Attainability Set at Instant for One-Side Turning Dubins Car

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Abstract: Three-dimensional attainability set “at instant” for a non-linear controlled system is studied, which is often called the “Dubins car”. A controlled object moves in the plane. It has linear velocity of a constant magnitude and bounded turn radius. The case is explored when the object can turn to one side only. Moreover, a rectilinear motion is prohibited by the constraints onto the control. We prove that the sections of the attainability set by planes orthogonal to the angle coordinate are convex. The geometric structure of these sections is analyzed.

Keywords: Dubins car, one-sided turn, three-dimensional attainability set, Pontryagin maximum principle, piecewise-constant control, convexity of attainability set sections.

1. INTRODUCTION

This paper deals with a study of attainability set at instant for a model of controlled motion in the plane, which is one of the most popular in the mathematical control theory and applied works. Dynamics of the motion with constant magnitude of the linear velocity and prescribed range of possible turn rate is defined by means of a system of three differential equations. Two phase variables describe the geometric location of the object in the plane, the third one is the velocity heading. The scalar control defines the current angular velocity of rotation of the velocity direction or, that is equivalent, the instantaneous turn radius. The value of the control parameter belongs to a closed segment.

Such a model is considered in works: Markov (1889); Isaacs (1951); Dubins (1957). Specialists in robotics call it the “Dubins car”. For this three-dimensional controlled system in the framework of time-optimal problem, a synthesis of the feedback control has been constructed for a control constraint, which is symmetric Pecsiaradi (1972) with respect to the origin or which is non-symmetric Bakolas and Tsiotras (2011). In work Berdyshev (2015), the traveling salesman problem is considered, when the moving object is the Dubins car.

Dynamics of the Dubins car is often used for constructing control of autonomous wheel robots (see, for example, Lau- mond (1998)), for computing flight trajectories in the system of civil aviation Pyatko and Krasov (2004), and, also, in applied works dealing with control of unmanned flying vehicles in the horizontal plane Meyer et al. (2015). In work Choi (2014) some aviation problems are shown, whose investigation can involve the Dubins model with one-side turn.

For the Dubins car, let us call the attainability set \( G(t_f) \) at instant \( t_f \) the collection of all points of three-dimensional phase space such that to each of them the system can hit at the instant \( t_f \) from the given initial state (which is meant to be the origin, without loss of generality) under some feasible control. The attainability set at instant should be distinguished from the attainability set \( up \) to instant. The latter is a union of all attainability sets at instant corresponding to the instants from 0 to \( t_f \).

Construction of attainability sets \( G(t_f) \) for the case when both left and right turns are possible is considered in papers Patsko et al. (2003); Fedotov et al. (2011).

In this paper, we study the sets \( G(t_f) \) for the case when the turn is possible to one side only and the rectilinear motion is impossible. Namely, it is supposed the the scalar control \( u \) belongs to a segment \([u_1, u_2]\) such that \( 0 < u_1 < u_2 \). The case \( u_1 = 0 \) is studied in works Fedotov and Patsko (2018); Patsko and Fedotov (2018). It was proved that the sections of the attainability set by planes orthogonal to the angular axis are convex in this case. The main objective of this paper is to show that the convexity maintains also for \( u_1 > 0 \). When studying the boundaries of the sets \( G(t_f) \), we use the Pontryagin maximum principle Pontryagin et al. (1962).

2. PROBLEM FORMULATION

Let the dynamics of a controlled object (Dubins car) in the plane \( x, y \) be described by the third order system of differential equations

\[
\begin{align*}
\dot{x} &= \cos \varphi, \\
\dot{y} &= \sin \varphi, \\
\varphi &= u, \quad u \in [u_1, u_2], \quad 0 < u_1 < u_2.
\end{align*}
\]  

(1)

Here, \( x, y \) are coordinates of the object geometric position, \( \varphi \) is the velocity direction angle counted counterclockwise from the axis \( x \) (Fig. 1), \( u \) is the scalar control. The magnitude of the linear velocity equals unit. We assume that \( u_2 = 1 \).

Representation (1) with \( u_2 = 1 \) can be obtained from any arbitrary controlled system of the third order with...
constant magnitude of the linear velocity and given range of the turn angular velocity (one-side turn). To do it, one should scale geometric coordinates and time. Without loss of generality, at the initial instant \( t_0 = 0 \), the initial phase state is the origin: \( x_0 = 0, \ y_0 = 0, \ \varphi_0 = 0 \).

As feasible controls \( u(\cdot) \), we consider measurable functions depending on the time and having their values \( u(\cdot) \) from the segment \([u_1, u_2]\). It is assumed that the angular coordinate \( \varphi \) takes its values in the interval \((-\infty, \infty)\).

3. PONTRYAGIN MAXIMUM PRINCIPLE

It is known Lee and Markus (1967) that the controls, which guide the system to the boundary of the attainability set \( G(t_f) \), obey the Pontryagin maximum principle (PMP). Let us write the relations of PMP for system (1).

Let \( u^*(\cdot) \) be some feasible control and \( (x^*(\cdot), y^*(\cdot), \varphi^*(\cdot)) \) be the motion of system (1) generated by the control in the interval \([t_0, t_f]\). The differential equation of the adjoint system are the following:

\[
\begin{align*}
\dot{\psi}_1 &= 0, \\
\dot{\psi}_2 &= 0, \\
\dot{\psi}_3 &= \psi_1 \sin \varphi^*(t) - \psi_2 \cos \varphi^*(t).
\end{align*}
\]

PMP means that a non-zero solution \( (\psi_1^*(\cdot), \psi_2^*(\cdot), \psi_3^*(\cdot)) \) of system (2) exists, for which a.e. in the interval \([t_0, t_f]\) the following condition is satisfied:

\[
\psi_3^*(t) u^*(t) = \max_{u \in [u_1, u_2]} \psi_3^*(t) u.
\]

One can see that the functions \( \psi_1^*(\cdot) \) and \( \psi_2^*(\cdot) \) are constant. Denote them by \( \psi_1^* \) and \( \psi_2^* \).

If \( \psi_1^* = 0 \) and \( \psi_2^* = 0 \), then \( \psi_3^*(t) \) is constant \( \neq 0 \) in the entire interval \([t_0, t_f]\). Therefore, in this case, we have that a.e. \( u^*(t) = u_1 \) or \( u^*(t) = u_2 \).

Let at least one of the numbers \( \psi_1^* \) and \( \psi_2^* \) is non-zero. Using the dynamics equations (1) and adjoint system equation (2), one can write the relation for \( \psi_3^*(t) \):

\[
\psi_3^*(t) = \psi_1^* y^*(t) - \psi_2^* x^*(t) + C.
\]

One obtains that \( \psi_3^*(t) = 0 \) iff the point \((x^*(t), y^*(t))\) of the geometric position at the instant \( t \) obeys the straight line equation

\[
\psi_1^* y - \psi_2^* x + C = 0.
\]

Since change of the sign of the function \( \psi_3^*(\cdot) \) implies switch of the control from one limit value to the another, line (4) is often called the switching line (or, shortly, SL).

Due to relation (3), if \( \psi_3^*(t) > 0 \) in some time interval, then \( u^*(t) = u_2 \) a.e. in that interval. Projection of the corresponding motion into the plane \( x, y \) goes counterclockwise along an arc of the circle with the radius \( 1/u_2 \). If \( \psi_3^*(t) < 0 \), then \( u^*(t) = u_1 \) and the motions go counterclockwise along an arc of the circle with the radius \( 1/u_1 \).

If \( \psi_3^*(t) = 0 \) in some interval, then in this interval the motion \((x^*(\cdot), y^*(\cdot))\) should go along SL (4). But it is impossible, because \( u_1 > 0 \).

Thus, when \( u_1 > 0 \), then the projection of a motion, which obeys PMP, into the plane \( x, y \) consists of circle arcs. On each such arc, the control is constant. Therefore, below, when we shall analyze the controls satisfying PMP, we can limit us to piecewise-constant controls (for which the right continuity is assumed in the discontinuity points).

A variant of the motion in the case when at least one of the constants \( \psi_1^* \) and \( \psi_2^* \) is non-zero is shown in Fig. 2. One has \( \psi_3^*(t) > 0 \) in the semi-plane \( \psi_1^* y - \psi_2^* x + C > 0 \) and \( \psi_3^*(t) < 0 \) in the semi-plane \( \psi_1^* y - \psi_2^* x + C < 0 \).

4. \( \varphi \)-SECTIONS OF ATTAINABILITY SET

Consider a motion of system (1) in the time interval \([t_0, t_f]\) \((t_0 = 0, \ t_f > 0)\) with the zero initial state. Possible values \( \varphi(t_f) \) of the coordinate \( \varphi \) at the instant \( t_f \) are in the interval \([\varphi_{u_1}, \varphi_{u_2}]\). The limit values of \( \varphi \) are obtained by means of controls \( u(\cdot) \equiv u_1 \) and \( u(\cdot) \equiv u_2 \). The corresponding limit points of a \( \varphi \)-section of the set \( G(t_f) \) are the following ones in the plane \( x, y \):

\[
\begin{pmatrix}
\sin(t_f \cdot u_i), \\
\cos(t_f \cdot u_i)
\end{pmatrix}
\]

Below, we fix the value \( \varphi \in (t_f \cdot u_1, t_f) \). The obtained \( \varphi \)-section of the set \( G(t_f) \) consists of more than one point.
The motions reaching the boundary of such a section obey PMP and should have not less than one switch (not less than two intervals of constant control). We use this for classification of the controls and corresponding points on the boundary of the section. Finally, we shall establish the structure of $\varphi$-sections and their convexity.

Introduce the following denotations. The symbol $t_1$ ($t_2$) denotes the length of the first (last) interval of control constancy, which joins to the instant $t_0$ ($t_f$).

The motions with controls obeying PMP under the constant $u_1$ during the first interval (that is, with motion along an arc of a large circle) and with the constant $u_2$ during the last interval (that is, with motion along an arc of a small circle) are attributed to the family “BS”. In the same way, we define the families “SB”, “BB”, “SS” with pairs of the controls $(u_2, u_1), (u_1, u_1), (u_2, u_2)$ during the first and last intervals. Each control obeying PMP belong to one and only one of these four families. The denotations will be used both for the controls and motions.

4.1 Boundary of $\varphi$-sections in the case $\varphi < 2\pi$

1. Consider a motion with one switch of the control $u$. Let the control during the first interval be equal $u_1$ and during the second one be equal $u_2 = 1$. The duration of the first interval is $t_1$, and the duration of the second one is $t_2$. One has $t_1 - u_1 < 2\pi, t_2 < 2\pi$. The relations are true

$$\varphi = t_1 \cdot u_1 + t_2, \quad t_f = t_1 + t_2.$$ 

From this, we obtain that for a fixed value $\varphi$, the values $t_1, t_2$, and, therefore, the switching instant are defined unambiguously. So, in $\varphi$-section of the set $G(t_f)$ with $\varphi < 2\pi$ only one point corresponds to the considered order of the controls $u_1, u_2$.

In the same way, to the sequence $u_2, u_1$ of controls, we get one point in the considered $\varphi$-section of the set $G(t_f)$.

2a) Consider a variant with two switches and a sequence $u_1, u_2, u_1$ of the controls. The durations of the intervals are $t_1, t_2, t_2$. One has

$$\varphi = t_1 \cdot u_1 + T_{u_2} + t_2 \cdot u_1, \quad t_f = t_1 + T_{u_2} + t_2.$$ 

It follows that

$$T_u = t_1 + t_2 = \frac{t_f - \varphi}{1 - u_1}, \quad T_{u_2} = \frac{t_f - t_1 \cdot u_1}{1 - u_1}.$$ 

Consequently, duration of the middle interval and the total duration of the first and last intervals are constant. The obtained family of controls is one-parameter. Let us take the duration $t_1$ as the parameter of the family; its interval is $t_1 \in (0, T_u)$, where $T_u < 2\pi/u_1$.

The corresponding points $(x_m[n], y_m[n])^T$ of the $\varphi$-sections of the set $G(t_f)$ after integration of equations (1) and some trigonometric transformations have the form

$$(x_m[n], y_m[n])^T = \frac{1}{u_1} \left( \begin{array}{c} \sin \varphi \\ 1 - \cos \varphi \end{array} \right) - 2 \left( \frac{1}{u_1} - 1 \right) \sin \left( \frac{T_{u_2}}{2} \right) \left( \begin{array}{c} \cos \left( t_1 \cdot u_1 + \frac{T_{u_2}}{2} \right) \\ \sin \left( t_1 \cdot u_1 + \frac{T_{u_2}}{2} \right) \end{array} \right).$$

(6)

The found collection of points $(x_m[n], y_m[n])^T$ formed by the sequence $u_1, u_2, u_1$ of controls under fixed values of $u_1, t_f, \varphi$ is a circle arc with the center at the point $\left( \frac{1}{u_1}, \sin \varphi \right)$. The radius of the circle is

$$2 \left( \frac{1}{u_1} - 1 \right) \sin \left( \frac{T_{u_2}}{2} \right),$$

and the angle of arc (6) is defined by the range of the value $u_1(t_f - \varphi)$. 

2b) Consider the second variant with two switches and the sequence $u_2, u_1, u_2$ of the controls. Durations of the corresponding intervals of control constancy are $t_1, T_{u_1}, t_2$.

Similarly to the case 2a, we have

$$T_u = t_1 + t_2 = \frac{\varphi - t_f \cdot u_1}{1 - u_1}, \quad T_{u_1} = t_f - \varphi.$$ 

The obtained collection of points is defined by the relations

$$(x_m[0], y_m[0]) = \left( \begin{array}{c} \sin \varphi \\ 1 - \cos \varphi \end{array} \right) - \frac{2}{u_1} \left( \begin{array}{c} \cos \left( t_1 \cdot u_1 + \frac{T_{u_1} \cdot u_1}{2} \right) \\ \sin \left( t_1 \cdot u_1 + \frac{T_{u_1} \cdot u_1}{2} \right) \end{array} \right).$$

(7)

and the range of possible values of the parameter $t_1 \in (0, T_u)$. This collection also is a circle arc with the center at the point $\left( \frac{\sin \varphi}{1 - \cos \varphi} \right)$ and the radius equal to

$$2 \left( \frac{1}{u_1} - 1 \right) \sin \left( \frac{T_{u_1} \cdot u_1}{2} \right).$$

The angle of the arc is defined by the range of the value $t_1$ and equal to $\frac{\varphi - t_f \cdot u_1}{1 - u_1}$.

3. One can easily establish that limit points of arcs (6) and (7) coincide. Namely,

$$(x_m[0], y_m[0]) = \left( \begin{array}{c} x_m[T_u] \\ y_m[T_u] \end{array} \right), \quad \left( \begin{array}{c} x_m[T_u] \\ y_m[T_u] \end{array} \right) = \left( \begin{array}{c} x_m[0] \\ y_m[0] \end{array} \right).$$

These limit points correspond to the motion with one switch considered earlier.

So, analyzing all possible variants of system (1) motions, which obey PMP, we get a locus in the plane $x, y$ as a closed curve consisting of two circle arcs joining by their limit points.

4. Consider a shift along arc (6) defined by the parameter $t_1$ under its growth from 0 to $T_u$. This shift is accompanied by clockwise rotation of the tangent vector. In the same way, a shift along arc (7) with increasing the parameter $t_2$ from 0 to $T_u$ also gives clockwise rotation of the tangent vector. The total angle of arcs (6) and (7) equals

$$\frac{(\varphi - t_f \cdot u_1)}{1 - u_1} + \frac{u_1(t_f - \varphi)}{1 - u_1} = \varphi.$$ 

According to the assumption $\varphi < 2\pi$ and taking into account that $\varphi \in (t_f \cdot u_1, t_f)$, we obtained the curve as the sought boundary description of the $\varphi$-section of the set $G(t_f)$. Such a section is strictly convex. Its boundary consists of two circle arcs.
In more details, computations of this subsection can be found in Patsko and Fedotov (2018) for the case $t_f \leq 2\pi$.

4.2 Boundary of $\phi$-sections in the case $\phi \geq 2\pi$

Properties of controls SB/BS. Motions of the types SB and BS have similar (accurately w.r.t. the reverse of the time) representations with even number of intermediate intervals (integer number of loops). For definiteness, let us consider relations connecting the parameters of motions of type SB (Fig. 2). Durations of the intervals of control constancy obey the relations

$$0 < t_1 \leq T_{u2} < 2\pi, \quad 0 < t_2 \leq T_{u1} < \frac{2\pi}{u_1}, \quad (8)$$

For fixed values $u_1, t_f, \phi$, we have

$$\phi = t_1 + n(T_{u1}u_1 + T_{u2}) + t_2u_1, \quad (9)$$

$$t_f = t_1 + n(T_{u1} + T_{u2}) + t_2. \quad (10)$$

Here, $n = \text{const} \geq 0$ is the number of loops:

$$n = \begin{cases} \frac{\phi}{2\pi} - 1 & \text{if} \ \phi \ \text{is multiple to} \ 2\pi, \\ \left[ \frac{\phi}{2\pi} \right] & \text{otherwise.} \end{cases} \quad (11)$$

Here, square brackets denote the integer part.

Properties of controls SS/BB. In the case SS (BB) between the first and last intervals, there are odd number of intervals of control constancy. They can be described as an integer number of loops and one additional interval with the control $u_1$ ($u_2$), which has the duration $T_{u1}$ ($T_{u2}$).

Consider controls of the type SS. Durations of the control constancy intervals obey the relations

$$0 < t_1 \leq T_{u2} < 2\pi, \quad 0 < t_2 \leq T_{u1} < \frac{2\pi}{u_1}. \quad (12)$$

For fixed values $u_1, t_f, \phi, n$, one has

$$\phi = t_1 + n(T_{u1}u_1 + T_{u2}) + T_{u1}u_1 + t_2, \quad (13)$$

$$t_f = t_1 + n(T_{u1} + T_{u2}) + T_{u1} + t_2. \quad (14)$$

Taking into account (5), (13), and (14), one can conclude that $T_{u1} = \text{const}, T_{u2} = \text{const},$ and $t_1 + t_2 = \text{const}$. Write the latter in more details

$$T_{u1} = t_1 + t_2 = \phi + T_{u2} - 2\pi(n + 1). \quad (15)$$

Relation (15) together with constraints-inequalities (12) defines the collection of controls of the type SS, which have the given number of loops equal to $n$ (Fig. 3). If this set is non-degenerated (that is, the parameter $t_1$ is taken from a non-degenerated interval), then the points $(x(t_f), y(t_f))^T$ defined by means of (15) in the $\phi$-section of the set $G(t_f)$ can be considered as a one-parameter curve.

In the non-degenerated case, it is a continuous smooth curve (a circle arc) in the plane $x$, $y$, namely

$$\begin{pmatrix} x_{BB}[n, t_1] \\ y_{BB}[n, t_1] \end{pmatrix} = \begin{pmatrix} \sin \phi \\ 1 - \cos \phi \end{pmatrix} + \frac{(n + 1)(1 - u_1)}{u_1} \begin{pmatrix} \sin(t_1 - T_{u2}) - \sin t_1 \\ \cos t_1 - \cos(t_1 - T_{u2}) \end{pmatrix}. \quad (16)$$

Let denote it by SS$_n$. Shifting along this arc is uniform on $t_1$ with clockwise rotation of the tangent vector; the angle of this rotation equals the increment of the parameter $t_1$. 

For a control of the type SS, the number $n$ of loops can be ambiguous. But the number of non-degenerated arcs of the type SS is not larger than two. With that, if there are two arcs of the type SS, then the difference between the number of loops in them is not larger than unit. To prove this, let us write relation (13) as

$$\phi = t_1 + t_2 - T_{u2} + 2\pi(n + 1). \quad (17)$$

One can see that the range of the parameter $n$ is defined by the range of possible values of the first summand. Taking into account relations (12), we get

$$-2\pi < (t_1 + t_2 - T_{u2}) < 2\pi.$$ 

From this, the necessary property of the number $n$ for arcs of the type SS follows immediately.

Similar properties are true for arcs of the type BB. Points of an arc of the type BB$_n$ look like

$$\begin{pmatrix} x_{BB}[n, t_1] \\ y_{BB}[n, t_1] \end{pmatrix} = \begin{pmatrix} \sin \phi \\ 1 - \cos \phi \end{pmatrix} + \frac{(n + 1)(1 - u_1)}{u_1} \begin{pmatrix} \sin(t_1 - T_{u2}) - \sin(t_1 - u_1 + T_{u2}) \\ \cos(t_1 - u_1 + T_{u2}) - \cos(t_1 - u_1) \end{pmatrix}. \quad (18)$$

The case when $\phi$ is multiple to $2\pi$. Below, we shall use the following denotations:

$$t_1^* = \frac{\phi - t_f - u_1}{(n + 1)(1 - u_1)}, \quad t_2^* = \frac{t_f - \phi}{(n + 1)(1 - u_1)}.$$ 

On the basis of (5) and (8) in this case, one can show that there is a unique SB control. It is defined by the durations of the first and last intervals $t_1 = t_1^*, t_2 = t_2^*$ and, also, (for $n > 0$) of the corresponding intermediate intervals $T_{u1} = t_2^*, T_{u2} = t_1^*$. The BS control is uniquely defined too: $t_1 = t_2^*, t_2 = t_1^*, T_{u1} = t_2^*, T_{u2} = t_1^*$.

The boundary of a $\phi$-section is formed by the SS and BB arcs. Due to (17), the number $n$ of loops is defined unambiguously. This case is very similar to the case when $\phi < 2\pi$. But here, the SS and BB arcs are located on the same circle with the radius $2\frac{(n + 1)(1 - u_1)}{u_1} \sin\left(\frac{T_{u2}}{2}\right)$ and the center at the origin. They have common endpoints, and their total angle is $2\pi$.

Thus, in the case when $\phi$ is multiple to $2\pi$, the boundary of a $\phi$-section of an attainability set is a circle consisting of two arcs of the type SS and BB. The points of join have the types SB and BS (see a schematic picture in Fig. 5).
The case when $\varphi > 2\pi$ and $\varphi$ is not multiple to $2\pi$.
Without loss of generality, let us consider arcs of the type SB. From two last inequalities of (8), we have the following linear dependence between the parameters $t_1$ and $t_2$:

$$t_1 + t_2 \cdot u_1 = \varphi - 2\pi n.$$  \hfill (19)

Moreover, taking into account (5), we have $n > 0$ and we can write relations defining the values $T_{u_2}$ and $T_{u_1}$ on the basis of $t_1$ and $t_2$, respectively,

$$T_{u_2} = \frac{1}{n} (t_2^*(n+1) - t_1), \quad T_{u_1} = \frac{1}{n} (t_2^*(n+1) - t_2).$$

Required family of feasible SB controls can be considered as one-parametric. The used values of $t_1$ and $t_2$ must satisfy the inequalities

$$0 < t_1 \leq t_1^*, \quad 0 < t_2 \leq t_2^*.$$  \hfill (20)

These inequalities define a family of SB controls as a segment of straight line (19) crossing rectangle (20) (Fig. 4).

So, we can conclude that under given conditions ($\varphi > 2\pi$ and $\varphi$ is not multiple to $2\pi$), there is a collection of SB controls, which can be defined by the parameter $t_1$ having its values in a non-degenerated segment.

Fig. 4. Feasible parameters of SB controls
Integrating system (1) for the shown SB controls, we obtain a description

$$x_{SB} [t_1], y_{SB} [t_1] = \left( \begin{array}{c}
\sin t_1 \\
1 - \cos t_1 \\
\frac{1}{u_1} \\
\cos t_1 - \cos \varphi \\
\frac{1}{n} + \frac{1}{u_1} - 1 \\
\sin (t_1 - T_{u_2}) - \sin t_1 \\
\cos t_1 - \cos (t_1 - T_{u_2})
\end{array} \right).$$  \hfill (21)

of the geometric position in the plane $x, y$ at the instant $t_f$.

The obtained curve is smooth.

Consider the tangent vector as the derivative on $t_1$ as follows:

$$x_{SB} [t_1]', y_{SB} [t_1]' = (n+1) \left( \begin{array}{c}
\frac{1}{u_1} - 1 \\
\cos (t_1 - T_{u_2}) - \cos t_1 \\
\sin (t_1 - T_{u_2}) - \sin t_1
\end{array} \right).$$

The last term can be rewritten as

$$2 \sin (T_{u_2}/2) \left( \begin{array}{c}
\sin \left( t_1 \frac{2n+1}{2n} - t_1 \frac{n+1}{2n} \right) \\
\cos \left( t_1 \frac{2n+1}{2n} - t_1 \frac{n+1}{2n} \right)
\end{array} \right).$$  \hfill (22)

From this, one can conclude that the shift along an SB arc with growth of the parameter $t_1$ has constant clockwise angular velocity $(1 + 1/2n)$.

For a BS arc in the same way, one can show that the shift with growth of the parameter $t_1$ has constant clockwise angular velocity $(1 + 1/2n) u_1$. With that the range of possible values of $t_1$ is $1/u_1$ times longer. So, the angle of the tangent vector rotation along SB and BS arcs is the same accounting the rotation direction.

Constructing boundary of $\varphi$-sections of attainability set.
Let us continue to consider the case when $\varphi > 2\pi$ and $\varphi$ is not multiple to $2\pi$.

Using the shown above properties of the motions obeying PMP, let us give a description of the corresponding end-points in the plane $x, y$ at the instant $t_f$ located in the considered $\varphi$-section.

Let us take SB arcs as the basis (see (21)). These parts of the curves are smooth and defined by the parameter $t_1$.

For definiteness, consider the case when

$$\varphi - 2\pi n < t_1^*, \quad \varphi - 2\pi n < t_2^* \cdot u_1.$$  

Here, the SB arc is defined by the range of the parameter $t_1$ from $0$ to $\varphi - 2\pi n$ (Fig. 4). The limit points of the arc for $t_1 \rightarrow 0$ and $t_1 \rightarrow (\varphi - 2\pi n)$ are the points generated by the BB and SS controls, respectively. These points are limit ones for the arcs BB, and SS (the variant 1 in Fig. 5)

$$x_{BB} [0], y_{BB} [0] = \left( \begin{array}{c}
\sin \varphi \\
\cos \varphi \\
n \cos \varphi \\
n \sin \varphi
\end{array} \right).$$

The BS arc has the same property.

Finally, we conclude that the collection of arcs is a closed curve consisting of parts of the types SB, SS, BS, and BB with growing the parameter $t_1$ in each part.

Joining of arcs is smooth. This can be easily established.

Let us take, for example, the pair of SB and SS arcs.

At the joining point

$$x_{SS} [n, 0], y_{SS} [n, 0] = \left( \begin{array}{c}
\sin \varphi + (n+1)(1-u_1) \frac{\sin T_{u_2}}{u_1} \\
1 - \cos \varphi + (n+1)(1-u_1) \frac{\cos T_{u_2}}{u_1}
\end{array} \right).$$

in the plane $x, y$, using (16) and (21), one gets the same values of the tangent vector

$$x_{SS} [n, 0]', y_{SS} [n, 0]' = \left( \begin{array}{c}
\frac{1}{u_1} (1-u_1) \cos T_{u_2} \\
\frac{1}{u_1} (1-u_1) \sin T_{u_2}
\end{array} \right).$$

Now, let us shortly show that the obtained curve is the boundary of the corresponding $\varphi$-section of the set $G(t_f)$.

For that, let us make sure that there are no motions obeying PMP except those we have already considered. All the motions of the type SB (BS) have been used for construction of the SB (BS) arcs. So, one need to consider only SS and BB arcs, for which there is ambiguity of choosing the number $n$: there can be two arcs of the same type, but with different $n$. Earlier, it has been shown that for any SS or BB arc its limit points coincide with the limit points of SB and BS arcs. Therefore, total number of the SS and BB arcs cannot be larger than two. So, the boundary of a $\varphi$-section of the set $G(t_f)$ is a subset of the constructed curve.

With that, we have noticed that during passage of the constructed curve that corresponds to growing the parameter $t_1$, the angle of the tangent vector changes monotonically clockwise. In each of four parts, the angular velocity of the tangent vector rotation is constant. So, having the lengths of the corresponding ranges of the parameter $t_1$,
one can compute the total rotation angle, which equals $2\pi$ for $\varphi \geq 2\pi$. Therefore, the constructed curve has no self-intersections.

From this, we conclude that $\varphi$-sections are convex and, in the problem under consideration, PMP is not only necessary, but also sufficient condition for reaching the boundary of an attainability set by a motion. Note, that in general, the three-dimensional set is not convex (see Patsko and Fedotov (2018)).

So, for the case when $\varphi > 2\pi$ and $\varphi$ is not multiple to $2\pi$, we have considered one of the variants of mutual location of straight line (19) and rectangle defined by $\pi$ to 2

the dependence of the structure of the $\varphi$-sections is investigated. Further, we plan to study

of the car is allowed to one side only. The structure of these sections is investigated. Further, we plan to study the dependence of the structure of the $\varphi$-sections on the angular coordinate.

![Fig. 5. Types of $\varphi$-sections of an attainability set](image)

5. CONCLUSION

In the paper, convexity of sections of the Dubins car three-dimensional attainability sets by planes orthogonal to the angular axis was established in the case when the turn of the car is allowed to one side only. The structure of these sections is investigated. Further, we plan to study the dependence of the structure of the $\varphi$-sections on the angular coordinate.

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