Advances in Mechanics

Dynamics and Control

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RUSSIAN ACADEMY OF SCIENCES
A.Yu. ISHLINSKY INSTITUTE FOR PROBLEMS IN MECHANICS

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Dynamics and Control

Proceedings of the 14th International Workshop on Dynamics and Control, Moscow–Zvenigorod, Russia May 28 – June 2, 2007

Dedicated to Professor Angelo Miele
on the occasion of his 85th Birthday

Editors:
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MOSCOW NAUKA 2008

FEEDBACK CONTROL IN PROBLEMS WITH UNKNOWN LEVEL OF DYNAMIC DISTURBANCE

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ABSTRACT. In the paper, methods of the differential game theory are applied to problems with an unkown *a priori* level of a dynamic disturbance. Problems with fixed terminal time and some prescribed geometric constraint for the useful control are considered. The objective of the useful control is to guide the system to some given terminal set at the terminal instant. A feedback control method called *adaptive* is suggested which provides leading the system to the terminal set if the disturbance is not greater than some critical level. With that, a "low-level" disturbance is parried by means of a "low-level" useful control. In a linearized formulation, a problem of interception of one weak-maneuvering object by another (of a missile by an anti-missile) is considered.

1. INTRODUCTION. In the theory of differential games [1–4], there are well-developed methods for problems, which settings include geometric constraints for both the first and second players. However, in practical mechanical problems, the constraints are often given only for the useful control (for the first player), but prescribing some constraint for the disturbance (for the second player) is not natural. Moreover, the optimal feedback control of the first player, which is obtained in the framework of the standard formalization of an antagonistic differential game, is directed to parry the worst disturbance and, therefore, uses the extremal values of control. But in real situations, the dynamic disturbance is usually not realized in its worst way.

One would like to have a feedback control method, which successfully works for disturbances from a wide range. With that, the "weaker" or "less optimal" the disturbance is, the "weaker" should be the parrying useful control. The aim of the work is to suggest such a control based on approaches of the differential game theory.

The pivotal concept from the theory of differential games which is used below is a stable bridge [1, 2]. This is a set in the space *time* $\times$ *phase variable*, where the first player governing the useful control can keep the system motion up to the terminal instant despite of the action of the second player (of the disturbance). When constructing such sets, one supposes that geometric constraints are given for both players' controls.

Consider a family of differential games where the geometric constraint for the second player's control depends on a scalar parameter. A constraint for the first player's control and, therefore, a stable bridge in the game space are connected to each value of the parameter too. Changing the parameter, we obtain a family of bridges, which is ordered by inclusion with increasing the parameter. The first player guarantees keeping the system in a tube using his control of the corresponding level if the second player's control obeys its corresponding constraint. This family allows us to construct a feedback control of the first player and to compute the guarantee provided by this control.

Assume that the system is influenced by a disturbance which does not exceed some level. Then, the system motion will cross the bridges from the family until it reaches (from above or below) the boundary of the bridge, which corresponds to this level of the disturbance. Further, the motion will come inside the bridge. So, the system tunes automatically the level of the useful control to the actual, but unknown level of the disturbance. Due to this, the control is called *adaptive*.

The idea described above is quite general. Its concrete realization depends on the opportunity to realize an algorithm for constructing stable bridges for systems of certain types. In connection with the algorithms, we mention works [4–9] dealing with the topic.

In the paper, we consider problems with linear dynamics, fixed terminal time, and convex compact terminal set, to which the first player tries to guide the system at the terminal instant. The useful control is bound by a geometric constraint which is a compact convex set. These properties of the system allow us to construct quite easily the family of stable bridges and the adaptive control. Namely, under these conditions, we need to compute some two bridges only, which are stored in memory and generate the ordered family. So, on the basis of these two bridges at any time instant, we can construct a suitable bridge from the family. Then the control is produced by the extremal shift [1, 2] to this bridge. Efficiency of this algorithm is provided by the fact that all the bridges have convex time sections (t-sections).

A corresponding software is developed now for the cases, when the terminal set is defined by two or three components of the phase vector at the terminal instant [10, 11].

In this paper, the method of adaptive control is applied to the problem of aerial intercept of one weak-maneuvering object by another one (of an aircraft or missile by an antimissile). Simulation results are given.

2. THE PROBLEM SETTING. Consider a linear differential game with fixed terminal time:

$$\dot{z} = A(t)z + B(t)u + C(t)\nu, \tag{1}$$

$$z \in R^m,\ t \in T,\ u \in P \subset R^p,\ \nu \in R^q.$$

Here, a convex compactum P is the constraint for the control u of the first player, $T = [\vartheta_0, \vartheta]$ is the time interval of the game. The matrix-valued functions A and C are continuous. The matrix-valued function B is Lipschitzian in the interval T. There is no any concrete compact constraint for the control ν of the second player. The set P should include the origin of its space.

The first player tries to guide system (1) at the instant ϑ to a terminal set M as closely to its center as possible. The terminal set M is a convex compactum in a space of some n chosen components of the phase vector z. It is assumed that M contains a neighborhood of the origin of the space.

One needs to suggest a method for constructing an adaptive feedback control for system (1).

Let us pass to a system, which right-hand side does not contain the phase vector:

$$\dot{\xi} = D(t)u + E(t)\nu, \tag{2}$$

$$\xi \in R^n, \ t \in T, \ u \in P, \ \nu \in R^q.$$

The passage is provided (see [1, p. 160] and [2, pp. 89–91]) by the following relations:

$$\xi(t) = X_{n,m}(\vartheta, t)z(t), \quad D(t) = X_{n,m}(\vartheta, t)B(t), \quad E(t) = X_{n,m}(\vartheta, t)C(t),$$

where $X_{n,m}(\vartheta, t)$ is a matrix combined of n rows of the fundamental Cauchy matrix for the system $\dot{z} = A(t)z,$ which correspond to the components of the vector z defining the terminal set M.

In this new problem, the first player, as before, tries to guide system (2) to the terminal set M at the terminal instant ϑ. The set M is now a convex compactum in R^n including a neighborhood of the origin.

Further reasoning is made for system (2). As the adaptive control $U(t, \xi)$ will be obtained in the framework of system (2), it can be applied to system (1) as $U(t, X_{n,m}(\vartheta, t)z)$.

3. CONSTRUCTING ADAPTIVE CONTROL. Below, the symbol $S(t) = \{\xi \in R^n : (t, \xi) \in S\}$ denotes the section of a set $S \subset T \times R^n$ at the instant $t \in T$. Let $O(\varepsilon) = \{\xi \in R^n : |\xi| \leq \varepsilon\}$ be a ball with radius ε in the space R^n.

3.1. Stable bridges. In the interval $T = [\vartheta_0, \vartheta]$, consider an antagonistic differential game with the terminal set M and geometric constraints P, Q for players' controls:

$$\dot{\xi} = D(t)u + E(t)\nu, \tag{3}$$

$$\xi \in R^n, \ t \in T, \ \mathrm{M}, \ u \in \mathrm{P}, \ \nu \in \mathrm{Q}.$$

Here, the matrices $D(t)$, $E(t)$ are the same as in system (2). The sets M, P and Q are assumed to be convex compacta. They are considered as parameters of the game.

Below, $u(\cdot)$ and $\nu(\cdot)$ denote measured functions of time with their values in the sets P and Q, respectively. A motion of system (3) (and, therefore, of system (2)) emanated from the point ξ_*, at the instant t_*, under controls $u(\cdot)$ and $\nu(\cdot)$ is denoted by the symbol $\xi(\cdot; t_*, \xi_*, u(\cdot), \nu(\cdot))$.

Following [1, 2], let us define stable and maximal stable bridges.

A set $W \subset T \times R^n$ is called a *stable bridge* for system (3) for some fixed parameters P, Q, and M if $W(\vartheta)$ = M and the following *stability* property holds: for any position $(t_*, \xi_*) \in W$ and for any control $\nu(\cdot)$ one can find a control $u(\cdot)$ such that the pair $(t, \xi(t)) = (t, \xi(t; t_*, \xi_*, u(\cdot)))$ stays in the set W for any instant $t \in (t_*, \vartheta]$. The set $W \subset T \times R^n$ such that it is maximal by inclusion, $W(\vartheta)$ = M, and possessing the stability property is called the *maximal stable bridge.*

The maximal stable bridge is [1, 2] a closed set. Its t-sections are convex ([2, p. 87]) due to linearity of system (3) and convexity of the set M.

3.2. Adaptive feedback control. Let us describe construction of the adaptive control for system (2).

1. Choose a set $Q_{\max} \subset R^q$, which is the "maximal" constraint for the second player's control, which the first player "agrees" to count reasonable for guiding system (1) to the set M. The set $Q_{\max}$ should include the origin of its space. Denote by W the maximal stable bridge for system (3) corresponding to the parameters $\mathrm{P} = P$, $\mathrm{Q} = Q_{\max}$ and $\mathrm{M} = M$.

Let us agree that the set $Q_{\max}$ is chosen in such a way that for some $\varepsilon > 0$ for any $t \in T$ the following inclusion holds:

$$O(\varepsilon) \subset W(t). \tag{4}$$

Below, the quantity ε is fixed.

2. Additionally, introduce a tube $W^{\#} \subset T \times R^n$ such that each its section $W^{\#}(t)$ is an attainability set for system (3) at the instant t with the initial set $O(\varepsilon)$ at the instant ϑ_0. When constructing the tube $W^{\#}$, it is supposed that the first player is absent ($u \equiv 0$) and the constraint for the second player's control ν equals $Q_{\max}$. It is evident that $W^{\#}$ is the maximal stable bridge for system (3) with $\mathrm{P} = \{0\}$, $Q = \mathrm{Q}_{\max}$, $\mathrm{M} = W^{\#}(\vartheta_0)$. For any $t \in T$, one has

$$O(\varepsilon) \subset W^{\#}(t). \tag{5}$$

3. Consider a family of sets W_k, which sections $W_k(t)$ are defined as follows:

$$W_k(t) = \begin{cases} kW(t), & 0 \le k \le 1, \\ W(t) + (k-1)W^{\#}(t), & k > 1. \end{cases}$$

The set $W_k(t)$ is closed and convex. For any numbers $0 \le k_1 < k_2 < k_3 < k_4$ according to (4), (5), the strict inclusions $W_{k_1}(t) \subset W_{k_2}(t) \subset W_{k_3}(t) \subset W_{k_4}(t)$ are true.

One can show [12] that the set W_k, when $0 \le k \le 1$, is the maximal stable bridge for system (3) corresponding to the constraint kP for the first player's control, the constraint $kQ_{\max}$ for the second player's control, and the terminal set kM. When $k > 1$, the set W_k is a stable bridge (but, maybe, non-maximal) for parameters $\mathrm{P} = P$, $\mathrm{Q} = kQ_{\max}$, $\mathrm{M} = M + (k-1)W^{\#}(\vartheta_0)$.

So, with increasing the parameter k, we get an increasing system of stable bridges, where each greater bridge corresponds to a larger constraint for the second player's control. The system is generated by the bridges W and $W^{\#}$ with the help of operations of addition and multiplication by a non-negative scalar.

Constructing the adaptive control $U(t, \xi)$ is implemented as follows. Let the current position (t, ξ) be outside some bridge W_k, but close to its boundary. Then, we can use this bridge for constructing some control which is constant in the next step of the discrete control scheme. This can be done by means of the extremal shift method [1, 2]. With that, the control is taken from the set P_k. Here, $P_k = kP$, when $0 \le k \le 1$, and $P_k = P$, when $k > 1$. If realization of the disturbance is weaker than the level $Q_k = kQ_{\max}$, then the motion will go inside the family to a bridge with less index k. And at the next step, we will use a weaker useful control. If, vice versa, the disturbance is stronger than the level Q_k, it can lead the system outside to a bridge with a larger index k. Respectively, at the next step, the useful control will be stronger (if there is possibility to strengthen the control). This is the way how the level of the useful control is tuned according to the actual level of the disturbance.

Application of the extremal shift method is standard: choose a bridge to be shifted to; find the nearest point at it and get the shift vector; among the points of the set P_k , choose such a vector, which gives maximal scalar product by the shift vector. This is the required control vector. We have some freedom for choosing the bridge to be shifted to. We do this in the following way: fix a number r; then the bridge, to be shifted to, is taken as the bridge separated from the current phase state in the distance r.

4. PROBLEM OF AERIAL INTERCEPT. As an illustration to the suggested method of adaptive control, let us consider a model setting of a problem of aerial intercept [13, 14].

In this problem, the pursuer P is an antimissile, the evader E is a weak-maneuvering aerial target (an aircraft or another missile). Here, the natural payoff is the miss, i.e., the closest distance between the objects.

The vectors of nominal velocities $(V_P)_{\text{nom}}$ and $(V_E)_{\text{nom}}$ are constant and directed such that there is exact collision along the nominal trajectories. The control accelerations of the objects are orthogonal to their current velocities. The maximal values of the lateral accelerations are bounded by constants a_P and a_E. The constant a_P is supposed to be given in the problem setting. The value a_E is not known exactly, it is only assumed that $a_P > a_E$. The evader governs its acceleration directly, and the pursuer governs its acceleration inertially with the time constant τ_P. Capabilities of the objects to change direction of their velocities are weak (these objects are weak-maneuvering).

The choice of the coordinates is the following. The origin O is put to the nominal position of the pursuer P_{nom} at the initial instant. The axis OX is directed along the nominal initial pursuer's line-of-sight. The axis OY is orthogonal to OX and lies in the plane defined by the vectors of objects' nominal velocities (Fig. 1). The axis OZ is orthogonal to the two first axes.

Since the deviations of the velocities $V_P(t)$ and $V_E(t)$ from their nominal values $(V_P)_{\text{nom}}$ and $(V_E)_{\text{nom}}$ are quite small, the relative motion along the axis OX can be considered as uniform, and the miss can be computed at the instant ϑ of the nominal collision as the lateral distance between objects in the plane YZ at this instant.

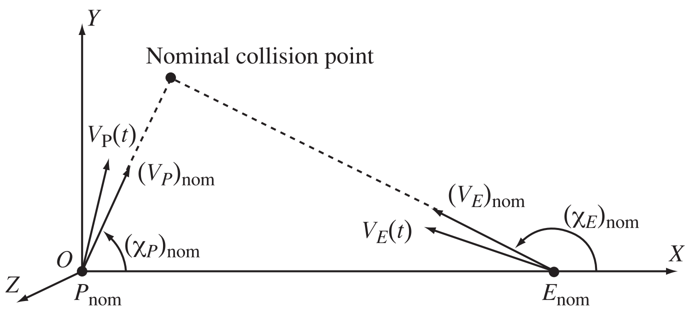

Fig. 1. Coordinate system in the aerial intercept problem

After linearization of the objects' dynamics along their nominal trajectories, one passes to the following system [13, 14]:

$$\begin{gathered} \ddot{x} = F, \\ \dot{F} = -(F-u)/\tau_P, \quad t \in [0, \vartheta], \; x, y \in R^2, \; u \in P, \; \nu \in R^2. \\ \ddot{x} = \nu, \end{gathered} \tag{6}$$

Here, x is the position vector of the first player (the pursuer), y is the position vector of the second player (the evader), τ_P is the time constant characterizing the inertiality of fulfillment of the first player's control. The set P, which is bounding the first player's control, is an ellipse

$$P = \left\{ u \in R^2 : \frac{u_1^2}{A_P^2} + \frac{u_2^2}{B_P^2} \le 1 \right\}.$$

The semiaxes A_P, B_P are parallel to the coordinate axes and are computed on the basis of the value a_P and cosine of the angle $(\chi_P)_{\text{nom}}$. The control ν of the second player is treated as a dynamic disturbance. The terminal time ϑ of the process is fixed. The terminal set is defined as a circle in the difference coordinates $x - y$.

To apply the adaptive method of control, one should introduce an auxiliary constraint $Q_{\max}$. To do this, let us take a reasonable value $a_{E\max}$ bounding the lateral acceleration of the evader. This value defines the constraint $Q_{\max}$ as an ellipse

$$Q_{\max} = \left\{ \nu \in R^2 : \frac{\nu_1^2}{A_E^2} + \frac{\nu_2^2}{B_E^2} \le 1 \right\},$$

where the semiaxes A_E, B_E are parallel to the coordinate axes and are computed on the basis of the value $a_{E\max}$ and cosine of the angle $(\chi_E)_{\text{nom}}$.

Let us show the simulation results for the case

$\tau_P = 1.0$ s, $\vartheta = 10.0$ s, $a_P = 1.3$ m/s^2, $(\chi_P)_{\text{nom}} = 47.94°$, $(\chi_E)_{\text{nom}} = 45°$.

The ellipse P, therefore, has the semiaxes equal to $A_P = 1.3$, $B_P = 0.87$. The radius of the terminal circle is taken to be equal to 2.

Let us choose the value $a_{E\max} = 1.0$ m/s^2. Then, the ellipse $Q_{\max}$ has semiaxes $A_E = 1.0$, $B_E = 0.71$.

To construct adaptive control, one should introduce also the parameter r. Let us take $r = 0.01$. The adaptive control U is applied in the discrete scheme with the time step $\Delta = 0.01$ s.

The initial phase vector in the difference coordinates is taken as $\Delta x_0 = x_0 - y_0 = (-3\text{ m}, 0\text{ m})$, $\Delta\dot{x}_0 = \dot{x}_0 - \dot{y}_0 = (0\text{ m/s}, 2\text{ m/s})$, $F = 0$. The disturbance control is generated as a piecewise-constant function, which values are in the ellipse $1.5Q_{\max}$ and which stays constant for a random time periods not longer than 3 s. The random procedure for choosing the next value from the ellipse is the following: at first, uniformly we choose an angle from the interval $[0,2\pi)$, then also uniformly in the radius-vector a point is chosen between the origin and the boundary of ellipse.

In Fig. 2, *a*, the phase trajectory of system (6) is shown in difference coordinates Δx_1, Δx_2. The initial point is denoted by an asterisk, the final one by a black circle. The circle of the terminal set is shown.

In Figs. 2, *b* and 2, *c*, one can see hodographs of realizations of the controls $u(t)$ and $\nu(t)$. The hodograph of the control $u(t)$ is inside the ellipse P, the initial and final

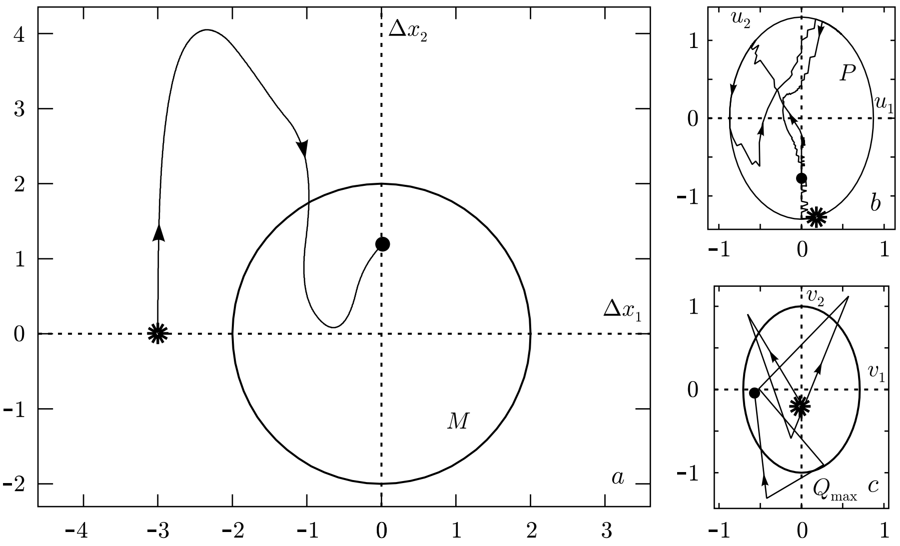

Fig. 2. Simulation results: a) the phase trajectory of the system in difference geometric coordinates; b) the hodograph of the useful control u; c) the hodograph of the disturbance v, the ellipse $Q_{\max}$ is shown

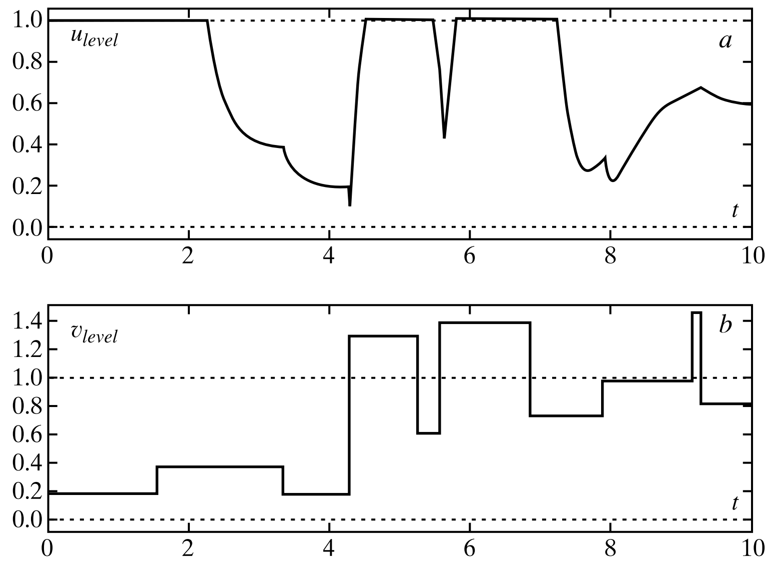

Fig. 3. Realizations of controls: a) the graph of the level of the useful control u; b) the graph of the level of the disturbance with respect to the chosen constraint $Q_{\max}$ (the level 1.0)

points are also marked by an asterisk and a black circle. The hodograph of the control $\nu(t)$ in some time intervals goes outside the ellipse $Q_{\max}$.

Figs. 3, *a* and 3, *b* show graphs of levels of the vector control $u(t)$ with respect to the ellipse P and of the vector disturbance $\nu(t)$ with respect to the ellipse $Q_{\max}$. There are two intervals of maximality of the useful control: at the beginning of the process, when the initial deviation is diminished; and in the middle of the process, when the disturbance is sufficiently outside the forecasted ellipse $Q_{\max}$. In other intervals the useful control level is less than maximally possible. One can see how the useful control reaches a level, which corresponds to the level of the disturbance in the next time interval.

From Fig. 2, *a*, one can also see that despite the disturbance realization is greater than the chosen level $Q_{\max}$, the process termination is successful: the system is guided inside the terminal set.

Acknowledgments. The work is supported by the Russian Foundation for Basic Research, projects 06-01-00414 and 07-01-96085.

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