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Control design in problems with an unknown level of dynamic disturbance☆

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Abstract

Linear differential games for two players with a fixed termination time are considered. The objective of player 1 is to bring the motion of the system into an assigned terminal set or fairly close to it at the termination time. Player 2 (the disturbance) opposes this. The control of player 1 is scalar and its absolute value is subject to a constraint. One special feature of the formulation is that no constraint on the control of player 2 is specified *a priori*. A method of designing the feedback control of player 1 that works satisfactorily over a broad range of disturbance levels and corresponds to a small magnitude of the control input at a low level of disturbance is proposed and verified. A numerical program is written for the case of small dimensionality of the phase variable. The results of the simulation of a system that describes a conflict-controlled pendulum are presented.

1. Introduction

In the theory of antagonistic differential games,[1–4] formulations that specify geometric constraints on both the active control input and the disturbance input are common. At the same time, assignment of a disturbance constraint is not natural in most cases.

A feedback control that ensures good quality by expending a control input of "small" magnitude at a "low" disturbance level that is not known *a priori* is called *robust*. As the disturbance level increases, an increase in the magnitude of the control input that guarantees good quality is allowed. This meaning of the term "robust control" is consistent with the literature (see, for example, Ref. 5).

The design of a linear robust control for H^∞ problems based on the theory of differential games with a linear-quadratic payoff functional is known.[6] Linear robust regulators have been investigated in L^1 optimization problems.[7–9]

This paper examines problems with a fixed termination time and an assigned geometric constraint on the control of player 1. Player 1 is interested in bringing the controlled system into the terminal set at the termination time, and player 2 attempts to prevent this. The concept of a robust control is refined in concise terms as follows:

1) if the control of player 2 is "weak", player 1 should bring the system into the terminal set (preferably, as close to its centre as possible), and, in addition, the control implemented by player 1 should also be "weak";

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2) if the control of player 2 is "stronger", player 1 should still ensure achievement of the game's objective by using his "stronger" control for this purpose;
3) if the control of player 2 is "very strong" and player 1, acting within his constraint, cannot guarantee that the system will be brought into the terminal set, a deviation from it is permissible, but this deviation must be as small as possible.

A similar concept of robustness was previously used in Ref. 10.

The methods of antagonistic differential games with geometric constraints on the controls of both players are applicable to the design of a robust control. The basic idea is as follows. Consider a family of differential games in which the constraint on the control of player 2 depends on a numerical parameter. Each value of the parameter is also associated with a constraint on the control of player 1 and a stable tube (a stable bridge in the terminology adopted in Refs. 2,4) in *time* × *phase vector* space. The family of tubes obtained is assumed to be ordered with increasing values of the parameter. Player 1 guarantees retention of the system within each of the tubes at the corresponding disturbance level using its control, which also lies within the corresponding constraint. This family can be treated as the assignment of some Lyapunov function in the game space, which, in turn, permits the design of the feedback control for player 1 and calculation of a guarantee provided by this control.

This idea is implemented below for problems with linear dynamics and a scalar control of player 1 that has a constraint on its absolute value. A method for constructing a family of nested stable tubes that are ordered with increasing values of the parameter that assigns the geometric constraint on the control of player 2 is proposed. A robust control method that relies on this system of tubes is described, and the guarantee theorem is proved.

For the numerical construction of a robust control, it is sufficient to store only one basic tube and a time-varying switching surface that determines the sign of the control input in the computer's memory.

2. Statement of the problem

Consider the linear differential game with a fixed termination time

$$\begin{aligned}&\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)u + \mathbf{C}(t)v\\&\mathbf{x} \in R^m, \quad t \in T, \quad u \in P = \{u \in R : |u| \le \mu\}, \quad v \in R^q\end{aligned} \qquad (2.1)$$

Here P is the constraint on the scalar control u of player 1, and $T = [\vartheta_0, \vartheta]$ is the duration of the game. The matrix-value functions $\mathbf{A}$ and $\mathbf{C}$ are continuous. The vector function $\mathbf{B}$ is Lipschitz-continuous in the interval T.

Player 1 is interested in bringing the system (2.1) into the terminal set M at the time ϑ, and player 2 has opposite interests. The terminal set M is assumed to be a convex compactum in the subspace $R^n \subset R^m$ of n selected components of the vector $\mathbf{x}$. We shall assume that M contains an empty neighbourhood of this subspace.

Unlike the standard differential game formulation,[1,2,4] in system (2.1) there is no constraint on the control v of player 2.

A method of designing a robust feedback control for system (2.1) is required.

3. A controlled system without a phase variable on the right-hand side

We will change to a system whose right-hand side does not have a phase variable:

$$\dot{x} = B(t)u + C(t)v, \quad x \in R^n, \quad t \in T, \quad u \in P, \quad v \in R^q. \qquad (3.1)$$

The change is made (Ref. 2, p. 160; Ref. 4, pp. 89–91) using the equalities

$$x(t) = X_{n,m}(\vartheta, t)\mathbf{x}(t), \quad B(t) = X_{n,m}(\vartheta, t)\mathbf{B}(t), \quad C(t) = X_{n,m}(\vartheta, t)\mathbf{C}(t),$$

where $X_{n,m}(\vartheta, t)$ is the n-row fundamental Cauchy matrix for the system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$, which corresponds to the subspace R^n that contains the terminal set M.

In the new problem, player 1, as before, attempts to bring system (3.1) into the terminal space M at the termination time ϑ, and player 2 tries to prevent this. The set M is a convex compactum in R^n, which includes an empty neighbourhood.

The ensuing arguments will relate to system (3.1). After obtaining a robust control method within system (3.1), we will reformulate it for system (2.1).

We use $E(t) = \{x \in R^n\colon (t, x) \in E\}$ to denote the intersection of sets $E \subset T \times R^n$ at the time $t \in T$.

Let $O(\varepsilon) = \{x \in R^n\colon |x| \le \varepsilon\}$ be a sphere of radius ε in the space R^n.

4. Stable bridges

In addition, let us consider a differential game with the terminal set $\mathcal{M}$ and the geometric constraints $\mathcal{P}$ and $\mathcal{Q}$ on the controls of the players in the time interval $T = [\vartheta_0, \vartheta]$:

$$\dot{x} = B(t)u + C(t)v, \quad x \in R^n, \quad t \in T, \quad \mathcal{M}, \quad u \in \mathcal{P}, \quad v \in \mathcal{Q}. \tag{4.1}$$

Here the matrices $B(t)$ and $C(t)$ are the same as in system (3.1). The sets $\mathcal{M}$, $\mathcal{P}$ and $\mathcal{Q}$ are assumed to be convex compacta. We shall treat them as parameters of the game.

Below we shall use $u(\cdot)$ and $v(\cdot)$ to denote measurable functions of time with values in the sets $\mathcal{P}$ and $\mathcal{Q}$, respectively. The motion of system (4.1) (and consequently of system (3.1)) that starts out from the point x_* at the time t_* under the action of the controls $u(\cdot)$ and $v(\cdot)$ will be denoted by $x(\cdot; t_*, x_*, u(\cdot), v(\cdot))$.

Following a well known approach,[2,4] we give definitions of a stable bridge and a maximum stable bridge.

Definition 1. The set $W \subset T \times R^n$ is called a *stable bridge* for system (4.1) with the fixed parameters $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{M}$ if $W(\vartheta) = \mathcal{M}$ and the following *stability* property holds: for any position $(t_*, x_*) \in W$ and any control $v(\cdot)$, there is a control $u(\cdot)$ which is such that the pair $(t, x(t)) = (t, x(t; t_*, x_*, u(\cdot), v(\cdot))$ remains in the set W at any time $t \in (t_*, \vartheta)$.

Definition 2. The maximum subset $W \subset T \times R^n$, $W(\vartheta) = \mathcal{M}$ that has the stability property is called the maximum stable bridge.

The maximum stable bridge is a closed set.[2,4] Its t sections are convex (Ref. 4, p. 87) by virtue of the linearity of system (4.1) and the convexity of $\mathcal{M}$.

5. Robust feedback control

5.1. Guarantee theorem

We will describe the design of the robust control for systems (3.1) and (2.1). Instead of the phrase “feedback control,” we shall sometimes use the word “strategy.”

1°. We choose the set $Q_{\max} \subset R^q$, i.e., the “maximum” constraint that can be imposed on the control of player 2, which player 1 “consents” to consider reasonable for the problem of bringing the system (3.1) into the set M. The set $Q_{\max}$ must include the null of its own space. We will use W to denote the maximum stable bridge for system (4.1) that corresponds to the parameters $\mathcal{P} = P$, $\mathcal{Q} = Q_{\max}$ and $\mathcal{M} = M$.

We will stipulate that $Q_{\max}$ is chosen so that the following relation holds for some $\varepsilon > 0$:

$$O(\varepsilon) \subset W(t), \quad t \in T. \tag{5.1}$$

The number ε will be assumed to be fixed.

2°. In addition, we introduce the tube $\hat{W} \subset T \times R^n$, each cross section $\hat{W}(t)$ of which is the attainability set of system (4.1) at the time t for the initial set $O(\varepsilon)$ at the time ϑ_0. In constructing $\hat{W}$ we assume that player 1 is absent ($u \equiv 0$) and that the constraint on the control v of player 2 is $Q_{\max}$. Clearly, $\hat{W}$ is the maximum stable bridge for system (4.1) when $\mathcal{P} = \{0\}$, $\mathcal{Q} = Q_{\max}$, $\mathcal{M} = \hat{W}(\vartheta)$. We have

$$O(\varepsilon) \subset \hat{W}(t), \quad t \in T. \tag{5.2}$$

3°. Consider the family of sets W_k, whose cross sections are defined by the formula

$$W_k(t) = \begin{cases} kW(t), & 0 \le k \le 1 \\ W(t) + (k-1)\hat{W}(t), & k > 1. \end{cases}$$

The sets $W_k(t)$ are closed and convex. For any numbers $0 \le k_1 < k_2 < 1 < k_3 < k_4$, the following strict inclusions hold by virtue of inclusions (5.1) and (5.2):

$$W_{k_1} \subset W_{k_2} \subset W \subset W_{k_3} \subset W_{k_4}.$$

It can be shown[1] that the set W_k, where $0 \le k \le 1$, is the maximum stable bridge for system (4.1) that corresponds to the constraint kP on the control of player 1, to the constraint $kQ_{\max}$ on the control of player 2 and to the terminal set kM. In the case when $k > 1$, the set W_k is a stable bridge (but not necessarily the maximum stable bridge) when

$$\mathcal{P} = P, \mathcal{Q} = kQ_{\max}, \mathcal{M} = M + (k-1)\hat{W}(\vartheta).$$

Thus, as the index k increases, we obtain an expanding system of stable bridges, in which each successive bridge corresponds to a greater constraint imposed on the control of player 2.

4°. We define the function V: $T \times R^n \to R$ as follows:

$$V(t, x) = \min\{k \ge 0 : (t, x) \in W_k\}.$$

The function $x \mapsto V(t, x)$ for any $t \in T$ is quasiconvex, i.e., its level sets $\{x \in R^n\colon V(t, x) \le k\} = W_k(t)$ are convex. We also note that by virtue of relations (5.1) and (5.2), it satisfies the Lipschitz condition with a constant $\lambda = 1/\varepsilon$.

Let $\mathcal{A}(t, x)$ denote a straight line in R^n parallel to the vector $B(t)$ and which passes through the point x:

$$\mathcal{A}(t, x) = \{z \in R^n : z = x + \alpha B(t), \alpha \in R\}.$$

Let

$$\mathcal{V}(t, x) = \min_{z \in \mathcal{A}(t, x)} V(t, z).$$

A minimum is reached, since the function $x \mapsto V(t, x)$ is continuous and goes to infinity as $|x| \to \infty$. Since this function is quasiconvex, a minimum is obtained either at a point or on a segment.

If $B(t) = 0$, we assume that $\mathcal{V}(t, x) \equiv V(t, x)$.

5°. For any $t \in T$, we set

$$\Pi(t) = \{x \in R^n : V(t, x) = \mathcal{V}(t, x)\}.$$

We also introduce the sets

$$\begin{aligned} \Pi_-(t) &= \{x \in R^n : x + \alpha B(t) \notin \Pi(t), \forall \alpha \ge 0\} \\ \Pi_+(t) &= \{x \in R^n : x + \alpha B(t) \notin \Pi(t), \forall \alpha \le 0\}. \end{aligned} \tag{5.3}$$

The set $\Pi(t)$ is closed, and the sets $\Pi_-(t)$ and $\Pi_+(t)$ are on different sides of it. These three sets divide the space R^n into three parts.

6°. We define the function

$$\bar{V}(t, x) = \min\{V(t, x), 1\}$$

[1] Ganebnyi SA, Kumkov SS, Patsko VS, Pyatko SG. *Robust Control in Game Problems with Linear Dynamics: Preprint*. Institute of Mathematics and Mechanics, Ekaterinburg: Ural Branch of the Russian Academy of Sciences; 2005.

and the multivalued function

$$\mathbf{U}^0(t,x) = \begin{cases} -\bar{V}(t,x)\mu, & x \in \Pi_-(t) \\ \bar{V}(t,x)\mu, & x \in \Pi_+(t) \\ [-\bar{V}(t,x)\mu, \bar{V}(t,x)\mu], & x \in \Pi(t). \end{cases} \tag{5.4}$$

As the strategy U of player 1, we take an arbitrary single-valued sample from the multivalued function $\mathbf{U}^0$:

$$U(t,x) \in \mathbf{U}^0(t,x) \quad (t,x) \in T \times R^n.$$

Thus, the control $U(t, x)$ "switches" on the set $\Pi(t)$. For simplicity, we will call $\Pi(t)$ the *switching surface* corresponding to the time t.

7°. A theorem specifying the guarantee provided to player 1 by the arbitrary single-valued sample $U \in \mathbf{U}^0$ is formulated below. To describe the influence of small inaccuracies in constructing the switching surface $\Pi(t)$, we shall consider the sets $\Pi^r(t) \supset \Pi(t)$, $r \geq 0$ and define the multivalued function $\mathbf{U}^r$ such that $\mathbf{U}^0(t, x) \subset \mathbf{U}^r(t, x)$.

If $B(t) \neq 0$, we set

$$\Pi^r(t) = \left\{ x \in R^n : x = z + \alpha \frac{B(t)}{|B(t)|}, z \in \Pi(t), |\alpha| \leq r \right\}.$$

The set $\Pi^r(t)$ is the geometric r-expansion of $\Pi(t)$, obtained using the vector $B(t)$. The following inequality holds for $x \in \Pi^r(t)$

$$V(t,x) \leq \mathcal{V}(t,x) + \lambda r. \tag{5.5}$$

If $B(t) = 0$, we stipulate that $\Pi^r(t) = \Pi(t) = R^n$.

We introduce the sets $\Pi^r_-(t)$, $\Pi^r_+(t)$, which are distinguished from the sets (5.3) by the replacement of $\Pi(t)$ by $\Pi^r(t)$, and we define the multivalued function $\mathbf{U}^r(t, x)$, which is distinguished from the function (5.4) by the replacement of $\Pi(t)$ by $\Pi^r(t)$ and of $\Pi_\pm(t)$ by $\Pi^r_\pm(t)$.

Let player 1 use some single-valued strategy $U \in \mathbf{U}^r$ in a control sampling scheme[2,4] with a spacing Δ. The control implemented in each interval of the sampling scheme is constant. Selecting the initial position (t_0, x_0) and the programmed control $v(\cdot)$ of player 2, we obtain the motion $t \mapsto x(t)$ of system (3.1).

Let β be the Lipschitz constant of $B(t)$, and let $\sigma = \max_{t \in T} |B(t)|$

The following guarantee theorem holds.

Theorem. Let $r \geq 0$, and let U be a strategy of player 1 such that

$$U(t,x) \in \mathbf{U}^r(t,x) \quad (t,x) \in T \times R^n.$$

We select arbitrary $t_0 \in T$, $x_0 \in R^n$ and $\Delta > 0$. We fix the number $k \geq 0$ and assume that the control of player 2 in the interval $[t_0, \vartheta]$ is restricted to the set $kQ_{\max}$. We introduce the notation

$$s = \max\{k, V(t_0, x_0)\}.$$

Let $x^*(\cdot)$ be the motion of system (3.1) from the point x_0 at the time t_0 under the action of the strategy U in a control sampling scheme with a spacing Δ and some control $v(t) \in kQ_{\max}$, $t \in [t_0, \vartheta]$. Then the control implemented by player 1 $u(t) = U(t, x^*(t))$ obeys the inclusion

$$u(t) \in \min\{s + \Lambda(t, t_0, \Delta, r), 1\} \cdot P, \quad t \in [t_0, \vartheta]. \tag{5.6}$$

The value $V(t, x^*(t))$ of the function V satisfies the inequality

$$V(t, x^*(t)) \leq s + \Lambda(t, t_0, \Delta, r), \quad t \in [t_0, \vartheta]. \tag{5.7}$$

Here

$$\Lambda(t, t_0, \Delta, r) = 2\lambda\sqrt{(2\sigma\mu\Delta + r)\beta\mu}(t - t_0) + 4\lambda\sigma\mu\Delta + \lambda r.$$

8°. Returning to system (2.1), we introduce the multivalued function

$$\tilde{\mathbf{U}}^0(t, \mathbf{x}) = \mathbf{U}^0(t, X_{n,m}(\vartheta, t)\mathbf{x}).$$

Any single-valued sample $\tilde{U}(t, \mathbf{x})$from it provides a robust control for system (2.1). The theorem allows of small errors in the numerical construction of the switching surface $\Pi(t)$.

We shall describe a procedure for constructing the robust control $\tilde{U}$. This procedure utilizes the fact that the stable sets W_k are ordered and relies on the construction of the time-dependent switching surface $\Pi(t)$. A proof of the theorem is presented in Section. 7. It largely follows that in Refs. 11,12 in which switching surfaces were used to construct the optimal feedback control of a minimizing player in linear antagonistic differential games with a fixed termination time and geometric constraints applied to the controls of both players.

For the numerical construction of the control $\tilde{U}$, it is necessary to preserve the cross sections $W(t)$ of the bridge W and the switching surface $\Pi(t)$ on some time grid $\{t_i\}$. Having the position $\mathbf{x}(t)$ of the system (2.1) at the time t, we can recalculate it in the coordinates of system (3.1) using the formula $x(t) = X_{n,m}(\vartheta, t)\mathbf{x}(t)$. The sign of $\tilde{U}(t, \mathbf{x}(t)) = U(t, x(t))$ is determined by the position of the point $x(t)$ with respect to the switching surface $\Pi(t)$. By analyzing the position of the point $x(t)$ relative to the boundary of the cross section $W(t)$ of the bridge W, we can calculate the absolute value of the control $|\tilde{U}(t, \mathbf{x}(t))|$. The similarity of the sets $W_k(t)$ for $k \leq 1$ is utilized here.

Remark. We do not claim that the proposed method for constructing a robust control implements any optimality criterion. Note also that there is some arbitrariness in the choice of the set $Q_{\max}$. After fixing $Q_{\max}$, our concern is to construct the feedback control for a dynamic disturbance that is subjected to a constraint of an assigned form, but of unknown level.

6. Construction of a robust control in the case of a two-dimensional system of type (3.1)

If the set M in the original controllable system (2.1) is defined by only two coordinates of the phase vector $\mathbf{x}$ (i.e., $n = 2$), on changing to system (3.1) we obtain a dimensionality of the phase vector x equal to 2. In this case, the sets $W(t)$ and $\hat{W}(t)$ are sets in a plane. We shall call the set $\Pi(t)$ the *switching line* for the time t.

The control method described above now appears as follows. Selection of the control input consists of two parts: selection of the sign and selection of the absolute value. As the time t we have the family of nested sets $W_k(t)$ in a plane (see Fig. 1, where $k_1 < 1 < k_2$). In each of these sets, we find points for which a straight line parallel to the vector $B(t)$ is a tangent to the set. By joining these points, we obtain the switching line $\Pi(t)$ that specifies the sign of the control. The absolute value of the control input is assigned by the formula

$$|U(t, x)| = \begin{cases} (l/L)\mu, & x \in W(t) \\ \mu, & x \notin W(t). \end{cases}$$

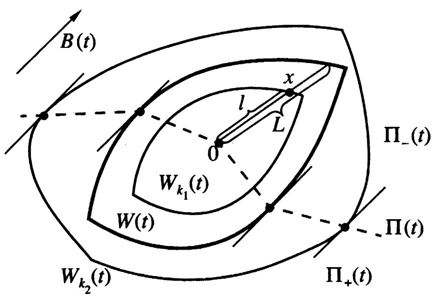

Fig. 1.

Here $l = |x|$, and L is the length of the segment that passes through the point x and joins the origin of coordinates to the boundary of $W(t)$.

Effective algorithms and programs for constructing the maximum stable bridges in linear antagonistic differential games have been developed for $n = 2$.[13,14] These programs are used to construct the cross sections $W(t)$ and $\hat{W}(t)$ of the maximum stable bridges W and $\hat{W}$.

The robust feedback control is constructed on the basis of the cross sections $W(t_i)$ of the set W and the switching line $\Pi(t_i)$, which are stored on some time grid $\{t_i\}$. In numerical constructions the cross sections $W(t_i)$ are convex polygons, and they are stored in a suitable form. Each switching line $\Pi(t_i)$ is a broken line with four linear segments. To store each such broken line in the computer memory, five of its vertices (one of which is the origin of coordinates) must be recorded.

7. Proof of the theorem

To write the variation of the function V along the motion $x(\cdot)$ of system (3.1) in some interval $[t_1, t_2]$, we introduce the notation

$$\mathrm{Var}(V, [t_1, t_2]) = V(t_2, x(t_2)) - V(t_1, x(t_1)). \tag{7.1}$$

Auxiliary assertions. For the compact sets X, Y in R^n, let

$$d(X, Y) = \max_{x \in X} \min_{y \in Y} |x - y|$$

be the Hausdorff deviation of the set X from Y.

Let $G_k(t; \bar{t}, \bar{x})$ for $k \geq 0$ denote the attainability set of system (3.1) at the time $t \geq \bar{t}$ from the state $\bar{x}$ at the time $\bar{t}$ using all the possible measurable programmed controls $u(t) \in P$, $v(t) \in kQ_{\max}$ in the interval $[\bar{t}, t]$. We set

$$\mathbf{G}_k(t; \bar{t}, \bar{x}) = G_k(t; \bar{t}, \bar{x}) + O(2(t - \bar{t})\sigma\mu).$$

Lemma 1. Let

$$k \geq 0, \quad \bar{t} \in T, \quad \bar{x} \notin \mathrm{int}W_k(\bar{t}), \quad \delta > 0, \quad \bar{t} + \delta \leq \vartheta. \tag{7.2}$$

Let $x^*(\cdot)$ be the motion of system (3.1) under the action of the programmed controls $u(t) \in P$, $v(t) \in kQ_{\max}$ at the time $\bar{t}$ from the point $\bar{x}$. Then the following estimate holds

$$\mathcal{V}(\bar{t} + \delta, x^*(\bar{t} + \delta)) \leq V(\bar{t}, \bar{x}) + \lambda\beta\mu\delta^2. \tag{7.3}$$

Proof. We introduce the notation $\hat{t} = \bar{t} + \delta$. Since $\bar{x} \notin \mathrm{int}W_k(\bar{t})$, we have $c_* = V(\bar{t}, \bar{x}) \geq k$. Therefore, using the properties of the function V, for the control $v(\cdot)$ we can find a control $u'(\cdot)$ such that $u'(t) \in \bar{c}\ \ P \subset P$, where $\bar{c} = \bar{V}(\bar{t}, \bar{x})$, and the motion $x'(\cdot)$ that starts out at the time $\bar{t}$ from the point $\bar{x}$ and is generated by the controls $u'(\cdot)$ and $v'(\cdot)$ satisfies the inclusion

$$x'(t) \in W_{c_*}(t), \quad t \in [\bar{t}, \hat{t}]. \tag{7.4}$$

We have

$$\begin{aligned} x^*(\hat{t}) - x'(\hat{t}) &= \int_{\bar{t}}^{\hat{t}} B(t)(u(t) - u'(t))dt = \\ &= \int_{\bar{t}}^{\hat{t}} (B(t) - B(\hat{t}))(u(t) - u'(t))dt + B(\hat{t})\int_{\bar{t}}^{\hat{t}} (u(t) - u'(t))dt. \end{aligned} \tag{7.5}$$

Let π denote the operator for orthogonal projection of the space R^n onto the space that is orthogonal to the vector $B(\hat{t})$.

Taking into account that the absolute values of the controls $u(t)$ and $u'(t)$ are constrained by the number μ and that $B(t)$ satisfies the Lipschitz condition with the constant β and that $\pi B(\hat{t}) = 0$, from (7.5) we obtain

$$|\pi x^*(\hat{t}) - \pi x'(\hat{t})| \le \beta\mu\delta^2. \tag{7.6}$$

Let $\tilde{x}$ be the point that is closest to the set $W_{c_*}(\hat{t})$ on the straight line $\mathcal{A}(\hat{t}, x^*(\hat{t}))$. From the inclusion $x'(\hat{t}) \in W_{c_*}(\hat{t})$, which derives from inequality (7.4), and the definition of the operator π, it follows that

$$d(\{\tilde{x}\}, W_{c_*}(\hat{t})) \le |\pi\tilde{x} - \pi x'(\hat{t})| = |\pi x^*(\hat{t}) - \pi x'(\hat{t})|.$$

Hence, taking into account the Lipschitz continuity of the function $x \to V(t, x)$ and the equality $c_* = V(\tilde{t}, \bar{x})$, we obtain

$$V(\hat{t}, \tilde{x}) \le c_* + \lambda d(\{\tilde{x}\}, W_{c_*}(\hat{t})) \le V(\bar{t}, \bar{x}) + \lambda|\pi x^*(\hat{t}) - \pi x'(\hat{t})|.$$

The required inequality (7.3) follows from inequality (7.6) and from the fact that

$$\mathcal{V}(\hat{t}, x^*(\hat{t})) \le V(\hat{t}, \tilde{x}). \quad \square$$

Lemma 2. Let the conditions (7.2) hold. We will assume that

$$\mathbf{G}_k(\bar{t} + \delta; \bar{t}, \bar{x}) \subset \Pi_+(\bar{t} + \delta) \quad (\mathbf{G}_k(\bar{t} + \delta; \bar{t}, \bar{x}) \subset \Pi_-(\bar{t} + \delta)).$$

Let $x^*(\cdot)$ be the motion of system (3.1) that starts out at the time $\bar{t}$ from the point $\bar{x}$ under the action of the constant control

$$u(t) \equiv \bar{V}(\bar{t}, \bar{x})\mu, \quad (u(t) \equiv -\bar{V}(\bar{t}, \bar{x})\mu) \tag{7.7}$$

and some control $v(t) \in kQ_{\max}$. Then the following estimate holds

$$V(\bar{t} + \delta, x^*(\bar{t} + \delta)) \le V(\bar{t}, \bar{x}) + \lambda\beta\mu\delta^2. \tag{7.8}$$

Proof. We introduce the notation $\hat{t} = \bar{t} + \delta$, $c_* = V(\bar{t}, \bar{x})$. As in the initial part of the proof of Lemma 1, for an assigned $v(\cdot)$ we select a control $u'(t) \in \bar{c}P$, where $\bar{c} = \bar{V}(\bar{t}, \bar{x})$ so that the corresponding motion $x'(\cdot)$ that starts out at the time $\bar{t}$ from the point $\bar{x}$ would satisfy the inclusion

$$x'(t) \in W_{c_*}(t), \quad t \in [\bar{t}, \hat{t}]. \tag{7.9}$$

Since $|u(t)| \equiv \bar{V}(\bar{t}, \bar{x})\mu$ and $\bar{c}P = \{u \in R : |u| \le \bar{V}(\bar{t}, \bar{x})\mu\}$, we have

$$|u(t)| \ge |u'(t)|, \quad t \in [\bar{t}, \hat{t}].$$

We set

$$\tilde{z} = x'(\hat{t}) + B(\hat{t})\int_{\bar{t}}^{\hat{t}}(u(t) - u'(t))dt. \tag{7.10}$$

We shall show that

$$V(\hat{t}, \tilde{z}) \le V(\hat{t}, x'(\hat{t})). \tag{7.11}$$

Let us consider the case

$$\mathbf{G}_k(\hat{t}; \bar{t}, \bar{x}) \subset \Pi_+(\hat{t}) \quad u(t) \equiv \bar{V}(\bar{t}, \bar{x})\mu.$$

We have $x'(\hat{t}) \in \Pi_+(\hat{t})$, $\tilde{z} \in \Pi_+(\hat{t})$. Since $u(t) \ge u'(t)$, $t \in [\bar{t}, \hat{t}]$, using representation (7.10), for $B(\hat{t}) \ne 0$ we find that the point $\tilde{z}$ lies on the straight line $\mathcal{A}(\hat{t}, x'(\hat{t}))$ relative to the point $x'(\hat{t})$ in the direction of the vector $B(\hat{t})$. Hence, taking into account the quasiconvexity of the function $x \mapsto V(t, x)$, we derive inequality (7.11).

In the case

$$\mathbf{G}_k(\hat{t}; \bar{t}, \bar{x}) \subset \Pi_-(\hat{t}) \quad u(t) \equiv -\bar{V}(\bar{t}, \bar{x})\mu$$

inequality (7.11) can be proved in a similar manner.

Since the right-hand side of inequality (7.11) does not exceed c_*, we obtain the inclusion $\tilde{z} \in W_{c_*}(\hat{t})$. Consequently,

$$d(\{x^*(\hat{t})\}, W_{c_*}(\hat{t})) \le |x^*(\hat{t}) - \tilde{z}|.$$

Using the definition of the vector $\tilde{z}$, we have

$$x^*(\hat{t}) - \tilde{z} = x^*(t) - x'(\hat{t}) - B(\hat{t})\int_{\bar{t}}^{\hat{t}}(u(t) - u'(t))dt = \int_{\bar{t}}^{\hat{t}}(B(t) - B(\hat{t}))(u(t) - u'(t))dt.$$

Therefore,

$$|x^*(\hat{t}) - \tilde{z}| \le \beta\mu\delta^2.$$

The required inequality (7.8) follows from the fact that

$$V(\hat{t}, x^*(\hat{t})) \le V(\hat{t}, \tilde{z}) + \lambda|x^*(\hat{t}) - \tilde{z}|, \quad V(\hat{t}, \tilde{z}) \le V(\bar{t}, \bar{x}). \quad \square$$

Lemma 3. Let

$$k \ge 0, \quad \bar{t} \in T, \quad \hat{t} \in (\bar{t}, \vartheta].$$

Suppose $x^*(\cdot)$ is the motion of system (3.1) that starts out at the time $\bar{t}$ from the point $\bar{x}$ under the action of the constant control (7.7) and some control $v(t) \in kQ_{\max}$.

We assume that

$$x^*(t) \in \Pi_+(t)\backslash \operatorname{int} W_k(t), \quad (x^*(t) \in \Pi\_(t)\backslash \operatorname{int} W_k(t))$$

for all $t \in [\bar{t}, \hat{t}]$. Then the following estimate holds

$$V(\hat{t}, x^*(\hat{t})) \le V(\bar{t}, \bar{x}). \tag{7.12}$$

Proof. Without loss of generality, let us consider the case when

$$u(t) \equiv \bar{V}(\bar{t}, \bar{x})\mu, \quad x^*(t) \in \Pi_+(t)\backslash\int W_k(t), \quad t \in [\bar{t}, \hat{t}].$$

We assume that

$$V(\hat{t}, x^*(\hat{t})) > V(\bar{t}, \bar{x}). \tag{7.13}$$

Let $\tilde{t}$ be the maximum time in the interval $[\bar{t}, \hat{t}]$, when $V(t, x^*(t)) = V(\bar{t}, \bar{x})$.

We divide the interval $[\tilde{t}, \hat{t}]$ at the times $t_1, t_2, \ldots, t_m (t_1 = t, t_m = \hat{t})$ with the spacing δ so that the following relation would hold for any $n = 1, 2, \ldots, m - 1$

$$\mathbf{G}_k(t_{n+1}; t_n, x^*(t_n)) \subset \Pi_+(t_{n+1}).$$

This can be done on the basis of an assumption regarding the location of the point $x^*(t)$ relative to the switching surface $\Pi(t)$. We note that for any n we have the inequality

$$V(t_n, x^*(t_n)) \ge V(\bar{t}, \bar{x}). \tag{7.14}$$

Let us consider the arbitrary time interval $[t_n, t_{n+1}]$. We use $\tilde{x}_n(\cdot)$ to denote the motion of system (3.1) that starts out at the time t_n from the point $x^*(t_n)$ under the action of the constant control $\tilde{u}_n(t) \equiv \bar{V}(t_n, x^*(t_n))\mu$ of player 1 and the control $v(\cdot)$ of player 2 specified in the formulation of the lemma.

By virtue of Lemma 2, we have the inequality

$$V(t_{n+1}, \tilde{x}(t_{n+1})) \le V(t_n, x^*(t_n)) + \lambda\beta\mu\delta^2. \tag{7.15}$$

We estimate $V(t_{n+1}, x^*(t_{n+1}))$ in terms of $V(t_{n+1}, \tilde{x}(t_{n+1}))$:

$$V(t_{n+1}, x^*(t_{n+1})) \le V(t_{n+1}, \tilde{x}(t_{n+1})) + \lambda|\tilde{x}(t_{n+1}) - x^*(t_{n+1})|. \tag{7.16}$$

Since

$$\tilde{x}(t_{n+1}) - x^*(t_{n+1}) = \int_{t_n}^{t_{n+1}} B(t)(\tilde{u}_n(t) - u(t))dt = \int_{t_n}^{t_{n+1}} B(t)(\bar{V}(t_n, x^*(t_n)) - \bar{V}(\bar{t}, \bar{x}))\mu dt,$$

we have

$$|\tilde{x}(t_{n+1}) - x^*(t_{n+1})| \le \sigma\mu(\bar{V}(t_n, x^*(t_n)) - \bar{V}(\bar{t}, \bar{x}))\delta. \tag{7.17}$$

Here we have used the inequality

$$\bar{V}(t_n, x^*(t_n)) \ge \bar{V}(\bar{t}, \bar{x}) \tag{7.18}$$

which follows from inequality (7.14).

By virtue of (7.16) and (7.17), we obtain

$$V(t_{n+1}, x^*(t_{n+1})) \le V(t_{n+1}, \tilde{x}(t_{n+1})) + \lambda\beta\mu\delta(\bar{V}(t_n, x^*(t_n)) - \bar{V}(\bar{t}, \bar{x})).$$

According to inequality (7.18),

$$\bar{V}(t_n, x^*(t_n)) - \bar{V}(\bar{t}, \bar{x}) \le V(t_n, x^*(t_n)) - V(\bar{t}, \bar{x}).$$

From inequality (7.15) and the last two inequalities, we have the inequality

$$V(t_{n+1}, x^*(t_{n+1})) \le V(t_n, x^*(t_n)) + \lambda\beta\mu\delta^2 + \lambda\sigma\mu\delta(V(t_n, x^*(t_n)) - V(\bar{t}, \bar{x})),$$

which we rewrite, using notation (7.1), in the form

$$\mathrm{Var}(V, [t_n, t_{n+1}]) \le \lambda\beta\mu\delta^2 + \lambda\sigma\mu\delta\mathrm{Var}(V, [\bar{t}, t_n]). \tag{7.19}$$

Note that

$$\begin{gathered}\mathrm{Var}(V, [\bar{t}, t_n]) = \mathrm{Var}(V, [t_1, t_n]) = \sum_{p=1}^{n-1} \mathrm{Var}(V, [t_p, t_{p+1}]) \\ \mathrm{Var}(V, [t_1, t_2]) \le \lambda\beta\mu\delta^2.\end{gathered} \tag{7.20}$$

Let us find an estimate for the quantity

$$V(\hat{t}, x^*(\hat{t})) - V(\bar{t}, \bar{x}) = \mathrm{Var}(V, [t_1, t_m]) = \sum_{n=1}^{m-1} \mathrm{Var}(V, [t_n, t_{n+1}]). \tag{7.21}$$

We consider the geometric progression

$$a_n = a_1 b^{n-1}, \quad k = 1, \dots, m-1; \quad a_1 = \lambda\beta\mu\delta^2, \quad b = 1 + \lambda\sigma\mu\delta. \tag{7.22}$$

From (7.19), (7.20) and (7.22) it can be seen that

$$\mathrm{Var}(V, [t_n, t_{n+1}]) \le a_n, \quad n = 1, \dots, m-1. \tag{7.23}$$

We have

$$\begin{gathered}\sum_{n=1}^{m-1} a_n = a_1 \frac{b^{m-1} - 1}{b - 1} = \\ = \frac{\beta\delta}{\sigma}((1 + \lambda\sigma\mu\delta)^{(\hat{t} - \bar{t})/\delta} - 1) \le \frac{\beta\delta}{\sigma}(e^{\lambda\sigma\mu(\hat{t} - \bar{t})} - 1) \le \frac{\beta\delta}{\sigma}(e^{\lambda\sigma\mu T} - 1).\end{gathered} \tag{7.24}$$

Here we have used the equality $m - 1 = (\hat{t} - \bar{t})/\delta$.

Relations (7.21), (7.23) and (7.24) lead to the inequality

$$V(\hat{t}, x^*(\hat{t})) \le V(\bar{t}, \bar{x}) + \frac{\beta\delta}{\sigma}(e^{\lambda\sigma\mu T} - 1), \tag{7.25}$$

which holds when the interval $[\bar{t}, \hat{t}]$ is divided with any sufficiently small spacing δ. A contradiction with assumption (7.13) has been obtained.

Thus, estimate (7.12) has been proved. □

The next lemma provides a trivial estimate of the variation of V along the motion of system (3.1) for some permissible programmed control of player 1 and constrained programmed control of player 2.

Lemma 4. Let

$$k \geq 0, \quad \bar{t} \in T, \quad \bar{x} \notin \operatorname{int} W_k(\bar{t}), \quad \hat{t} \in (\bar{t}, \vartheta].$$

Let $x^*(\cdot)$ be the motion of system (3.1) that starts out at the time $\bar{t}$ from the point $\bar{x}$ under the action of the programmed controls $u(t) \in P$ and $v(t) \in kQ_{\max}$. The following estimate holds

$$V(\hat{t}, x^*(\hat{t})) \leq V(\bar{t}, \bar{x}) + 2\lambda\sigma\mu(\hat{t} - \bar{t}). \tag{7.26}$$

Proof. Let $c_* = V(t, \bar{x})$. For the control $v(\cdot)$ we construct a control $u'(\cdot)$ such that $u'(t) \in P$ and the motion $x'(\cdot)$ that starts out at the time $\bar{t}$ from the point $\bar{x}$ under the action of $u'(\cdot)$ and $v'(\cdot)$ satisfies the inclusion

$$x'(t) \in W_{c_*}(t), \quad t \in [\bar{t}, \hat{t}].$$

Consequently,

$$V(\hat{t}, x'(\hat{t})) \leq V(\bar{t}, \bar{x}). \tag{7.27}$$

Since

$$x^*(\hat{t}) - x'(\hat{t}) = \int_{\bar{t}}^{\hat{t}} B(t)(u(t) - u'(t))dt,$$

we can write

$$|x^*(\hat{t}) - x'(\hat{t})| \leq 2\sigma\mu(\hat{t} - \bar{t}). \tag{7.28}$$

Taking into account the Lipschitz condition for the function $x \mapsto V(t, x)$, we have

$$V(\hat{t}, x^*(\hat{t})) - V(\hat{t}, x'(\hat{t})) \leq \lambda|x^*(\hat{t}) - x'(\hat{t})|.$$

This inequality, together with inequalities (7.27) and (7.28), gives inequality (7.26). □

7.1. Completion of the proof of the theorem

Let us establish the validity of inequality (5.7). When this inequality holds, inclusion (5.6) also holds, since

$$|U(t, x)| \leq \bar{V}(t, x)\mu = \min\{V(t, x), 1\}\mu.$$

In the time interval $[t_0, \vartheta]$ we identify closed intervals on which $x^*(t) \notin \operatorname{int} W_k(t)$. Outside these intervals we have $V(t, x^*(t)) < k \leq s$, and inequality (5.7) automatically holds.

Let $[\xi, \zeta]$ be an interval that is arbitrarily selected from the intervals identified. We assume that it cannot be extended to the left with compliance to the condition $x^*(t) \notin \operatorname{int} W_k(t)$. Then either $V(\xi, x^*(\xi)) = k$ or $V(\xi, x^*(\xi)) > k$. The latter case is possible only for $\xi = t_0$.

We shall prove relation (5.7) in the interval $[\xi, \zeta]$.

1°. Let $\beta > 0$, $\sigma > 0$. We set

$$h = \sqrt{(2\sigma\mu\Delta + r)/\beta\mu}. \tag{7.29}$$

A. Along the motion $x^*(\cdot)$, we identify "loops" that are associated with entry into the set $\Pi^r(t)$. We also define free intervals.

Moving from ξ to ζ, we find the first time t when $x^*(t) \in \Pi^r(t)$. We call this time the starting time of the first loop, and we denote it by t_1. Next, we mark the ending time $\tilde{t}_1$ of the first loop as the last time t in the interval $[t_1, t_1 + h] \cap [\xi, \zeta]$ when $x^*(t) \in \Pi^r(t)$. The instant $\tilde{t}_1$, in particular, can coincide with t_1.

As the starting time t_2 of the second loop we take the first time $t \in [t_1 + h, \zeta]$ when $x^*(t) \in \Pi^r(t)$. Then we mark the ending time $\tilde{t}_2$ of the second loop as the last time t in the interval $[t_2, t_2 + h] \cap [\xi, \zeta]$ when $x^*(t) \in \Pi^r(t)$.

Continuing this process, we obtain a set of loops in $[\xi, \zeta]$.

From $[\xi, \zeta]$ we remove the interior of the intervals of the loops constructed. We obtain an ordered set of time segments. We call each of them a free interval. A segment can be degenerate, i.e., it can consist of one point.

If there are no loops in $[\xi, \zeta]$, we consider $[\xi, \zeta]$ to be a free interval.

B. Let $[\tau, \eta]$ be some free interval. We shall show that the increment of V on it is described by the inequality

$$\mathrm{Var}_f(V, [\tau, \eta]) \le 2\lambda\sigma\mu\Delta. \tag{7.30}$$

The subscript f stresses that the variation of V is calculated in a free interval.

In the interior of the free interval, the motion $x^*(\cdot)$ is on one side of the set

$$\Pi^r = \{(t, x) : t \in T, x \in \Pi^r(t)\},$$

and consequently it is on one side of the set

$$\Pi = \{(t, x) : t \in T, x \in \Pi(t)\}$$

At the starting time t_Δ of the successive time sampling when

$$x^*(t_\Delta) \in \Pi^r_+(t_\Delta) \quad (x^*(t_\Delta) \in \Pi^r_-(t_\Delta))$$

the control

$$u(t_\Delta) = \bar{V}(t_\Delta, x^*(t_\Delta))\mu \quad (u(t_\Delta) = -\bar{V}(t_\Delta, x^*(t_\Delta))\mu)$$

is selected, and this control acts until the beginning of the next sampling. By virtue of Lemma 3, we have $\mathrm{Var}(V, [t_\Delta, t_\Delta + \Delta]) \le 0$, if $t_\Delta + \Delta \le \eta$, and $\mathrm{Var}(V, [t_\Delta, \eta]) \le 0$, if $t_\Delta + \Delta > \eta$.

Performing the summation for all the time samplings that begin in the half-open interval $[\tau, \eta)$, we obtain

$$\mathrm{Var}(V, [t_\Delta^{(1)}, \eta]) \le 0.$$

Here $t_\Delta^{(1)}$ is the beginning of the first sampling in $[\tau, \eta)$.

By virtue of Lemma 4, for the interval $[\tau, t_\Delta^{(1)}]$ we find

$$\mathrm{Var}(V, [\tau, t_\Delta^{(1)}]) \le 2\lambda\sigma\mu(t_\Delta^{(1)} - \tau) \le 2\lambda\sigma\mu\Delta.$$

Combining the last two inequalities, we arrive at estimate (7.30).

C. We shall say that $[\tau, \eta]$ is an interval of type E_1 if it is composed of some loop $[t_i, \tilde{t}_i]$ and a free interval that is adjacent to it on the right. We shall call the interval $[\tau, \eta]$ of type E_1 with the additional condition $\tau + h \le \eta$ an interval of type E_2.

Let us evaluate the increment of the function V along the motion $x^*(\cdot)$ in an interval of type E_1.

Consider the loop interval $[t_i, \tilde{t}_i]$. Applying Lemma 1, for $\delta = \tilde{t}_i - t_i$ we obtain

$$\mathcal{V}(\tilde{t}_i, x^*(\tilde{t}_i)) \le V(t_i, x^*(t_i)) + \lambda\beta\mu(\tilde{t}_i - t_i)^2.$$

Since $\tilde{t}_i - t_i \le h$, the second term on the right-hand side can be replaced by $\lambda\beta\mu h(\tilde{t}_i - t_i)$. Taking into account the inequality

$$V(\tilde{t}_i, x^*(\tilde{t}_i)) \le \mathcal{V}(\tilde{t}_i, x^*(\tilde{t}_i)) + \lambda r$$

which holds by virtue of the inequality (5.5) and the inclusion $x^*(\tilde{t}_i) \in \Pi^r(\tilde{t}_i)$, we arrive at the relation

$$\mathrm{Var}(V, [t_i, \tilde{t}_i]) \le \lambda\beta\mu h(\tilde{t}_i - t_i) + \lambda r. \tag{7.31}$$

In the free interval $[\tilde{t}_i, \eta]$ we have inequality (7.30) when $\tau = \tilde{t}_i$. Combining this inequality with inequality (7.31) and taking the relation $\tilde{t}_i - t_i \le \eta - \tau$ into account, we obtain

$$\mathrm{Var}_1(V, [\tau, \eta]) \le \lambda\beta\mu h(\eta - \tau) + 2\lambda\sigma\mu\Delta + \lambda r. \tag{7.32}$$

The subscript 1 emphasizes that the calculation of the increment of V is observed in an interval of type E_1.

We will now estimate the increment Var_2 of V along the motion $x^*(\cdot)$ in an interval of type E_2. Since $\eta - \tau \geq h$ in this case, from (7.29) we obtain the inequality

$$2\lambda\sigma\mu\Delta + \lambda r \leq \lambda\beta\mu h(\eta - \tau).$$

Invoking inequality (7.32), we obtain

$$\mathrm{Var}_2(V, [\tau, \eta]) \leq 2\lambda\beta\mu h(\eta - \tau). \tag{7.33}$$

D. Consider the interval $[\xi, t]$ $(t \leq \xi)$. We will represent it as being composed of the first free interval $[\xi, \bar{t}]$, a finite number of successive intervals of type E_2 from the time $\tilde{t}$ to some time $\hat{t}$ (their summed time interval is $[\bar{t}, \hat{t}]$) and the remaining interval $[\hat{t}, t]$ of type E_1. Successively applying estimates (7.30), (7.33) and (7.32), we have

$$\mathrm{Var}(V, [\xi, t]) = \mathrm{Var}_f(V, [\xi, \bar{t}]) + \mathrm{Var}(V, [\bar{t}, \hat{t}]) + \mathrm{Var}_1(V, [\hat{t}, t]) \leq$$
$$\leq 2\lambda\sigma\mu\Delta + 2\lambda\beta\mu h(\hat{t} - \bar{t}) + \lambda\beta\mu h(t - \hat{t}) + 2\lambda\sigma\mu\Delta + \lambda r = 2\lambda\beta\mu h(t - \bar{t}) + 4\lambda\sigma\mu\Delta + \lambda r.$$

Substituting the expression for h from formula (7.29), we obtain

$$\mathrm{Var}(V, [\xi, t]) \leq \Lambda(t, \xi, \Delta, r). \tag{7.34}$$

2°. Suppose $\beta = 0$, and $\sigma \geq 0$. Moving from ξ to t $(t \leq \zeta)$, we find the first time when $x^*(t) \in \Pi^r(t)$. We denote it by $\bar{t}$. Let $\hat{t}$ be the last time in $[\xi, t]$ when $x^*(t) \in \Pi^r(t)$.

We have

$$x^*(t) \notin \Pi^r(t), \quad t \in [\xi, \bar{t}) \cup (\hat{t}, t].$$

For the intervals $[\xi, \tilde{t}]$ and $[\hat{t}, t]$, based on Lemmas 3 and 4 (just as in the derivation of inequality (7.30)), we obtain

$$\mathrm{Var}(V, [\xi, \bar{t}]) \leq 2\lambda\sigma\mu\Delta, \quad \mathrm{Var}(V, [\hat{t}, t]) \leq 2\lambda\sigma\mu\Delta. \tag{7.35}$$

For the interval $[\bar{t}, \hat{t}]$, applying Lemma 1 with $\beta = 0$, we have

$$\mathcal{V}(\hat{t}, x^*(\hat{t})) \leq V(\bar{t}, x^*(\bar{t})).$$

Therefore, taking into account the inequality

$$V(\hat{t}, x^*(\hat{t})) \leq \mathcal{V}(\hat{t}, x^*(\hat{t})) + \lambda r,$$

we arrive at the estimate

$$\mathrm{Var}(V, [\bar{t}, \hat{t}]) \leq \lambda r. \tag{7.36}$$

Combining inequalities (7.35) and (7.36), we obtain

$$\mathrm{Var}(V, [\xi, t]) \leq 4\lambda\sigma\mu\Delta + \lambda r. \tag{7.37}$$

3°. Based on inequality (7.34) for the case when $\beta > 0$ and $\sigma > 0$ and on equality (7.37) for the case when $\beta = 0$ and $\sigma \geq 0$, we have the estimate

$$V(t, x^*(t)) \leq V(\xi, x^*(\xi)) + \Lambda(t, \xi, \Delta, r). \tag{7.38}$$

At the instant ξ, the value $V(\xi, x^*(\xi))$ of the function V equals either k or $V(t_0, x_0)$. Thus,

$$V(\xi, x^*(\xi)) \leq \max\{k, V(t_0, x_0)\} = s.$$

Substituting this inequality into (7.38) and bearing in mind that

$$\Lambda(t, t_0, \Delta, r) \geq \Lambda(t, \xi, \Delta, r),$$

we obtain inequality (5.7). □

8. Example. A conflict-controlled pendulum

Suppose a system describing a linearized conflict-controlled pendulum is specified as follows:

$$\dot{\mathbf{x}}_1 = \mathbf{x}_2 + v, \quad \dot{\mathbf{x}}_2 = -\mathbf{x}_1 + u. \tag{8.1}$$

Here u and v are the scalar controls of Player 1 and Player 2 (the active control and the disturbance). The absolute value of the control of Player 1 is subject to the constraint: $|u| \leq 1$. This inequality defines the set P. No geometrical constraint on the control of Player 2 is specified in the formulation of the problem.

The behaviour of the system will be studied in the time interval $T = [0,10]$. In the plane of the phase variables $\mathbf{x}_1$, $\mathbf{x}_2$ we define the terminal set M in the form of a circle of radius 2 with its centre at the origin of coordinates. Player 1 tries to bring the system (8.1) into M at the termination time $\vartheta = 10$.

An auxiliary constraint $Q_{\max}$ on the control of Player 2 should be selected to construct the robust control $\tilde{U}$. We take this constraint in the form $|v| \leq 1$.

Fig. 2 presents cross sections $W(t)$ of the maximum stable bridge W corresponding to the sets P, $Q_{\max}$ and M for the times $t = 0$, 2, 4, 6, 8, 10. The dashed lines indicate the switching lines $\Pi(t)$. The plus and minus

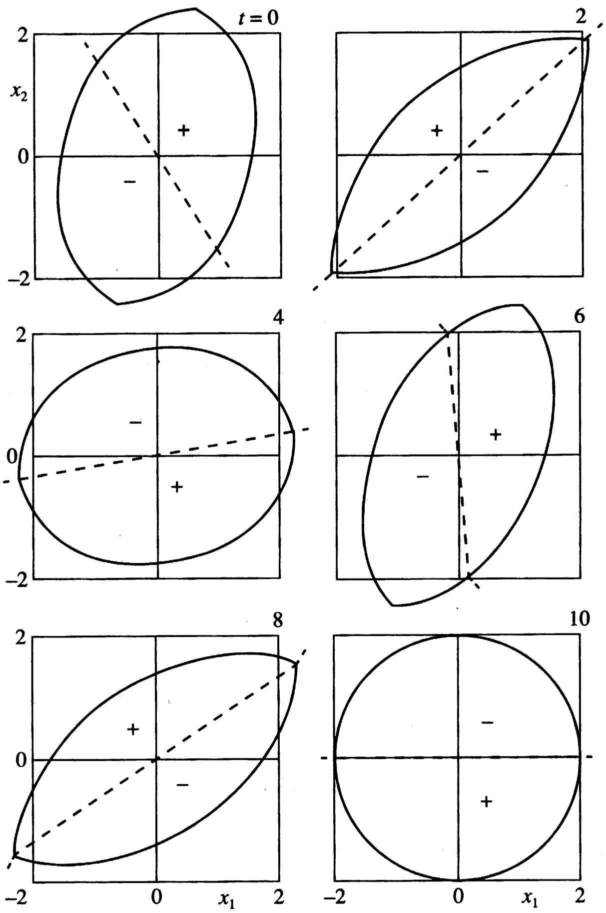

Fig. 2.

signs denote the sign of the control in the applicable region. The images are given in x_1, x_2 coordinates of game (3.1).

We will assume that the robust control $\tilde{U}(t, \mathbf{x})$ is implemented in a control sampling scheme with a spacing $\Delta = 0.05$. We take $\mathbf{x}(0) = (0, 1)$ as the initial point. It can be verified that

$$x(0) = X_{2,2}(10, 0)\mathbf{x}(0) \in W(0).$$

To compare the results obtained, we also consider the optimal control methods (optimal strategies) of Player 1 and Player 2 in an antagonistic differential game, in which the constraint $\mathfrak{Q}$ of Player 2 is specified *a priori*. We take

$$\mathfrak{Q} = 1.3Q_{\max}. \tag{8.2}$$

We leave the control of Player 1 unchanged: $|u| \leq 1$. Let $\varphi(\mathbf{x}) = 0.5|\mathbf{x}|$ be the terminal payoff function in such a game. The optimal strategy of the players can be assigned[11,12,15] using switching surfaces. We use S_1 to denote the optimal strategy of player 1 and S_2 to denote the optimal strategy of Player 2. We set the spacing of the control sampling schemes equal to 0.05 when the strategies S_1 and S_2 are used.

Two types of control of Player 2 were taken in the simulation. One was the programmed sinusoidal control $v(t) = 1.3 \sin(0.8\pi t)$. The other was the strategy S_2 that was obtained from the antagonistic differential game described above with the *a priori* specified constraint (8.2). Thus, in both cases the maximum disturbance level does not exceed that which is built into $Q_{\max}$.

Fig. 3 shows the results of a calculation for the case of a sinusoidal disturbance: phase trajectories in the $\mathbf{x}_1$, $\mathbf{x}_2$ plane and realizations of the control $u(t)$ and the sinusoidal disturbance $v(t)$. Curves 1 correspond to the robust control $\tilde{U}$, and curves 2 were obtained using the strategy S_1. The maximum value of the control $u(t)$ for $\tilde{U}$ only slightly exceeds the 0.5 level, while the values $u(t) = \pm 1$ are obtained for S_1. The position of the phase point at the termination time $\vartheta = 10$ for $\tilde{U}$ is somewhat better than the position for S_1. There is no contradiction here: the sinusoidal disturbance considered is not optimal from the point of view of Player 2.

The results of modelling for a disturbance formed according to the feedback principle using the strategy S_2 are shown in Fig. 4. It can be seen that the results obtained are much better than in the case of the sinusoidal disturbance. The control strategy S_1 of Player 1, which is oriented to constraint (8.2), leads to a better result than does the robust control $\tilde{U}$.

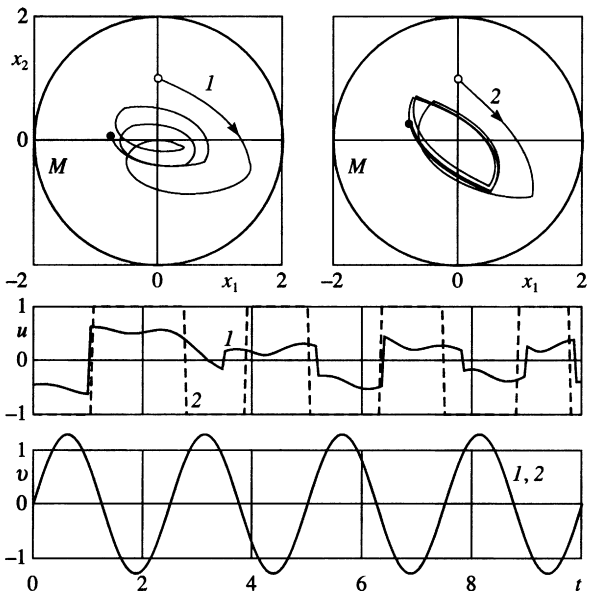

Fig. 3.

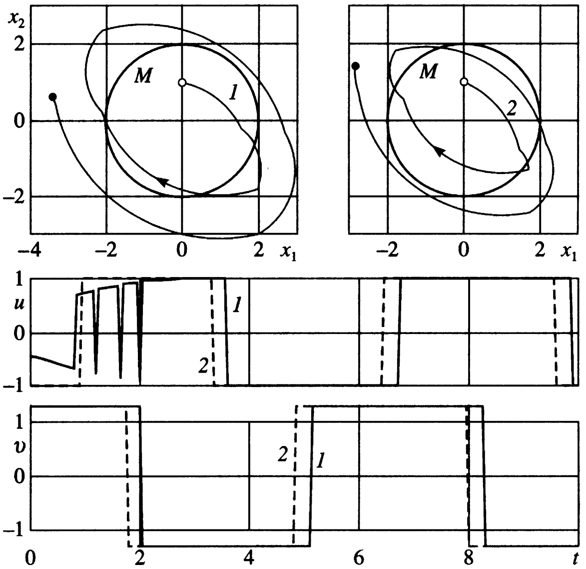

Fig. 4.

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