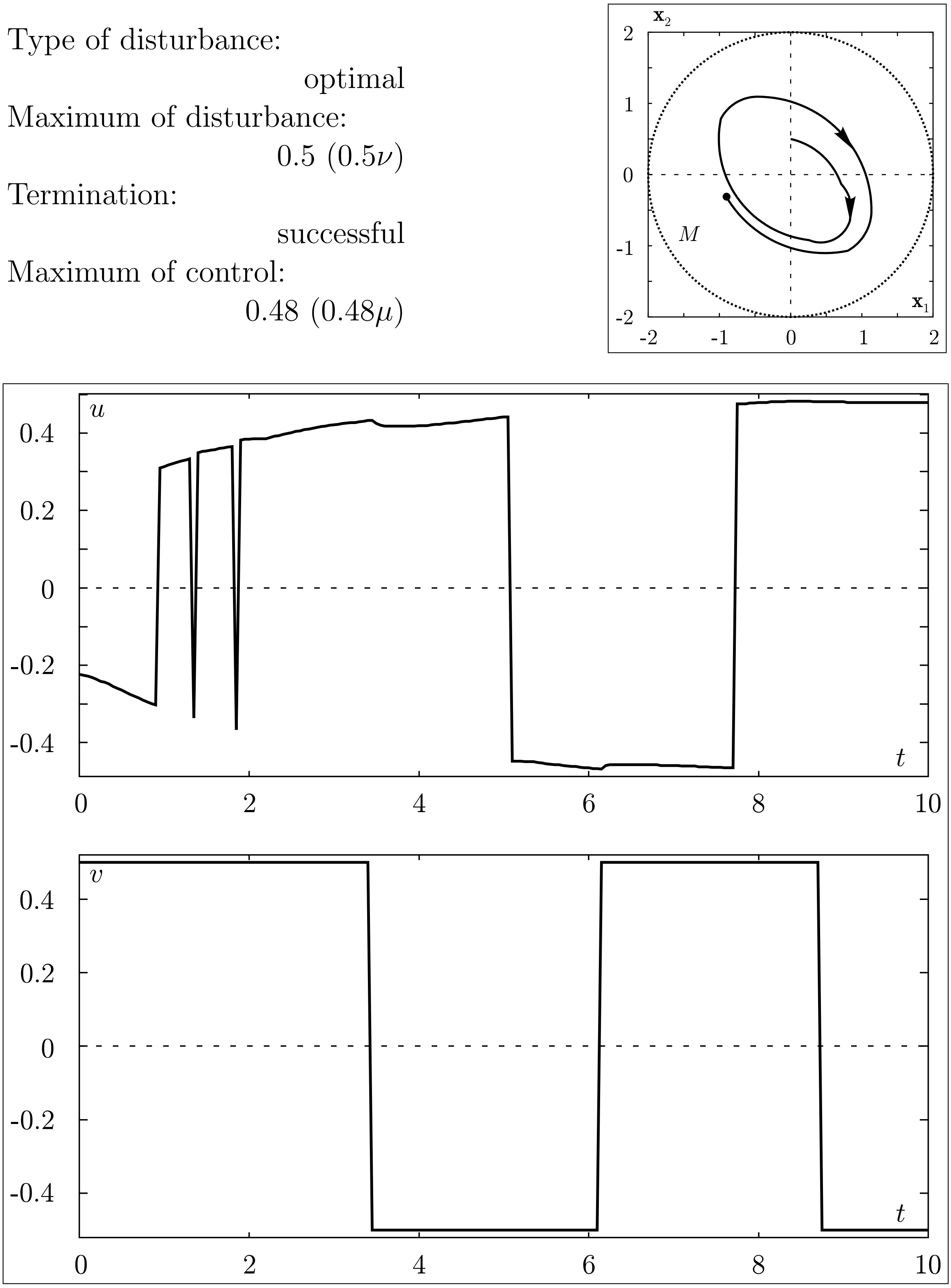
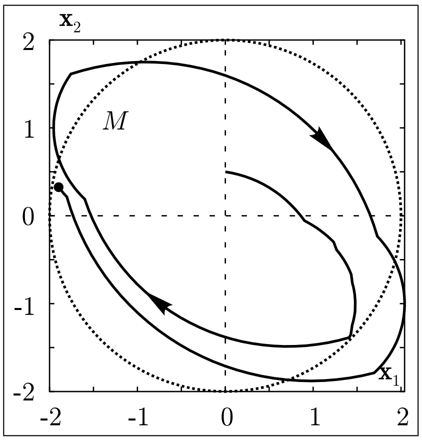


Figure 6: Example 1. Trajectory of the system (in the original coordinates $\mathbf{x}_1$, $\mathbf{x}_2$) and its state at the terminal instant (the circle in the upper figure); realizations of the players' controls: of the first player (the middle figure) and second one (the lower figure)

Type of disturbance:
optimal
Maximum of disturbance:
1.0 (1.0ν)
Termination:
successful
Maximum of control:
0.98 (0.98μ)



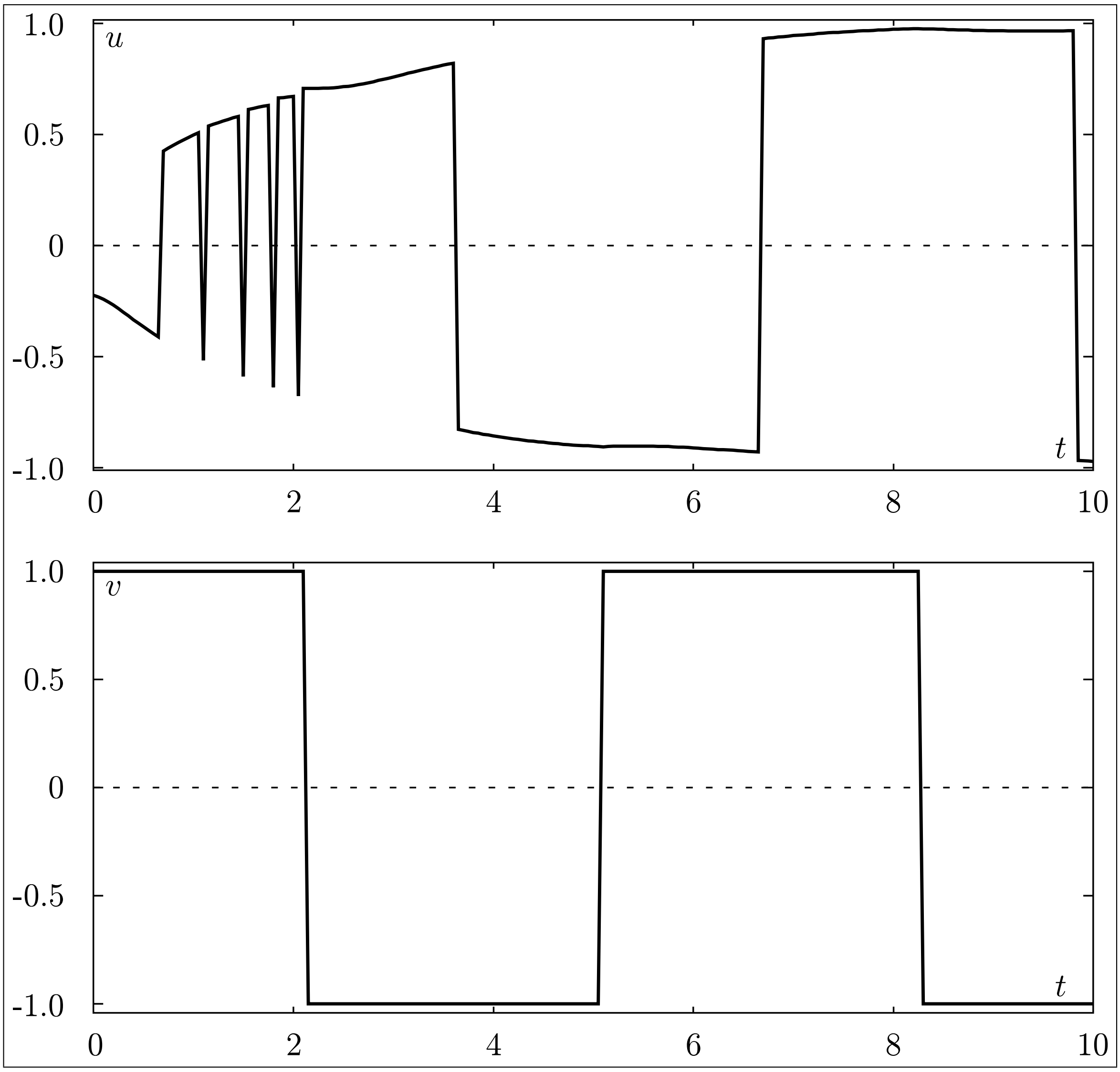


Figure 7: Example 1. Trajectory of the system (in the original coordinates $\mathbf{x}_1$, $\mathbf{x}_2$) and its state at the terminal instant (the circle in the upper figure); realizations of the players' controls: of the first player (the middle figure) and second one (the lower figure)

Type of disturbance:

optimal

Maximum of disturbance:

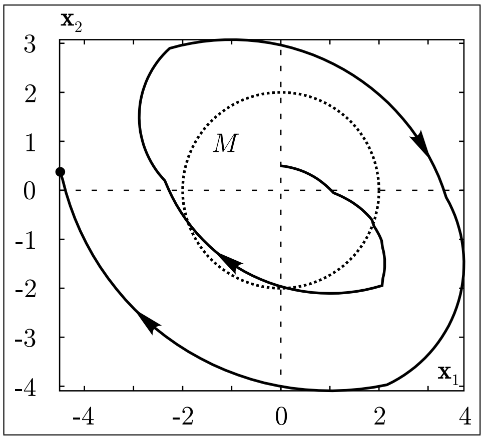
1.5 (1.5ν)

Termination:

unsuccessful

Maximum of control:

1.0 (1.0μ)



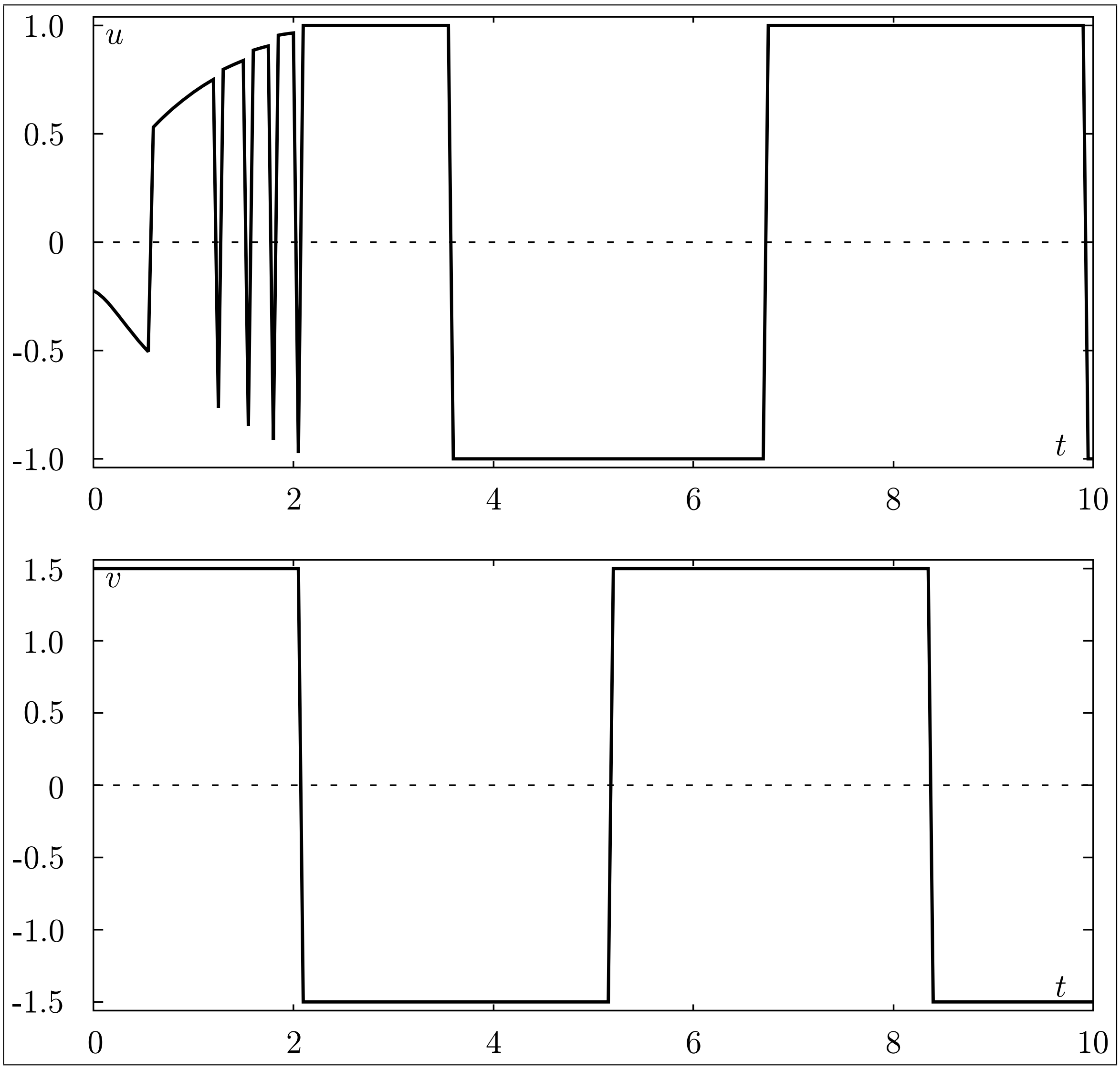


Figure 8: Example 1. Trajectory of the system (in the original coordinates $\mathbf{x}_1$, $\mathbf{x}_2$) and its state at the terminal instant (the circle in the upper figure); realizations of the players' controls: of the first player (the middle figure) and second one (the lower figure)

4 Example 2. Robust control in problem of aircraft landing

4.1 Formulation of problem

During the last 20 years, there were a lot of publications dealing with application of methods of modern control theory and differential game theory to problems of landing and take-off an aircraft under wind disturbances (see, for example, [Miele et al., 1986; Miele et al., 1988; Leitmann and Pandey, 1991; Bulirsch et al., 1991; Patsko et al., 1994; Seube et al., 2000] and references therein).

In this paper, the problem of lateral motion of an average transport aircraft during final stage of landing is considered. A linear approximation of the motion dynamics is described [Kein et al., 1980; Botkin et al., 1983; Botkin et al., 1984] by the following system of differential equations:

$$\begin{aligned}
\dot{\mathbf{x}}_1 &= \mathbf{x}_2, \\
\dot{\mathbf{x}}_2 &= -0.0762\mathbf{x}_2 - 5.34\mathbf{x}_3 + 9.81\mathbf{x}_5 + 0.0762v, \\
\dot{\mathbf{x}}_3 &= \mathbf{x}_4, \\
\dot{\mathbf{x}}_4 &= -0.0056\mathbf{x}_2 - 0.392\mathbf{x}_3 - 0.0889\mathbf{x}_4 - 0.0378\mathbf{x}_5 - 0.17\mathbf{x}_6 + \\
&\quad\; 0.0378\mathbf{x}_7 + 0.0056v, \\
\dot{\mathbf{x}}_5 &= -\mathbf{x}_5 + \mathbf{x}_7, \\
\dot{\mathbf{x}}_6 &= -0.0129\mathbf{x}_2 - 0.9016\mathbf{x}_3 - 0.2045\mathbf{x}_4 - 0.0869\mathbf{x}_5 - 0.89\mathbf{x}_6 + \\
&\quad\; 0.0869\mathbf{x}_7 + 0.0129v, \\
\dot{\mathbf{x}}_7 &= -\mathbf{x}_7 + u.
\end{aligned} \tag{47}$$

Components of the phase vector $\mathbf{x}$ have the following physical sense:

$\mathbf{x}_1$ is the lateral deviation of the mass center of the aircraft from the central axis of runway;

$\mathbf{x}_2$ is the velocity of the lateral deviation;

$\mathbf{x}_3$ is the yaw angle counted clockwise from the runway axis;

$\mathbf{x}_4$ is the angular velocity of yaw angle;

$\mathbf{x}_5$ is the bank angle;

$\mathbf{x}_6$, $\mathbf{x}_7$ are auxiliary variables.

The control u can be treated as the required bank angle. The disturbance parameter v is the lateral component of the wind velocity. The lateral deviation is measured in meters, all angles are measured in radians, the time is measured in seconds.

The system is investigated in the time interval $[0, \vartheta]$, where ϑ is the instant of passing the runway threshold. In the examples below, $\vartheta = 15\,\text{sec}$.

The required bank angle is bounded:

$$|u| \leqslant \mu = 0.2613\,\text{rad}.$$

This constraint defines the set P.

In the subspace of phase variables $\mathbf{x}_1$ and $\mathbf{x}_2$, let us define a set

$$M = \left\{ (\mathbf{x}_1, \mathbf{x}_2) \colon \frac{\mathbf{x}_1^2}{216} - \frac{2\mathbf{x}_1}{9} - \frac{3}{2} \leqslant \mathbf{x}_2 \leqslant -\frac{\mathbf{x}_1^2}{216} - \frac{2\mathbf{x}_1}{9} + \frac{3}{2} \right\}.$$

If at the instant ϑ the lateral deviation $\mathbf{x}_1(\vartheta)$ and its velocity $\mathbf{x}_2(\vartheta)$ are such that $\big(\mathbf{x}_1(\vartheta), \mathbf{x}_2(\vartheta)\big) \in M$, then it is supposed that successful final landing is provided after the terminal instant. Otherwise, if $\big(\mathbf{x}_1(\vartheta), \mathbf{x}_2(\vartheta)\big) \notin M$, there is no such a guarantee. So, the set M is the tolerance for the coordinates $\mathbf{x}_1$, $\mathbf{x}_2$ at the instant of passing runway threshold.

As the constraint $Q_{\max}$, let us take

$$|v| \leqslant \nu = 10\,\text{m/sec}.$$

In Fig. 9, the sections of the maximal stable bridge W corresponding to the parameters P, $Q_{\max}$, and M are drawn for the instants $t = 0$, 3, 6, 9, 12, and 15 sec together with the switching lines defining the sign of the first player's control; corresponding domains are marked by symbols "+" and "−". These figures use the coordinates x_1, x_2 of game (2).

4.2 Simulation of motions

It is supposed that the control u in system (47) is generated by the robust feedback law $\widetilde{U}(t, \mathbf{x})$ in a discrete scheme of control with some time step Δ.

To model a motion of the system, one should put the initial position and the second player's control.

In the following simulations, the initial point has the lateral deviation equal to 30 m:

$$\mathbf{x}(0) = (30, 0, 0, 0, 0, 0, 0).$$

One can check that

$$x(0) = X_{2,7}(15, 0)\mathbf{x}(0) \in W(0).$$

Here, $X_{2,7}$ is the matrix for passing to system of type (2). It consists of two first rows of the fundamental Cauchy matrix for system (47).

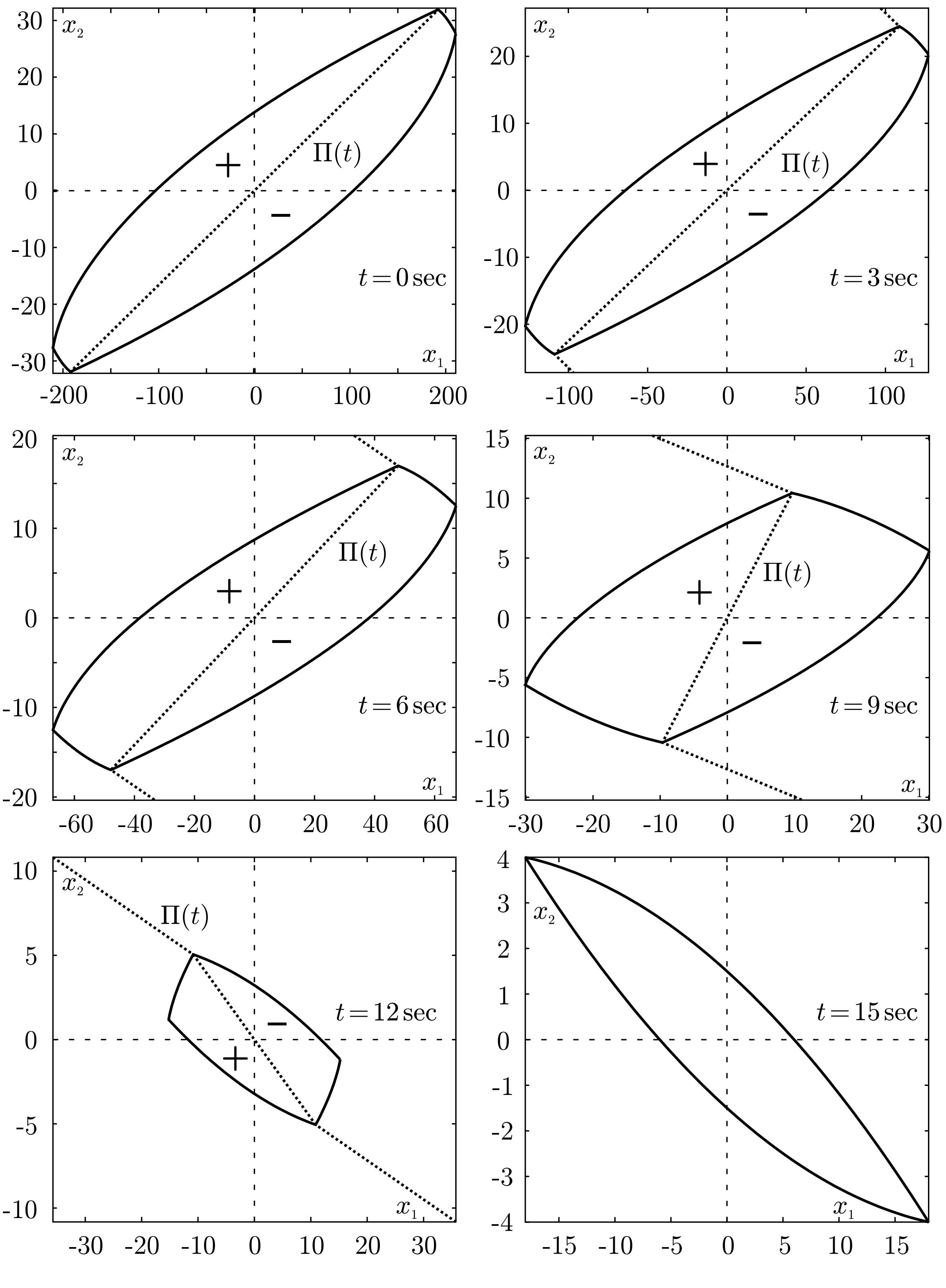


Figure 9: Example 2. Sections $W(t)$ of the maximal stable bridge W shown together with switching lines $\Pi(t)$

The simulations have been carried out for two types of the second player's control: taken as sinusoids having the same frequency and different amplitudes, and as an optimal strategy from an auxiliary differential game. This game has fixed terminal time, given constraints for both players' controls, and the terminal payoff function of Minkowski type generated by the set M.

The simulation results are given for three levels of the disturbance:

- 5 m/sec, a level less than the chosen maximal one equal to 10 m/sec;
- 10 m/sec, the level equal to the chosen maximal one;
- 15 m/sec, a level greater than the chosen maximal one; so, in this case there is no guarantee of reaching the terminal set M.

In total, there are 6 variants of the disturbance. For these variants, Figs. 10–15 show the trajectory in the plane of phase variables $\mathbf{x}_1$ and $\mathbf{x}_2$, the system position $\big(\mathbf{x}_1(\vartheta), \mathbf{x}_2(\vartheta)\big)$ at the terminal instant $\vartheta = 15\,\mathrm{sec}$, and the realizations of players' controls.

Figures 10–12 concern the situation of sinusoidal disturbance, Figs. 13 –15 correspond to the extremal (i.e., optimal for the correspondent disturbance level) one generated by a feedback strategy taken from an auxiliary differential game.

When modeling motions, the time step Δ in the discrete scheme of control was taken equal to 0.05 sec.

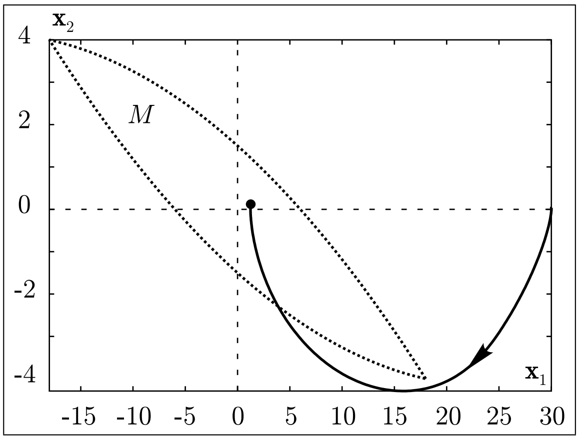
4.3 Discussion of simulation results

The simulation results show that the suggested robust control parries successfully the sinusoidal disturbance, including the case when it is stronger than the chosen maximal level. With that, when the disturbance level is less than the maximal one (i.e., for the variant 5 m/sec), the maximal value of the first player's control $u(t)$ is sufficiently less than the maximal level 0.2613 rad. In the case 15 m/sec, the reaching of the extremal value of the first player's control appears only in the final stage of process close to the terminal instant $\vartheta = 15\,\mathrm{sec}$.

The results become worse with changing the sinusoidal disturbance by the extremal law. In this case when the disturbance level is 15 m/sec, the phase coordinates $\mathbf{x}_1(\vartheta)$, $\mathbf{x}_2(\vartheta)$ at the terminal instant $\vartheta = 15\,\mathrm{sec}$ do not belong to the terminal set M (see Fig. 15). However, the deviation from the terminal set is not too large. Again, the realization of the first player's control reaches its extremal value 0.2613 rad at the final stage of the process only.

For the variants with extremal second player's control, the realization of control u has frequent switches. This means that the phase trajectory in the space of system (2) goes near the switching surface and passes from one its side to another. But since $u(t)$ is the required (command) bank angle, which affects the angle of ailerons via some servo-mechanism, there is nothing bad in these switches. They are smoothed in the mechanism.

Type of disturbance:
sinusoidal
Maximum of disturbance:
5 m/sec (0.5ν)
Termination:
successful
Maximum of control:
0.08 (0.31μ)



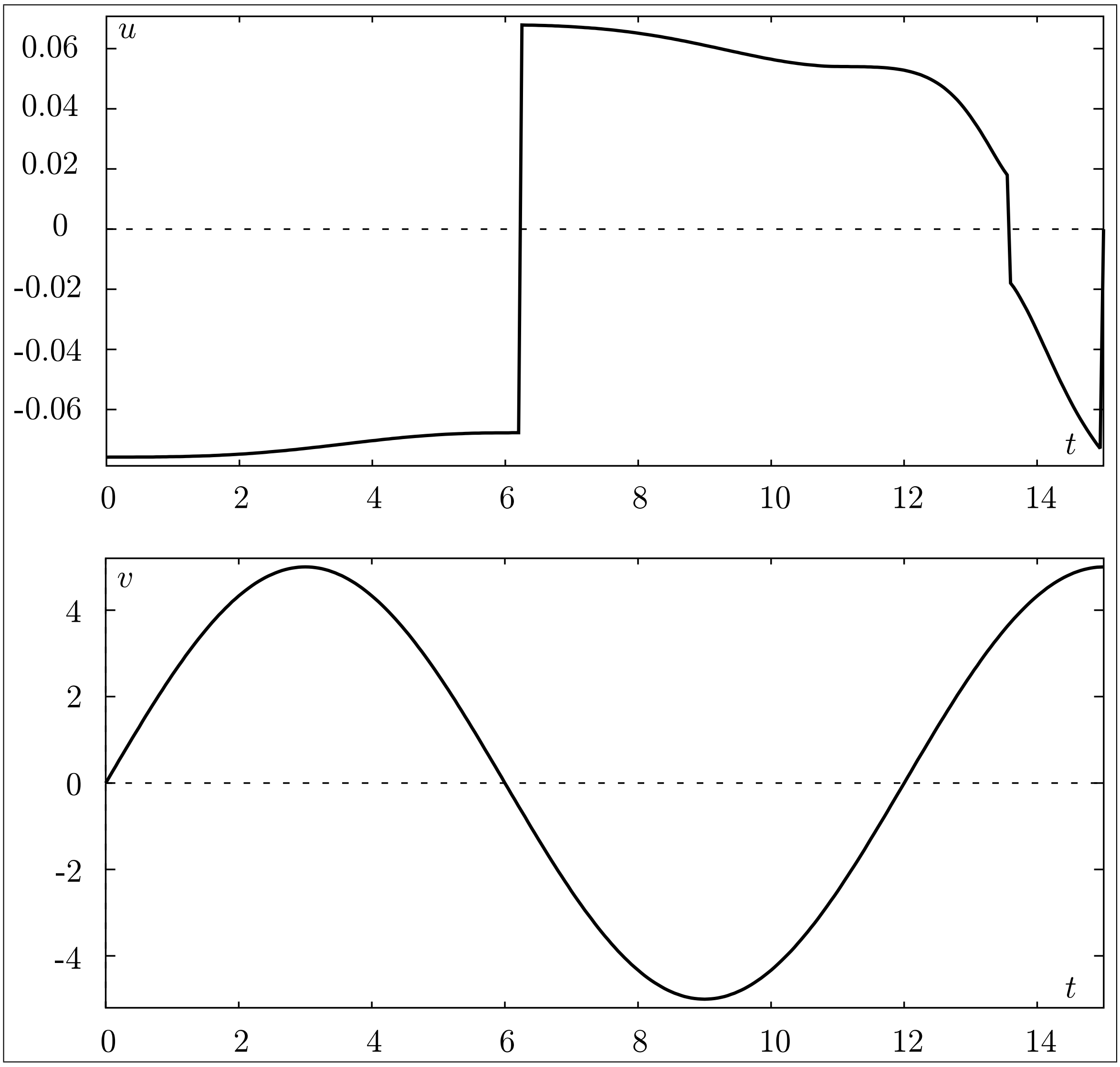
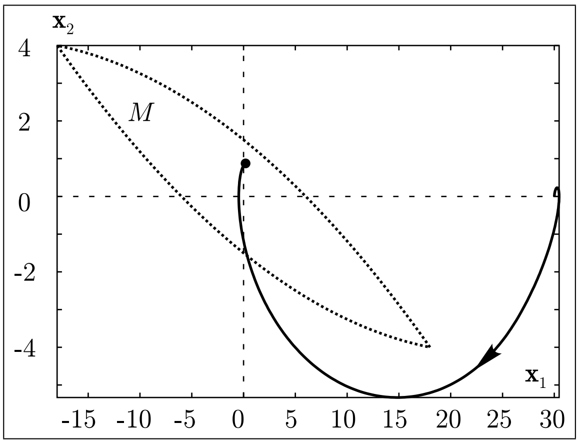


Figure 10: Example 2. Trajectory of the system (in the coordinates of lateral deviation $\mathbf{x}_1$ and lateral velocity $\mathbf{x}_2$) and its state at the terminal instant (the circle in the upper figure); realizations of the players' controls: of the first player (the middle figure) and second one (the lower figure)

Type of disturbance:
sinusoidal
Maximum of disturbance:
10 m/sec (1.0ν)
Termination:
successful
Maximum of control:
0.16 (0.62μ)



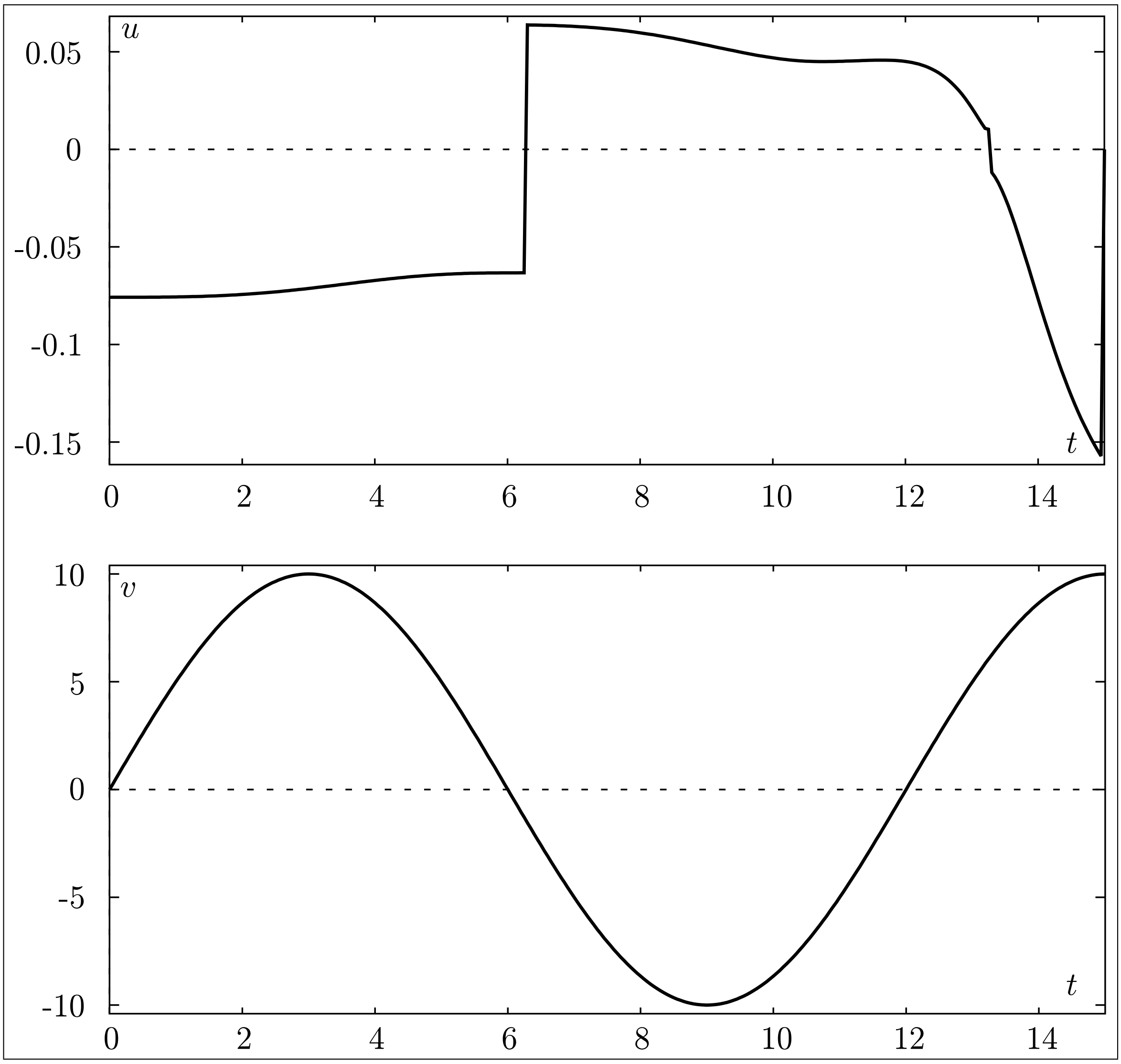
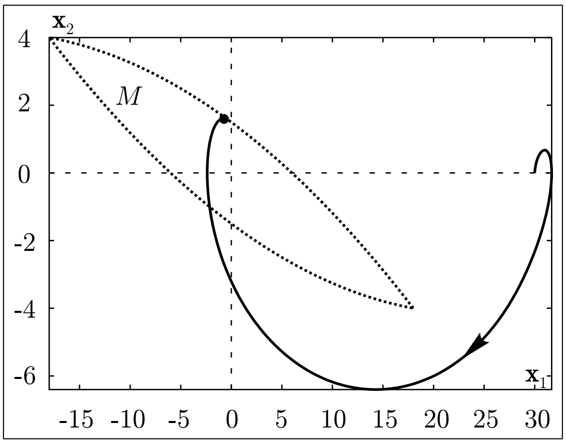


Figure 11: Example 2. Trajectory of the system (in the coordinates of lateral deviation $\mathbf{x}_1$ and lateral velocity $\mathbf{x}_2$) and its state at the terminal instant (the circle in the upper figure); realizations of the players' controls: of the first player (the middle figure) and second one (the lower figure)

Type of disturbance:
sinusoidal
Maximum of disturbance:
$15\,\mathrm{m/sec}$ (1.5ν)
Termination:
successful
Maximum of control:
0.25 (0.96μ)



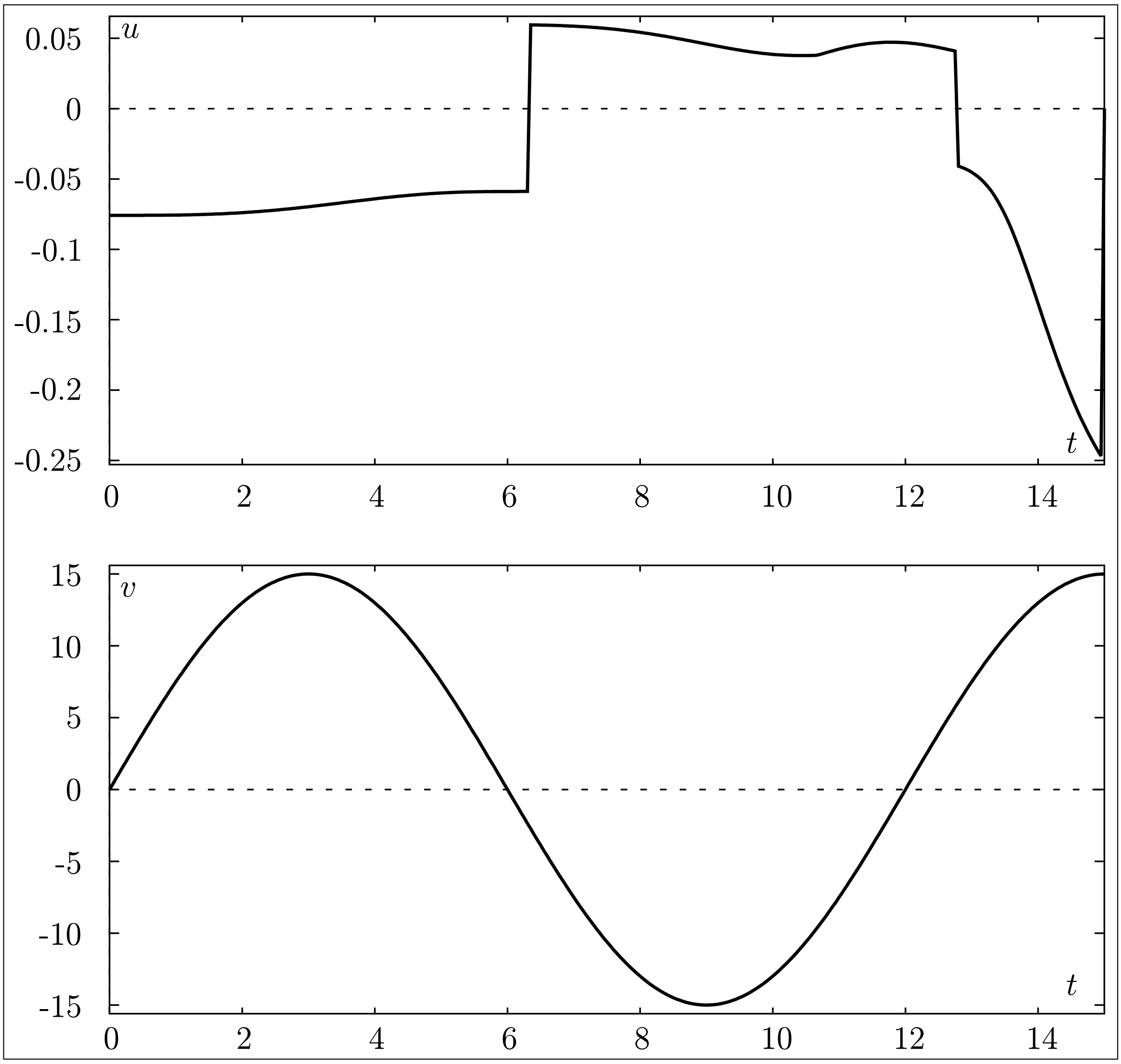
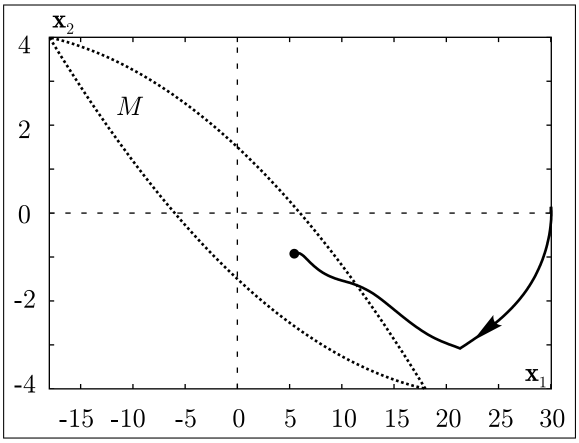


Figure 12: Example 2. Trajectory of the system (in the coordinates of lateral deviation $\mathbf{x}_1$ and lateral velocity $\mathbf{x}_2$) and its state at the terminal instant (the circle in the upper figure); realizations of the players' controls: of the first player (the middle figure) and second one (the lower figure)

Type of disturbance:
extremal
Maximum of disturbance:
5 m/sec (0.5ν)
Termination:
successful
Maximum of control:
0.11 (0.42μ)



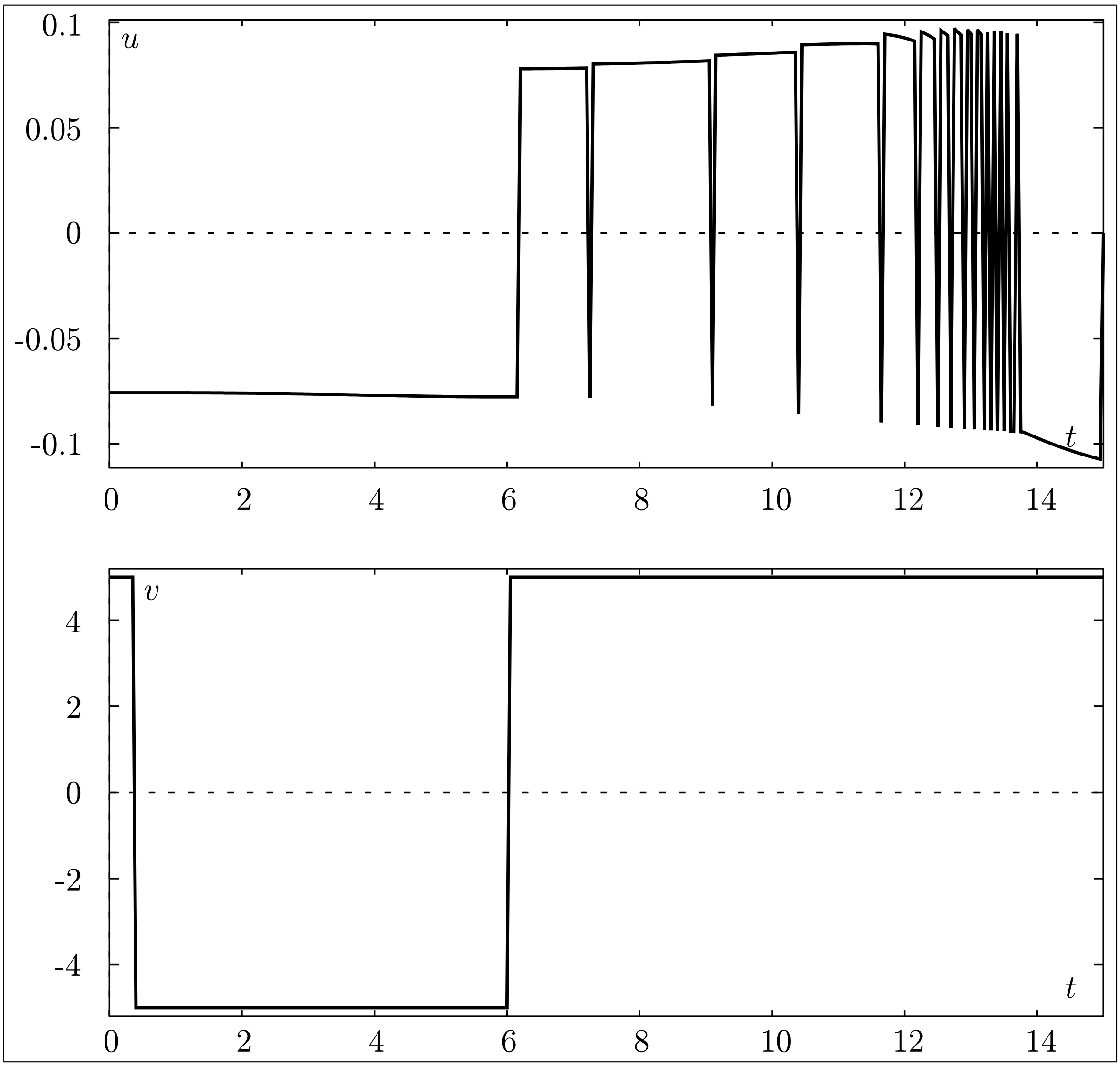


Figure 13: Example 2. Trajectory of the system (in the coordinates of lateral deviation $\mathbf{x}_1$ and lateral velocity $\mathbf{x}_2$) and its state at the terminal instant (the circle in the upper figure); realizations of the players' controls: of the first player (the middle figure) and second one (the lower figure)

Type of disturbance:

extremal

Maximum of disturbance:

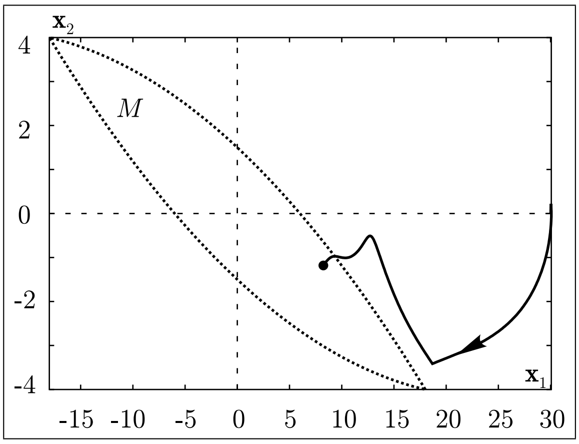
$10\,\mathrm{m/sec}$ (1.0ν)

Termination:

successful

Maximum of control:

0.19 (0.73μ)



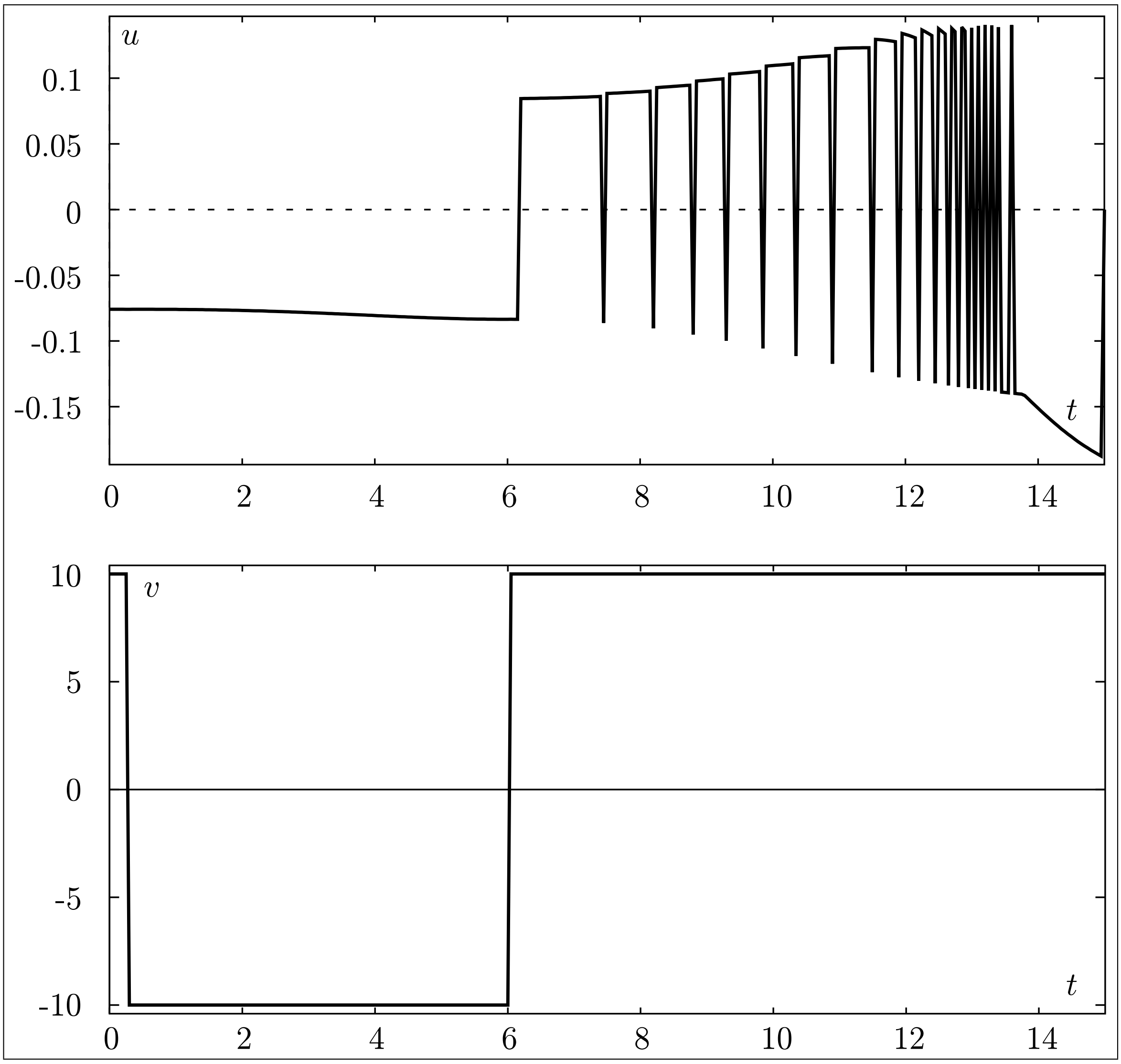
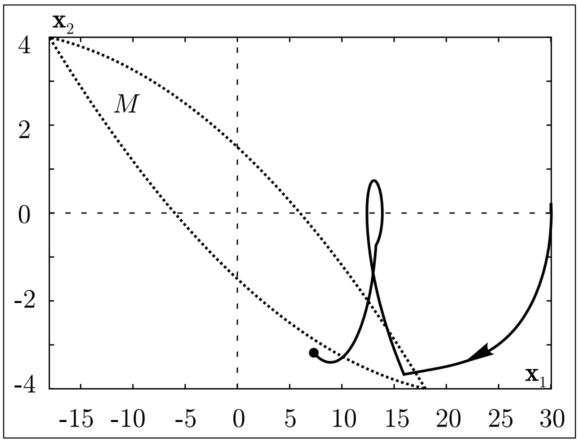


Figure 14: Example 2. Trajectory of the system (in the coordinates of lateral deviation $\mathbf{x}_1$ and lateral velocity $\mathbf{x}_2$) and its state at the terminal instant (the circle in the upper figure); realizations of the players' controls: of the first player (the middle figure) and second one (the lower figure)

Type of disturbance:
extremal
Maximum of disturbance:
15 m/sec (1.5ν)
Termination:
almost successful
Maximum of control:
0.26 (1.0μ)



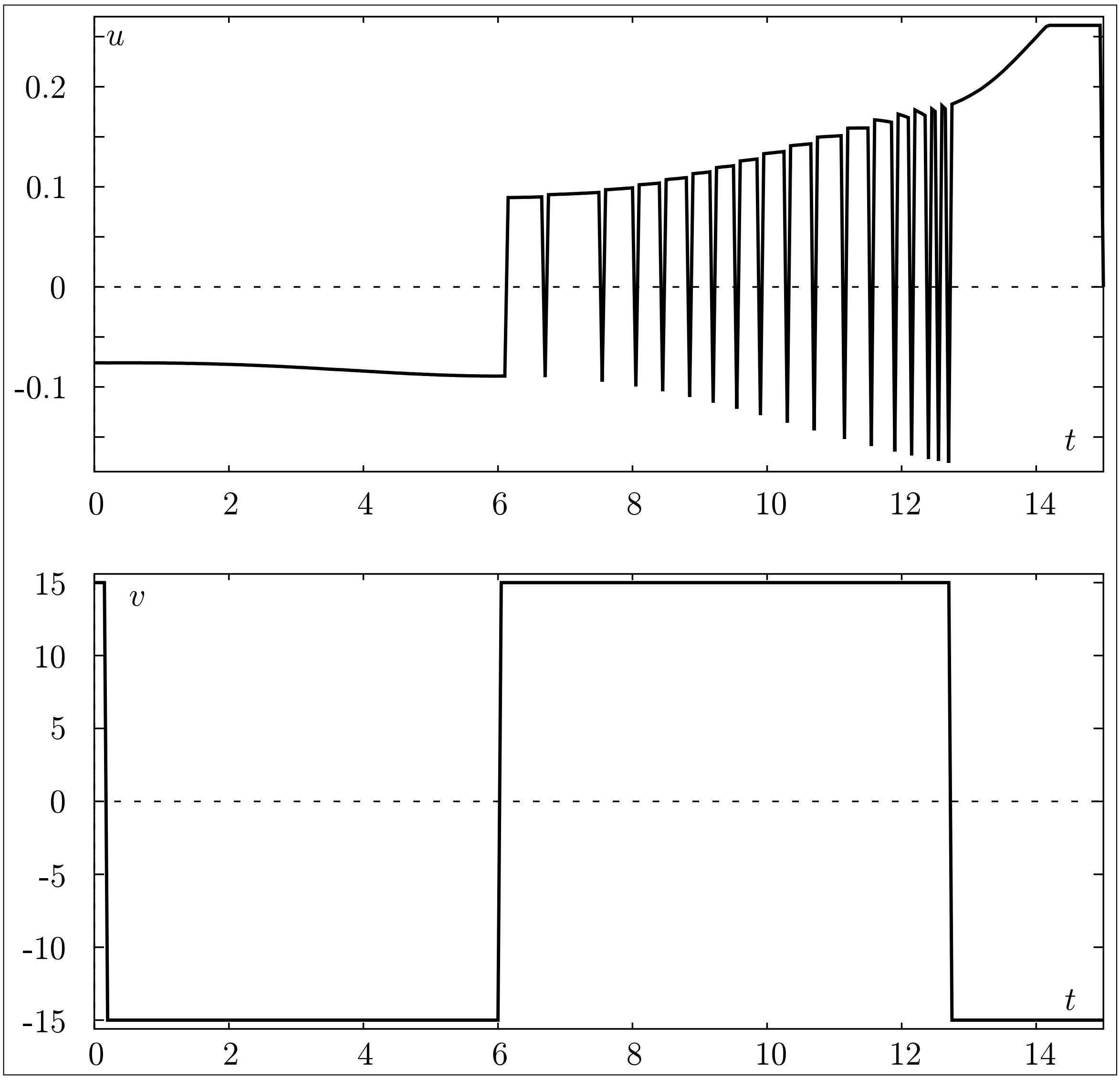


Figure 15: Example 2. Trajectory of the system (in the coordinates of lateral deviation $\mathbf{x}_1$ and lateral velocity $\mathbf{x}_2$) and its state at the terminal instant (the circle in the upper figure); realizations of the players' controls: of the first player (the middle figure) and second one (the lower figure)

Conclusion

In the modern theory of antagonistic differential games, there are well-developed areas dealing with problems, where controls of both the first player (the minimizer) and second player (the maximizer) are geometrically constrained. At the same time, practical engineer problems are typical, where the geometric constraint is given only for the first player, and it is difficult or even impossible to define any reasonable constraint for the disturbance.

To cover problems of this type, it is natural to consider a family of differential games, where the constraint for the second player's control depends on a numerical parameter. To each its value, we connect a stable set (a "tube") in the space *time* $\times$ *phase vector*. The first player guarantees keeping the motion of the system inside any such a tube. Then one constructs a family of these tubes ordered by increasing the parameter. This family can be treated as a definition of a Lyapunov function in the game space, which in its turn allows to build a feedback control of the first player and to compute the guarantee provided by the control. The constructed control is called robust, because it is designed for a wide range of disturbances.

The question is crucial whether it is necessary to store the whole family of the tubes (in some small time grid) to produce the current value of the robust control during the motion of the system. Often, it can be difficult.

In this work, a problem with linear dynamics and bounded scalar control of the first player is considered. The instant of termination assumed to be fixed. The objective of the first player is to lead the system to a given terminal set at the terminal instant (as close to its center as it is possible). If such a transfer is not guaranteed, then the first player tries to minimize the miss from the terminal set.

A method is suggested for constructing a family of stable tubes inserted to each other and ordered by the parameter defining the constraint for the second player. This family determines the guarantee of the first player. An approach for building a robust feedback control of the first player providing this guarantee is described.

To realize the corresponding algorithm it is necessary to store in the computer memory only one basic tube and a switching surface, which changes in time and determines the sign of the control.

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