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MATHEMATICS

Reachable Set of the Dubins Car with an Integral Constraint on Control

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Abstract—A three-dimensional reachable set for a nonlinear controlled object "Dubins car" is investigated. The control is the angular velocity of rotation of the linear velocity vector. An integral quadratic constraint is imposed on the control. Based on the Pontryagin maximum principle, a description of the motions generating the boundary of the reachable set is given. The motions leading to the boundary are optimal Euler elasticae. Simulation results are presented.

Keywords: Dubins car, integral constraint on control, three-dimensional reachable set, Pontryagin maximum principle, Euler elasticae, numerical constructions

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1. INTRODUCTION

By the mathematical "Dubins car," we mean an object moving on a plane with a constant value of linear velocity. The phase state includes two coordinates of the geometric position and the angle of the direction of the velocity vector. The scalar control u has the meaning of the instantaneous angular velocity of the turn. The control is restricted on the interval $[0, t_f]$ by an integral quadratic constraint

$$\int_0^{t_f} u^2(t)dt \le \mu \tag{1}$$

with a given value $\mu > 0$. The purpose of the paper is to numerically study a three-dimensional reachable set $G(t_f)$ at time t_f.

Any non-zero control leading to the boundary of the set $G(t_f)$ delivers the minimal value (equal to μ) of the functional

$$J(u(\cdot)) = \int_0^{t_f} u^2(t)dt \tag{2}$$

under fixed boundary conditions. Extreme motions corresponding to functional (2) were classified by Euler [3] and are called Euler elasticae.

Studying the problem of constructing the set $G(t_f)$, we rely on the experience [1, 2] of analytical description and numerical construction of the reachable set for the case of geometric constraints $|u(t)| \le \mu$. The fundamental difference is that in the case of the geometric constraints, many calculations can be performed explicitly using elementary functions, while in the case of integral constraints, analytical calculations are difficult due to the need to use special elliptic functions. Nevertheless, numerical constructions of the reachable set $G(t_f)$ are possible. The results obtained when constructing the boundary of the set $G(t_f)$ complement the researches of Zelikin [4], Sachkov and Ardentov [5], associated with Euler elasticae.

2. STATEMENT OF THE PROBLEM

Let the motion of a controlled object on a plane be described by a system of differential equations

$$\dot{x} = \cos\varphi, \quad \dot{y} = \sin\varphi, \quad \dot{\varphi} = u. \tag{3}$$

Here x, y are the coordinates of the geometric position, φ is the angle of inclination of the velocity vector measured counterclockwise from the positive direction of the axis x. The speed is equal to one. We consider the values of the angle φ on the interval $(-\infty, \infty)$. The initial instant t_0 is equal to zero. Initial values $x(t_0), y(t_0), \varphi(t_0)$ are also considered zero. Admissible controls are measurable integrable functions $u(\cdot)$, satisfying the constraint (1).

The reachable set $G(t_f)$ for $t_f > t_0$ is the collection of all points $(x, y, \varphi)^\top$, into each of which it is possible to transfer system (3) at the time t_f with an admissible

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control. Denote by $G_\varphi(t_f)$ the two-dimensional cross-section of the set $G(t_f)$ corresponding to the value φ of the angular coordinate. Let ∂ be the symbol of the boundary of a set. If some point $(x,y)^\top$ belongs to $\partial G_\varphi(t_f)$, then the point $(x,y,\varphi)^\top$ belongs to $\partial G(t_f)$. The converse, generally speaking, is not true.

It is required to construct the three-dimensional reachable set $G(t_f)$. For the sake of brevity, we put $z = (x,y,\varphi)^\top$. We denote by $z^0(t_f)$ the point on $\partial G(t_f)$ to which the control $u(t) \equiv 0$ leads.

3. SYMMETRY OF CROSS-SECTIONS OF THE SET $G(t_f)$ ALONG THE ANGULAR COORDINATE

Let $t \to u(t)$ be an admissible control leading at the instant t_f to some point $z(t_f)$ of the set $G(t_f)$. We introduce a "reverse" control $u^\#(t) = u(t_f - t)$, $t \in [0,t_f]$. Obviously, this new control is admissible with the old value of the integral of the squared control.

Consider the motion $t \to z^\#(t)$ under the control $u^\#(\cdot)$. We have

$$\begin{gathered}\varphi(t) = \int_0^t u(\tau)d\tau,\\ \varphi^\#(t) = \int_0^t u^\#(\tau)d\tau = \int_0^t u(t_f-\tau)d\tau = \int_{t_f-t}^{t_f} u(\tau)d\tau.\end{gathered}\tag{4}$$

Therefore, $\varphi^\#(t_f) = \varphi(t_f)$. We draw an auxiliary axis X through the origin of the system x, y at the angle $\varphi(t_f)/2$ with respect to the direction of the axis x. Assume that the axis Y is orthogonal to the axis X. By the symbols $(X(t_f),Y(t_f))^\top$ and $(X^\#(t_f),Y^\#(t_f))^\top$ we denote the positions of the points $(x(t_f),y(t_f))^\top$ and $(x^\#(t_f),y^\#(t_f))^\top$ in the auxiliary coordinate system X,Y.

Lemma 1. *The relations* $X^\#(t_f) = X(t_f)$, $Y^\#(t_f) = -Y(t_f)$ *are valid.*

Proof. From the formulas (4), we get $\varphi^\#(t) = \varphi(t_f) - \varphi(t_f - t)$. Let us introduce the angles measured from the axis X:

$$\varphi_X(t) = \varphi(t) - \frac{\varphi(t_f)}{2},$$

$$\begin{aligned}\varphi_X^\#(t) &= \varphi^\#(t) - \frac{\varphi(t_f)}{2} = \varphi(t_f) - \varphi(t_f - t) - \frac{\varphi(t_f)}{2}\\ &= -\left(\varphi(t_f - t) - \frac{\varphi(t_f)}{2}\right).\end{aligned}$$

Then

$$Y(t_f) = \int_0^{t_f} \sin\left(\varphi(t) - \frac{\varphi(t_f)}{2}\right)dt,$$

$$\begin{aligned}Y^\#(t_f) &= \int_0^{t_f} \sin\left[-\left(\varphi(t_f - t) - \frac{\varphi(t_f)}{2}\right)\right]dt\\ &= -\int_0^{t_f} \sin\left(\varphi(s) - \frac{\varphi(t_f)}{2}\right)ds = -Y(t_f).\end{aligned}$$

Replacing sin in the integrals with cos and taking into account that cos is an even function, we get $X^\#(t_f) = X(t_f)$. □

We have from Lemma 1 that any φ-section $G_\varphi(t_f)$ is symmetric with respect to the axis X of the auxiliary coordinate system. Note also a symmetry of φ-sections $G_\varphi(t_f)$ for $\varphi > 0$ and $\varphi < 0$. Namely, the φ-section $G_{\bar\varphi}(t_f)$ for $\bar\varphi < 0$ is related to the φ-section $G_{\tilde\varphi}(t_f)$, where $\tilde\varphi = -\bar\varphi$, by mirror reflection about the axis x. It follows from the fact that the motions of system (3) from the initial zero point $z(t_0)$ under controls $\bar u(\cdot)$ and $\tilde u(\cdot) = -\bar\varphi$ are related by $\bar x(\cdot) = \tilde x(\cdot)$, $\bar y(\cdot) = -\tilde y(\cdot)$, $\bar\varphi(\cdot) = -\tilde\varphi(\cdot)$.

4. PONTRYAGIN MAXIMUM PRINCIPLE

It follows from general results of the mathematical control theory that the set $G(t_f)$ is closed and bounded. In [6] it is shown that for any point $z(t_f) \neq z^0(t_f)$ on $\partial G(t_f)$, the Pontryagin maximum principle (PMP) is satisfied for the problem of minimizing the functional (2) on the motions of system (3) with fixed boundary conditions $z(t_0) = 0$ and $z(t_f)$. Herewith, the minimum of the functional is equal to μ.

Let us write the PMP relations for the problem of minimizing the functional (2) under fixed boundary conditions for system (3) (see, for example, [4, 5]). Let $u(\cdot)$ be an admissible control not identically zero and $(x(\cdot),y(\cdot),\varphi(\cdot))^\top$ be the corresponding motion of system (3) on the interval $[t_0,t_f]$. The differential equations of the adjoint system have the form

$$\begin{gathered}\dot\psi_1 = 0, \quad \dot\psi_2 = 0,\\ \dot\psi_3 = \psi_1 \sin\varphi(t) - \psi_2\cos\varphi(t).\end{gathered}\tag{5}$$

The PMP means that if $u(\cdot)$ is a minimizing control, then there is a nonzero solution $(\psi_1(\cdot),$

$\psi_2(\cdot), \psi_3(\cdot))^\top$ of system (5), for which the equality $u(t) = \psi_3(t)/2$ is fulfilled on $[t_0, t_f]$. In what follows, a control that satisfies the PMP is assumed to be continuous.

The functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ are constants. We denote them by ψ_1 and ψ_2. If $\psi_1 = 0$ and $\psi_2 = 0$, then $\psi_3(t) = \text{const} \neq 0$ on $[t_0, t_f]$. Therefore, in this case $u(t) \equiv \text{const}$ and it is given by the formula $u(t) = \pm\sqrt{\mu/t_f}$. This constant control determines the one-point section $G_\varphi(t_f)$ for the extreme value $\varphi = \pm\sqrt{\mu t_f}$.

Now let at least one of the numbers ψ_1, ψ_2 be not equal to zero. Based on (3) and (5), we can write the expression $\psi_3(t) = \psi_1 y(t) - \psi_2 x(t) + C$. It implies that $\psi_3(t) = 0$ if and only if the point $(x(t), y(t))^\top$ of the geometric position at the instant t satisfies the equation of a straight line

$$\psi_1 y - \psi_2 x + C = 0. \tag{6}$$

The straight switching line (6) is not universal: when the control that satisfies the PMP is changed, the switching line also changes. In what follows, we write SSL instead of "straight switching line."

Complementing the systems (3) and (5) with the relation

$$u(t) = \psi_3(t)/2, \tag{7}$$

we get a closed system of differential equations, for which the standard conditions for the existence of a unique solution are satisfied. Therefore, in particular, on the plane x, y there cannot be motions, tangentially approaching the SSL in a finite time. Likewise, there cannot be motions leaving the SSL after they have been moving along the SSL for some time. It is only possible to cross the SSL at a non-zero angle, or leave it at the initial instant (respectively, enter it at the last instant) with a non-zero angle. Considering the values ψ_1, ψ_2, $\psi_3(t_0)$ in addition to the fixed initial condition $z(t_0) = 0$, we obtain a collection of motions $t \to z(t)$, among which there must be all the motions leading to $\partial G(t_f)$.

Taking into account (7), we write the equations for $\dot\varphi$ and $\dot\psi_3$ (with the specified constants ψ_1 and ψ_2) in the form of one equation of the second order:

$$\ddot\varphi(t) = \rho \sin(\varphi(t) - \beta).$$

Here ρ is the length of the vector with the components $\psi_1/2$ and $\psi_2/2$, β is the slope angle of this vector measured counterclockwise from the axis x. Thus, consideration of the constants ψ_1, ψ_2, $\psi_3(0)$ can be replaced by consideration of the constants ρ, β, and $\dot\varphi(0) = \psi_3(0)/2$.

Applying a relation from [4] gives

$$\frac{d\,(\dot\varphi(t))^2}{dt} = 2\dot\varphi(t)\ddot\varphi(t) = 2\dot\varphi(t)\rho\sin(\varphi(t) - \beta).$$

Therefore,

$$(\dot\varphi(t))^2 = c_* - 2\rho\cos(\varphi(t) - \beta). \tag{8}$$

Thus, for $\dot\varphi(t) \neq 0$ we have

$$\dot\varphi(t) = \pm\sqrt{c_* - 2\rho\cos(\varphi(t) - \beta)}. \tag{9}$$

We use this formula on the intervals of the motion where $\dot\varphi(t) \neq 0$. The sign "+" corresponds to the control $u(t) > 0$, the sign "−"means that $u(t) < 0$. Bearing in mind expression (6) for SSL and considering the equality $\psi_3(t)/2 = \dot\varphi(t)$, we get that the sign "+" in front of the root corresponds to one half-plane defined by the SSL, and the sign "−" corresponds to the other half-plane. Let us agree to choose the direction of the SSL in such a way that the half-plane, where $u > 0$, lies on the left, and the half-plane with $u < 0$ is on the right. The angle β is equal to the angle (measured counterclockwise) between the direction of the axis x and the direction of the SSL. The constants C in (6) and c_* in (8) are related by the equality $c_* = 2\rho\cos\beta + C^2/4$.

From (9), we have

$$dt = \frac{d\varphi}{\pm\sqrt{c_* - 2\rho\cos(\varphi - \beta)}}. \tag{10}$$

Formula (10) allows one to replace an integration over t with an integration over φ in half-planes with a constant control sign.

5. SIMPLE CONSEQUENCES FROM THE PMP. THEOREM ON CONTROLS LEADING TO THE BOUNDARY OF THE REACHABLE SET

In the statements below, it is assumed that the consumption of the integral resource under the considered admissible control is equal to μ.

Proposition 1. *Let the motion $z(\cdot)$ of system* (3) *on the interval $[t_0, t_f]$ be generated by a continuous control $u(\cdot)$* (*not equal to zero identically*) *and the PMP be satisfied. Then the control $u(\cdot)$ changes sign at most a finite number of times. Furthermore*:

(a) *the points of the geometric position of system* (3) *on the plane x, y at the instants of sign change of the control $u(\cdot)$ lie on the SSL*;

(b) *if $z(\cdot)$ is such that the motion $(x(\cdot), y(\cdot))^\top$ intersects the SSL at least three times, then the intervals between consecutive crossings of the SSL are the same*; *the absolute values of the corresponding increments of the angle are also the same*;

(c) *if* $z(\cdot)$ *is such that the motion* $(x(\cdot), y(\cdot))^\top$ *intersects the SSL at least once, then the accumulated angle in absolute value does not exceed* 2π *on each interval of control sign constancy.*

Lemma 2. *Let the motion* $z(\cdot)$ *of system* (3) *on the interval* $[t_0, t_f]$ *be generated by a continuous control* $u(\cdot)$ *satisfying the PMP with two instants* t_1, t_2 *of sign change of the control, where* $t_0 < t_1 < t_2 < t_f$. *Assume that*

$$\left|(\varphi(t_1) - \varphi(t_0)) + (\varphi(t_f) - \varphi(t_2))\right| > \left|\varphi(t_2) - \varphi(t_1)\right|. \quad (11)$$

Then $z(t_f) \in \operatorname{int} G(t_f)$.

Proof. Without loss of generality, we accept the following sequence of signs of the control $u(\cdot)$: –, +, –. Then the condition (11) can be written in the form

$$-(\varphi(t_1) - \varphi(t_0)) - (\varphi(t_f) - \varphi(t_2)) > \varphi(t_2) - \varphi(t_1).$$

To prove the lemma by contradiction, we assume that $z(t_f) \in \partial G(t_f)$. Then any control leading to this point satisfies the PMP.

We choose instants $\bar{t} \in (t_0, t_1)$ and $\hat{t} \in (t_2, t_f)$ such that the equality

$$-(\varphi(t_1) - \varphi(\bar{t})) - (\varphi(\hat{t}) - \varphi(t_2)) = \varphi(t_2) - \varphi(t_1).$$

holds. The possibility of this choice follows from the continuity of $\varphi(t)$. We have $\varphi(\bar{t}) = \varphi(\hat{t})$.

Consider the reverse control $u^\#(t) = u(\hat{t} - t)$ on the interval $[\bar{t}, \hat{t}]$. Replacing the initial condition $z(t_0) = 0$ in Lemma 1 with $z(\bar{t})$ and taking into account the equality $\varphi(\bar{t}) = \varphi(\hat{t})$, we get $(x(\hat{t}), y(\hat{t}))^\top = (x^\#(\hat{t}), -y^\#(\hat{t}))^\top$. Let us now take the control $\tilde{u}(t) = -u^\#(t)$, $t \in [\bar{t}, \hat{t}]$. For the corresponding motion starting from the point $z(\bar{t})$, we get $\tilde{z}(\hat{t}) = z(\hat{t})$. We extend the control $\tilde{u}(\cdot)$ and the corresponding motion $\tilde{z}(\cdot)$ onto the interval $[t_0, t_f]$ by setting $\tilde{u}(t) = u(t)$ for $t \in [t_0, \bar{t}) \cup (\hat{t}, t_f]$. The integral resource consumption under the control $\tilde{u}(\cdot)$ on $[t_0, t_f]$ coincides with the resource consumption under the control $u(\cdot)$. We have $\tilde{z}(t_f) = z(t_f)$. Therefore, the control $\tilde{u}(\cdot)$ also leads to $\partial G(t_f)$. However, it does not satisfy the PMP since it is discontinuous at the instants $\bar{t}, \hat{t}$. □

Lemma 3. *Let the motion* $z(\cdot)$ of system (3) *on the interval* $[t_0, t_f]$ *be generated by a continuous control* $u(\cdot)$ *satisfying the PMP with three instants* t_1, t_2, t_3 *of sign change of the control, where* $t_0 < t_1 < t_2 < t_3 < t_f$. *Then* $z(t_f) \in \operatorname{int} G(t_f)$.

Proof. By Proposition 1b, we have $\varphi(t_2) - \varphi(t_1) = \varphi(t_2) - \varphi(t_3)$. Therefore, on the interval $[t_1, t_f]$, the conditions of Lemma 2 are satisfied (with the integral constraint equal to the difference between the original constraint and the integral of the squared control over $[t_0, t_1]$). Therefore, the motion $z(\cdot)$ comes at the instant t_f to the interior of the reachable set constructed on the interval $[t_1, t_f]$ from the initial state $z(t_1)$. Hence $z(t_f) \in \operatorname{int} G(t_f)$. □

Let us list all possible types of continuous controls $u(\cdot)$ with at most two instants of sign change. The type U_1 is characterized by the inequality $u(t) > 0$ satisfied on the entire interval $[t_0, t_f]$. Similarly, we define the type U_4 with the positive control replaced by a negative one. The type U_3 has one instant of sign change of the control, with the sign "+"coming first, then "–." The type U_2 also has one instant of sign change, but from "–" to "+." The type U_5 is given by two instants of sign change with the sequence +, –, +. The type U_6 has two instants of sign change of the control with the sequence –, +, –.

Theorem 1. *For any point* $z(t_f) \neq z^0(t_f)$ *on* $\partial G(t_f)$, *there is a continuous control, leading to this point*, that *satisfies the PMP and belongs to one of the types* U_1–U_6. *There are no other variants of control leading to the boundary.*

If $\varphi(t_f) > 0$, *then there are only four types* U_1, U_2, U_3, U_6 *left in the list of possible control types. In the case* $\varphi(t_f) < 0$, *four types* U_2, U_3, U_4, U_5 *are possible. If* $\varphi(t_f) = 0$, *there are four types* U_2, U_3, U_5, U_6 *in the list*; *in this case*, *controls of the types* U_5 *and* U_6 *generate the same set of points*.

Proof. For any point $z(t_f) \neq z^0(t_f)$ on $\partial G(t_f)$, there is a control (leading to this point) that satisfies the PMP. By virtue of Proposition 1, the control has at most a finite number of sign change instants.

To prove the theorem by contradiction, we assume that there is a point $\hat{z}$ on $\partial G(t_f)$, transfer to which is possible using a control with three or more instants of sign change. If there are several such controls, then we take the control $u^\lozenge(\cdot)$ with the least number of instants of sign change. We denote by $z^\lozenge(\cdot)$ the motion under the control $u^\lozenge(\cdot)$. Let us consider the motion $z^\lozenge(\cdot)$ on the last four intervals of constancy of the control sign. By virtue of Lemma 3, we obtain $z^\lozenge(t_f) \in \operatorname{int} G(t_f)$.

Thus, to any point $z(t_f) \neq z^0(t_f)$ on $\partial G(t_f)$, we can pass using a control related to one of the types U_1–U_6. Taking Lemmas 1 and 2 into account, we refine this fact depending on the sign of the angle φ for the point $z(t_f) = \left(x(t_f), y(t_f), \varphi(t_f)\right)^{\top}$ under consideration as follows.

Any control of the type U_1 leads to a point with $\varphi(t_f) > 0$. For controls of the type U_4, we have $\varphi(t_f) < 0$. Therefore, the types U_1 and U_4 are excluded for $\varphi(t_f) = 0$. Lemma 1 implies that the controls U_5 and U_6 generate the same set of points $(x(t_f), y(t_f))^{\top}$ for the case $\varphi(t_f) = 0$.

Let $\varphi(t_f) > 0$. Controls of the type U_4 are excluded. Controls of the type U_5 are also excluded since such controls lead to the interior of the reachable set by Lemma 2.

The case of $\varphi(t_f) < 0$ is treated similarly. Here we also get four control variants: U_2, U_3, U_4, U_5. □

6. RELATIONS FOR CALCULATING MOTIONS UNDER CONTROLS OF THE TYPES U_1–U_6

To describe the curves from which the boundary of the φ-section for $\varphi = \varphi(t_f) \in [0, \sqrt{\mu t_f})$ is formed, we will use the curves A_1, A_2, A_3, and A_6, which are generated by controls of the types U_1, U_2, U_3, and U_6.

(1) The curve A_1 consists of points, to each of which a positive control leads. For such controls, the following relations are valid:

$$t_f = \int_0^{\varphi(t_f)} \frac{d\varphi}{\sqrt{c_* - 2\rho\cos(\varphi - \beta_1)}}, \tag{12}$$

$$\mu = \int_0^{\varphi(t_f)} \sqrt{c_* - 2\rho\cos(\varphi - \beta_1)}\,d\varphi. \tag{13}$$

Here β_1 is the angle measured from the axis x (along which the velocity vector is directed at the initial instant) counterclockwise up to the direction of the SSL. Let us represent the constant c_* in the form $2\rho k_1$, where $k_1 > 0$. This allows, considering the multiplication $t_f \cdot \mu$, to obtain from formulas (12) and (13) for a fixed β_1 an equation with one unknown k_1:

$$t_f \cdot \mu = \int_0^{\varphi(t_f)} \frac{d\varphi}{\sqrt{k_1 - \cos(\varphi - \beta_1)}} \times \int_0^{\varphi(t_f)} \sqrt{k_1 - \cos(\varphi - \beta_1)}\,d\varphi. \tag{14}$$

The first and second integrals in this relation are reduced by a simple transformation to elliptic integrals of the first and second kind [7]. Having determined k_1, we find ρ from relation (13). Next, we integrate the first two equations of system (3) over $[0, t_f]$, taking into account (9). We get a motion on the plane x, y, the end of which is denoted by $A_1(\beta_1)$ and referred to the curve A_1 for the considered β_1.

If $\beta_1 = \varphi(t_f)/2$, then the corresponding motion comes onto the axis X. With that, the direction of the SSL coincides with the direction of the axis X. The construction of the curve A_1 is convenient to start from this very point by going over β_1 in the range $\left[\frac{\varphi(t_f)}{2}, \frac{\varphi(t_f)}{2} + \pi\right]$. We increment β_1 from the value $\varphi(t_f)/2$. By the symbol $\tilde{\beta}_1$, we denote the largest β_1 for which there is a solution to Eq. (14). Geometrically, the value $\tilde{\beta}_1$ stands out by the fact that the corresponding SSL passes through the point $A_1(\tilde{\beta}_1)$. For any $\beta_1 \in \left[\frac{\varphi(t_f)}{2}, \tilde{\beta}_1\right)$, the motion leading to the point $A_1(\beta_1)$ is located strictly to the left of the SSL on the interval $[0, t_f]$. If $\tilde{\beta}_1 < \frac{\varphi(t_f)}{2} + \pi$, then the angle between the velocity vector at point $A_1(\tilde{\beta}_1)$ and the direction of the SSL is greater than zero. Let us denote it by $\tilde{\beta}_3$. If $\tilde{\beta}_1 = \frac{\varphi(t_f)}{2} + \pi$, this angle is zero. The value $\tilde{\beta}_1 = \frac{\varphi(t_f)}{2} + \pi$ is realized only for $\varphi(t_f) = 2\pi$, i.e., when the direction of the axis X is opposite to the axis x.

When constructing numerically, the range $\left[\frac{\varphi(t_f)}{2}, \tilde{\beta}_1\right]$ is divided into three parts, in each of which the features of elliptic integrals are taken into account in a special way [7].

Having constructed the described part of the curve A_1, we reflect it relative to the axis X (according to Lemma 1) and obtain the symmetric part. The union of such two parts forms the curve A_1. When $\tilde{\beta}_1 = \frac{\varphi(t_f)}{2} + \pi$, the curve A_1 becomes closed. If $\varphi(t_f) = 0$, then the curve A_1 is not constructed.

(2) When constructing the curve A_3, we go through all the controls with one instant of sign change from "+" to "–," which provide at the instant t_f the angle value equal to $\varphi(t_f)$. Each trajectory is completely determined by the values t_f, μ, $\varphi(t_f)$, and the angle β_3 of the trajectory inclination at the instant of crossing the SSL (the angle is measured counterclockwise from the direction of the velocity vector to the direction of the SSL). Therefore, taking an auxiliary starting point on the SSL, we can separately consider the part of the

motion from this point in direct time under a negative control satisfying the PMP and with a given angle β_3. The considered part of the motion continues until the modulus of the change of the angle φ reaches the fixed meaning $\varphi_3 \geq 0$. After that, in reverse time, from the same auxiliary point at the same angle β_3, we consider a motion under the positive control on the time interval, on which the change of the angle φ is $\varphi_3 + \varphi(t_f)$. The value φ_3 for the fixed β is chosen in such a way that the total time on these two intervals is equal to t_f and the integral resource consumption under the control is equal to μ. Gluing the two resulting trajectories together at their common starting point, we obtain one trajectory on the interval $[0, t_f]$ with a total change in angle equal to $\varphi(t_f)$. Transferring the initial point of the trajectory to the origin of the system x, y and aligning the direction of the velocity vector at the initial instant with the direction of the axis x, we obtain the required motion. Let us denote its end point by $A_3(\beta_3)$. We form the curve A_3 by increasing the angle $\beta_3 \geq \tilde{\beta}_3$. In addition, we make sure that the branch of the curve constructed in reverse time does not go back to the SSL. The latter determines the largest angle $\hat{\beta}_3 \in [\tilde{\beta}_3, \pi]$ of the inclination of the velocity vector to the SSL at the auxiliary starting point.

The above conditions lead to the following system of relations:

$$t_f = \int_0^{\varphi_3} \frac{d\varphi}{\sqrt{c_* - 2\rho\cos(\varphi + \beta_3)}} + \int_0^{\varphi_3 + \varphi(t_f)} \frac{d\varphi}{\sqrt{c_* - 2\rho\cos(\varphi + \beta_3)}}, \tag{15}$$

$$\mu = \int_0^{\varphi_3} \sqrt{c_* - 2\rho\cos(\varphi + \beta_3)}\,d\varphi + \int_0^{\varphi_3 + \varphi(t_f)} \sqrt{c_* - 2\rho\cos(\varphi + \beta_3)}\,d\varphi. \tag{16}$$

Taking into account $\psi_3(0) = 0$ at the auxiliary starting point on the SSL, we have $c_* = 2\rho\cos\beta_3$. Multiplying Eqs. (15) and (16), we get an equation for φ_3. For each $\beta_3 \in [\tilde{\beta}_3, \hat{\beta}_3]$, we find the only solution φ_3 to this equation, and then the only value ρ from the relation (16). Based on the found values of φ_3 and ρ, we construct two branches of the desired geometric curve. Both of these branches get out from the auxiliary starting point on the SSL. We transfer the initial point of the glued curve to the origin of the system x, y.

Going through the values $\beta_3 \in [\tilde{\beta}_3, \hat{\beta}_3]$ gives the curve A_3. This curve corresponds to a curve A_2 symmetrical with respect to the axis X, to each point of which a reverse control leads. The curve A_3 is not constructed if $\tilde{\beta}_3 = \hat{\beta}_3$.

(3) Let $\hat{\varphi}_{6,1} > 0$ be the value of φ_3 obtained for $\beta_3 = \hat{\beta}_3$. When constructing the curve A_6, we take $\varphi_{6,1}$ as a one-dimensional parameter, decreasing it from the value $\hat{\varphi}_{6,1}$. A motion leading to the point $A_6(\varphi_{6,1})$ for the considered $\varphi_{6,1}$ consists of three parts, whose angular coordinate changes are equal, respectively, to $-\varphi_{6,2}$ (the part lies to the right of the SSL), $+\varphi_{6,2} + \varphi_{6,1} + \varphi(t_f)$ (this part lies to the left of the SSL), and $-\varphi_{6,1}$ (to the right of the SSL). Here $\varphi_{6,2}, \varphi_{6,1}, \varphi(t_f)$ are assumed to be positive. The following equality holds:

$$2\pi - 2\beta_6 - \varphi_{6,2} - \varphi_{6,1} = \varphi(t_f). \tag{17}$$

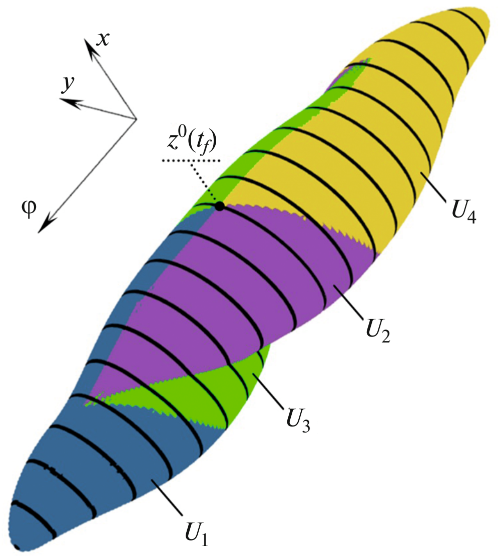
The angle β_6 is defined in the same way as the angle β_3, but it corresponds to the second hit of the motion on the SSL.

Let us write the following relations:

$$t_f = \int_0^{\varphi_{6,2}} \frac{d\varphi}{\sqrt{c_* - 2\rho\cos(\varphi + \beta_6)}} + \int_0^{\varphi_{6,1}} \frac{d\varphi}{\sqrt{c_* - 2\rho\cos(\varphi + \beta_6)}} + \int_0^{\varphi_{6,2} + \varphi_{6,1} + \varphi(t_f)} \frac{d\varphi}{\sqrt{c_* - 2\rho\cos(\varphi + \beta_6)}},$$

$$\mu = \int_0^{\varphi_{6,2}} \sqrt{c_* - 2\rho\cos(\varphi + \beta_6)}\,d\varphi + \int_0^{\varphi_{6,1}} \sqrt{c_* - 2\rho\cos(\varphi + \beta_6)}\,d\varphi + \int_0^{\varphi_{6,2} + \varphi_{6,1} + \varphi(t_f)} \sqrt{c_* - 2\rho\cos(\varphi + \beta_6)}\,d\varphi.$$

We set $c_1 = 2\rho\cos\beta_6$. The above relations together with Eq. (17) form a system of equations for the unknowns β_6 and ρ with a given $\varphi_{6,1}$. We construct the curve A_6 until it hits the axis X with the equality $\varphi_{6,1} = \varphi_{6,2}$. By Lemma 1, considering a curve symmetric with respect to the axis X, we obtain the combined curve A_6. The curve A_6 connects the ends of the curves A_3 and A_2, being smoothly conjugated with them. The curve A_6 is not constructed if $\hat{\varphi}_{6,1} = 0$.



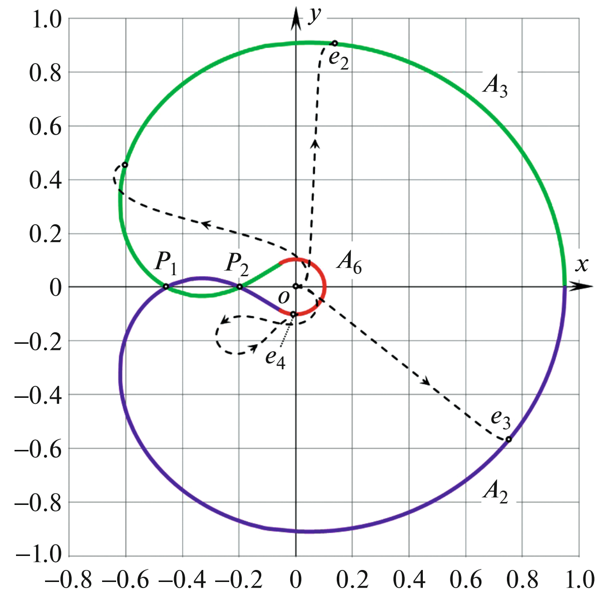


Fig. 1. Three-dimensional reachable set $G(t_f)$ for $\mu = 100$ and $t_f = 0.95$, and its φ-section at $\varphi = 0$.

7. SIMULATION RESULTS

The three-dimensional reachable set $G(t_f)$ calculated for $\mu = 100$ and $t_f = 0.95$ is shown on the left of Fig. 1. The parts of the boundary to which various types of controls lead are highlighted in color: U_1 is a positive control (blue), U_4 is a negative control (yellow), U_3 is a control with one instant of sign change from "+" to "–" (green), and U_2 is a control with one instant of sign change from "–" to "+"(purple). The point $z^0(t_f)$, for which the control is identically equal to zero, lies at the junction of the four indicated parts. The parts corresponding to U_3 and U_2 are not smoothly glued together. The black lines mark the contours of the cross-sections of the three-dimensional set $G(t_f)$ with some step along the axis φ. The set $G(t_f)$ is not simply connected: there is a cavity that does not belong to the set. The cavity is not visible when we look at the set from the outside.

To show the cavity, the φ-section $G_\varphi(t_f)$ of the set $G(t_f)$ at $\varphi = 0$ is presented in Fig. 1 on the right. Since $\varphi = 0$, the auxiliary axis X, with respect to which the φ-section is symmetric, coincides with the axis x. There is no curve A_1 on $\partial G_\varphi(t_f)$. The point $z^0(t_f)$ belongs to this φ-section and is located on the axis x. The curves A_3 and A_2 are symmetric to each other and depart from the axis x. Their arcs up to the point P_1 of the first intersection give the "outer" boundary of the φ-section. The arcs of the curves A_3 and A_2 from the point P_1 up to the point P_2 of the second intersection lie in the interior of the φ-section. The curve A_6 and the adjacent arcs of the curves A_3 and A_2 after the point P_2 form the boundary of the "hole" that does not belong to $G_\varphi(t_f)$. The dashed lines show the trajectories of four motions on the plane x, y, leading onto the boundary of the φ-section (and, therefore, onto the boundary of the three-dimensional reachable set). The trajectories leading to the points e_1, e_2, and e_3 have one inflection point (the sign change point of the control). Such curves represent globally optimal Euler elasticae. The trajectory leading to the point e_4 on the curve A_6 has two inflection points (two sign change points of the control) and this is a locally optimal elastica.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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