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IMAGES OF CONTROL AND DIFFERENTIAL GAME PROBLEMS

Mathematical objects in the theory of optimal control and differential games may have complicated topological structure.

The presented drawings are supplied with short explanations of the dynamics and problem data.

The pictures correspond to conceptual problems considered in the works by R.Isaacs, J.Breakwell, A.Merz, J.Lewin, G.Olsder, J.Shinar, and the authors of this presentation.

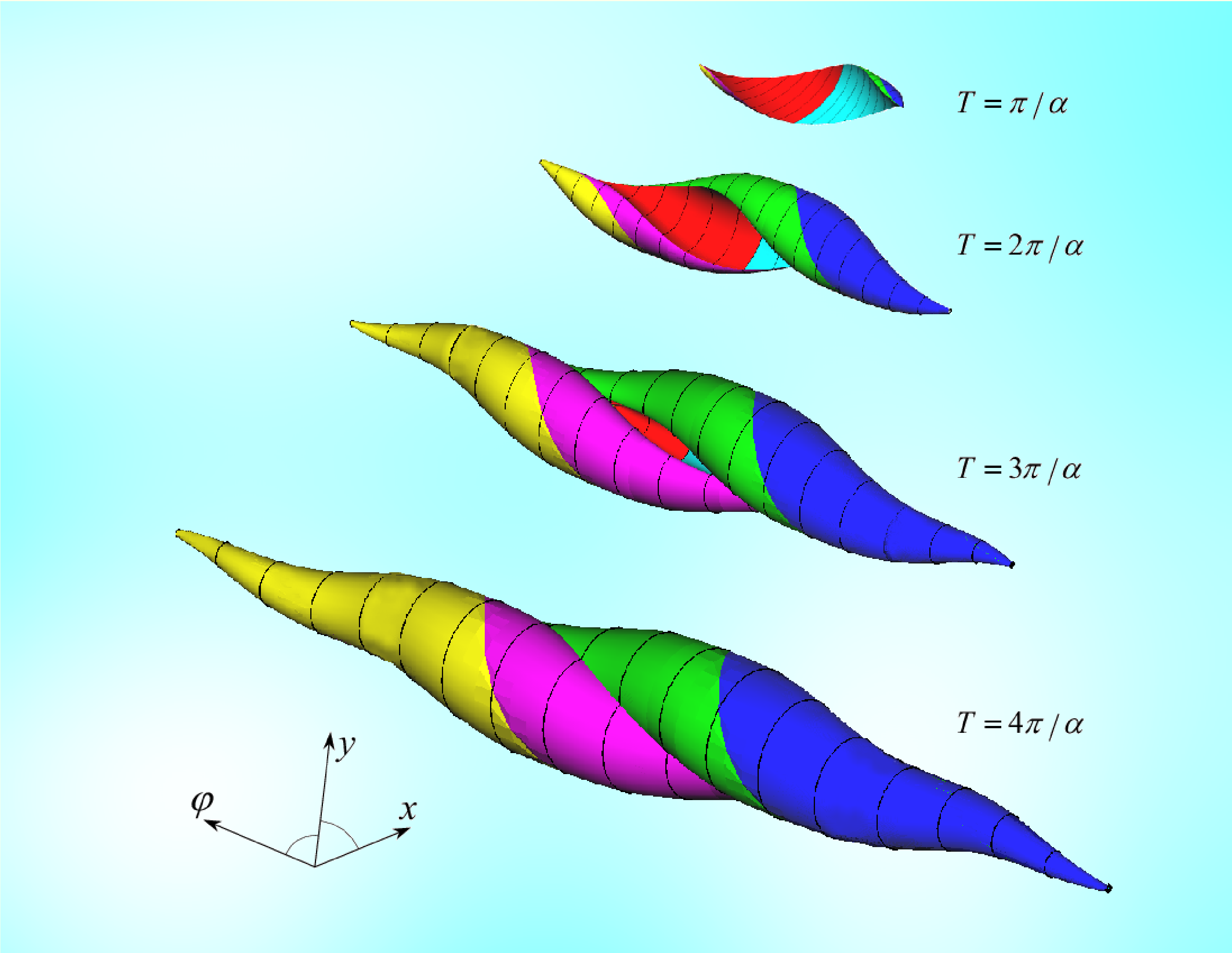
Vizualization tools developed by V.L.Averbukh, A.I.Zenkov, O.A.Pykhteev, and D.A.Yurtaev are used.

EVOLUTION OF ATTAINABILITY SET

$$\dot{x} = V\cos\varphi,$$
$$\dot{y} = V\sin\varphi,$$
$$\dot{\varphi} = \frac{k}{V}u,$$
$$|u| \leq 1,$$
$$\alpha = \frac{k}{V}.$$

Structure of controls, leading motions to the boundary:

- Blue: $1, 0, 1$
- Magenta: $-1, 0, 1$
- Green: $1, 0, -1$
- Yellow: $-1, 0, -1$
- Red: $1, -1, 1$
- Cyan: $-1, 1, -1$



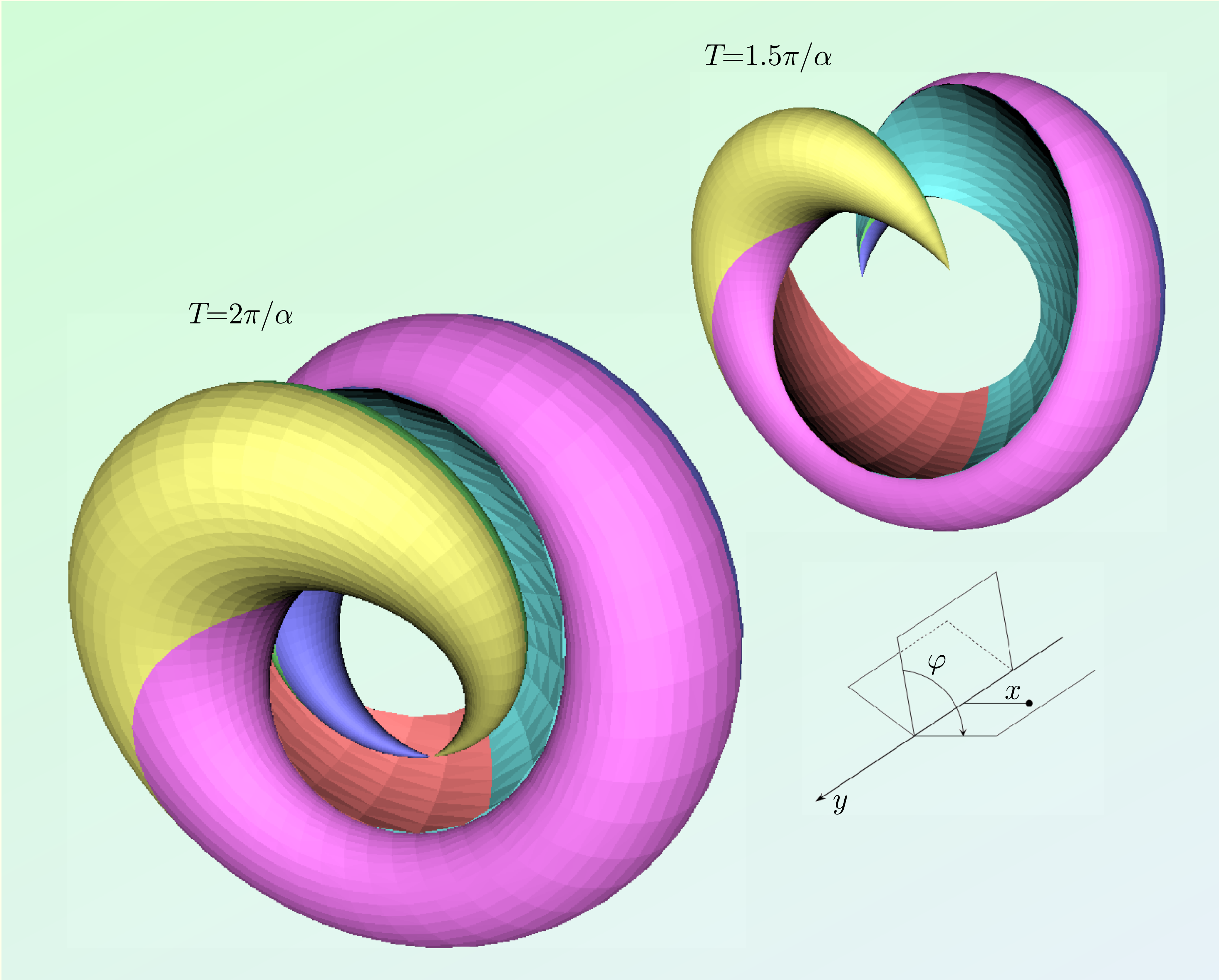
EVOLUTION OF ATTAINABILITY SET
(CYLINDRICAL SYSTEM OF COORDINATES)

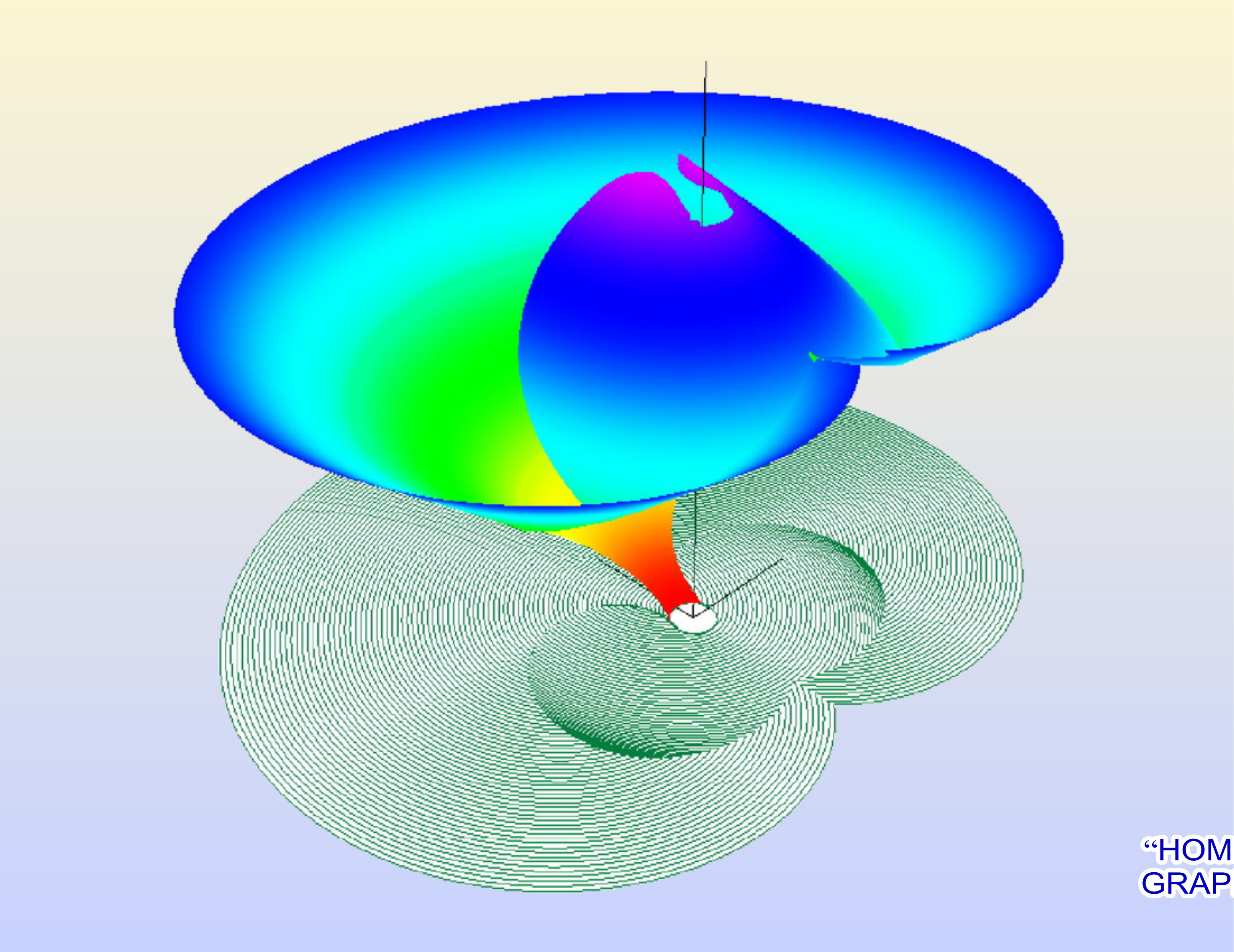
$$\dot{x} = V\cos\varphi,$$
$$\dot{y} = V\sin\varphi,$$
$$\dot{\varphi} = \frac{k}{V}u,$$

$$|u| \leq 1,$$
$$\alpha = \frac{k}{V}.$$

Structure of controls,
leading motions
to the boundary:

- $1, 0, 1$
- $-1, 0, 1$
- $1, 0, -1$
- $-1, 0, -1$
- $1, -1, 1$
- $-1, 1, -1$



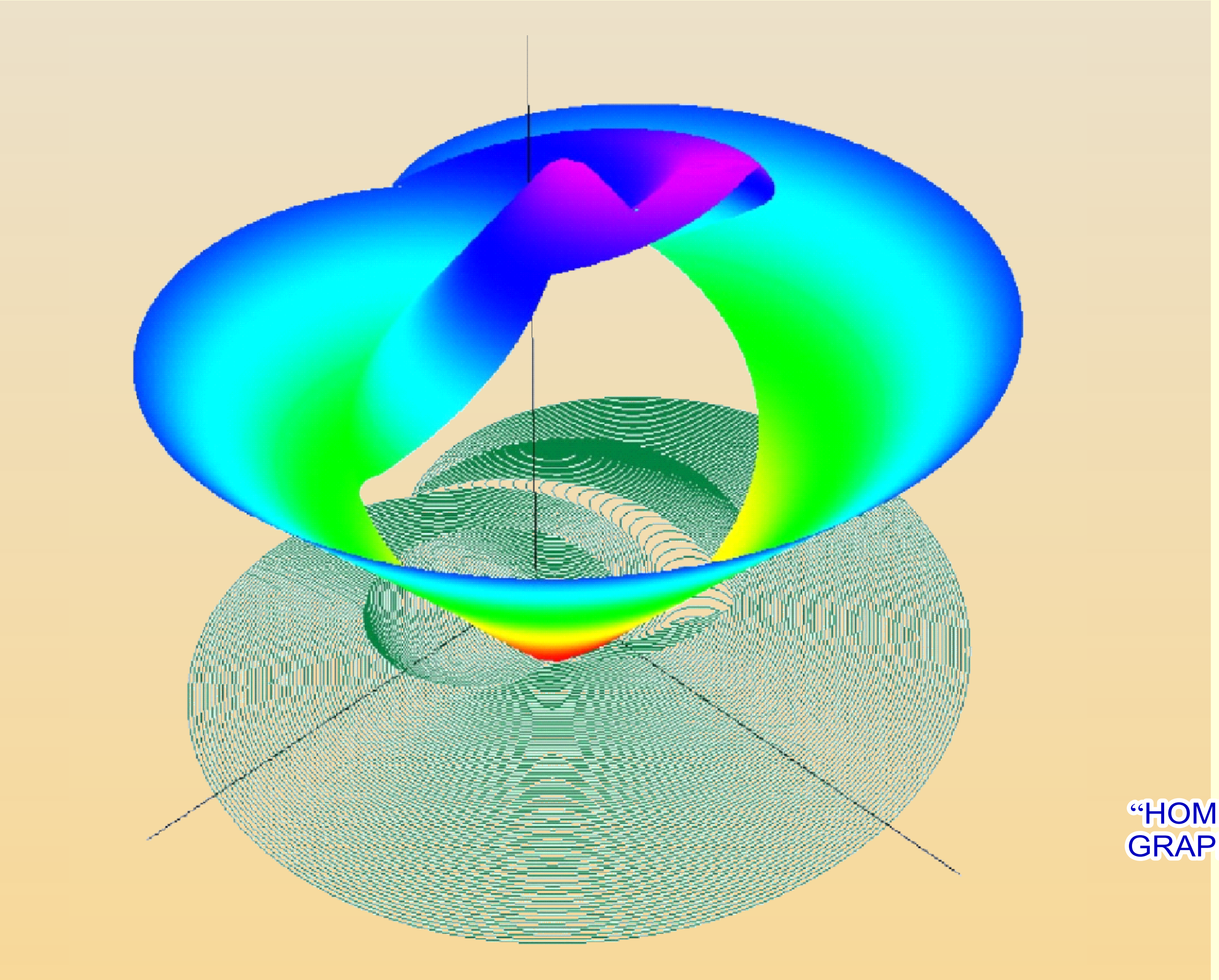


“HOMICIDAL CHAUFFEUR” GAME,
GRAPH OF THE VALUE FUNCTION

$$\dot{x} = -wyu/R + v_1,$$
$$\dot{y} = wxu/R + v_2 - w,$$

$$|u| \leq 1, \ |v| \leq \nu, \ v = (v_1, v_2).$$

Parameters of the problem: $R = 3$, $\nu = 1$, $w = 3$.
The radius of the terminal circle is $r = 1$.

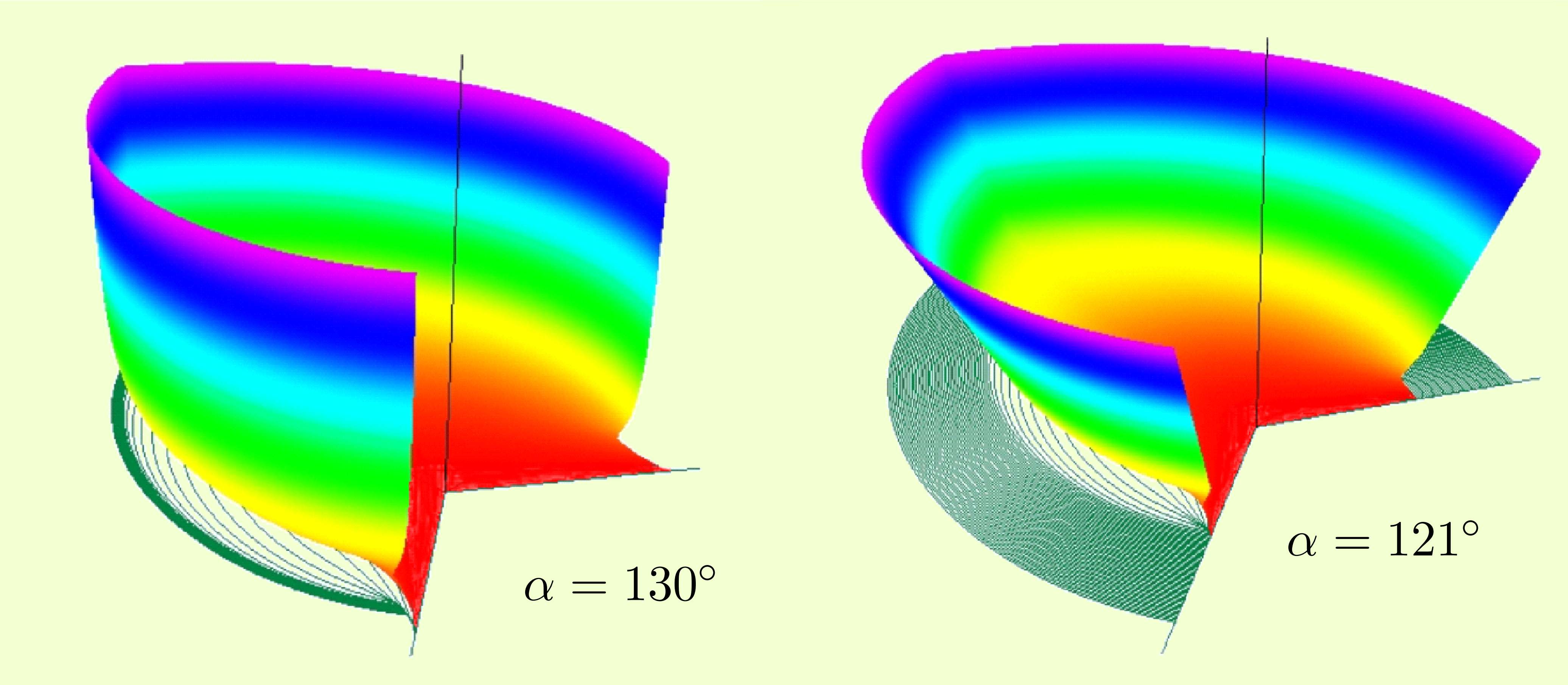


"HOMICIDAL CHAUFFEUR" GAME, GRAPH OF THE VALUE FUNCTION

$$\dot{x} = -wyu/R + v_1,$$
$$\dot{y} = wxu/R + v_2 - w,$$

$$|u| \leq 1,\ |v| \leq \nu,\ v = (v_1, v_2).$$

Parameters of the problem: $R = 0.2$, $\nu = 0.6$, $w = 2$.
Target set is a small circle in the first quadrant.



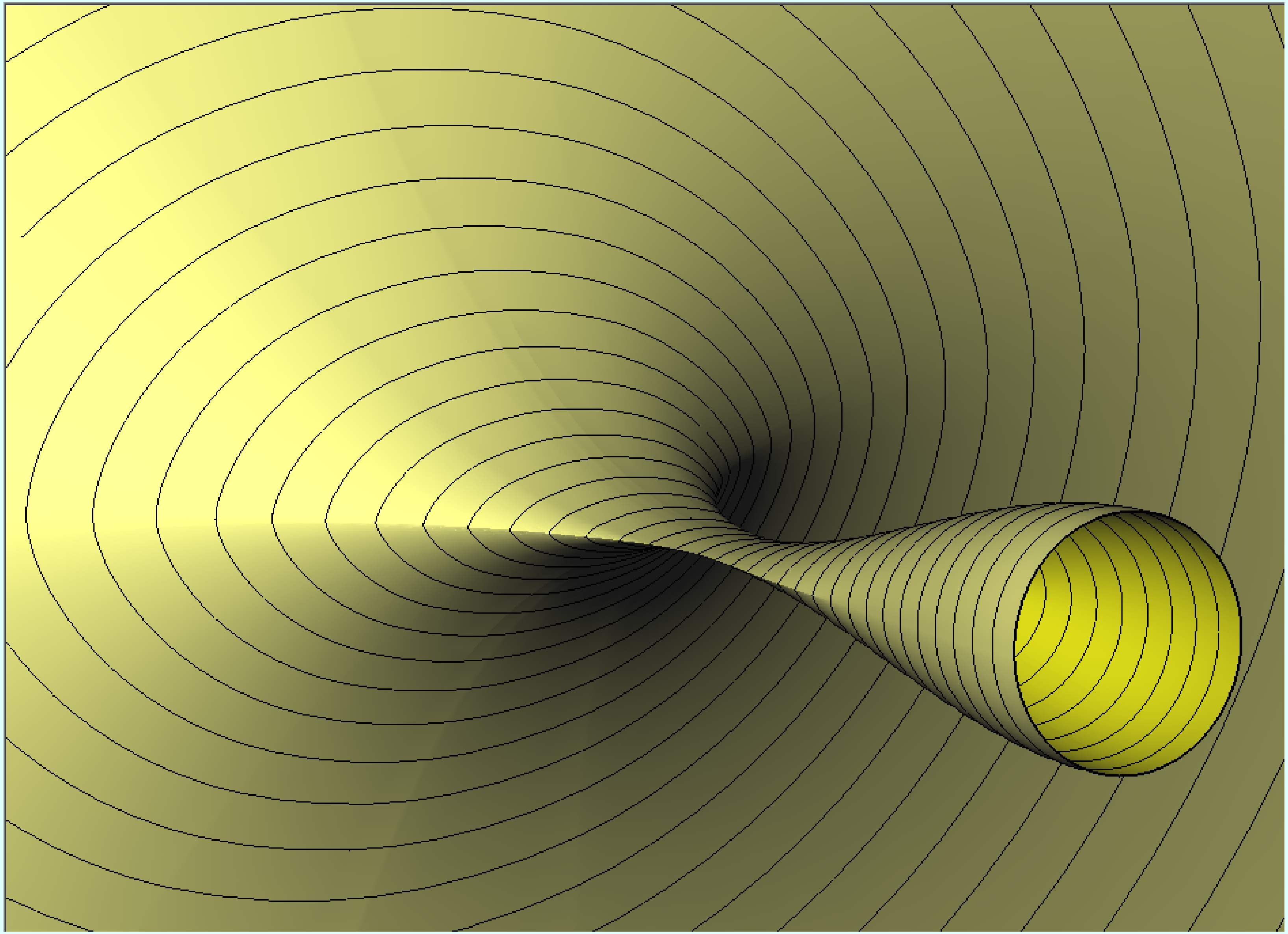
VALUE FUNCTION
IN THE SURVEILLANCE-EVASION GAME

$$\dot{x} = -wyu/R + v_1,$$
$$\dot{y} = wxu/R + v_2 - w,$$

$$|u| \le 1,\ |v| \le \nu,\ v = (v_1, v_2).$$

Control u tries to keep the system in the nonconvex detection cone, the aim of v is opposite.
Parameters of the problem: $\nu = 1$, $R = 1$, $w = 1.7$.
Quantity α is the semi-angle of the nonconvex detection cone.

“J.SHINAR'S WINEGLASS”



$$\ddot{x} = F,$$
$$\dot{F} = -(F - u)/\tau_P,$$
$$\ddot{y} = v,$$

$t \in [0, T], \quad x, y \in \mathbb{R}^2,$

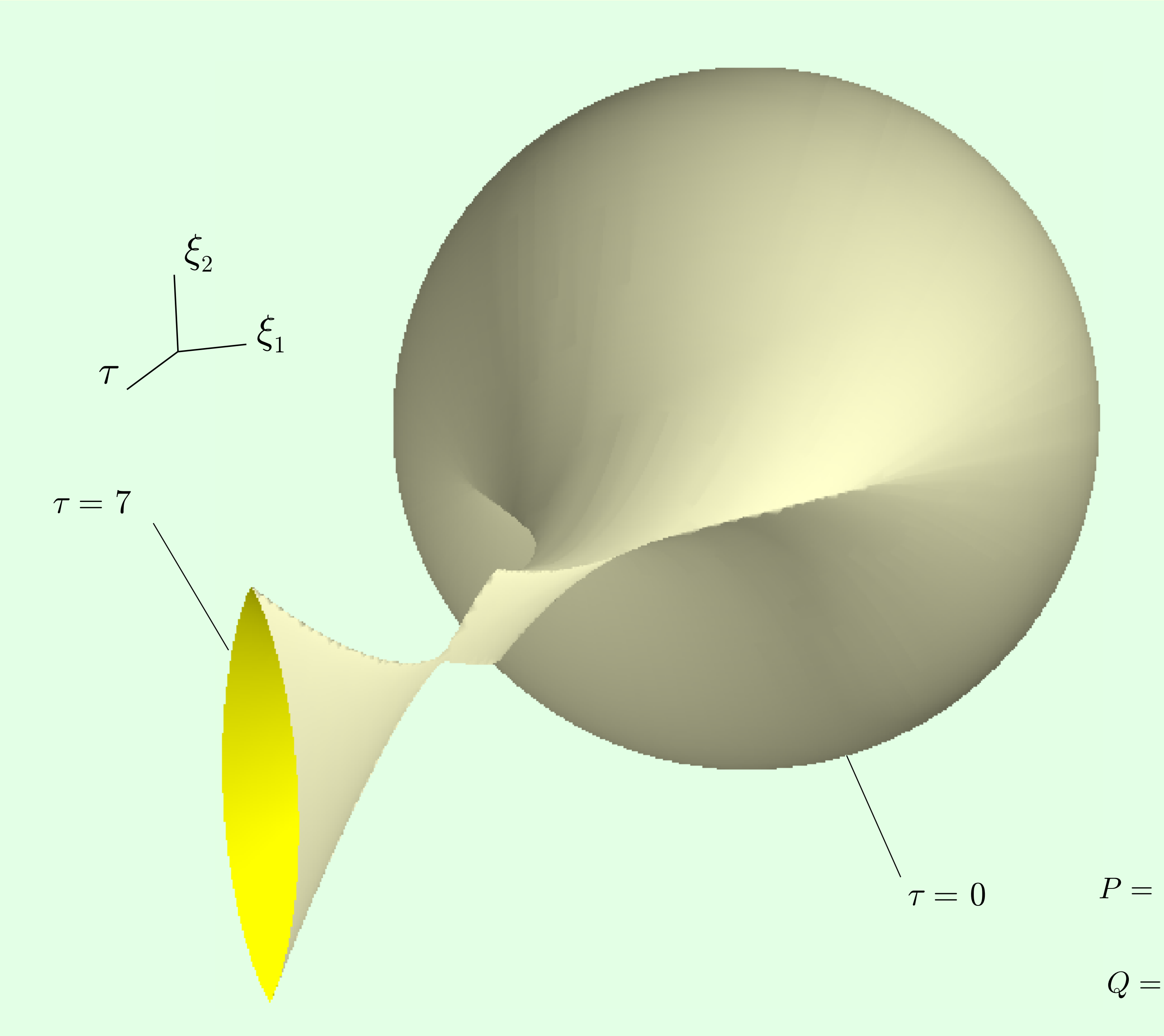
$u \in P, \quad v \in Q,$

$\varphi\left(x(T), y(T)\right) = |x(T) - y(T)|,$

$T = 2.0, \quad \tau_P = 1,$

$$P = \left\{ u \in \mathbb{R}^2 : \ \frac{u_1^2}{0.87^2} + \frac{u_2^2}{1.00^2} \le 5.0^2 \right\}, \quad Q = \left\{ v \in \mathbb{R}^2 : \ \frac{v_1^2}{0.66^2} + \frac{v_2^2}{1.00^2} \le 1.0^2 \right\}.$$

Maximal stable bridge (level set of the value function) for the payoff value 0.141.



THE "NARROW THROAT"

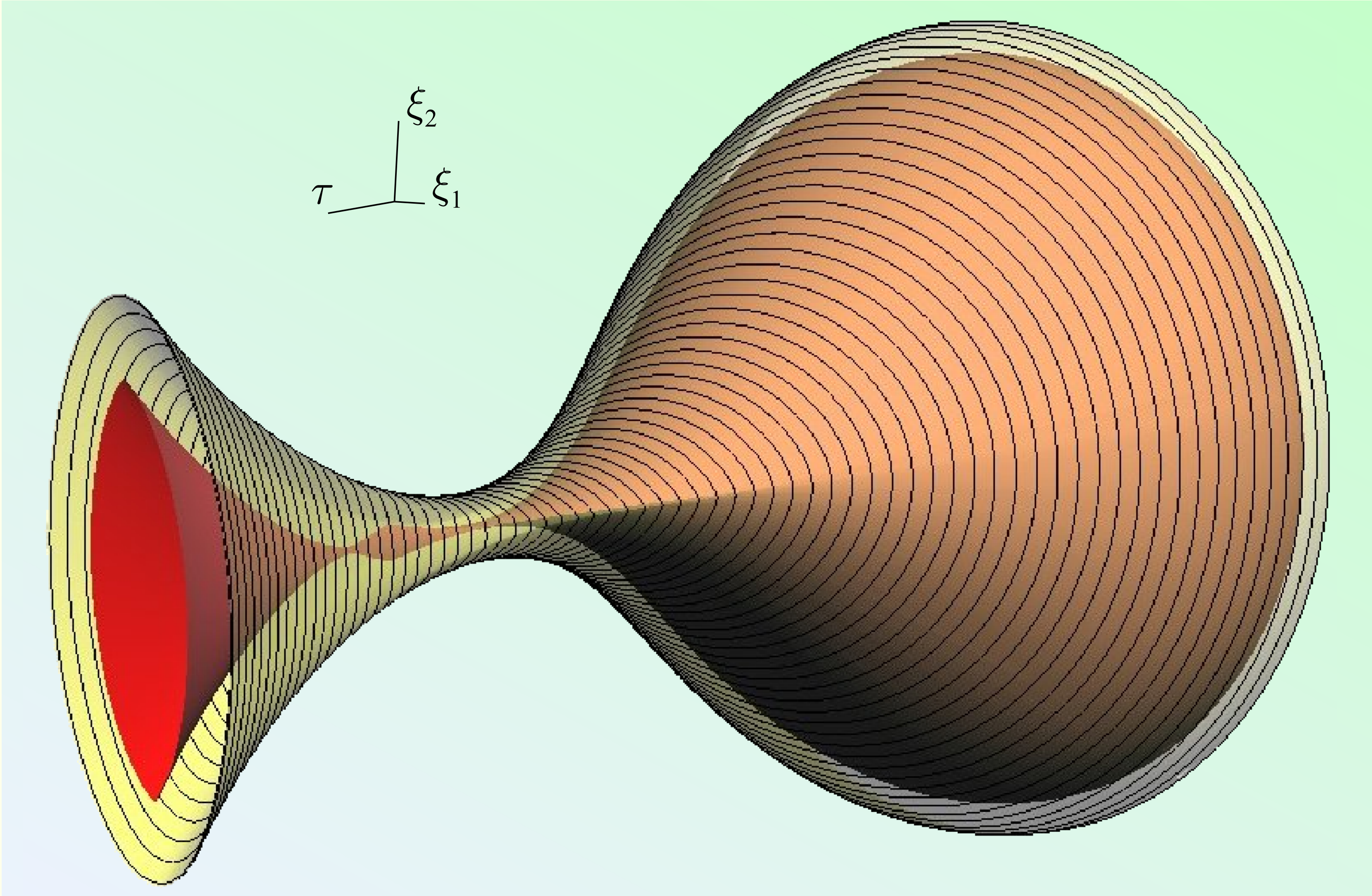
$$\ddot{x} = F,$$
$$\dot{F} = -(F - u)/\tau_P,$$
$$\ddot{y} = v,$$

$$t \in [0, T], \quad x, y \in \mathbb{R}^2,$$
$$u \in P, \quad v \in Q, \quad \tau_P = 1,$$
$$\varphi\,(x(T), y(T)) = |x(T) - y(T)|,$$

$$P = \left\{ u \in \mathbb{R}^2 : \ \frac{u_1^2}{0.67^2} + \frac{u_2^2}{1.00^2} \le 1.30^2 \right\},$$

$$Q = \left\{ v \in \mathbb{R}^2 : \ \frac{v_1^2}{0.71^2} + \frac{v_2^2}{1.00^2} \le 1.0^2 \right\}.$$

Maximal stable bridge (level set of the value function) for the value of payoff 1.546;
$\tau = T - t$ is backward time.



SMOOTHNESS
WITH
LEVEL GROWTH

$\ddot{x} = F,$
$\dot{F} = -(F - u)/\tau_P,$
$\ddot{y} = v,$

$t \in [0, T],\ \ x, y \in \mathbb{R}^2,\quad u \in P,\ \ v \in Q,$

$\varphi\,(x(T), y(T)) = |x(T) - y(T)|,$

$T = 7.0,\ \ \tau_P = 1,$

$$P = \left\{ u \in \mathbb{R}^2 :\ \frac{u_1^2}{0.67^2} + \frac{u_2^2}{1.00^2} \le 1.30^2 \right\},\ \ Q = \left\{ v \in \mathbb{R}^2 :\ \frac{v_1^2}{0.71^2} + \frac{v_2^2}{1.00^2} \le 1.0^2 \right\}.$$

Maximal stable bridges (level sets of the value function) for the values of payoff 1.546 (the red set) and 1.67 (the yellow transparent set).

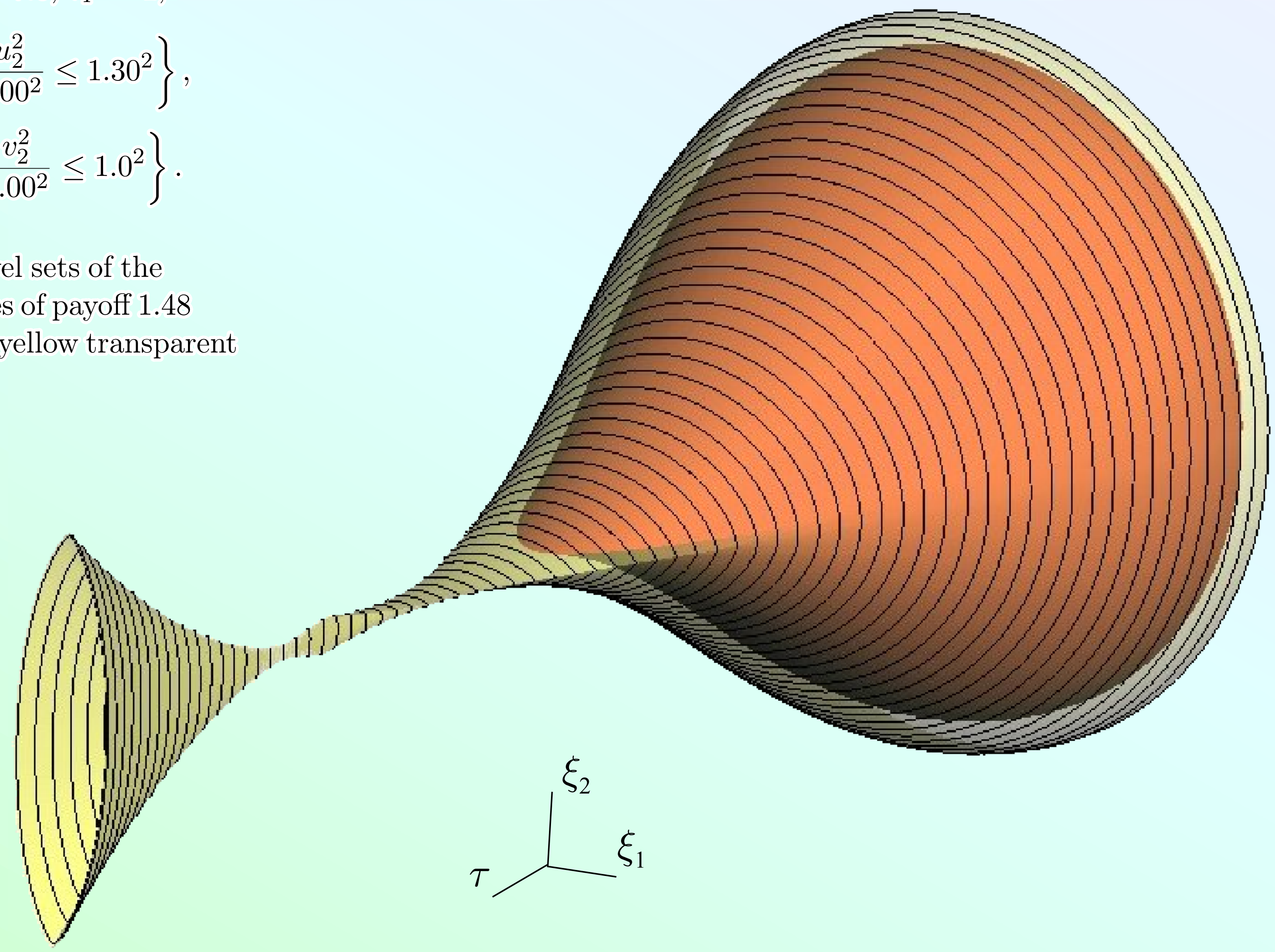
BREAK NEAR THE “THROAT”

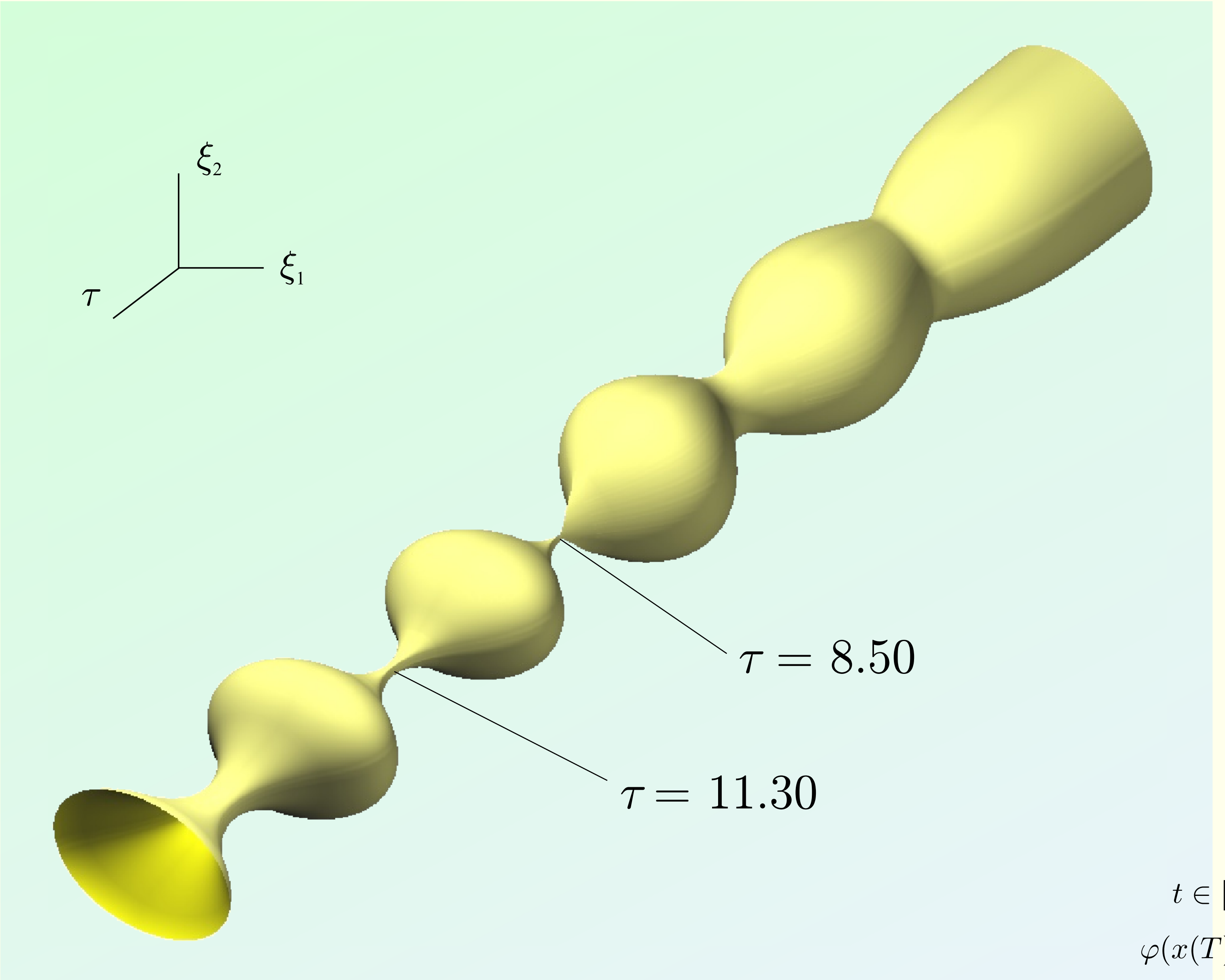
$$\ddot{x} = F, \qquad t \in [0,T],\ x, y \in \mathbb{R}^2,\ u \in P,\ v \in Q,$$
$$\dot{F} = -(F-u)/\tau_P, \qquad \varphi\,(x(T), y(T)) = |x(T) - y(T)|,$$
$$\ddot{y} = v, \qquad T = 7.0,\ \tau_P = 1,$$

$$P = \left\{ u \in \mathbb{R}^2 :\ \frac{u_1^2}{0.67^2} + \frac{u_2^2}{1.00^2} \le 1.30^2 \right\},$$

$$Q = \left\{ v \in \mathbb{R}^2 :\ \frac{v_1^2}{0.71^2} + \frac{v_2^2}{1.00^2} \le 1.0^2 \right\}.$$

Maximal stable bridges (level sets of the value function) for the values of payoff 1.48 (the red set) and 1.546 (the yellow transparent set).





TWO “NARROW THROATS”

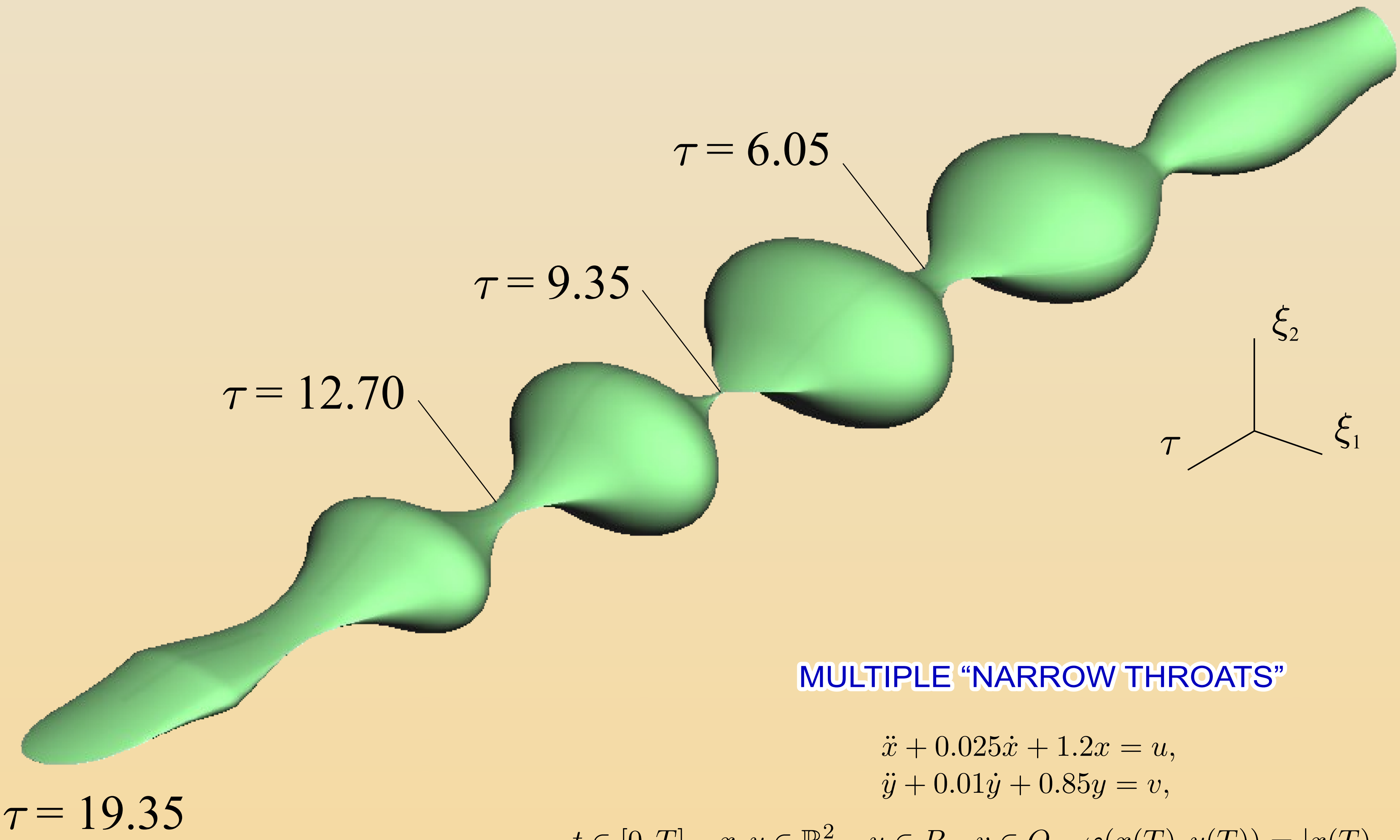
$$\ddot{x} - 0.025\dot{x} + 1.3x = u,$$
$$\ddot{y} + y = v,$$

$$t \in [0, T],\ \ x, y \in \mathbb{R}^2,\ \ u \in P,\ \ v \in Q,$$

$$\varphi(x(T), y(T)) = |x(T) - y(T)|,\ \ T = 16\,,$$

$$P = Q = \left\{ v \in \mathbb{R}^2 : \frac{v_1^2}{1.5^2} + \frac{v_2^2}{1.05^2} \leq 1 \right\}.$$

Maximal stable bridge (level set of the value function) for the value of payoff 1.2; $\tau = T - t$ is the backward time.



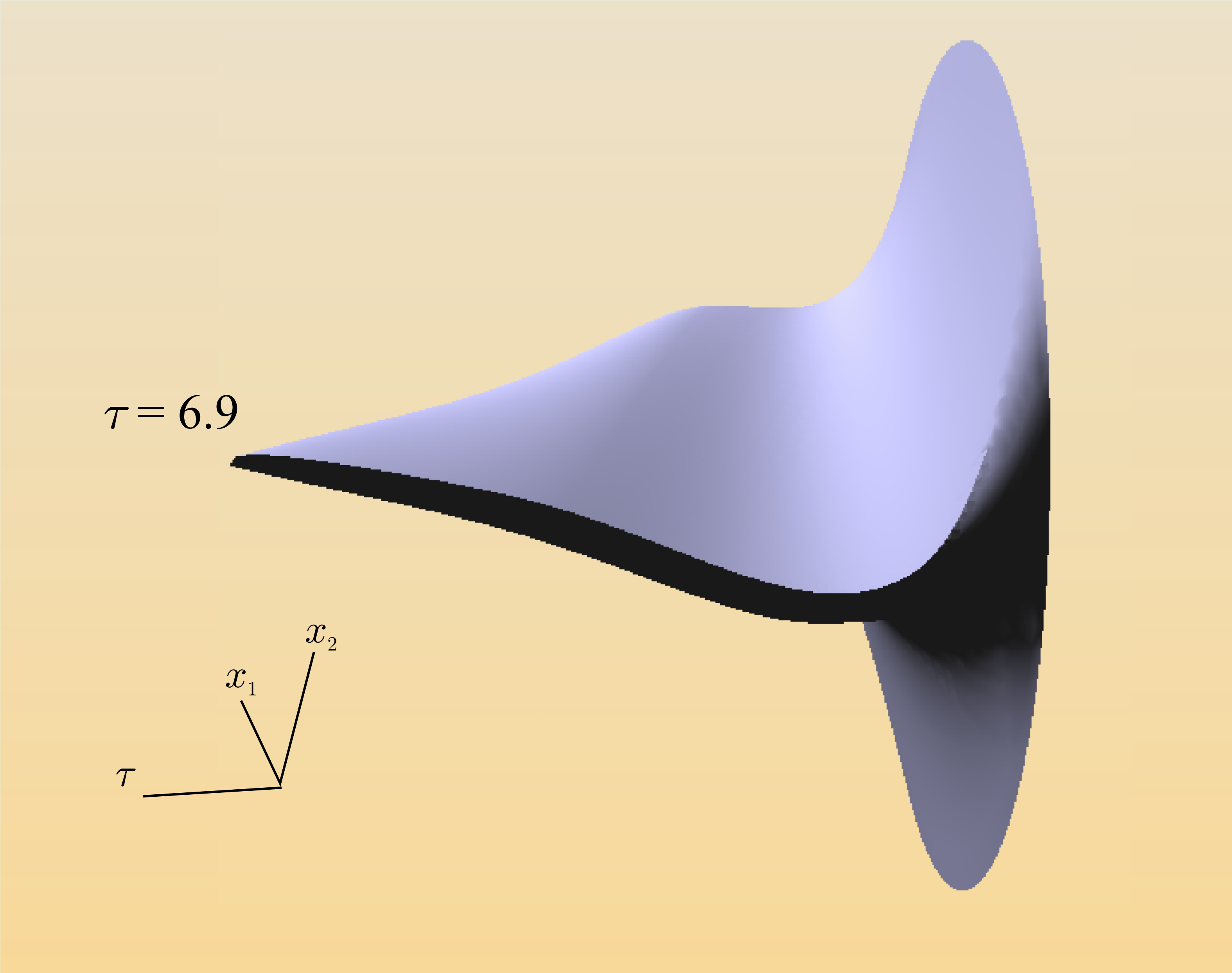
MULTIPLE "NARROW THROATS"

$$\ddot{x} + 0.025\dot{x} + 1.2x = u,$$
$$\ddot{y} + 0.01\dot{y} + 0.85y = v,$$

$$t \in [0, T], \quad x, y \in \mathbb{R}^2, \quad u \in P, \quad v \in Q, \quad \varphi(x(T), y(T)) = |x(T) - y(T)|,$$

$$P = \left\{ u \in \mathbb{R}^2 : \frac{u_1^2}{2.0^2} + \frac{u_2^2}{1.3^2} \le 1 \right\}, \quad Q = \left\{ v \in \mathbb{R}^2 : \frac{v_1^2}{1.5^2} + \frac{v_2^2}{1.05^2} \le 1 \right\}.$$

Maximal stable bridge (level set of the value function) for the value of payoff 0.397; $\tau = T - t$ is the backward time.



"DRAWING-PIN"

$$\dot{x}_1 = x_1 + 2x_2 + v,$$
$$\dot{x}_2 = x_2 + u,$$

$$|u| \leq 1, \ \ |v| \leq 0.9, \ \ \varphi(x_1(T), x_2(T)) = x_1^2(T) + x_2^2(T).$$

Maximal stable bridge (level set of the value function) for the value of payoff 7.0; $\tau = T - t$ is the backward time.