Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 51-32 (2018) 201–206

Attainability Set at Instant for One-Side Turning Dubins Car ⋆

V. S. Patsko * A. A. Fedotov *

* Krasovskii Institute of Mathematics and Mechanics, Ekaterinburg, 620990 Russia (e-mail: patsko@imm.uran.ru).

Abstract: Three-dimensional attainability set "at instant" for a non-linear controlled system is studied, which is often called the "Dubins car". A controlled object moves in the plane. It has linear velocity of a constant magnitude and bounded turn radius. The case is explored when the object can turn to one side only. Moreover, a rectilinear motion is prohibited by the constraints onto the control. We prove that the sections of the attainability set by planes orthogonal to the angle coordinate are convex. The geometric structure of these sections is analyzed.

Keywords: Dubins car, one-sided turn, three-dimensional attainability set, Pontryagin maximum principle, piecewise-constant control, convexity of attainability set sections.

1. INTRODUCTION

This paper deals with a study of attainability set at instant for a model of controlled motion in the plane, which is one of the most popular in the mathematical control theory and applied works. Dynamics of the motion with constant magnitude of the linear velocity and prescribed range of possible turn rate is defined by means of a system of three differential equations. Two phase variables describe the geometric location of the object in the plane, the third one is the velocity heading. The scalar control defines the current angular velocity of rotation of the velocity direction or, that is equivalent, the instantaneous turn radius. The value of the control parameter belongs to a closed segment.

Such a model is considered in works: Markov (1889); Isaacs (1951); Dubins (1957). Specialists in robotics call it the "Dubins car". For this three-dimensional controlled system in the framework of time-optimal problem, a synthesis of the feedback control has been constructed for a control constraint, which is symmetric Pecsvaradi (1972) with respect to the origin or which is non-symmetric Bakolas and Tsiotras (2011). In work Berdyshev (2015), the traveling salesman problem is considered, when the moving object is the Dubins car.

Dynamics of the Dubins car is often used for constructing control of autonomous wheel robots (see, for example, Laumond (1998)), for computing flight trajectories in the system of civil aviation Pyatko and Krasov (2004), and, also, in applied works dealing with control of unmanned flying vehicles in the horizontal plane Meyer et al. (2015). In work Choi (2014) some aviation problems are shown, whose investigation can involve the Dubins model with one-side turn.

For the Dubins car, let us call the *attainability set* $G(t_f)$ *at instant* t_f the collection of all points of *three-dimensional phase space* such that to each of them the system can hit at the instant t_f from the given initial state (which is meant to be the origin, without loss of generality) under some feasible control. The attainability set at instant should be distinguished from the attainability set *up to instant*. The latter is a union of all attainability sets at instant corresponding to the instants from 0 to t_f.

Construction of attainability sets $G(t_f)$ for the case when both left and right turns are possible is considered in papers Patsko et al. (2003); Fedotov et al. (2011).

In this paper, we study the sets $G(t_f)$ for the case when the turn is possible *to one side only* and the rectilinear motion is impossible. Namely, it is supposed the the scalar control u belongs to a segment $[u_1, u_2]$ such that $0 < u_1 < u_2$. The case $u_1 = 0$ is studied in works Fedotov and Patsko (2018); Patsko and Fedotov (2018). It was proved that the sections of the attainability set by planes orthogonal to the angular axis are convex in this case. The main objective of this paper is to show that the convexity maintains also for $u_1 > 0$. When studying the boundaries of the sets $G(t_f)$, we use the Pontryagin maximum principle Pontryagin et al. (1962).

2. PROBLEM FORMULATION

Let the dynamics of a controlled object (Dubins car) in the plane x, y be described by the third order system of differential equations

$$\begin{aligned} \dot{x} &= \cos\varphi, \\ \dot{y} &= \sin\varphi, \\ \dot{\varphi} &= u, \qquad u \in [u_1, u_2], \quad 0 < u_1 < u_2. \end{aligned} \tag{1}$$

Here, x, y are coordinates of the object geometric position, φ is the velocity direction angle counted counterclockwise from the axis x (Fig. 1), u is the scalar control. The magnitude of the linear velocity equals unit. We assume that $u_2 = 1$.

Representation (1) with $u_2 = 1$ can be obtained from any arbitrary controlled system of the third order with

⋆ This work was supported by the Russian Foundation for Basic Research under project no. 18-01-00410.

10.1016/j.ifacol.2018.11.381

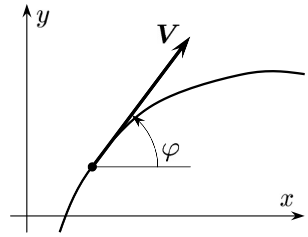

Fig. 1. The coordinate system, $\boldsymbol{V} = (\dot{x}, \dot{y})^{\mathsf{T}}$

constant magnitude of the linear velocity and given range of the turn angular velocity (one-side turn). To do it, one should scale geometric coordinates and time. Without loss of generality, at the initial instant $t_0 = 0$, the initial phase state is the origin: $x_0 = 0$, $y_0 = 0$, $\varphi_0 = 0$.

As feasible controls $u(\cdot)$, we consider measurable functions depending on the time and having their values $u(t)$ from the segment $[u_1, u_2]$. It is assumed that the angular coordinate φ takes its values in the interval $(-\infty, \infty)$.

3. PONTRYAGIN MAXIMUM PRINCIPLE

It is known Lee and Markus (1967) that the controls, which guide the system to the boundary of the attainability set $G(t_f)$, obey the Pontryagin maximum principle (PMP). Let us write the relations of PMP for system (1).

Let $u^*(\cdot)$ be some feasible control and $\big(x^*(\cdot), y^*(\cdot), \varphi^*(\cdot)\big)^{\mathsf{T}}$ be the motion of system (1) generated by the control in the interval $[t_0, t_f]$. The differential equation of the adjoint system are the following:

$$\begin{aligned} \dot{\psi}_1 &= 0, \\ \dot{\psi}_2 &= 0, \\ \dot{\psi}_3 &= \psi_1 \sin\varphi^*(t) - \psi_2 \cos\varphi^*(t). \end{aligned} \tag{2}$$

PMP means that a *non-zero* solution $\big(\psi_1^*(\cdot), \psi_2^*(\cdot), \psi_3^*(\cdot)\big)^{\mathsf{T}}$ of system (2) exists, for which a.e. in the interval $[t_0, t_f]$ the following condition is satisfied:

$$\psi_3^*(t)u^*(t) = \max_{u \in [u_1, u_2]} \psi_3^*(t)u. \tag{3}$$

One can see that the functions $\psi_1^*(\cdot)$ and $\psi_2^*(\cdot)$ are constant. Denote them by ψ_1^* and ψ_2^*.

If $\psi_1^* = 0$ and $\psi_2^* = 0$, then $\psi_3^*(t) = \text{const} \neq 0$ in the entire interval $[t_0, t_f]$. Therefore, in this case, we have that a.e. $u^*(t) = u_1$ or $u^*(t) = u_2$.

Let at least one of the numbers ψ_1^* and ψ_2^* is non-zero. Using the dynamics equations (1) and adjoint system equation (2), one can write the relation for $\psi_3^*(t)$:

$$\psi_3^*(t) = \psi_1^* y^*(t) - \psi_2^* x^*(t) + C.$$

One obtains that $\psi_3^*(t) = 0$ iff the point $\big(x^*(t), y^*(t)\big)^{\mathsf{T}}$ of the geometric position at the instant t obeys the straight line equation

$$\psi_1^* y - \psi_2^* x + C = 0. \tag{4}$$

Since change of the sign of the function $\psi_3^*(\cdot)$ implies switch of the control from one limit value to the another, line (4) is often called the *switching line* (or, shortly, SL).

Due to relation (3), if $\psi_3^*(t) > 0$ in some time interval, then $u^*(t) = u_2$ a.e. in that interval. Projection of the corresponding motion into the plane x, y goes counterclockwise along an arc of the circle with the radius $1/u_2$. If $\psi_3^*(t) < 0$, then $u^*(t) = u_1$ and the motions go counterclockwise along an arc of the circle with the radius $1/u_1$.

If $\psi_3^*(t) = 0$ in some interval, then in this interval the motion $\big(x^*(\cdot), y^*(\cdot)\big)^{\mathsf{T}}$ should go along SL (4). But it is impossible, because $u_1 > 0$.

Thus, when $u_1 > 0$, then the projection of a motion, which obeys PMP, into the plane x, y consists of circle arcs. On each such arc, the control is constant. Therefore, below, when we shall analyze the controls satisfying PMP, we can limit us to *piecewise-constant* controls (for which the right continuity is assumed in the discontinuity points).

A variant of the motion in the case when at least one of the constants ψ_1^* and ψ_2^* is non-zero is shown in Fig. 2. One has $\psi_3^*(t) > 0$ in the semi-plane $\psi_1^* y - \psi_2^* x + C > 0$ and $\psi_3^*(t) < 0$ in the semi-plane $\psi_1^* y - \psi_2^* x + C < 0$.

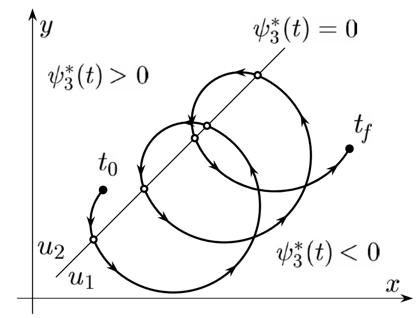

Fig. 2. A trajectory according to PMP

The parts of the motions, which *begin* and *finish* on SL and are located in the semi-plane $\psi_1^* y - \psi_2^* x + C \leq 0$, have the same lengths. Denote it by T_{u1}. Also, the parts in the semi-plane $\psi_1^* y - \psi_2^* x + C \geq 0$ also have the same lengths. Denote this length by T_{u2}. The parts with controls u_1 and u_2 alternate and join on the SL (Fig. 2). Motion along two such neighbor parts gives increasing the angular coordinate φ equal to 2π, that is,

$$T_{u1} \cdot u_1 + T_{u2} = 2\pi. \tag{5}$$

The time necessary for passing such a "loop" is, evidently, larger than 2π. Therefore, a control that leads a motion to the boundary of the attainability set has a finite number of switches.

4. φ-SECTIONS OF ATTAINABILITY SET

Consider a motion of system (1) in the time interval $[t_0, t_f]$ ($t_0 = 0$, $t_f > 0$) with the zero initial state. Possible values $\varphi(t_f)$ of the coordinate φ at the instant t_f are in the interval $\big[t_f \cdot u_1, t_f\big]$. The limit values of φ are obtained by means of controls $u(t) \equiv u_1$ and $u(t) \equiv u_2 = 1$. The corresponding limit points of a φ-section of the set $G(t_f)$ are the following ones in the plane x, y:

$$\left(\frac{\sin(t_f \cdot u_i)}{u_i}, \frac{-\cos(t_f \cdot u_i)}{u_i}\right)^{\mathsf{T}}, \quad i = 1, 2.$$

Below, we *fix* the value $\varphi \in \big(t_f \cdot u_1, t_f\big)$. The obtained φ-section of the set $G(t_f)$ consists of more than one point.

The motions reaching the boundary of such a section obey PMP and should have not less than one switch (not less than two intervals of constant control). We use this for classification of the controls and corresponding points on the boundary of the section. Finally, we shall establish the structure of φ-sections and their convexity.

Introduce the following denotations. The symbol t_1 (t_2) denotes the length of the first (last) interval of control constancy, which joins to the instant t_0 (t_f).

The motions with controls obeying PMP under the constant u_1 during the first interval (that is, with motion along an arc of a large circle) and with the constant u_2 during the last interval (that is, with motion along an arc of a small circle) are attributed to the family “BS”. In the same way, we define the families “SB”, “BB”, “SS” with pairs of the controls (u_2, u_1), (u_1, u_1), (u_2, u_2) during the first and last intervals. Each control obeying PMP belong to one and only one of these four families. The denotations will be used both for the controls and motions.

4.1 Boundary of φ-sections in the case $\varphi < 2\pi$

1. Consider a motion with *one switch* of the control u. Let the control during the first interval be equal u_1 and during the second one be equal $u_2 = 1$. The duration of the first interval is t_1, and the duration of the second one is t_2. One has $t_1{\cdot}u_1 < 2\pi$, $t_2 < 2\pi$. The relations are true

$$\varphi = t_1{\cdot}u_1 + t_2, \qquad t_f = t_1 + t_2.$$

From this, we obtain that for a fixed value φ, the values t_1, t_2, and, therefore, the switching instant are defined unambiguously. So, in φ-section of the set $G(t_f)$ with $\varphi < 2\pi$ only one point corresponds to the considered order of the controls u_1, u_2.

In the same way, to the sequence u_2, u_1 of controls, we get one point in the considered φ-section of the set $G(t_f)$.

2a) Consider a variant with *two switches* and a sequence u_1, u_2, u_1 of the controls. The durations of the intervals are t_1, T_{u2}, t_2. One has

$$\varphi = t_1{\cdot}u_1 + T_{u2} + t_2{\cdot}u_1, \qquad t_f = t_1 + T_{u2} + t_2.$$

It follows that

$$T_{\rm B} = t_1 + t_2 = \frac{t_f - \varphi}{1 - u_1}, \qquad T_{u2} = \frac{\varphi - t_f{\cdot}u_1}{1 - u_1}.$$

Consequently, duration of the middle interval and the total duration of the first and last intervals are constant. The obtained family of controls is one-parameter. Let us take the duration t_1 as the parameter of the family; its interval is $t_1 \in (0, T_{\rm B})$, where $T_{\rm B} < 2\pi/u_1$.

The corresponding points $\big(x_{\rm BB}[t_1], y_{\rm BB}[t_1]\big)^{\mathsf T}$ of the φ-sections of the set $G(t_f)$ after integration of equations (1) and some trigonometric transformations have the form

$$\begin{pmatrix} x_{\rm BB}[t_1] \\ y_{\rm BB}[t_1] \end{pmatrix} = \frac{1}{u_1}\begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix}$$
$$- 2\left(\frac{1}{u_1} - 1\right)\sin\left(\frac{T_{u2}}{2}\right)\begin{pmatrix} \cos\left(t_1{\cdot}u_1 + \dfrac{T_{u2}}{2}\right) \\ \sin\left(t_1{\cdot}u_1 + \dfrac{T_{u2}}{2}\right) \end{pmatrix}. \tag{6}$$

The found collection of points $\big(x_{\rm BB}[t_1], y_{\rm BB}[t_1]\big)^{\mathsf T}$ formed by the sequence u_1, u_2, u_1 of controls under fixed values of u_1, t_f, φ is a circle arc with the center at the point $\left(\dfrac{1}{u_1}\right)\begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix}$. The radius of the circle is $2\left(\dfrac{1}{u_1} - 1\right)\sin\left(\dfrac{T_{u2}}{2}\right)$, and the angle of arc (6) is defined by the range of the value t_1 and equal to $\dfrac{u_1(t_f - \varphi)}{1 - u_1}$.

2b) Consider the second variant with *two switches* and the sequence u_2, u_1, u_2 of the controls. Durations of the corresponding intervals of control constancy are t_1, T_{u1}, t_2. Similarly to the case 2a, we have

$$T_{\rm S} = t_1 + t_2 = \frac{\varphi - t_f{\cdot}u_1}{1 - u_1}, \qquad T_{u1} = \frac{t_f - \varphi}{1 - u_1}.$$

The obtained collection of points is defined by the relations

$$\begin{pmatrix} x_{\rm SS}[t_1] \\ y_{\rm SS}[t_1] \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix}$$
$$+2\left(\frac{1}{u_1} - 1\right)\sin\left(\frac{T_{u1}{\cdot}u_1}{2}\right)\begin{pmatrix} \cos\left(t_1 + \dfrac{T_{u1}{\cdot}u_1}{2}\right) \\ \sin\left(t_1 + \dfrac{T_{u1}{\cdot}u_1}{2}\right) \end{pmatrix} \tag{7}$$

and the range of possible values of the parameter $t_1 \in (0, T_{\rm S})$. This collection also is a circle arc with the center at the point $\begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix}$ and the radius equal to $2\left(\dfrac{1}{u_1} - 1\right)\sin\left(\dfrac{T_{u1}{\cdot}u_1}{2}\right)$. The angle of the arc is defined by the range of the value t_1 and equal to $\dfrac{\varphi - t_f{\cdot}u_1}{1 - u_1}$.

3. One can easily establish that limit points of arcs (6) and (7) coincide. Namely,

$$\begin{pmatrix} x_{\rm BB}[0] \\ y_{\rm BB}[0] \end{pmatrix} = \begin{pmatrix} x_{\rm SS}[T_{\rm S}] \\ y_{\rm SS}[T_{\rm S}] \end{pmatrix}, \qquad \begin{pmatrix} x_{\rm BB}[T_{\rm B}] \\ y_{\rm BB}[T_{\rm B}] \end{pmatrix} = \begin{pmatrix} x_{\rm SS}[0] \\ y_{\rm SS}[0] \end{pmatrix}.$$

These limit points correspond to the motion with one switch considered earlier.

So, analyzing all possible variants of system (1) motions, which obey PMP, we get a locus in the plane x, y as a closed curve consisting of two circle arcs joining by their limit points.

4. Consider a shift along arc (6) defined by the parameter t_1 under its growth from 0 to $T_{\rm B}$. This shift is accompanied by clockwise rotation of the tangent vector. In the same way, a shift along arc (7) with increasing the parameter t_1 from 0 to $T_{\rm S}$ also gives clockwise rotation of the tangent vector. The total angle of arcs (6) and (7) equals

$$\frac{(\varphi - t_f{\cdot}u_1)}{1 - u_1} + \frac{u_1(t_f - \varphi)}{1 - u_1} \;=\; \varphi.$$

According to the assumption $\varphi < 2\pi$ and taking into account that $\varphi \in \big(t_f{\cdot}u_1, t_f\big)$, we obtained the curve as the sought boundary description of the φ-section of the set $G(t_f)$. Such a section is strictly convex. Its boundary consists of two circle arcs.

In more details, computations of this subsection can be found in Patsko and Fedotov (2018) for the case $t_f \le 2\pi$.

4.2 Boundary of φ-sections in the case $\varphi \ge 2\pi$

Properties of controls SB/BS. Motions of the types SB and BS have similar (accurately w.r.t. the reverse of the time) representations with even number of intermediate intervals (integer number of loops). For definiteness, let us consider relations connecting the parameters of motions of type SB (Fig. 2). Durations of the intervals of control constancy obey the relations

$$0 < t_1 \le T_{u2} < 2\pi, \quad 0 < t_2 \le T_{u1} < \frac{2\pi}{u_1}. \tag{8}$$

For fixed values u_1, t_f, φ, we have

$$\varphi = t_1 + n(T_{u1}{\cdot}u_1 + T_{u2}) + t_2{\cdot}u_1, \tag{9}$$

$$t_f = t_1 + n(T_{u1} + T_{u2}) + t_2. \tag{10}$$

Here, $n = \text{const} \ge 0$ is the number of loops:

$$n = \begin{cases} \dfrac{\varphi}{2\pi} - 1 & \text{if } \varphi \text{ is multiple to } 2\pi, \\ \left[\dfrac{\varphi}{2\pi}\right] & \text{otherwise.} \end{cases} \tag{11}$$

Here, square brackets denote the integer part.

Properties of controls SS/BB. In the case SS (BB) between the first and last intervals, there are odd number of intervals of control constancy. They can be described as an integer number of loops and one additional interval with the control u_1 (u_2), which has the duration T_{u1} (T_{u2}).

Consider controls of the type SS. Durations of the control constancy intervals obey the relations

$$0 < t_1 \le T_{u2} < 2\pi,\ 0 < t_2 \le T_{u2} < 2\pi,\ 0 < T_{u1} < \frac{2\pi}{u_1}. \tag{12}$$

For fixed values u_1, t_f, φ, n, one has

$$\varphi = t_1 + n{\cdot}(T_{u1}{\cdot}u_1 + T_{u2}) + T_{u1}{\cdot}u_1 + t_2, \tag{13}$$

$$t_f = t_1 + n{\cdot}(T_{u1} + T_{u2}) + T_{u1} + t_2. \tag{14}$$

Taking into account (5), (13), and (14), one can conclude that $T_{u1} = \text{const}$, $T_{u2} = \text{const}$, and $t_1 + t_2 = \text{const}$. Write the latter in more details

$$T_{\text{S}} = t_1 + t_2 = \varphi + T_{u2} - 2\pi(n+1). \tag{15}$$

Relation (15) together with constraints-inequalities (12) defines the collection of controls of the type SS, which have the given number of loops equal to n (Fig. 3). If this set is non-degenerated (that is, the parameter t_1 is taken from a non-degenerated interval), then the points $\big(x(t_f), y(t_f)\big)^{\mathsf{T}}$ defined by means of (15) in the φ-section of the set $G(t_f)$ can be considered as a one-parameter curve.

In the non-degenerated case, it is a continuous smooth curve (a circle arc) in the plane x, y, namely

$$\begin{pmatrix} x_{\text{SS}}[n,t_1] \\ y_{\text{SS}}[n,t_1] \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ 1-\cos\varphi \end{pmatrix} + \frac{(n+1)(1-u_1)}{u_1} \begin{pmatrix} \sin(t_1 - T_{u2}) - \sin t_1 \\ \cos t_1 - \cos(t_1 - T_{u2}) \end{pmatrix}. \tag{16}$$

Let denote it by SS_n. Shifting along this arc is uniform on t_1 with clockwise rotation of the tangent vector; the angle of this rotation equals the increment of the parameter t_1.

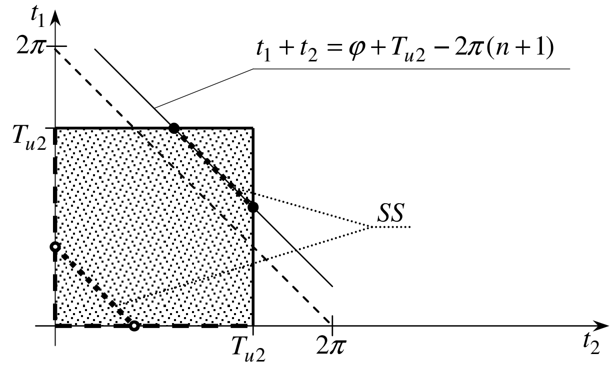

Fig. 3. Feasible parameters of a control of the type SS

For a control of the type SS, the number n of loops can be ambiguous. But the number of non-degenerated arcs of the type SS is not larger than two. With that, if there are two arcs of the type SS, then the difference between the number of loops in them is not larger than unit. To prove this, let us write relation (13) as

$$\varphi = t_1 + t_2 - T_{u2} + 2\pi(n+1). \tag{17}$$

One can see that the range of the parameter n is defined by the range of possible values of the first summand. Taking into account relations (12), we get

$$-2\pi < (t_1 + t_2 - T_{u2}) < 2\pi.$$

From this, the necessary property of the number n for arcs of the type SS follows immediately.

Similar properties are true for arcs of the type BB. Points of an arc of the type BB_n look like

$$\begin{pmatrix} x_{\text{BB}}[n,t_1] \\ y_{\text{BB}}[n,t_1] \end{pmatrix} = \frac{1}{u_1}\begin{pmatrix} \sin\varphi \\ 1-\cos\varphi \end{pmatrix} + \frac{(n+1)(1-u_1)}{u_1} \begin{pmatrix} \sin(t_1{\cdot}u_1) - \sin(t_1{\cdot}u_1 + T_{u2}) \\ \cos(t_1{\cdot}u_1 + T_{u2}) - \cos(t_1{\cdot}u_1) \end{pmatrix}. \tag{18}$$

The case when φ is multiple to 2π. Below, we shall use the following denotations:

$$t_1^* = \frac{\varphi - t_f{\cdot}u_1}{(n+1)(1-u_1)}, \quad t_2^* = \frac{t_f - \varphi}{(n+1)(1-u_1)}.$$

On the basis of (5) and (8) in this case, one can show that there is a unique SB control. It is defined by the durations of the first and last intervals $t_1 = t_1^*$, $t_2 = t_2^*$ and, also, (for $n > 0$) of the corresponding intermediate intervals $T_{u1} = t_2^*$, $T_{u2} = t_1^*$. The BS control is uniquely defined too: $t_1 = t_2^*$, $t_2 = t_1^*$, $T_{u1} = t_2^*$, $T_{u2} = t_1^*$.

The boundary of a φ-section is formed by the SS and BB arcs. Due to (17), the number n of loops is defined unambiguously. This case is very similar to the case when $\varphi < 2\pi$. But here, the SS and BB arcs are located on the same circle with the radius $2\left(\dfrac{(n+1)(1-u_1)}{u_1}\right)\sin\left(\dfrac{T_{u2}}{2}\right)$ and the center at the origin. They have common endpoints, and their total angle is 2π.

Thus, in the case when φ is multiple to 2π, the boundary of a φ-section of an attainability set is a circle consisting of two arcs of the type SS and BB. The points of join have the types SB and BS (see a schematic picture in Fig. 5).

The case when $\varphi > 2\pi$ and φ is not multiple to 2π. Without loss of generality, let us consider arcs of the type SB. From two last inequalities of (8), we have the following linear dependence between the parameters t_1 and t_2:

$$t_1 + t_2 \cdot u_1 = \varphi - 2\pi n. \tag{19}$$

Moreover, taking into account (5), we have $n > 0$ and we can write relations defining the values T_{u2} and T_{u1} on the basis of t_1 and t_2, respectively,

$$T_{u2} = \frac{1}{n}\left(t_1^*(n+1) - t_1\right), \quad T_{u1} = \frac{1}{n}\left(t_2^*(n+1) - t_2\right).$$

Required family of feasible SB controls can be considered as one-parametric. The used values of t_1 and t_2 must satisfy the inequalities

$$0 < t_1 \le t_1^*, \quad 0 < t_2 \le t_2^*. \tag{20}$$

These inequalities define a family of SB controls as a segment of straight line (19) crossing rectangle (20) (Fig. 4). So, we can conclude that under given conditions ($\varphi > 2\pi$ and φ is not multiple to 2π), there is a collection of SB controls, which can be defined by the parameter t_1 having its values in a non-degenerated segment.

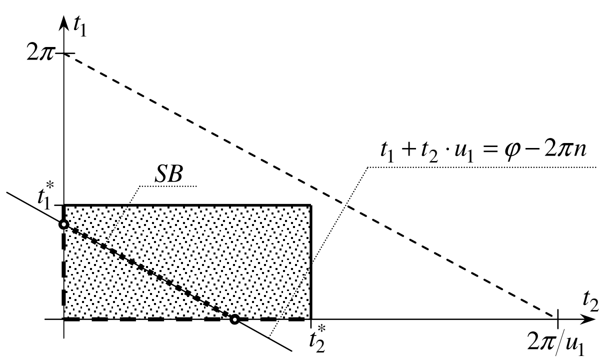

Fig. 4. Feasible parameters of SB controls

Integrating system (1) for the shown SB controls, we obtain a description

$$\begin{pmatrix} x_{\text{SB}}[t_1] \\ y_{\text{SB}}[t_1] \end{pmatrix} = \begin{pmatrix} \sin t_1 \\ 1 - \cos t_1 \end{pmatrix} + \frac{1}{u_1}\begin{pmatrix} \sin\varphi - \sin t_1 \\ \cos t_1 - \cos\varphi \end{pmatrix} + n\left(\frac{1}{u_1} - 1\right)\begin{pmatrix} \sin(t_1 - T_{u2}) - \sin t_1 \\ \cos t_1 - \cos(t_1 - T_{u2}) \end{pmatrix} \tag{21}$$

of the geometric position in the plane x, y at the instant t_f. The obtained curve is smooth.

Consider the tangent vector as the derivative on t_1 as follows:

$$\begin{pmatrix} x_{\text{SB}}[t_1] \\ y_{\text{SB}}[t_1] \end{pmatrix}'_{t_1} = (n+1)\left(\frac{1}{u_1} - 1\right)\begin{pmatrix} \cos(t_1 - T_{u2}) - \cos t_1 \\ \sin(t_1 - T_{u2}) - \sin t_1 \end{pmatrix}.$$

The last term can be rewritten as

$$2\sin(T_{u2}/2)\begin{pmatrix} \sin\left(t_1\dfrac{2n+1}{2n} - t_1^*\dfrac{n+1}{2n}\right) \\ \cos\left(t_1\dfrac{2n+1}{2n} - t_1^*\dfrac{n+1}{2n}\right) \end{pmatrix}. \tag{22}$$

From this, one can conclude that the shift along an SB arc with growth of the parameter t_1 has constant clockwise angular velocity $(1 + 1/2n)$.

For a BS arc in the same way, one can show that the shift with growth of the parameter t_1 has constant clockwise angular velocity $(1 + 1/2n)u_1$. With that the range of possible values of t_1 is $1/u_1$ times longer. So, the angle of the tangent vector rotation along SB and BS arcs is the same accounting the rotation direction.

Constructing boundary of φ-sections of attainability set. Let us continue to consider the case when $\varphi > 2\pi$ and φ is not multiple to 2π.

Using the shown above properties of the motions obeying PMP, let us give a description of the corresponding endpoints in the plane x, y at the instant t_f located in the considered φ-section.

Let us take SB arcs as the basis (see (21)). These parts of the curves are smooth and defined by the parameter t_1. For definiteness, consider the case when

$$\varphi - 2\pi n < t_1^*, \quad \varphi - 2\pi n < t_2^* \cdot u_1.$$

Here, the SB arc is defined by the range of the parameter t_1 from 0 to $\varphi - 2\pi n$ (Fig. 4). The limit points of the arc for $t_1 \to 0$ and $t_1 \to (\varphi - 2\pi n)$ are the points generated by the BB and SS controls, respectively. These points are limit ones for the arcs BB_n and SS_n (the variant 1 in Fig. 5)

$$\begin{pmatrix} x_{\text{SB}}[0] \\ y_{\text{SB}}[0] \end{pmatrix} = \begin{pmatrix} x_{\text{BB}}[n, T_{\text{S}}] \\ y_{\text{BB}}[n, T_{\text{S}}] \end{pmatrix}, \qquad \begin{pmatrix} x_{\text{SB}}[\varphi - 2\pi n] \\ y_{\text{SB}}[\varphi - 2\pi n] \end{pmatrix} = \begin{pmatrix} x_{\text{SS}}[n, 0] \\ y_{\text{SS}}[n, 0] \end{pmatrix}.$$

The BS arc has the same property.

Finally, we conclude that the collection of arcs is a closed curve consisting of parts of the types SB, SS_n, BS, and BB_n with growing the parameter t_1 in each part.

Joining of arcs is smooth. This can be easily established. Let us take, for example, the pair of SB and SS_n arcs. At the joining point

$$\begin{pmatrix} x_{\text{SS}}[n, 0] \\ y_{\text{SS}}[n, 0] \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix} + \frac{(n+1)(1-u_1)}{u_1}\begin{pmatrix} -\sin T_{u2} \\ 1 - \cos T_{u2} \end{pmatrix}$$

in the plane x, y, using (16) and (21), one gets the same values of the tangent vector

$$\begin{pmatrix} x_{\text{SS}}[n, 0] \\ y_{\text{SS}}[n, 0] \end{pmatrix}'_{t_1} = \frac{(n+1)(1-u_1)}{u_1}\begin{pmatrix} \cos T_{u2} - 1 \\ -\sin T_{u2} \end{pmatrix}.$$

Now, let us shortly show that the obtained curve is the boundary of the corresponding φ-section of the set $G(t_f)$. For that, let us make sure that there are no motions obeying PMP except those we have already considered. All the motions of the type SB (BS) have been used for construction of the SB (BS) arcs. So, one need to consider only SS and BB arcs, for which there is ambiguity of choosing the number n: there can be two arcs of the same type, but with different n. Earlier, it has been shown that for any SS or BB arc its limit points coincide with the limit points of SB and BS arcs. Therefore, total number of the SS and BB arcs cannot be larger than two. So, the boundary of a φ-section of the set $G(t_f)$ is a subset of the constructed curve.

With that, we have noticed that during passage of the constructed curve that corresponds to growing the parameter t_1, the angle of the tangent vector changes monotonically clockwise. In each of four parts, the angular velocity of the tangent vector rotation is constant. So, having the lengths of the corresponding ranges of the parameter t_1, one can compute the total rotation angle, which equals 2π for $\varphi \geq 2\pi$. Therefore, the constructed curve has no self-intersections.

From this, we conclude that φ-sections are convex and, in the problem under consideration, PMP is not only necessary, but also sufficient condition for reaching the boundary of an attainability set by a motion. Note, that in general, the three-dimensional set is not convex (see Patsko and Fedotov (2018)).

So, for the case when $\varphi > 2\pi$ and φ is not multiple to 2π, we have considered one of the variants of mutual location of straight line (19) and rectangle defined by ranges (20) of possible values of t_1 and t_2 (Fig. 4). This variant is schematically shown with number 1 in Fig. 5. Here, the boundary of a φ-section of the set $G(t_f)$ consists of four non-degenerated SB, SS, BS, and BB arcs joining smoothly. Other variants can be considered similarly, and one also obtains convexity of attainability sets and uniqueness of the control leading the system to a given point of the boundary of the φ-section. In Figure 5, all possible variants are schematically shown. The variants 2–4 have the boundary of a φ-section, which consists of four non-degenerated arcs. In contrast to the variant 1, these variants have another types and/or order of the arcs. The variants 5–8 on of the SS or BB arcs degenerates to a point. In the variant 9, both SS and BB arcs degenerate. In Figure 5, one can see also the scheme of a φ-section for the case $\varphi < 2\pi$ (the variant 10) and for the case, when φ is multiple to 2π (the variant 11). In these cases, the SB and BS arcs degenerate.

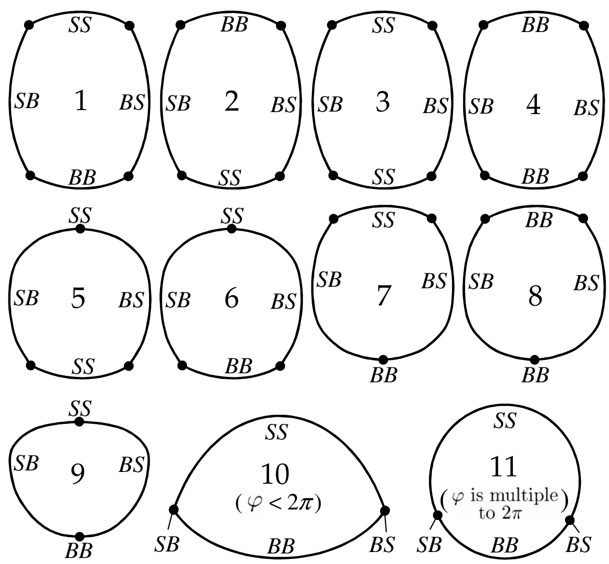

Fig. 5. Types of φ-sections of an attainability set

5. CONCLUSION

In the paper, convexity of sections of the Dubins car three-dimensional attainability sets by planes orthogonal to the angular axis was established in the case when the turn of the car is allowed to one side only. The structure of these sections is investigated. Further, we plan to study the dependence of the structure of the φ-sections on the angular coordinate.

ACKNOWLEDGEMENTS

The authors are grateful to S.S. Kumkov for the help in preparing English text of the paper.

REFERENCES

Bakolas, E. and Tsiotras, P. (2011). Optimal synthesis of the asymmetric sinistral/dextral Markov-Dubins problem. *Journal of Optimization Theory and Applications*, 150(2), 233–250.

Berdyshev, Y. (2015). *Nonlinear Problems in Sequential Control and Their Application.* IMM UB RAS, Ekaterinburg. (in Russian).

Choi, H. (2014). *Time-Optimal Paths for a Dubins Car and Dubins Airplane with a Unidirectional Turning Constraint.* Ph.D. thesis, University of Michigan.

Dubins, L. (1957). On curves of minimal length with a constraint on average curvature and with prescribed initial and terminal positions and tangents. *American Journal of Mathematics*, 79(3), 497–516.

Fedotov, A., Patsko, V., and Turova, V. (2011). Reachable sets for simple models of car motion. In A. Topalov (ed.), *Recent Advances in Mobile Robotics*, chapter 8, 147–172. InTech, Rijeka, Croatia.

Fedotov, A. and Patsko, V. (2018). Investigation of reachable set at instant for the Dubins' car. In *Proceedings of the 58th Israel Annual Conference on Aerospace Sciences*, 1655–1669. Tel-Aviv & Haifa.

Isaacs, R. (1951). Games of pursuit. Technical report, Scientific report of the RAND Corporation, Santa Monica.

Laumond, J.P. (ed.) (1998). *Robot Motion Planning and Control*, volume 229 of *Lecture Notes in Control and Information Sciences.* Springer-Verlag, Berlin Heidelberg.

Lee, E. and Markus, L. (1967). *Foundations of Optimal Control Theory.* Wiley & Sons, New York.

Markov, A. (1889). A few examples of solving special problems on the largest and smallest values. *The communications of the Kharkov mathematical society. Series 2*, 1(2), 250–276. (in Russian).

Meyer, Y., Shima, T., and Isaiah, P. (2015). On Dubins paths to intercept a moving target. *Automatica*, 256–263.

Patsko, V. and Fedotov, A. (2018). Reahable set at a certain time for a Dubins car in the case of a one-sided turn. *Proceedings of the Institute of Mathematics and Mechanics*, 24(1), 143–155. (in Russian).

Patsko, V., Pyatko, S., and Fedotov, A. (2003). Three-dimensional reachability set for a nonlinear control system. *Journal of Computer and Systems Sciences International*, 42(3), 320–328.

Pecsvaradi, T. (1972). Optimal horizontal guidance law for aircraft in the terminal area. *IEEE Trans. on Automatic Control*, 17(6), 763–772.

Pontryagin, L., Boltyanskii, V., Gamkrelidze, R., and Mischenko, E. (1962). *The Mathematical Theory of Optimal Processes.* Interscience, New York.

Pyatko, S. and Krasov, A. (eds.) (2004). *Avtomatizirovannye sistemy upravleniya vozdushnym dvizheniem: Novye informatsionnye tekhnologii v aviatsii (Automated Air Traffic Control Systems: New Informational Technologies in Aviation).* Politekhnika, St. Petersburg. (in Russian).