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Localization with Several Instants of Signal Transmission in Multilateration Systems ⋆

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Abstract: The problem of object localization in multilateration systems is considered in the case when measurements (times of arrival, or TOA) of several consecutive signal transmissions are processed together. The suggested solution of the problem is based on minimization of the sum of residuals between TOA and their model. We proposed an effective numerical method for this optimization task, which accuracy is close to the Cramer–Rao lower bound for the corresponding observation equations. The results of work of the algorithm on simulated data with real locations of the receiving stations are presented.

Keywords: Multilateration (MLAT), Nonlinear Least Squares, Cramer–Rao Lower Bound, Levenberg–Marquardt Algorithm

1. PROBLEM OF MULTILATERATION

The problem of multilateration (MLAT) is as follows: at some unknown time instant t the object (aircraft) under observation, which is at location r, transmits a radio signal. It can be a reply of the airborne transponder to the secondary radar request or a signal of the ADS-B system. This signal is received by several stations (there are m stations) with known coordinates $\{r_i\}_{i=1}^m$. The receiving station i records the time of arrival (TOA) t_i of this signal with a random measurement error w_i. We can write the following observation equation (c is the speed of light):

$$\begin{cases} t_i = t + \dfrac{1}{c}\|r - r_i\| + w_i, \\ i = 1, \dots, m. \end{cases} \tag{1}$$

Multilateration algoritm have to produce an estimate of the object position $\hat{r}$ so that the estimation error would be as less as possible. We assume the estimation error as mean squared error (MSE) $\mathbf{E}\{(\hat{r} - r)^2\}$.

Note that in the literature (for example, I.A.Mantilla-Gaviria et al. (2015)) the problem of multilateration is often considered for the case when the measurements are not times of arrival (TOA) $\{t_i\}_{i=1}^m$, but the time differences of arrival (TDOA) $\{t_i - t_j\}_{i,j\in P}$ for some pairs P of receivers. However, taking into account the features of architecture of the system, where a practical application is possible, we are interested in the statement with the TOA measurements. For this case, the mathematical statement of the multilateration problem almost completely coincides with the statement of the problem of localization in the global positioning system (GPS). But there are differences. So, in the case of GPS, the unknown variable is not the transmission time t, but the time bias between the receiver clock and the satellite clock. Also, the accuracy of time measurement in the receivers and the typical values of the station coordinates r_i are different. In multilateration, the stations are usually located on the Earth surface, and the observed object is usually an aircraft, which is also near the surface (in comparison with the satellites in the case of GPS). As a consequence, all the vectors $r - r_i$ are close to the plane of the local horizon, which makes the task difficult from the view point of numerical methods, and even makes the solution impossible in some cases. The estimation error along the vertical direction is especially large.

Another feature of multilateration is bad detection of a signal that often arises due to the shading of the propagation path by obstacles on the Earth surface. If this happens at some station (for example, i), there is no corresponding measurement t_i. If the number of stations that received the signal is less than 4, it is impossible to make the estimate using the remaining measurements. These peculiarities led us to the idea of combining the measurements obtained from several successive instants of signal transmission and making a joint estimate. It would be possible to use all the measurements available from the beginning of the observation, and to perform filtering using the model of the moving object (see Bar-Shalom et al. (2004)). But such a solution does not look universal; for different aircraft, it would be necessary to use different filters with specific settings. The combination of measurements into small batches and using the simplest assumption of straight line and steady speed motion looks as a more useful and straightforward approach. But this easy method can already reduce the effect of lost measurements and increase the accuracy of estimation.

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2. MODEL OF OBSERVATIONS. CRAMER–RAO LOWER BOUND

The observation model for the problem with several signal transmission instants is based on equations (1). Let us denote by t the time of last transmission and accept that it is the time for which it is necessary to provide the estimate $\hat{r}$. Assume that the instants of signal transmission are $t^j = t + \Delta^j$, the number of them is n. As t corresponds to the last transmission, following equations are fulfilled: $t = t^n$, $\Delta^n = 0$. The measurement instants and their random errors are denoted by t_i^j and $\tilde{w}_i^j$, respectively. We assume that the random errors $\tilde{w}_i^j$ are independent and identically distributed for all i, j, and have zero mean: $\mathbf{E}\left\{\tilde{w}_i^j\right\} = 0$. The following equations are fulfilled

$$\begin{cases} t_i^j = t + \Delta^j + \dfrac{1}{c}\|r + v\Delta^j - r_i\| + \tilde{w}_i^j, \\ i \in I^j, \quad j = 1, \dots, n. \end{cases} \tag{2}$$

Here, I^j is the set of indices of the stations that have received the signal at the time t^j (they can be not all stations, and at each time the set can change); r and v are the position and velocity of the object at the instant t.

For simplicity, it is useful to multiply equations (2) to c. Let us denote $\tau_i^j = ct_i^j$, $\tau = ct$, $\delta^j = c\Delta^j$, and $w_i^j = c\tilde{w}_i^j$, then equation (2) is modified as follows

$$\begin{cases} \tau_i^j = \tau + \delta^j + \|r + v\Delta^j - r_i\| + w_i^j, \\ i \in I^j, \quad j = 1, \dots, n. \end{cases} \tag{3}$$

In the observation model (3), there are $7 + n$ independent variables: $r, v \in \mathbb{R}^3$, $\tau, \delta^1, \dots, \delta^n \in \mathbb{R}$. Equality $\delta^n = 0$ removes only one of them, so total number of the variables is significantly large. Often, in practice, the differences $\{\delta^j\}_{j=1}^n$ of transmission instants can be easily recovered. Let us consider the case of known the differences δ^j (and Δ^j). For sake of brevity, we denote the vector of the parameters as a $\theta = \begin{bmatrix} r^\mathsf{T} & v^\mathsf{T} & \tau \end{bmatrix}^\mathsf{T}$.

For the observation model (3), it is possible to construct the Cramer–Rao lower bound of accuracy (see Chernyak (2008)), which shows potentially achievable accuracy of unbiased estimators. Let us introduce unit vectors of the direction from the location r_i of the station i to the object location r (the position of the object at the instant t)

$$e_i^j(\theta) = \frac{r + v\Delta^j - r_i}{\|r + v\Delta^j - r_i\|}.$$

Below, we refer this quantity as e_i^j without the symbol θ in brackets. The probability density function $\rho_t(\tau_i^j)$ of the measurements τ_i^j is expressed through the density $\rho_w(\cdot)$ of the random variable w_i^j (it is the same for all i, j because the errors w_i^j are independent and identically distributed)

$$\rho_t(\tau_i^j|\theta) = \rho_w\Big(f_i^j(\tau_i^j;\theta)\Big) = \rho_w\Big(\tau_i^j - g_i^j(\theta)\Big)$$

where f_i^j are residual functions

$$f_i^j(\tau_i^j;\theta) = \tau_i^j - \tau - \delta^j - \|r + v\Delta^j - r_i\| \tag{4}$$

between the observations τ_i^j and their models

$$g_i^j(\theta) = \tau + \Delta^j + \|r + v\Delta^j - r_i\|.$$

The gradient of residual function (we suppose that gradients are columns) have the form

$$\begin{aligned} \nabla f_i^j(\tau_i^j;\theta) &= -\nabla g_i^j(\theta) \\ &= -\begin{bmatrix} \dfrac{\partial}{\partial r}\|r + v\Delta^j - r_i\| \\ \dfrac{\partial}{\partial v}\|r + v\Delta^j - r_i\| \\ 1 \end{bmatrix} = -\begin{bmatrix} e_i^j \\ \Delta^j e_i^j \\ 1 \end{bmatrix} \end{aligned} \tag{5}$$

and does not depend on τ_i^j. Using equalities

$$\frac{\partial}{\partial r}\, e_i^j = \frac{1}{\|r + v\Delta^j - r_i\|}\left(I - e_i^j e_i^{j\mathsf{T}}\right), \ \frac{\partial}{\partial v}\, e_i^j = \Delta^j \frac{\partial}{\partial r}\, e_i^j,$$

we can write the hessian of f_i^j too

$$\begin{aligned} \nabla^2 f_i^j(\tau_i^j;\theta) &= -\nabla^2 g_i^j(\theta) \\ &= -\frac{1}{\|r + v\Delta^j - r_i\|}\begin{bmatrix} E_i^j & \Delta^j E_i^j & 0 \\ \Delta^j E_i^j & (\Delta^j)^2 E_i^j & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{6}$$

Here, we denote the projection matrix $\left(I - e_i^j e_i^{j\mathsf{T}}\right)$ as E_i^j.

Let us to calculate the Fisher information matrix

$$\begin{aligned} I(\theta) &= \sum_{j=1}^n \sum_{i \in I^j} \mathbf{E}\left\{\left(\frac{\partial}{\partial\theta}\ln\rho_t(\tau_i^j|\theta)\right)\left(\frac{\partial}{\partial\theta}\ln\rho_t(\tau_i^j|\theta)\right)^\mathsf{T}\right\} \\ &= \sum_{j=1}^n \sum_{i \in I^j} \int \left(-\frac{\dot{\rho}_w(\tau_i^j - g_i^j(\theta))}{\rho_w(\tau_i^j - g_i^j(\theta))}\nabla g_i^j(\theta)\right) \\ &\times \left(-\frac{\dot{\rho}_w(\tau_i^j - g_i^j(\theta))}{\rho_w(\tau_i^j - g_i^j(\theta))}\nabla g_i^j(\theta)\right)^\mathsf{T} \rho_w(\tau_i^j - g_i^j(\theta))\, d\tau_i^j \\ &= \kappa \sum_{j=1}^n \sum_{i \in I^j} \begin{bmatrix} e_i^j e_i^{j\mathsf{T}} & \Delta^j e_i^j e_i^{j\mathsf{T}} & e_i^j \\ \Delta^j e_i^j e_i^{j\mathsf{T}} & (\Delta^j)^2 e_i^j e_i^{j\mathsf{T}} & \Delta^j e_i^j \\ e_i^{j\mathsf{T}} & \Delta^j e_i^{j\mathsf{T}} & 1 \end{bmatrix} \\ &=: \kappa D(\theta)^{-1}. \end{aligned} \tag{7}$$

Here

$$\kappa = \int \left(\dot{\rho}_w(w)^2/\rho_w(w)\right) dw$$

is a constant depending on the properties of the random measurement errors w_i^j only, and the matrix $D(\theta)$ expresses the geometric properties of the observation problem and has name the Dilution of Precision (DOP) matrix. The Cramer–Rao inequality (see Borovkov (1984))

$$\mathbf{E}\left\{\left(\hat{\theta} - \theta\right)\left(\hat{\theta} - \theta\right)^\mathsf{T}\right\} \geq I(\theta)^{-1} = \kappa^{-1} D(\theta) \tag{8}$$

stadns that the accuracy of an estimate $\hat{\theta}$ of θ cannot be better than $\kappa^{-1}D(\theta)$ in the case of unbiased estimates (see Borovkov (1984)). The inequality in (8) should be considered in the sense of positive semidefiniteness of the matrices difference. The estimation accuracy of a part of the vector θ (for example, r, which is important in our case) is connected with the corresponding part of the matrix $I(\theta)^{-1}$.

Given the characteristics of the multilateration system (location of the receiving stations $\{r_i\}_{i=1}^m$, characteristics of the errors w_i^j), it is possible to build the accuracy boundary at each position in the observation zone using formula (8). The vectors e_i^j are almost identical for

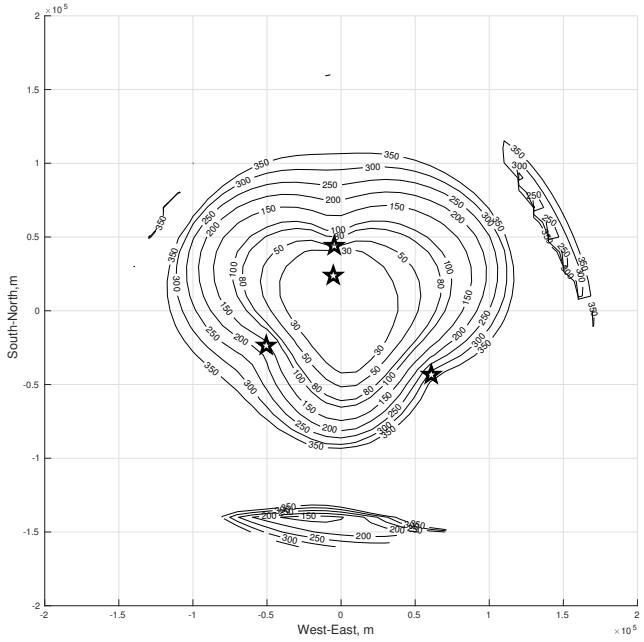

Fig. 1. Fig. 1. Levels for the Cramer–Rao lower bound of accuracy (in meters) of estimation for the horizontal coordinates of the aircraft for the specific location of the receiving stations (marked with asterisks).

different j in some reasonable range of v and Δ^j. As a consequence, the dependence of matrix (7) on v is very weak and it can be neglected in calculations.

Figure 1 shows the levels of the accuracy lower bound (according to equation (8)) for the horizontal coordinates. They are calculated for specific locations of the receiving stations on the ground (shown by asterisks) and the normally distributed errors w_i^j with zero mean and standard deviation of 1 μs; the aircraft altitude is 2000 m.

3. NONLINEAR LEAST SQUARES ESTIMATE. MINIMIZATION OF THE FUNCTIONAL

The solution of the multilateration problem and the development of estimates of r can be made in different ways, but apparently the most promising is maximum likelihood estimates (see I.A.Mantilla-Gaviria et al. (2015)). In the particular case of normally distributed observation errors w_i^j, such an estimate corresponds to the minimization of mean square of residuals (4)

$$J(\theta) = \frac{1}{N}\sum_{j=1}^{n}\sum_{i\in I^j}\left(f_i^j(\tau_i^j;\theta)\right)^2 = \frac{1}{N}\sum_{j=1}^{n}\sum_{i\in I^j}\left(\tau_i^j - g_i^j(\theta)\right)^2. \quad (9)$$

where $N = \sum_j |I^j|$ is the total number of measurements.

This estimate reaches accuracy bound (8) in the limit (asymptotic efficiency) when the number of measurements increases (see Borovkov (1984)). Even if the distribution of the errors w_i^j is not normal, the estimate $\hat{\theta} = \operatorname{argmin}_\theta J(\theta)$ has good properties.

Functional (9) is not convex, so its optimization is complicated. Different scales of variables also make significant difficulties. The gradient descent method used for minimization showed an extremely low rate of convergence, requiring a large number of iterations (about `1e4`). And convergence is not always achieved in experiments. In a sufficiently large percentage of cases, the numerical procedure makes fluctuations and "jumps".

Second-order methods based on Newton's method have a very high rate of convergence (see Bakhvalov et al. (1987)), but their application to the optimization of functional (9) requires regularization. Thus, the classical Newton's method (see Bakhvalov et al. (1987)) has the form

$$\theta_{k+1} = \theta_k - \left(\nabla^2 J(\theta_k)\right)^{-1}\nabla J(\theta_k)\,, \quad (10)$$

where $\nabla^2 J(\theta_k)$ is the matrix of second derivatives calculated at the current approximation point θ_k. However, the main condition of convergence of procedure (10) is positive definiteness of $\nabla^2 J(\theta_k)$, which is not fulfilled in the case of functional (9). To overcome this difficulty Levenberg–Marquardt modification (see Bakhvalov et al. (1987)) of Newton's method is used, which is constructed as follows. Consider the matrix of the second derivatives

$$\nabla^2 J(\theta) = \frac{1}{N}\sum_{j=1}^{n}\sum_{i\in I^j}\Big(\nabla f_i^j(\tau_i^j;\theta)\nabla f_i^j(\tau_i^j;\theta)^{\mathsf{T}} + f_i^j(\tau_i^j;\theta)\nabla^2 f_i^j(\tau_i^j;\theta)\Big) = \frac{1}{N}\sum_{j=1}^{n}\sum_{i\in I^j}\left(\nabla g_i^j(\theta)\nabla g_i^j(\theta)^{\mathsf{T}} - f_i^j(\tau_i^j;\theta)\nabla^2 g_i^j(\theta)\right).$$

Only the second term in the brackets is responsible for the violation of positive definiteness of $\nabla^2 J(\theta)$ because the matrices $\nabla g_i^j \nabla g_i^{j\,\mathsf{T}}$ are always positive semidefinite. Let us consider the second term more closely using (6):

$$f_i^j(\tau_i^j;\theta)\nabla^2 g_i^j(\theta) = -\frac{f_i^j(\tau_i^j;\theta)}{\|r + v\Delta^j - r_i\|}\begin{bmatrix} E_i^j & \Delta^j E_i^j & 0 \\ \Delta^j E_i^j & (\Delta^j)^2 E_i^j & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The values in the matrix are bounded by $(\Delta^j)^2$, since the total "magnitude" of the term is determined by multiplier in front of the matrix, which has an approximation near the true value of θ:

$$\frac{f_i^j(\tau_i^j;\theta)}{\|r + v\Delta^j - r_i\|} = \frac{\tau_i^j - g_i^j(\theta)}{\|r + v\Delta^j - r_i\|} \approx \frac{w_i^j}{\|r + v\Delta^j - r_i\|}\,.$$

This value are small if the magnitude of the random errors less than typical observation distance. In consequence, the second term is less then the first one and can be neglected. Another argument for neglecting is that the residuals f_i^j become smaller while J tends to minimal value.

Therefore, we can make an approximate matrix Q_λ for which positive definiteness is true

$$Q_\lambda = \frac{1}{N}\sum_{j=1}^{n}\sum_{i\in I^j}\nabla g_i^j(\theta)\nabla g_i^j(\theta)^{\mathsf{T}} + \lambda I > 0\,,$$

and which can be used instead of $\nabla^2 J(\theta_k)$ in (10):

$$\theta_{k+1} = \theta_k - Q_\lambda^{-1}\nabla J(\theta_k)\,. \quad (11)$$

This method is known as the Levenberg–Marquardt algorithm. Method (11), directly applied to the optimization,

shows a good rate of convergence with the constant λ of the order of `1e-3` in some experiments, but in another situations it stops far from the minimum of functional (9). The most probable reason why it happens is as follows. The condition of stable work of (11) is $\nabla^2 J(\theta) \leq Q_\lambda$. But if the initial point θ_0 is far enough from the true value of θ this condition can be violated.

This hypotesis suggests that this situation can be overcome by dynamic changing the constant of regularization λ. If the functional J ceases decreasing, but its value remains large, then the constant should be increased: $\lambda := 2\lambda$. If the step length is small, but there is a stable decrease of the functional on several iterations, an attempt should be made, on the contrary, to decrease λ (for example, by the rule $\lambda := \lambda/2$).

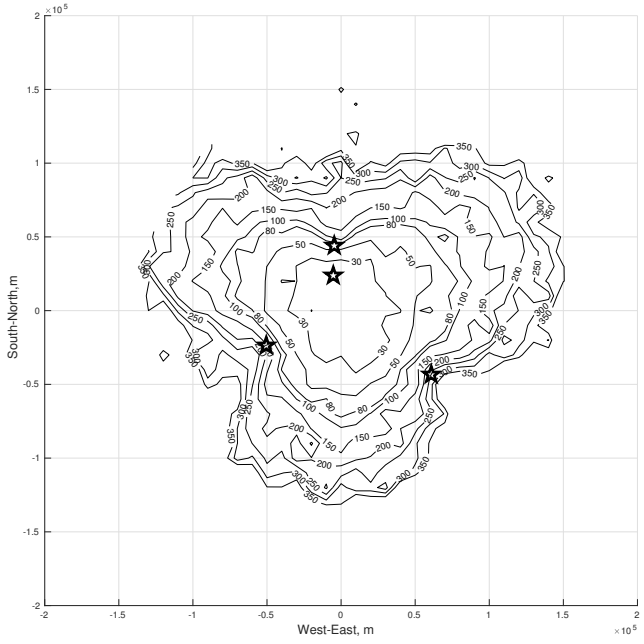

Fig. 2. Levels for the root mean squared deviation of estimation for the horizontal coordinates of the aircraft for the specific location of the receiving stations (marked with asterisks).

With such a modification, the Levenberg–Marquardt algorithm has shown a good performance on simulated data in terms of recovery of the horizontal components of the coordinates of r. In Fig. 2, the levels of accuracy of the method are shown for the same configuration of the receiving stations as in Fig. 1. The empirical standard deviation $\hat{\sigma} = \sqrt{\sum_{i=1}^{K} (\hat{r} - r)^2 / K}$ over K realizations is taken as the accuracy estimate. The comparison of Figs. 2 and 1 shows that the accuracy of the estimate obtained by minimizing the functional (9) using the method (11) is comparable to the Cramer–Rao lower bound of accuracy (8).

REFERENCES

Bakhvalov, N., Zhidkov, N., and Kobelkov, G. (1987). *Chislennye metody (Numerical methods).* Nauka, Moscow. (in Russian).

Bar-Shalom, Y., Li, X., and Kirubarajan, T. (2004). *Estimation with Applications to Tracking and Navigation: Theory Algorithms and Software.* John Wiley & Sons.

Borovkov, A. (1984). *Matematicheskaya statistika (Statistics).* Nauka, Moscow. (in Russian).

Chernyak, V. (2008). Using potential accuracy of object localisation with multilateration systems. In *Proceedings of ESAV'08.* Capri, Italy.

I.A.Mantilla-Gaviria, M.Leonardi, G.Galati, and J.V.Balbastre-Tejedor (2015). Localization algorithms for multilateration (mlat) systems in airport surface surveillance. *Signal Image and Video Processing*, (9), 1549–1558.