Investigation of Reachable Set at Instant for the Dubins' Car

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For a control system of the third order, a reachable set at instant is investigated. Such a system is often called the Dubins' car. An object moves in the plane with constant linear velocity and symmetric or asymmetric bounds onto the right and left turns. We also consider the case of constraints on the control, under which the turn is possible only to one side (to the left or to the right). Examples are given with results of numerical construction of three-dimensional reachable sets.

I. Introduction

In applied works based on the mathematical control theory, there is a popular model of a controlled object, which is described by a nonlinear system of differential equations of the third order. Two phase variables characterize the object geometric state in the plane, the third variable is the direction of the velocity vector. The magnitude of the linear velocity is supposed to be constant. The control is the instantaneous angular velocity of rotation of the linear velocity direction. The control is restricted by some constraint. Under constant linear velocity, the control determines the current radius of turn. So, this is a controlled object (for instance, an aircraft or a car) with the simplest model of motion in the horizontal plane.

In 1957, American mathematician L. Dubins had published theoretical work [1] about a curve of the minimal length that has the constrained radius of curvature and connects two points in the plane with a prescribed direction of leaving from the first point and with a prescribed direction of reaching the second one. The results obtained by L. Dubins became very useful for investigating the motions of objects with bounded radius of turn and constant linear velocity. This is why such objects are used to call the Dubins' car. Later, it was found that similar problems were investigated in 1889 by Russian mathematician A.A. Markov in work [2] devoted to some aspects of constructing railroads. Dynamics of the simplest car was also used by R. Isaacs in his works on differential games [3,4]. The Dubins' models are used for control of wheel robots [5,6], for calculation of aircraft trajectories (by air traffic managers in civil aviation) [7], and, also, in applied works for constructing trajectories of unmanned vechicles in the horizontal plane [8]. In

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book [9], Yu.I. Berdyshev used the Dubins' model in problems of time-optimal sequential visiting several points in the plane.

We shall mean the reachable set $G(t_f)$ *at instant* t_f as a totality of all phase states in the three-dimensional space, which are reachable exactly at the instant t_f from the given initial phase state using some admissible control.

Traditionally, the bounds onto the radii of the left and right turns are given. The case with equal bounds for both turns is called symmetric. If the left and right turns are possible, but bounds for these turns are not equal, we call this case asymmetric. In the common case, the constraints onto the scalar angular velocity given in the form of a segment can mean a prohibition on the left turn or right turn and, also, a prohibition on straight line motion (if such a segment does not contain the zero point).

The goal of this paper is investigation of boundary of the reachable set *at instant* for the Dubins' car in the general case. For this aim, we use the Pontryagin's Maximum Principle [12] for controls carrying the object onto the reachable set boundary [13]. Three-dimensional reachable sets at instant for symmetric and asymmetric cases were earlier investigated by the authors in works [17, 18, 20]. Results for the case when there is a prohibition on the left turn or right turn are new.

From analysis of structure of the reachable set at instant, some useful facts follow for the time-optimal problem (for instance, an estimate of switch number and character of switches of the optimal control). But in whole, investigation of boundary of a reachable set at instant and solution of time-optimal problem (particularly, constructing the optimal synthesis) are separate problems. For the Dubins' car, synthesis of the optimal control is known in the time-optimal control problem for the symmetric (see [10]) and asymmetric (see [11]) cases.

It is necessary to distinct reachable sets *at instant* and *up to instant*. In the second case, the terminal instant is not fixed, but must belong to the interval $[0, t_f]$. For symmetric and asymmetric cases, the construction of reachable sets *up to instant* was considered in works [5,15,18]. It is established that reachable sets at instant and up to instant coincide when the instant t_f is sufficiently large. Papers [16,19] describe the material construction of reachable sets with usage of 3D-printing.

II. Problem formulation

Let the controlled object (the Dubins' car) motion in the plane x, y be described by the following differential equation system:

$$\begin{aligned} \dot{x} &= \cos\varphi, \\ \dot{y} &= \sin\varphi, \\ \dot{\varphi} &= u, \qquad u \in [u_1, u_2], \quad u_1 < u_2, \end{aligned} \tag{1}$$

where x, y are geometric coordinates of the state, φ is an angle of velocity heading (Fig. 1), u is a control. The velocity magnitude is equal to one. In the sequel, we shall suppose that $u_2 = 1$.

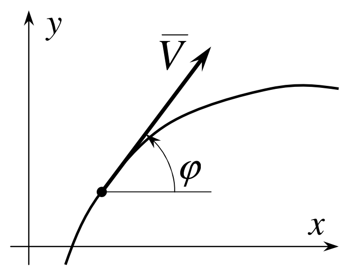


Fig. 1: Coordinate system

The three-dimensional system (defining the motion with the constant linear velocity) can be reduced to form (1) with $u_2 = 1$ by rescaling of the geometric coordinates and time. At the initial instant $t_0 = 0$, we take the initial phase state of system (1) equal to $x_0 = 0$, $y_0 = 0$, $\varphi_0 = 0$.

As admissible open-loop controls $u(\cdot)$, we shall consider measurable functions of time with values $u(t) \in [u_1, u_2]$. The angle φ is considered in the interval $(-\infty, \infty)$.

In the paper, we consider constructing the reachable sets $G(t_f)$ for the cases $u_1 < 0$, $u_1 = 0$, and $u_1 > 0$.

III. Pontryagin's Maxixmum Principle

It is known [13] that controls that carry a system onto the reachable set boundary satisfy the Pontryagin's Maximum Principle (PMP). We write relations of the PMP for system (1).

Let $u^*(\cdot)$ be some admissible control and $(x^*(\cdot), y^*(\cdot), \varphi^*(\cdot))$ be the corresponding motion of system (1) on the interval $[t_0, t_f]$. Differential equations for the adjoint system have the form

$$\begin{aligned} \dot{\psi}_1 &= 0, \\ \dot{\psi}_2 &= 0, \\ \dot{\psi}_3 &= \psi_1 \sin\varphi^*(t) - \psi_2 \cos\varphi^*(t). \end{aligned} \tag{2}$$

The PMP means that a nonzero solution $(\psi_1^*(\cdot), \psi_2^*(\cdot), \psi_3^*(\cdot))$ of system (2) exists, for which almost everywhere (a.e.) on the interval $[t_0, t_f]$, the following condition is satisfied:

$$\psi_1^*(t)\cos\varphi^*(t) + \psi_2^*(t)\sin\varphi^*(t) + \psi_3^*(t)u^*(t)$$

$$= \max_{u\in[u_1,u_2]} [\psi_1^*(t)\cos\varphi^*(t) + \psi_2^*(t)\sin\varphi^*(t) + \psi_3^*(t)u].$$

Thus, the maximum condition takes the form

$$\psi_3^*(t)u^*(t) = \max_{u\in[u_1,u_2]} \psi_3^*(t)u, \quad \text{a.e. } t \in [t_0, t_f]. \tag{3}$$

Note that the functions $\psi_1^*(\cdot)$ and $\psi_2^*(\cdot)$ are constants. Denote these constants by ψ_1^* and ψ_2^*. If $\psi_1^* = 0$ and $\psi_2^* = 0$, then $\psi_3^*(t) = \text{const} \neq 0$ on the interval $[t_0, t_f]$. Therefore, in this case we have either a.e. $u^*(t) = u_1$ or a.e. $u^*(t) = u_2$.

Now, let at least one of the numbers ψ_1^*, ψ_2^* is not equal to zero. Basing on (1) and (2), one can write an expression for $\psi_3^*(t)$:

$$\psi_3^*(t) = \psi_1^* y^*(t) - \psi_2^* x^*(t) + C.$$

From this, it follows that $\psi_3^*(t) = 0$ iff the point $(x^*(t), y^*(t))$ of the geometric state at instant t satisfies the equation of a straight line

$$\psi_1^* y - \psi_2^* x + C = 0. \tag{4}$$

Line (4) is used in many works (see, for instance, [21,22]), in which the PMP was analyzed for system (1).

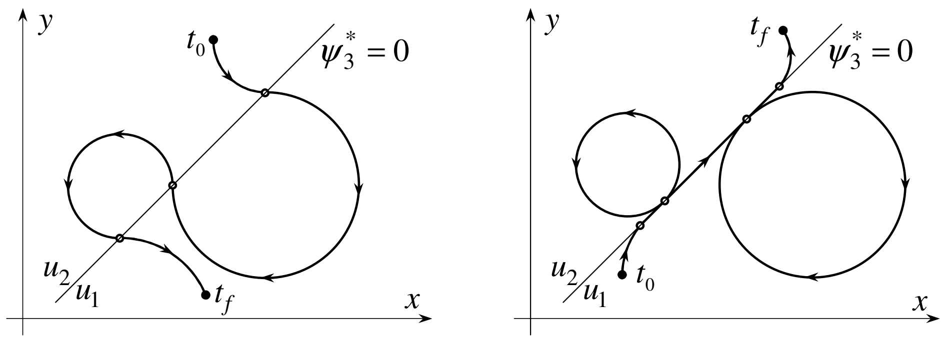
By virtue of relation (3), if $\psi_3^*(t) > 0$ on some time interval, then $u^*(t) = u_2$ a.e. on this interval. With that, the corresponding motion (in projection onto the plane x, y) goes over a circle arc with the radius $1/u_2$ counter-clockwise in the half-plane $\psi_1^* y - \psi_2^* x + C > 0$. If $\psi_3^*(t) < 0$, then $u^*(t) = u_1$. In this case, a motion in the half-plane $\psi_1^* y - \psi_2^* x + C < 0$ goes over a circle arc with the radius $1/|u_1|$ clockwise for $u_1 < 0$, over a straight line for $u_1 = 0$, and over a circle arc with the radius $1/u_1$ counter-clockwise for $u_1 > 0$.

A part of the motion, on which a.e. $u^*(t) = u_2$ or a.e. $u^*(t) = u_1 \neq 0$, and the angle varies by 2π, we shall call a *cycle*. The motion trajectory on such a part in projection onto the plane x, y is a circular curve (circumference).

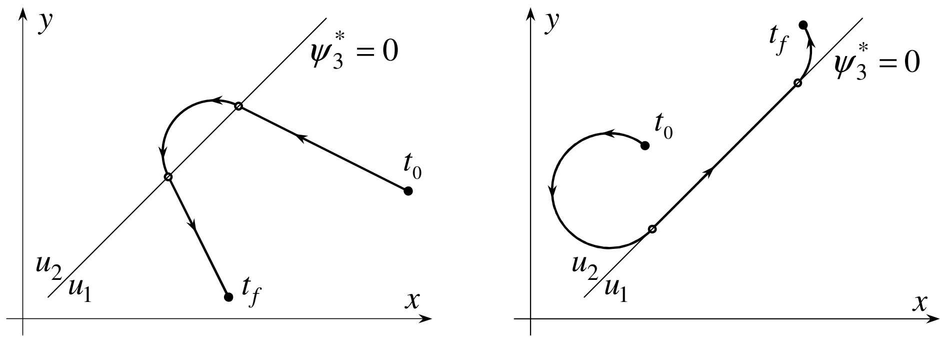
If $\psi_3^*(t) = 0$ on some time interval, then on this interval the motion $(x^*(\cdot), y^*(\cdot))$ goes over a straight line (4). Thus, $\varphi^*(t) = \text{const}$. So, $u^*(t) = 0$ a.e. on this interval. Such a case is not possible under $u_1 > 0$.

It is evident that controls carrying the state onto the boundary of the reachable set (and satisfying the PMP) can change their value only on line (4), which we shall call the *straight switching line* (SSL).

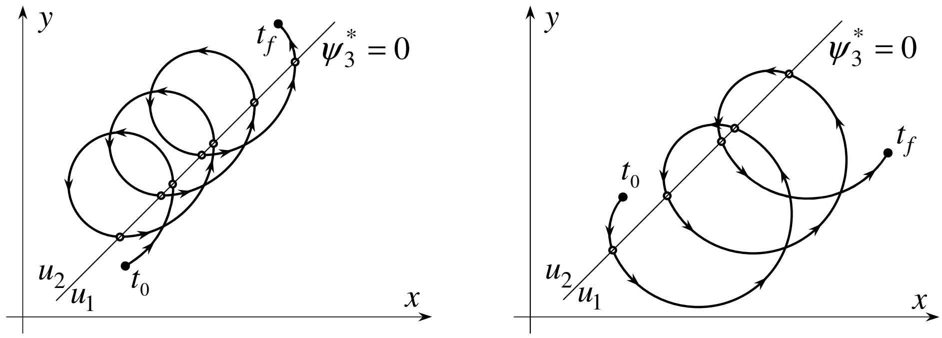
Having considered variants of possible mutual location of the motion trajectory $(x^*(\cdot), y^*(\cdot))$ and the SSL (Fig. 2), we can formulate the following proposition.



a) Trajectories $(x^*(\cdot),\ y^*(\cdot))$ for the case $u_1 < 0$ (the example with $u_1 = -0.5$)



b) Trajectories $(x^*(\cdot),\ y^*(\cdot))$ for the case $u_1 = 0$



c) Trajectory $(x^*(\cdot),\ y^*(\cdot))$ for the case $u_1 > 0$ (the example with $u_1 = 0.5$)

Fig. 2: Trajectories of the maximum principle and the straight switching line.

Proposition 1. *Let a motion of system* (1) *obey the PMP. Then the corresponding trajectory consists of finite number of circle arcs and rectilinear parts. The latter is possible only if* $0 \in [u_1, u_2]$.

Sketch of the Proof.

Let us use the PMP with the maximum condition written as (3). Consider the motion *between* the first and the last instants, when it is on the SSL. The parts of this trajectory located *outside* the SSL are circle arcs. Intersections of these parts with the SSL (including the case of tangency) happen with the same angle (Fig. 2), and time duration of them is the same for all parts at each side from the SSL. Therefore, the number of these parts is finite. Thus, we obtain the boundedness of the number of switches and of the number of parts with constant control (including the parts of rectilinear motion) along the whole trajectory on the interval $[t_0, t_f]$. $\square$

If the condition of the PMP (3) is fulfilled, then on the base of Proposition 1 the function $\psi_3^*(\cdot)$ on the interval $[t_0, t_f]$ can change its sign only a finite number of times. Therefore, we can take the control $u^*(\cdot)$ (generating the motion to the boundary of the reachable set $G(t_f)$) to be piecewise–constant with a finite number of switches on the interval $[t_0, t_f]$. For certainty, let us admit that such a control is piecewise–constant from the right, and the instant t_f is not included in the collection of switching instants.

IV. Case $u_1 < 0$

In works [17, 18, 20], symmetric (with the bounds $u_1 = -1$, $u_2 = 1$) and asymmetric ($u_1 < 0$, $u_2 = 1$) cases are investigated. Theorems have been proved on a finite number of switches and on the character of switches of controls carrying the motion onto the boundary of the reachable set. Namely, it is proved that it is possible to get any boundary point of the reachable set $G(t_f)$ by means of a control having not more than two switches. With that, it is possible to consider only six variants of the control sequences:

$$\begin{array}{lll} 1)\ u_2, 0, u_2; & 2)\ u_1, 0, u_2; & 3)\ u_2, 0, u_1; \\ 4)\ u_1, 0, u_1; & 5)\ u_2, u_1, u_2; & 6)\ u_1, u_2, u_1. \end{array} \tag{5}$$

To construct the boundary of the reachable set $G(t_f)$, we look over all controls of the forms 1–6 from list (5) with two switching instants t_1, t_2. The parameter t_1 is chosen from the interval $[0, t_f]$, and the parameter t_2 is taken from the interval $[t_1, t_f]$. Controls with one switching and without it are considered also. Taking some specific variant of switches and choosing parameters t_1, t_2 for it over some sufficiently accurate grid, we obtain a collection of points creating a surface in the three-dimensional space x, y, φ.

Each of the six variants in list (5) gives the corresponding surface in the three-dimensional space. The boundary of the reachable set $G(t_f)$ comprises of parts of these surfaces. Then, six surfaces are loaded into a visualization program without any additional processing. With its help, we form an image of the boundary of the reachable set. Some surfaces (partially or completely) can get inside the reachable set. Then during visualization of the boundary, such parts are invisible.

Figure 3 shows (from two points of view) the boundary of the set $G(t_f)$ for the instant $t_f = 1.5\pi$. Different parts of the boundary are marked by their own colors. With some step along the axis φ, cross-sections (by planes $\varphi = \text{const}$) of the reachable set are drawn.

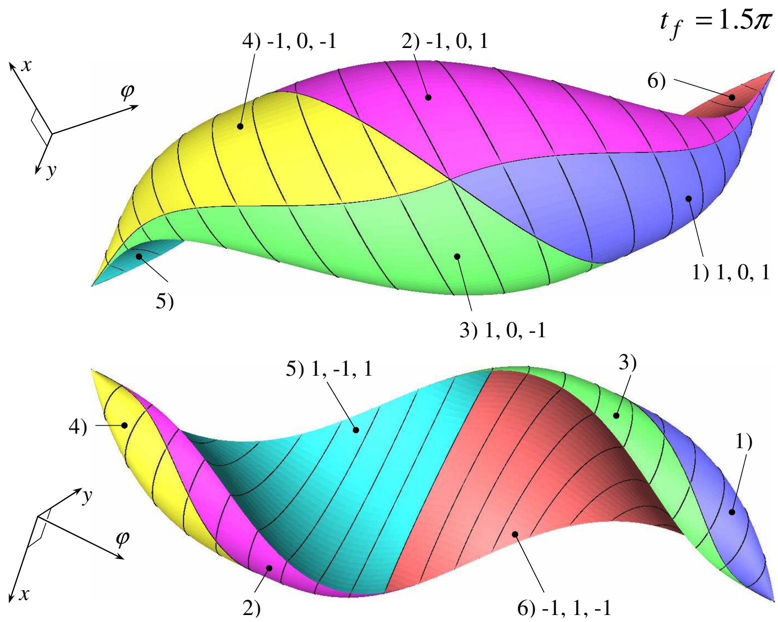


Fig. 3: Reachable set $G(t_f)$ at instant $t_f = 1.5\pi$ for $u_1 = -1$ and $u_2 = 1$

The control identically equal to zero leads to the point of joining the parts 1–4. Controls with only one switching get the motion to points of the lines locating on junctions of parts 1–2, 1–3, 2–4, 2–5, 2–6, 3–4, 3–5, and 3–6. Two motions come at any point of the line that is common for the parts 5 and 6. Each of these two motions has two control switches. On such a line, the parts 5 and 6 have nonsmooth connection.

Figure 4 shows (from the same point of view) the reachable sets $G(t_f)$ for four instants t_f. Variation of structure of the reachable set boundary is cleary seen. As the time increases, the frontal part of the boundary (comprised of parts 1–4) "embraces" the rear part composed of parts 5 and 6.

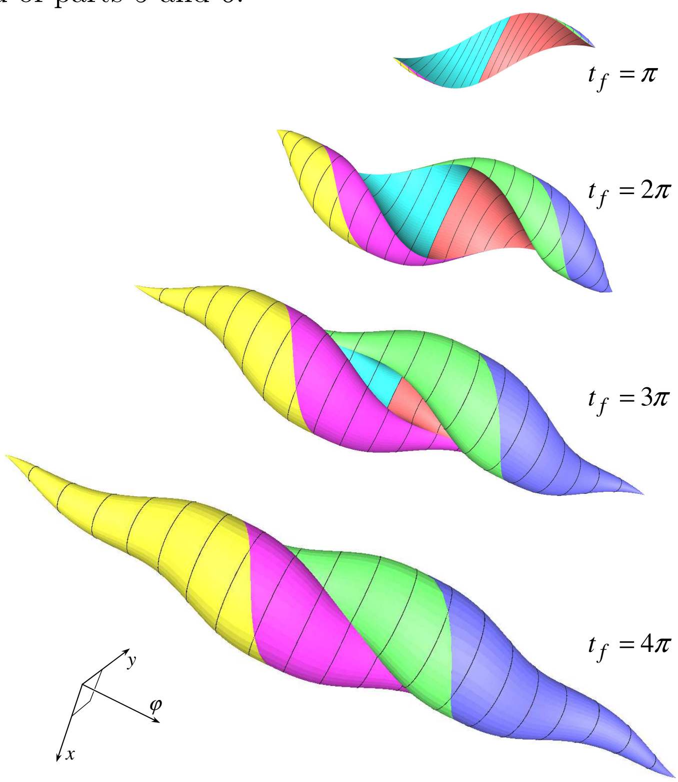


Fig. 4: Evolution of reachable sets $G(t_f)$ for $t_f = \pi, 2\pi, 3\pi, 4\pi$ in the symmetric case

As it was noticed in the problem formulation, we consider possible variations of the angle φ in the interval $(-\infty, \infty)$. In engineering practice, the angle φ is usually considered in an interval of size 2π. In this case, φ-sections for the reachable sets at instant represent the union of φ-sections of the original reachable sets, which are overlaid over each other for values of φ coinciding by modulo 2π. Corresponding reachable set is easily constructed on the basis of the considered approach. In Fig. 5, examples of reachable sets are given for three instants when the angle φ is considered on the interval $(-\pi, \pi]$.

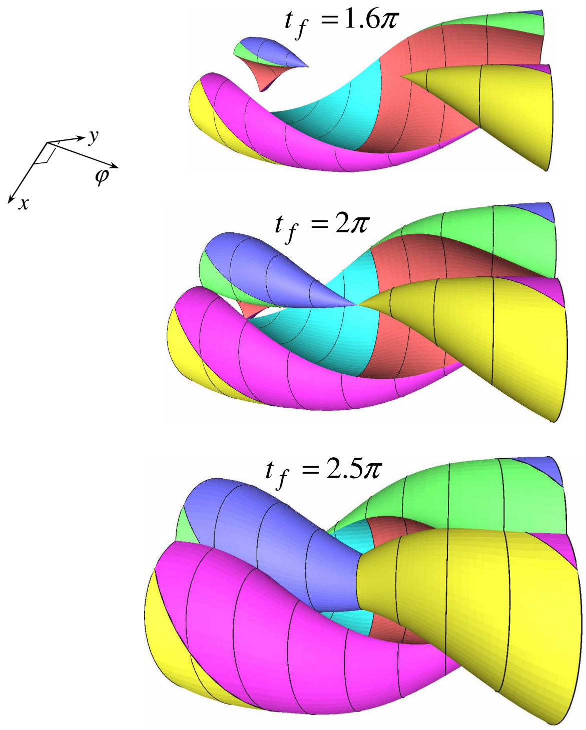


Fig. 5: Reachable sets $G(t_f)$ with φ computed by modulo 2π for the instants $t_f = 1.6\pi, 2\pi, 2.5\pi$ in the symmetric case

In Fig. 6, the picture is shown for the control bounds $u_1 = -0.25$ and $u_2 = 1$ (asymmetric case), the instant t_f is equal to 6π.

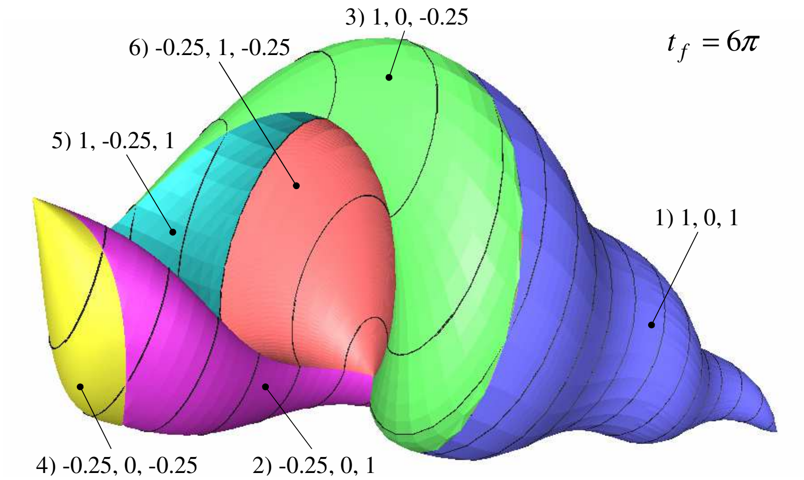


Fig. 6: Reachable sets $G(t_f)$ with $u_1 = -0.25$ for $t_f = 6\pi$ in the asymmetric case

Figures 3−6 are taken from the work [18].

V. Case $u_1 = 0$

In this case, the controls (given by the maximum principle (3)) carrying the motion onto the boundary of the reachable set $G(t_f)$ take the marginal values $u_1 = 0, u_2 = 1$. The main peculiarity of this case is that the rectilinear parts can appear in two situations:

1) When the part is located at the side of SSL, where $\psi_3^*(t) < 0$. Then the maximum in condition (3) is attained when $u(t) = 0$;

2) When the system moves upon SSL.

If for a motion that carries the system to the boundary of the set $G(t_f)$, one has $\psi_3^*(t_0) < 0$ at the initial instant t_0, then either the motion is rectilinear (to the direction of the initial velocity vector) in the entire interval $[t_0, t_f]$, or it reaches SSL with some non-zero angle at some instant and further turns to a circle arc in the half-plane $\psi_3^*(t) > 0$. If earlier than the instant t_f the trajectory reaches SSL again, then since the entrance angle is the same, it passes to the half-plane $\psi_3^*(t) < 0$ and further is rectilinear. As a result, it has no more than two switches.

Now, let $\psi_3^*(t_0) > 0$. Then in the initial period of time, the motion goes along a circle arc. When it reaches SSL with a non-zero angle, it passes to the half-plane $\psi_3^*(t) < 0$, where the motion continues rectilinearly. If the first reach of SSL is tangent (Fig. 7), then in further the motion goes either along a circle arc, or upon SSL with possible transition to a circle arc in the half-plane $\psi_3^*(t) > 0$. If the instant t_f is sufficiently large, then cycles are possible, which were mentioned in Section III. Such cycles can be translated to the initial or, vice versa, final part of the motion with final hit of the same point of the set $G(t_f)$ (as it is shown in Fig. 7). With that, the motion has no more than two switches.

In the same way, the case $\psi_3^*(t_0) = 0$ can be considered.

Thus, the following statement is true:

Proposition 2. *Let $u_1 = 0$. Then any point on the boundary of the set $G(t_f)$ can be reached by a motion, which has no more than two switches. With that, only two variants of control sequences are possible : u_1, u_2, u_1; u_2, u_1, u_2. For the first variant, the duration of the second trajectory part, where $u(t) = u_2 = 1$, is less than 2π.*

In this proposition, variants with only one switching and without it are also considered. For this, the length of corresponding one or two intervals are equal to zero.

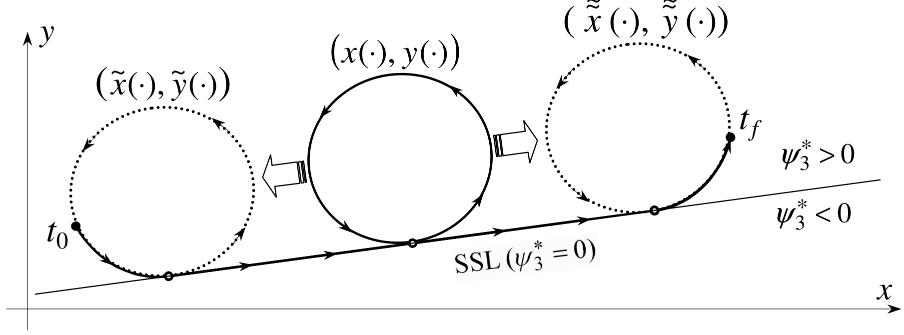


Fig. 7: Case $u_1 = 0$. Translation of a cycle to the beginning or to the end of the motion for a variant of a tangency with SSL. Three motions reach the same point of the set $G(t_f)$.

Due to the mentioned property (by analogy with the previous constructions for $u_1 < 0$), the boundary of the reachable set $G(t_f)$ can be built by means of two surfaces generated by the variants of control shown above. In Fig. 8, examples of sets $G(t_f)$ are presented for instants $t_f = 4\pi, 6\pi$.

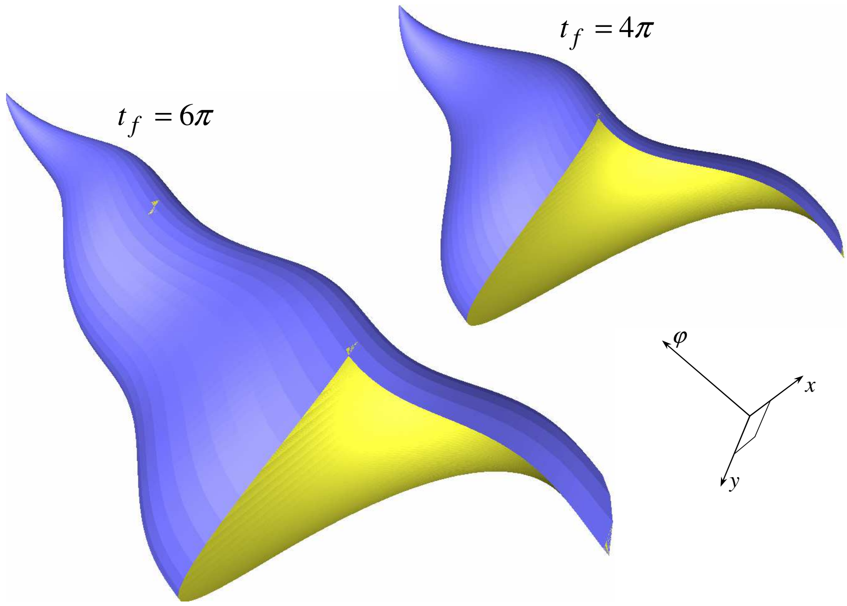


Fig. 8: Reachable sets $G(t_f)$ with $u_1 = 0$ for the instants $t_f = 4\pi, 6\pi$

Let us show that cross-sections of the sets $G(t_f)$ on the coordinate φ are convex and have the form of a circle or a part of circle which is cut off by a chord. We call this figure circle segment.

Consider a motion on the time interval $[t_0, t_f]$ with two instants t_1 and t_2 $(t_0 \leqslant t_1 < t_2 \leqslant t_f)$ of switching and with three intervals of control constancy. The initial state (as before) is regarded to be the origin at the initial instant $t_0 = 0$. The control is constant on the intervals $[t_0, t_1)$, $[t_1, t_2)$, $[t_2, t_f]$. With that, control values on the first and third intervals coincide. Denote the length of intervals with the control constancy as follows: $\Delta t_1 = t_1 - t_0$, $\Delta t_2 = t_2 - t_1$, $\Delta t_3 = t_f - t_2$.

We write formulas for phase states of system (1) at the instant t_f for the variant with the control sequence $0, 1, 0$:

$$x(t_f) = \Delta t_1 + \sin(\Delta t_2) + \cos(\Delta t_2) \cdot \Delta t_3,$$

$$y(t_f) = (1 - \cos(\Delta t_2)) \; + \; \sin(\Delta t_2) \cdot \Delta t_3,$$

$$\varphi(t_f) = \quad \Delta t_2.$$

It is evident that for a fixed value of $\varphi(t_f)$, motions coming onto the boundary of the reachable set are characterized by a constant value Δt_2. Further, taking into account the relation $\Delta t_3 = t_f - \Delta t_2 - \Delta t_1$, one can conclude that the values $x(t_f)$ and $y(t_f)$ depend linearly on Δt_1. Totality of such points comprises a segment in the plane x, y for admissible collection of values Δt_1: $0 \leqslant \Delta t_1 \leqslant (t_f - \Delta t_2)$.

Note that the variant $0, 1, 0$ of the control sequence is considered only for $\Delta t_2 < 2\pi$. It follows from Proposition 2.

Consider now the variant with the control sequence $1, 0, 1$. Similarly, integrating dynamic equations (1), we obtain the following relations defining system states at the instant t_f:

$$\begin{aligned} x(t_f) &= \sin(\Delta t_1 + \Delta t_3) + \cos(\Delta t_1) \cdot \Delta t_2, \\ y(t_f) &= (1 - \cos(\Delta t_1 + \Delta t_3)) + \sin(\Delta t_1) \cdot \Delta t_2, \\ \varphi(t_f) &= \Delta t_1 + \Delta t_3. \end{aligned}$$

Here for a fixed value $\varphi(t_f)$, the motions carrying onto the boundary of the reachable set are characterized by a constant value $\Delta t_1 + \Delta t_3$. The value $\Delta t_2 = t_f - \Delta t_1 - \Delta t_3$ is also a constant.

Totality of points $x(t_f), y(t_f)$ (obtained in such a way) in the plane x, y is defined by variation of the parameter Δt_1 in the limits from 0 up to $(t_f - \Delta t_2)$. Corresponding hodograph satisfies the equation of a circle. The circle radius is equal to Δt_2.

Having considered possible collection of values of the parameter Δt_1, we obtain either a circle arc or the whole circle (circumference). The marginal points of the circle arc correspond to controls $0, 1$ and $1, 0$. These points coincide with the ends of the straight line interval (in the plane x, y) obtained earlier for variant with the control sequence $0, 1, 0$. Thus, we have a description of the φ-sections of the reachable set that actually represents either a circle segment (for $\varphi < 2\pi$) or a whole circle (for $\varphi \geqslant 2\pi$).

In Fig. 9, the variant is shown, in which values of the coordinate φ are taken by modulo 2π.

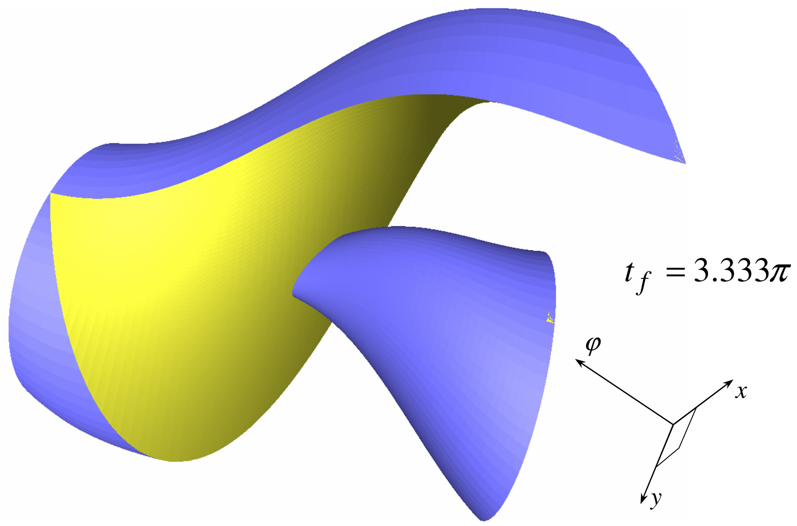


Fig. 9: Reachable set $G(t_f)$ for $t_f = 3.333\pi$ with φ computed by modulo 2π in the case $u_1 = 0$

VI. Case $u_1 > 0$

In this case, rectilinear motions disappear and controls carrying the motion onto the boundary of the reachable set produce a collection of circle arcs with radii $1/u_1$ and $1/u_2$, and the number of switches is finite.

Consider two neighbor parts of the motion satisfying the PMP and suppose that their time intervals do not touch the boundaries of the interval $[0, t_f]$. Total variation of the angle φ during these parts is equal to 2π. Corresponding total duration in time is constant and belongs to the interval $(2\pi/u_2, 2\pi/u_1)$.

Using this property, we can formulate the following statement.

Proposition 3. *Let $u_1 > 0$. Then an estimate from above for a number of switches of the control carrying onto the boundary of the reachable set $G(t_f)$ is valid*:

$$\begin{cases} \dfrac{t_f{\cdot}u_2}{\pi} & \text{if } t_f{\cdot}u_2 \text{ is multiple to } 2\pi, \\ 2\left[\dfrac{t_f{\cdot}u_2}{2\pi}\right] + 2 & \text{otherwise.} \end{cases} \tag{6}$$

Here, the square brackets denote the integer part of a real number. It is seen that as t_f grows, the number of switches grows also in contrast to two previous cases with $u_1 < 0$ and $u_1 = 0$.

Numerical constructions of the reachable sets are based on consideration of two surfaces. The first surface corresponds to the initial control $u = u_1$, the second one corresponds to the initial control $u = u_2$. In both cases, the motion (satisfying the PMP) is defined by the first two switching instants. Each of the mentioned surfaces is a two-parameter family of points in the three-dimensional phase space. In Fig. 10, the examples of reachable sets $G(t_f)$ (constructed by such a way) are shown for the instants $t_f = 8\pi, 16\pi$ in the case $u_1 = 0.5$.

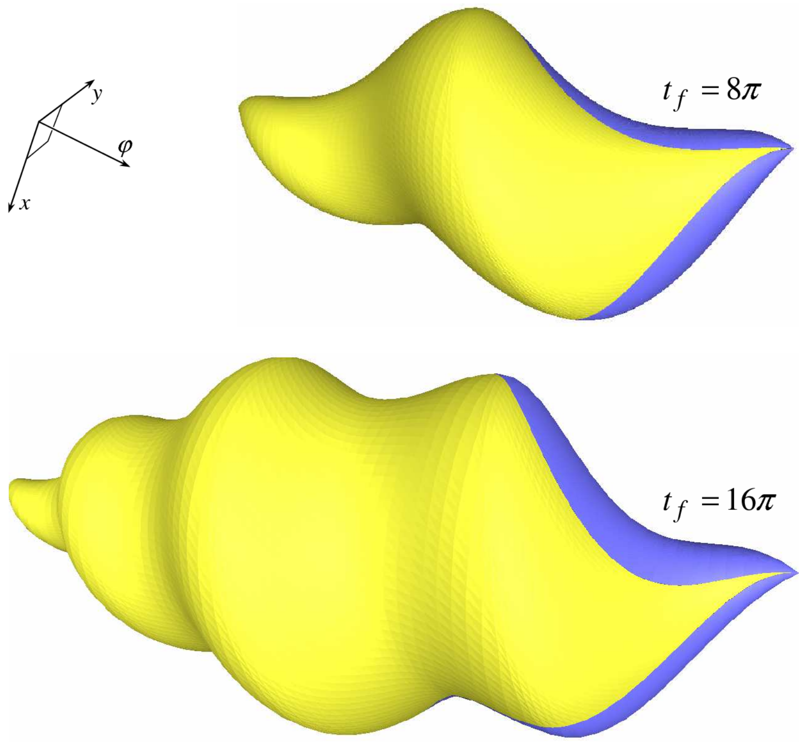


Fig. 10: Reachable sets $G(t_f)$ for the instants $t_f = 8\pi, 16\pi$
in the case $u_1 = 0.5$

During the process of numerical construction of the set $G(t_f)$ for $u_1 > 0$, an *experimental* fact has been discovered on convexity of the φ-sections. But now there is no any theoretical proof of this fact.

VII. Projections of reachable sets at instant onto the plane of geometric coordinates

Investigation of reachable sets for the Dubins' car in the plane of geometric coordinates x, y has a particular interest. Denote a reachable set at instant in the plane x, y by $G_{x,y}(t_f)$. For the symmetric case the reachable sets $G_{x,y}(t_f)$ are described in paper [14]. In the symmetric case, the controls carrying the system to the boundary $\partial G_{x,y}(t_f)$ of the set $G_{x,y}(t_f)$ has no more than one switching. The variants of these controls are the following: -1, 0; 1, 0; -1, 1; 1, -1. Controls in the asymmetric case has similar structure. Namely, instead of -1, one should take u_1. In the case $u_1 = 0$, from studying the projections of the three-dimensional sets $G(t_f)$ into the plane x, y, one can obtain the following structure of the controls generating motions going to $\partial G_{x,y}(t_f)$: 1, 0; 0, 1. When $u_1 > 0$, the number of switchings of controls guiding the system to $\partial G_{x,y}(t_f)$ grows with growth of t_f. A peculiarity is also that for a fixed t_f, motions that going to different points on $\partial G_{x,y}(t_f)$ can have different number of switchings. The authors did not study this question in details yet. In Figs. 11, 12, the sets $G_{x,y}(t_f)$, which correspond to the case $u_1 > 0$, are obtained by projections of three-dimensional sets $G(t_f)$ into the plane x, y. The boundaries of the sets $G_{x,y}(t_f)$ for $u_1 = -1$, $u_1 < 0$, and $u_1 = 0$ are computed with usage of the described property of the extremal motions going to the boundary.

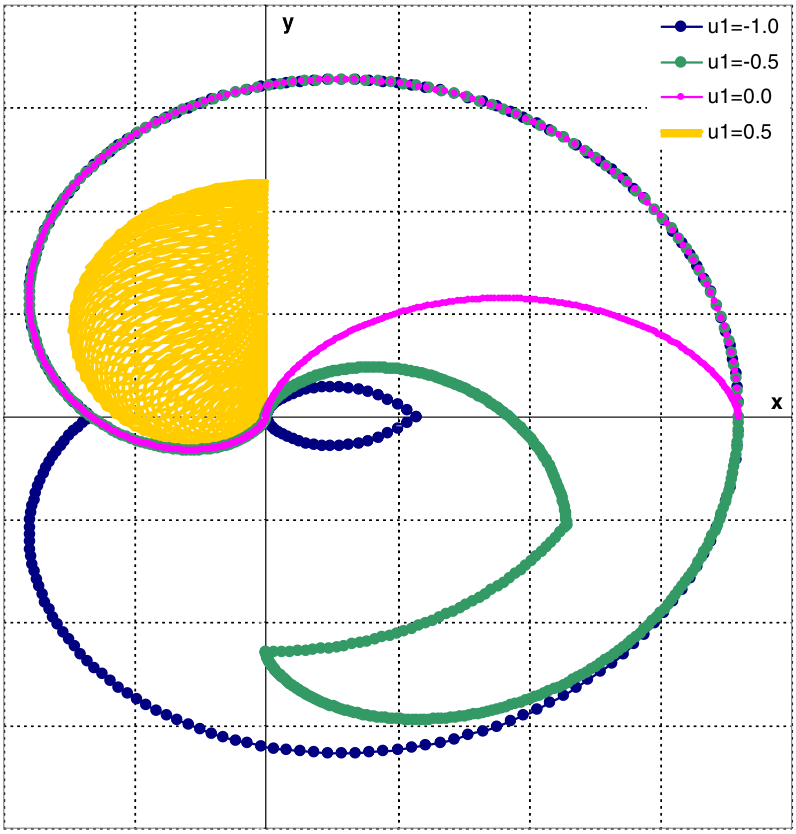


Fig. 11: Reachable sets $G_{x,y}(t_f)$ for the instant $t_f = 2\pi$

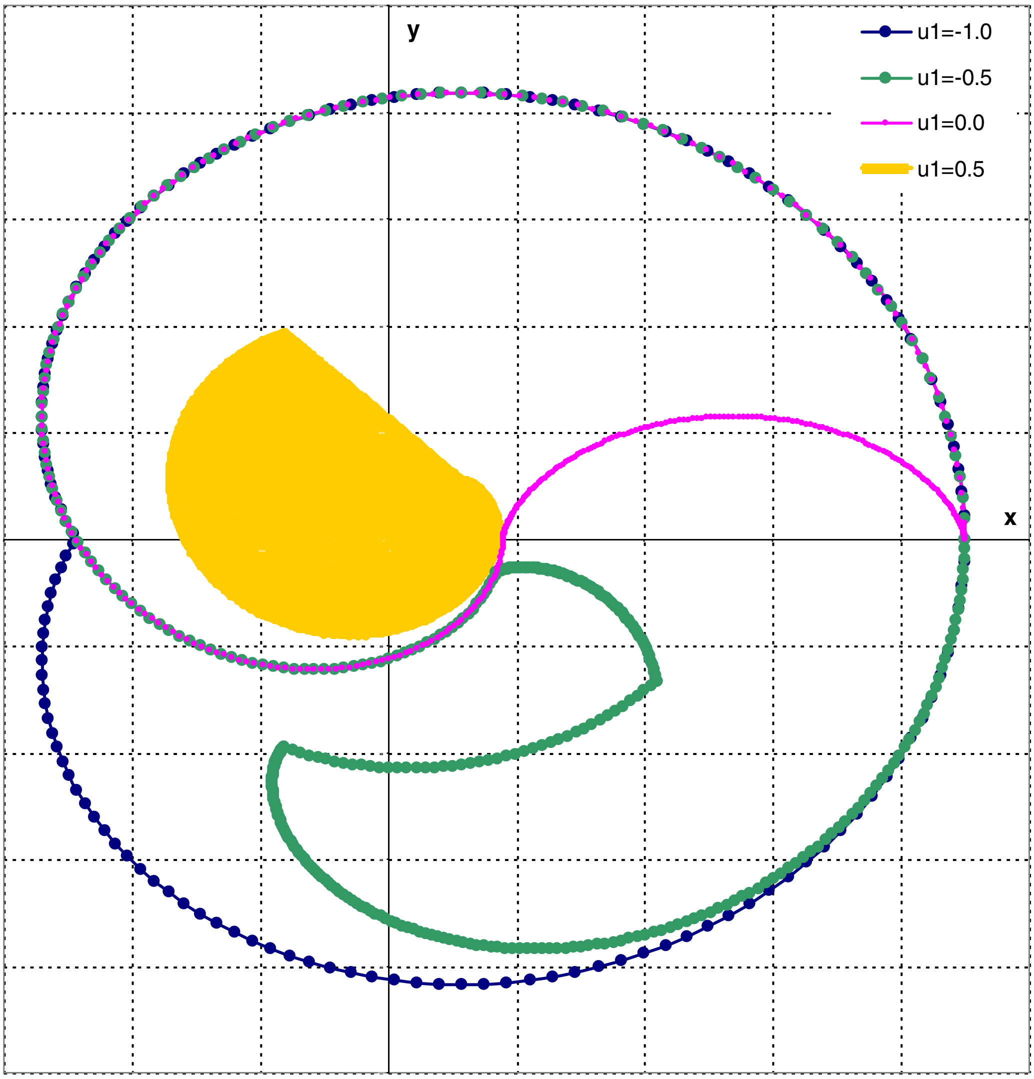


Fig. 12: Reachable sets $G_{x,y}(t_f)$ for the instant $t_f = 2.5\pi$

VIII. Conclusion

The paper presents results of numerical studies of three-dimensional reachable sets at instant for the Dubins' car. The authors proved the convexity of the cross-sections (orthogonal to the angular axis) of the reachable set for the case when zero lies on the edge of the interval of admissible values of the control. In this case, the car can turn only to one side with the possibility of rectilinear segments of motion. It is discovered that the convexity of the cross-sections orthogonal to the angular axis is also kept in the case when the control constraints exclude the possibility of rectilinear motion. In the future, an attempt will be made to prove theoretically such a property.

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