Three-Dimensional Reachable Set at Instant for the Dubins Car: Properties of Extremal Motions

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For the Dubins car model, the paper highlights the cases when the Pontryagin Maximum Principle (PMP) is both necessary and sufficient condition for the motions leading onto the boundary of the reachable set at a given instant. Some relation is revealed between the PMP sufficient property and convexity of the reachable set sections on the angular coordinate. Uniqueness of the extremal controls leading onto the boundary is analyzed in the class of piecewise-constant controls.

I. Introduction

Let the dynamics of a controllable object (the Dubins car) in the plane x, y be described by the following system of ordinary differential equations of the third order

$$\begin{aligned} \dot{x} &= \cos\varphi, \\ \dot{y} &= \sin\varphi, \\ \dot{\varphi} &= u, \qquad u \in [u_1, u_2], \quad -u_2 \leqslant u_1 < u_2. \end{aligned} \tag{1}$$

Here, x, y are geometric coordinates of the car in the plane; φ is an angle of the velocity vector measured counter-clockwise from the axis x (Fig. 1); u is a scalar control. We suppose that the angular coordinate φ takes its values from the interval $(-\infty, +\infty)$. The linear velocity value is constant and equal to one.

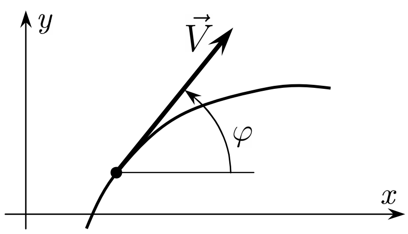

Fig. 1: The coordinate system, $\vec{V} = (\dot{x}, \dot{y})^{\mathsf{T}}$

Further, we assume that $u_2 = 1$, while the value u_1 is considered as a parameter of the problem. Any arbitrary system of the third order describing the object motion

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with a constant linear velocity and a given interval of the angular turn velocity can be transformed to system (1) with $u_2 = 1$ by a proper normalization. Without loss of generality, we assume the zero state $x_0 = 0$, $y_0 = 0$, $\varphi_0 = 0$ of the system at the initial instant $t_0 = 0$.

Any measurable functions $u(\cdot)$ of time are considered as admissible controls with their values $u(t)$ from the interval $[u_1, u_2]$. We define a reachable set $G(t_f)$ *at the instant* t_f all those phase states in the three-dimensional space, which are reachable exactly at the instant t_f from the given initial phase state using some admissible control. In this work, we show using the Pontryagin Maximum Principle (PMP) [23] that it will be enough to consider only piecewise-constant controls for the reachable set analysis.

We distinguish the following four cases:
1) $0 < u_1 < u_2 = 1$; 2) $u_1 = 0$, $u_2 = 1$; 3) $u_1 = -1$, $u_2 = 1$; 4) $-1 < u_1 < 0$, $u_2 = 1$.

In case 1), we deal with the situation of a strictly one-side turn. Case 2) corresponds to the one-side turn with possible motion along a straight line. In cases 3) and 4), the turn is possible to both sides. Case 3) is called symmetric, and case 4) is asymmetric. Two-dimensional cross-sections of the reachable set on the angular coordinate φ are called φ-sections.

The Dubins model is very often used to analyze the horizontal motion of an aircraft or an unmanned aerial vehicle (see, for instance, [1, 5, 13–15, 22]). A possible application of the one-side turn is described in [4]. The Dubins model is also used to simplify the description of motion dynamics for some carts in control problems in the plane [9, 10]. In works [2,3], new time-optimal problems are considered. They are related to a sequential passage of the Dubins car through several points in the plane.

In paper [6], a complete description of the reachable set projection into the plane x, y is given for case 3). Some questions connected with investigation and construction of the three-dimensional reachable sets "at the instant" for cases 1) — 4) were considered in the previous papers [8, 17–19, 21]. For cases 1) and 2), an analytical description of φ-sections of the reachable set is presented in [16, 20].

The main purpose of this paper is to find out a relation between the PMP and the convexity of φ-sections of the three-dimensional reachable set. More precisely, we are interested in the following topics:

1. The Pontryagin Maximum Principle for controls leading onto the boundary is a necessary condition. Is it also a sufficient condition?

2. Are φ-sections of the reachable set convex and simply connected?

3. In what cases is the extremal control leading onto the reachable set boundary unique?

We will give answers to these questions for each of cases 1) – 4).

II. Pontryagin Maximum Principle as the necessary condition for controls leading onto the reachable set boundary

It is known [12] that the PMP is a necessary condition, which is satisfied by motions leading onto the boundary of the reachable set. Repeating [19], let us give the PMP formulas without going into details of each of cases 1) – 4).

Let $u^*(\cdot)$ be some admissible control with values from the interval $[u_1, u_2]$ and $(x^*(\cdot), y^*(\cdot), \varphi^*(\cdot))^{\mathsf{T}}$ be the motion of system (1) generated by this control in $[0, t_f]$.

The adjoint differential equations [1, 10, 21] are

$$\begin{aligned} \dot{\psi}_1 &= 0, \\ \dot{\psi}_2 &= 0, \\ \dot{\psi}_3 &= \psi_1 \sin\varphi^*(t) - \psi_2 \cos\varphi^*(t). \end{aligned} \tag{2}$$

The PMP means that there is a *non-zero* solution $(\psi_1^*(\cdot), \psi_2^*(\cdot), \psi_3^*(\cdot))^{\mathsf{T}}$ to system (2), for which the condition

$$\psi_1^*(t)\cos\varphi^*(t) + \psi_2^*(t)\sin\varphi^*(t) + \psi_3^*(t)u^*(t)$$

$$= \max_{u\in[u_1,u_2]} [\psi_1^*(t)\cos\varphi^*(t) + \psi_2^*(t)\sin\varphi^*(t) + \psi_3^*(t)u]$$

holds almost everywhere (a.e.) in the interval $[0, t_f]$ or what is the same,

$$\psi_3^*(t)u^*(t) = \max_{u\in[u_1,u_2]} \psi_3^*(t)u \quad \text{a.e. in } [0, t_f]. \tag{3}$$

The functions $\psi_1^*(\cdot)$ and $\psi_2^*(\cdot)$ are constants. Denote them by ψ_1^* and ψ_2^*.

If $\psi_1^* = 0$ and $\psi_2^* = 0$, then $\psi_3^*(t) = \text{const} \neq 0$ over the interval $[0, t_f]$. In this case, one has $u^*(t) = u_1$ a.e. in $[0, t_f]$ or $u^*(t) = u_2$ a.e. in $[0, t_f]$.

Let us assume that at least one of the numbers ψ_1^*, ψ_2^* does not equal to zero. Using equations (1) and (2), one can write the following relation for $\psi_3^*(t)$:

$$\psi_3^*(t) = \psi_1^* y^*(t) - \psi_2^* x^*(t) + C.$$

Hence, $\psi_3^*(t) = 0$ if and only if the point $(x^*(t), y^*(t))^{\mathsf{T}}$ of the geometric position at the instant t obeys the linear equation

$$\psi_1^* y - \psi_2^* x + C = 0. \tag{4}$$

So, $\psi_3^*(t) > 0$ in the half plane $\psi_1^* y - \psi_2^* x + C > 0$, and $\psi_3^*(t) < 0$ in the half plane $\psi_1^* y - \psi_2^* x + C < 0$. Since change of the sign of $\psi_3^*(\cdot)$ implies change of the control from one extremal value to another, the line defined by (4) is often called *the switching line.*

Due to relation (3), if $\psi_3^*(t) > 0$ in some interval, then $u^*(t) = u_2$ a.e. in this interval. The projection of the corresponding motion into the plane x, y goes counter-clockwise along a circular arc of radius $1/u_2$. If $\psi_3^*(t) < 0$, then $u^*(t) = u_1$. The projection of the corresponding motion goes clockwise along a circular arc of radius $1/|u_1|$ when $u_1 < 0$, counter-clockwise when $u_1 > 0$, and along a straight line when $u_1 = 0$.

If $\psi_3^*(t) = 0$ in some interval, then the motion $(x^*(\cdot), y^*(\cdot))^{\mathsf{T}}$ in this interval goes along the switching line (4). With that, $u^*(t) = 0$ a.e. in the interval. Such a case is impossible when $u_1 > 0$.

Thus, for system (1), the projection of any motion obeying the PMP into the plane x, y consists of circular arcs and straight line segments. Within each of the arcs or segments, the control can be considered as a constant. So, analyzing the controls obeying PMP, one can consider only *piecewise-constant* controls (assuming their right-continuity at the points of discontinuity). The number of switchings in the interval $[0, t_f]$ is finite.

III. Case of a strictly one-side turn with $u_1 > 0$

A. Extremal motions in the plane x, y

The projections into the plane x, y of a strictly one-side turn motion satisfying the PMP are formed by motions over the circular arcs of radii $1/u_1$ and $1/u_2$ under the controls u_1 and u_2.

In Fig. 2, a variant of the motion satisfying the PMP is shown in the plane x, y and the corresponding switching straight line is also presented. The motion along the circular arcs is performed counter-clockwise with increasing time t.

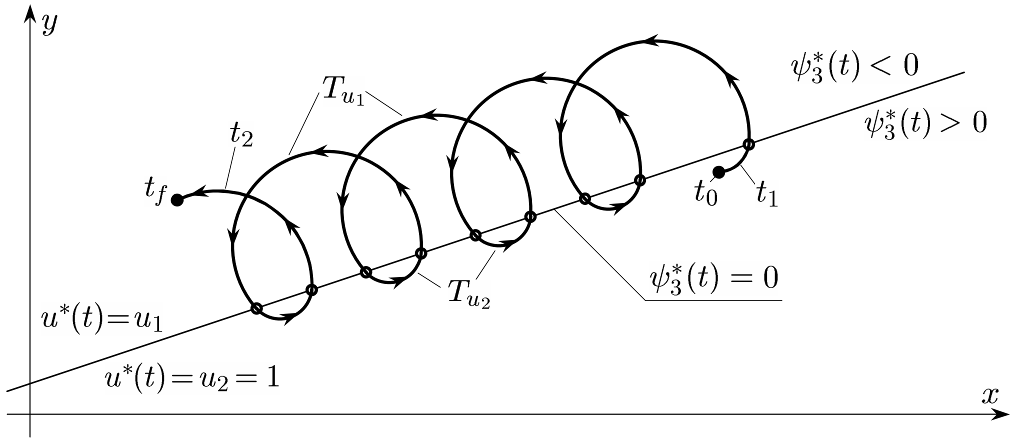

Fig. 2: The case of a strictly one-side turn. The trajectory satisfies the PMP. The straight line is the corresponding switching line

Those parts of the motion that *begin* and *end* on the switching line and pass in the half plane $\psi_1^* y - \psi_2^* x + C \leqslant 0$ have equal time-length, which we denote by T_{u_1}. The intermediate parts (passing in the half plane $\psi_1^* y - \psi_2^* x + C \geqslant 0$) also have equal time-lengths, which we denote by T_{u_2}.

Note that the switching line and, as a result, the intermediate parts of the lengths T_{u_1} and T_{u_2} are connected with a particular motion. For another motion satisfying the PMP, their lengths can be different.

Intermediate parts with the controls u_1 and u_2 go one after another and join on the switching line (Fig. 2). The motion along any such *neighbour* arcs gives an increment of the angular coordinate φ that is equal to 2π; *i.e.*, $T_{u_1} \cdot u_1 + T_{u_2} \cdot u_2 = 2\pi$. According to the statement of the problem, we have $u_2 = 1$. So,

$$T_{u_1} \cdot u_1 + T_{u_2} = 2\pi. \tag{5}$$

Since $u_1 < 1$, the time-length $T_{u_1} + T_{u_2}$ exceeds 2π that is necessary for implementation of one "loop". Thus, in the interval $[0,\, t_f]$, any control leading onto the boundary of the reachable set either has no switchings or their number is finite.

Consider the motion of system (1) over the time interval $[t_0, t_f]$ ($t_0 = 0$, $t_f > 0$) with the zero initial phase state. Possible values $\varphi(t_f)$ of the coordinate φ at the instant t_f are in the interval $[t_f{\cdot}u_1, t_f]$. The lower and upper values of φ are implemented under the controls $u(t){\equiv}u_1$ and $u(t){\equiv}u_2 = 1$, which have no switchings. The corresponding φ-sections of the set $G(t_f)$ are points in the plane x, y:

$$\left(\frac{\sin(t_f{\cdot}u_i)}{u_i}, \frac{-\cos(t_f{\cdot}u_i)}{u_i}\right)^{\mathsf{T}}, \quad i = 1, 2.$$

Further, suppose that $\varphi \in (t_f{\cdot}u_1, t_f)$. The motions arriving onto the boundary of such a φ-section satisfy the PMP and must have more than one switching (*i.e.*, they have at least two parts of the control constancy).

We introduce the following notation. The symbol t_1 (respectively t_2) means the length of the first (last) part of the control constancy joining to the instant t_0 (t_f) (Fig. 2).

We say that a motion and its generating control belong to the BS type if the control satisfies the PMP and is equal to u_1 on the first part (where the motion goes along an arc of the big circle) and equal to u_2 on the last part (where the motion goes along an arc of the small circle).

Similarly, we define the SB, BB, and SS types of motions and their controls by the control pairs (u_2, u_1), (u_1, u_1), and (u_2, u_2) on the first and last parts of the control constancy. Any control satisfying the PMP belongs to one and only one of the mentioned four types. For example, the trajectory shown in Fig. 2 is generated by the control of the SB type.

B. Boundary of the φ-sections in the case $\varphi < 2\pi$

First of all, note that the controls satisfying the PMP have no more than two switchings for $\varphi < 2\pi$. Otherwise, relation (5) implies $\varphi > 2\pi$.

a) Let us consider a motion with one switching of the control u. Assume the control is equal to $u_2 = 1$ on the first part of motion, and the control coincides with u_1 on the second part (the SB type). The following relations are valid:

$$\varphi = t_1 + t_2{\cdot}u_1, \qquad t_f = t_1 + t_2.$$

From this, we obtain that the values t_1, t_2 (and as a consequence, the switching instant) are determined uniquely by fixed values u_1, t_f, and φ. Therefore, only one point corresponds to the sequence of controls u_2, u_1 in any φ-section of the set $G(t_f)$.

Similarly, for the sequence of the controls u_1, u_2 (the BS type), we obtain the coordinates of the point in the φ-section of the set $G(t_f)$.

b) Now we investigate the version with two switchings and the sequence of controls u_2, u_1, u_2 (the SS type). The length of corresponding parts (of the control constancy) are t_1, T_{u_1}, t_2. So, we have

$$\varphi = t_1 + T_{u_1}{\cdot}u_1 + t_2, \qquad t_f = t_1 + T_{u_1} + t_2,$$

which yields

$$T_{u_1} = \frac{t_f - \varphi}{1 - u_1}\,, \qquad t_1 + t_2 = \frac{\varphi - t_f{\cdot}u_1}{1 - u_1}\,.$$

Therefore, the length of the middle part and the summary length of the first and last parts are the constant values in the present case. The obtained family of the admissible controls is one-parametric. We take the value t_1 as a parameter with the range $(0, \mathcal{T}_{\mathrm{s}})$, where $\mathcal{T}_{\mathrm{s}} = (\varphi - t_f{\cdot}u_1)\,/(1 - u_1)$.

Integrating equations (1) over the mentioned intervals with the constant values of the controls and using trigonometric transformations, we obtain the points $(x_{\mathrm{SS}}[t_1], y_{\mathrm{SS}}[t_1])^{\mathsf{T}}$

of the φ-section of the set $G(t_f)$:

$$\begin{pmatrix} x_{\text{SS}}[t_1] \\ y_{\text{SS}}[t_1] \end{pmatrix} = \begin{pmatrix} \sin t_1 \\ 1-\cos t_1 \end{pmatrix} + \frac{1}{u_1}\begin{pmatrix} \sin(t_1{+}T_{u_1}\cdot u_1) - \sin t_1 \\ \cos t_1 - \cos(t_1{+}T_{u1}\cdot u_1) \end{pmatrix} + \begin{pmatrix} \sin\varphi - \sin(t_1{+}T_{u_1}\cdot u_1) \\ \cos(t_1{+}T_{u_1}\cdot u_1) - \cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \sin\varphi \\ 1-\cos\varphi \end{pmatrix} \ + \ 2\left(\frac{1}{u_1}-1\right)\sin\left(\frac{T_{u_1}\cdot u_1}{2}\right)\begin{pmatrix} \cos\left(t_1+\dfrac{T_{u_1}\cdot u_1}{2}\right) \\ \sin\left(t_1+\dfrac{T_{u_1}\cdot u_1}{2}\right) \end{pmatrix}, \qquad t_1 \in (0,\ \mathcal{T}_{\text{S}})\,. \tag{6}$$

In the plane x, y, such points compose a circle arc with center

$$\begin{pmatrix} \sin\varphi \\ 1-\cos\varphi \end{pmatrix} \tag{7}$$

and radius

$$2\left(\frac{1}{u_1}-1\right)\sin\left(\frac{T_{u_1}\cdot u_1}{2}\right). \tag{8}$$

The arc (6) subtends the angle, which is determined by the range t_1 and is equal to $\mathcal{T}_{\text{S}}$.

c) Finally consider the second version with two switchings and the sequence of controls u_1, u_2, u_1 (the BB type). The lengths of corresponding parts of the control constancy are t_1, T_{u_2}, t_2. By analogy with the previous version, we have

$$\varphi = t_1\cdot u_1 + T_{u_2} + t_2\cdot u_1, \qquad t_f = t_1 + T_{u_2} + t_2,$$

which implies

$$T_{u_2} = \frac{\varphi - t_f\cdot u_1}{1-u_1}, \qquad t_1 + t_2 = \frac{t_f - \varphi}{1-u_1}.$$

Integrating equations (1) over the mentioned intervals with constant values of the controls, we get the points $(x_{\text{BB}}[t_1], y_{\text{BB}}[t_1])^{\mathsf{T}}$ of the φ-section of the set $G(t_f)$:

$$\begin{pmatrix} x_{\text{BB}}[t_1] \\ y_{\text{BB}}[t_1] \end{pmatrix} = \frac{1}{u_1}\begin{pmatrix} \sin(t_1\cdot u_1) \\ 1-\cos(t_1\cdot u_1) \end{pmatrix} + \begin{pmatrix} \sin(t_1\cdot u_1+T_{u_2}) - \sin(t_1\cdot u_1) \\ \cos(t_1\cdot u_1) - \cos(t_1\cdot u_1+T_{u_2}) \end{pmatrix} + \frac{1}{u_1}\begin{pmatrix} \sin\varphi - \sin(t_1\cdot u_1+T_{u_2}) \\ \cos(t_1\cdot u_1+T_{u_2}) - \cos\varphi \end{pmatrix}$$

$$= \frac{1}{u_1}\begin{pmatrix} \sin\varphi \\ 1-\cos\varphi \end{pmatrix} \ - \ 2\left(\frac{1}{u_1}-1\right)\sin\left(\frac{T_{u_2}}{2}\right)\begin{pmatrix} \cos\left(t_1\cdot u_1+\dfrac{T_{u_2}}{2}\right) \\ \sin\left(t_1\cdot u_1+\dfrac{T_{u_2}}{2}\right) \end{pmatrix}, \qquad t_1 \in (0,\ \mathcal{T}_{\text{B}})\,, \tag{9}$$

where $\mathcal{T}_{\text{B}} = (t_f - \varphi)\,/(1-u_1)$. This is also a circle arc with center

$$\frac{1}{u_1}\begin{pmatrix} \sin\varphi \\ 1-\cos\varphi \end{pmatrix} \tag{10}$$

and radius

$$2\left(\frac{1}{u_1}-1\right)\sin\left(\frac{T_{u_2}}{2}\right). \tag{11}$$

The arc (9) subtends the angle, which is equal to $\mathcal{T}_{\rm B}{\cdot}u_1$.

In the case of a strictly one-side turn under consideration, the following state is valid.

Theorem 1 Assume $\varphi < 2\pi$. Then the φ-section of the reachable set $G(t_f)$ is a strictly convex set in the plane x, y. Its boundary is composed of two arcs of circles with centers (7) and (10), radii (8) and (11), respectively.

Proof. It is easy to prove that the arcs (6) and (9) coincide at the extreme points, that is

$$\begin{pmatrix} x_{\rm SS}[0] \\ y_{\rm SS}[0] \end{pmatrix} = \begin{pmatrix} x_{\rm BB}[\mathcal{T}_{\rm B}] \\ y_{\rm BB}[\mathcal{T}_{\rm B}] \end{pmatrix}, \qquad \begin{pmatrix} x_{\rm SS}[\mathcal{T}_{\rm S}] \\ y_{\rm SS}[\mathcal{T}_{\rm S}] \end{pmatrix} = \begin{pmatrix} x_{\rm BB}[0] \\ y_{\rm BB}[0] \end{pmatrix}.$$

The arc extreme points correspond to the case with one switching that was considered in item a) of this subsection.

Analyzing possible variants of motions of system (1) satisfying the PMP, we obtain a set of positions in the plane x, y, which has the form of a closed curve. The curve is composed of two circle arcs joining at the extreme points.

Consider the motion along arc (6). This motion is determined by the parameter t_1. As the parameter t_1 grows from 0 up to $\mathcal{T}_{\rm S}$, the motion along arc (6) makes a counter-clockwise turn of the tangent vector.

Similarly, as the parameter t_1 grows from 0 up to $\mathcal{T}_{\rm B}$, the motion along arc (9) makes also a counter-clockwise turn of the tangent vector.

The sum of angles subtended by arcs (6) and (9) is equal to

$$\frac{u_1(t_f-\varphi)}{1-u_1}+\frac{(\varphi-t_f{\cdot}u_1)}{1-u_1} \;=\; \varphi\,.$$

The set bounded by arcs (6) and (9) is the intersection of two circles. So, the assumption $\varphi < 2\pi$ implies strict convexity of this set. □

C. Boundary of the φ-sections in the case when φ is a multiple of 2π

In the case $\varphi < 2\pi$ each φ-section is convex and its boundary is composed of two arcs of circles. As the value φ approaches 2π from below, the circle centers come to the origin and their radii have the same value

$$2\left(\frac{1-u_1}{u_1}\right)\sin\left(\frac{2\pi-t_f{\cdot}u_1}{2\,(1-u_1)}\right).$$

Therefore, if $\varphi = 2\pi$, then the φ-section is the circle of the mentioned radius. This property is generalized to the case when φ is a multiple of 2π.

If φ is a multiple of 2π (*i.e.*, $\varphi = 2\pi k$ for some natural k), then the number n of loops for the controls of the SS type is uniquely determined and equal to $k-1$.

Similarly, for the controls of the BB type, we obtain $n = k-1$. The boundary of the φ-section is composed of the arcs $\mathrm{SS}_{(n)}$ and $\mathrm{BB}_{(n)}$. The arc extreme points correspond to controls of the SB and BS type. The arcs $\mathrm{SS}_{(n)}$ and $\mathrm{BB}_{(n)}$ lie just on the same circle with center at the origin and radius

$$2\left(\frac{k(1-u_1)}{u_1}\right)\sin\left(\frac{\varphi-t_f{\cdot}u_1}{2k(1-u_1)}\right). \tag{12}$$

These arcs are joined at their extreme points and they together form the angle equal to 2π.

Thus, the following statement is valid.

Theorem 2. Let φ be a multiple of 2π. Then the φ-section of the reachable set $G(t_f)$ is a circle with center at the origin and radius (12) in the plane x, y.

D. Boundary of the φ-sections in the case when $\varphi > 2\pi$ and φ is not a multiple of 2π

Here, we confine only by the statement of the result.

Theorem 3. Assume $\varphi > 2\pi$ and φ is not a multiple of 2π. Then the φ-section of the reachable set $G(t_f)$ is strictly convex and has a smooth boundary in the plane x, y. Possible variants of arcs composing the boundary are marked in Fig. 3 by numbers $1-9$.

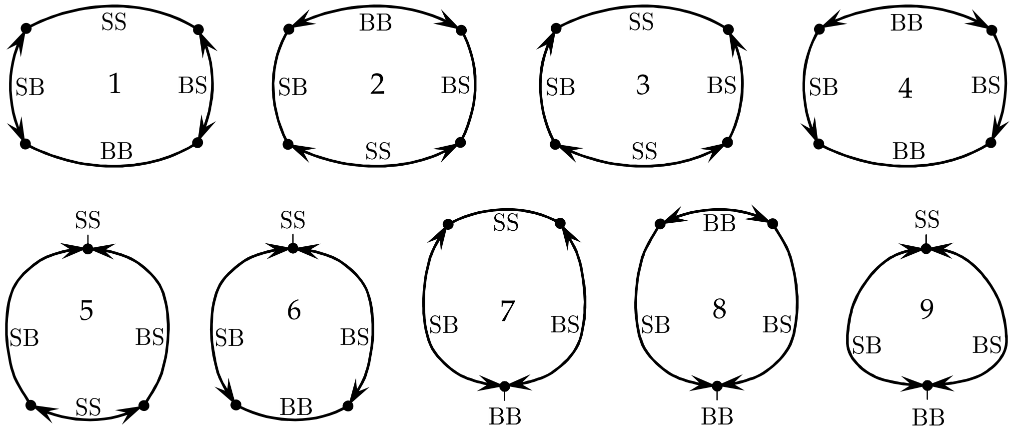

Fig. 3: Variants of the φ-sections of the reachable set in the case of a strictly one-side turn when $\varphi > 2\pi$ and φ is not a multiple of 2π

Proof of this theorem is long enough and completely given in [20] where some analytical formulas are established for all the arcs included in variants $1-9$.

E. Properties of extremal motions

The proofs of Theorems 1 – 3 also show that the constructed boundary of the reachable set contains ends of all the motions satisfying the PMP. Therefore, in the case of a strictly one-side turn, the PMP satisfaction is a sufficient condition for getting onto the reachable set boundary. Moreover, only a unique extremal control leads to each point on the boundary in the class of the piece-constant controls.

For the case of a strictly one-side turn, Fig. 5 shows three-dimensional reachable sets (from two points of view) for two instants $t_f = 6\pi$ and $t_f = 20\pi$. The colors of the surface parts correspond to different control types. The same color is used several times, since the number of switchings changes as the angle φ changes.

For fixed values t_f and φ, the switching number is the same for the arcs BS and SB.

Under chosen direction of moving along the boundary (clockwise or counter-clockwise), four variants of arcs connecting are possible: 1) SB, BB, BS, SS; 2) SB, SS, BS, BB; 3) SB, SS, BS, SS; 4) SB, BB, BS, BB. Depending on t_f and φ, some arcs can degenerate. The arcs BS and SB degenerate simultaneously. In work [20], the authors proved that there can exist 11 types of the φ-sections. Nine of them correspond to the cases when $\varphi > 2\pi$ and φ is not a multiple of 2π. Other two ones correspond to the cases when $\varphi < 2\pi$ and φ is a multiple of 2π.

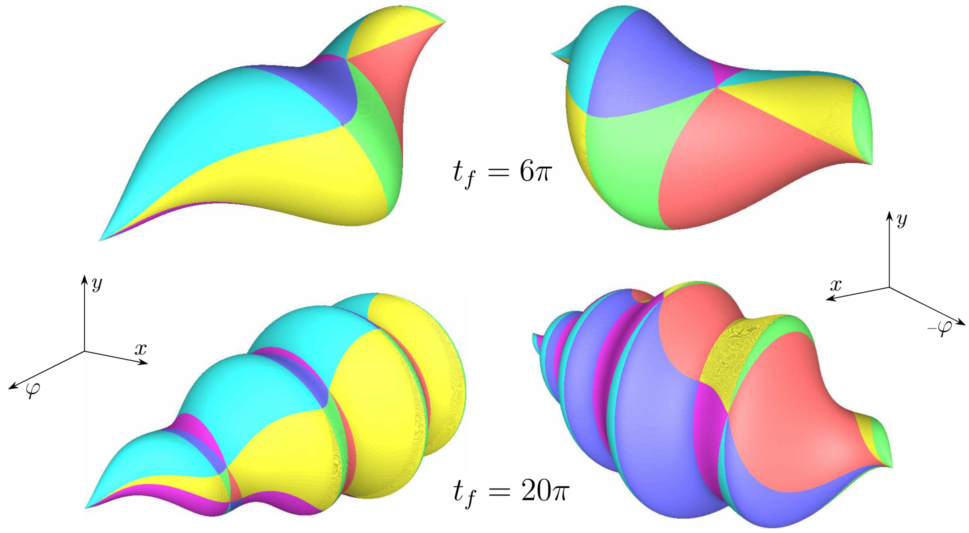

Fig. 4: Three-dimensional reachable sets in the case of a strictly one-side turn at the instants $t_f = 6\pi$ и $t_f = 20\pi$

IV. Case of the one-side turn with $u_1 = 0$

A. Extremal motions in the plane x, y

In this case, the controls satisfying the PMP (3) take the extreme values $u_1 = 0$ and $u_2 = 1$. Here, the main feature is that two variants of straight line motion are possible (Fig. 2):

1) on the time intervals such that $\psi_3^*(t) < 0$;
2) along the switching line ($\psi_3^*(t) = 0$).

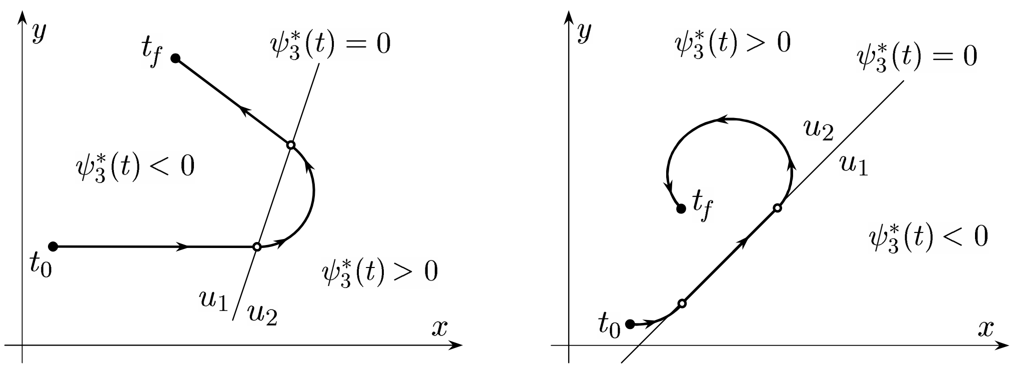

Fig. 5: Extremal trajectories $(x^*(\cdot), y^*(\cdot))^{\mathsf{T}}$ for the case of the one-side turn

The motion part under control $u^*(t) \equiv u_2$ and with the angle $\varphi^*(t)$ changing by 2π is called *the cycle*. The motion trajectory on such a part is a complete circle in projection into the plane x, y.

Assume that the extremal trajectory comes to the switching line by the tangent slope (Fig. 6). Then the further motion continues either along a circle arc or along the switching line with opportunity to leave it at any instant into the half plane $\psi_3^*(t) > 0$ along the arc of a circle of radius 1. If the value t_f is large enough, then cyclic motions can appear.

But such cycles can be "transferred" to the beginning or, contrarily, to the last part of the motion with getting into the same point on the boundary of the reachable set $G(t_f)$ at the instant t_f. Thus, in general for $u_1 = 0$, the uniqueness is absent for the extremal motions leading to the same point on the reachable set boundary.

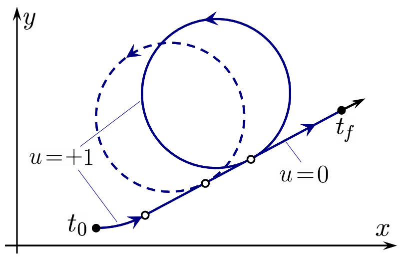

Fig. 6: Non-uniqueness of the extremal motions by virtue of the cycle "transfer"

B. Boundary of the φ-sections

In work [16], it was shown that in the case of the one-side turn, the φ-sections of the set $G(t_f)$ on the angular coordinate φ are convex and for $\varphi \in (0, t_f)$ have the shape of a circle or segment of a circle.

The following statement is proved.

Theorem 4. Assume $u_1 = 0$. Then a piecewise-constant control with no more than two switchings leads to any point on the set $G(t_f)$ boundary. Each φ-section $\varphi \in [0, t_f]$ of the set $G(t_f)$ is convex. For the extreme values $\varphi = 0$ and $\varphi = t_f$, the φ-section is a point. For $\varphi \in (0, t_f)$, the boundary of the φ-section is composed of a circle arc and a linear segment if $\varphi < 2\pi$, and the φ-section is a circle if $\varphi \geqslant 2\pi$.

C. Properties of the extremal motions

In work [16], it was also proved that the constructed boundary of the reachable set contains the end of *all* the motions satisfying the PMP. Therefore, in the case of the one-side turn, the PMP satisfaction is a sufficient condition for getting the reachable set boundary. But unlike the case $u_1 > 0$, in the case $u_1 = 0$, the uniqueness is absent for the extremal motions under $t_f > 2\pi$. The reason is the possibility of the cycle "transfer" along the switching line.

Figure 7 shows the three-dimensional reachable set (from two points of view) in the case of the one-side turn for the instant $t_f = 6\pi$. The colors of the boundary parts correspond to the following two types of extremal controls. The blue color marks the boundary part formed by the controls $1, 0, 1$. The yellow color corresponds to controls of the type $0, 1, 0$.

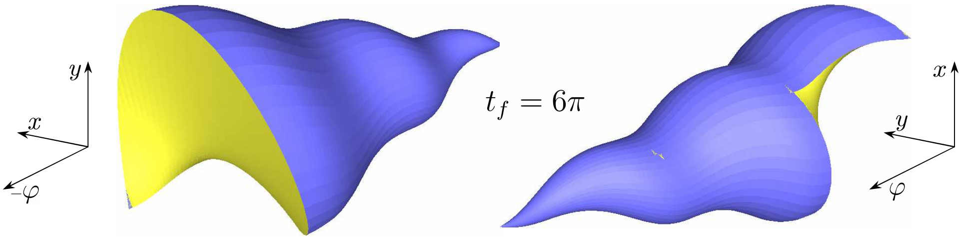

Fig. 7: Three-dimensional reachable sets in the case of the one-side turn at $t_f = 6\pi$

V. Case of the two-side turn $u_1 < 0$

It is evident that if $u_1 = -u_2$, the φ-sections of the reachable set $G(t_f)$ for the values φ and $-\varphi$ are symmetric w.r.t. the initial direction. But such symmetric property is lost if $u_1 \in (-u_2, 0)$. Other properties of the reachable sets in the symmetric and asymmetric cases are similar.

The main difference of the cases 3) and 4) from the considered above cases 1) and 2) is in the fact that, in general, the φ-sections of the reachable sets are not convex. Moreover, for some t_f, the sets $G(t_f)$ are not simply connected.

In work [21], it was shown that in the symmetric case for analysis of controls satisfying the PMP and leading onto the reachable set boundary, it is sufficient to consider only piecewise-constant controls of the following six types:

$$+1, 0, +1; \quad +1, 0, -1; \quad -1, 0, -1; \quad -1, 0, +1; \quad +1, -1, +1; \quad -1, +1, -1.$$

The writing $+1,\ 0, -1$, for example, means that the interval $[0, t_f]$ is divided into three parts. The control $u = +1$ acts on the first part, the control $u = 0$ operates on the second one, and the control $u = -1$ is applied on the third part.

The mentioned six types of piecewise-constant controls coincide with the six types of controls that were described in paper [7]. But note that paper [7] deals with the time-optimal problem and, thus, with the reachable set "up to the instant".

In the symmetric case, the whole boundary of the reachable set is formed of by no more than six two-dimensional surfaces. Each surface corresponds to one mentioned type of the controls. On the lines of touch of two different surfaces, one of the three intervals can disappear. At the point of touch of three and more surfaces, two intervals (of the control constancy) can disappear, and only one can remain. Similar properties are also valid for the asymmetric case [8].

For the instant $t_f = 1.5\pi$, Fig. 8 shows the set $G(t_f)$ in the symmetric case (from two points of view). In the asymmetric case for $u_1 = -0.25$ and $t_f = 6\pi$, the set $G(t_f)$ is presented in Fig. 9. The next Fig. 10 clarifies the violation of the simple connectedness of the set $G(t_f)$ in the symmetric case. These figures are taken from the previous publications of the authors. Without these illustrations, it would be difficult to explain the further text.

So far the authors have not constructed an analytical description of the φ-sections for both symmetric and asymmetric cases. The most difficult is to describe such sections for $|\varphi| < 2\pi$. By experiment, it was found out that a φ-section is a circle for $|\varphi| \geqslant 2\pi$.

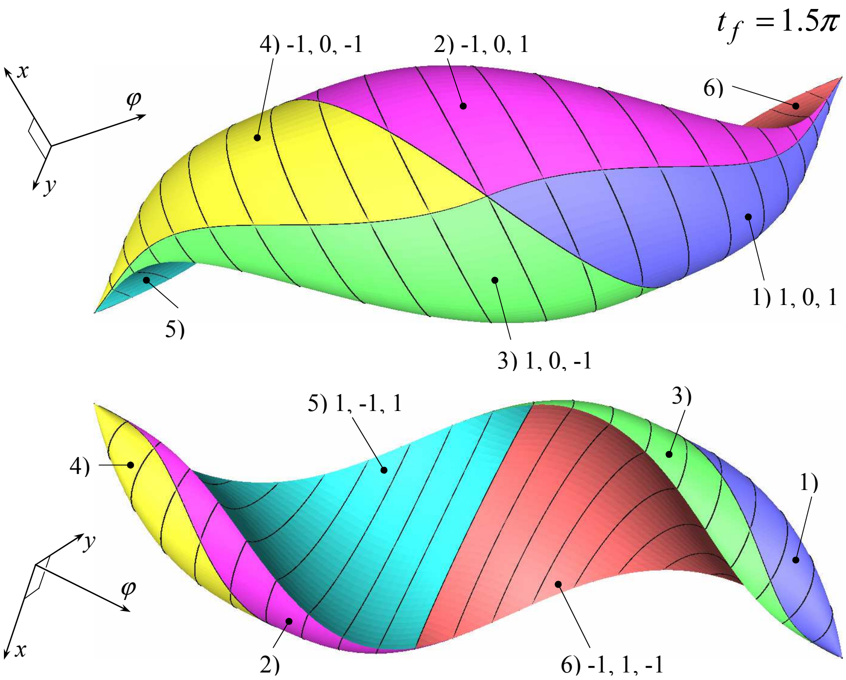

Fig. 8: Reachable set $G(t_f)$ at instant $t_f = 1.5\pi$ for $u_1 = -1$ and $u_2 = 1$

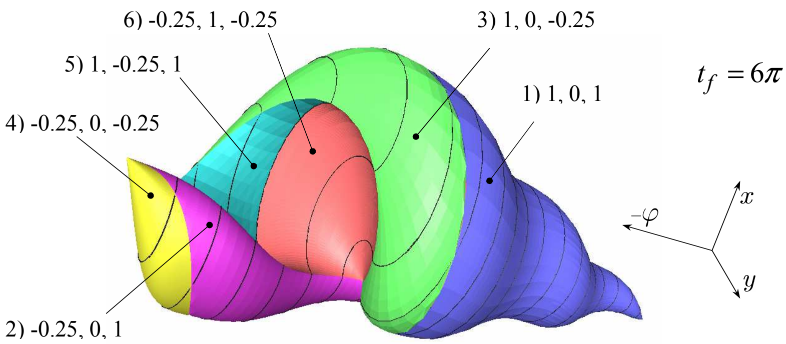

Fig. 9: Reachable sets $G(t_f)$ with $u_1 = -0.25$ for $t_f = 6\pi$ in the asymmetric case

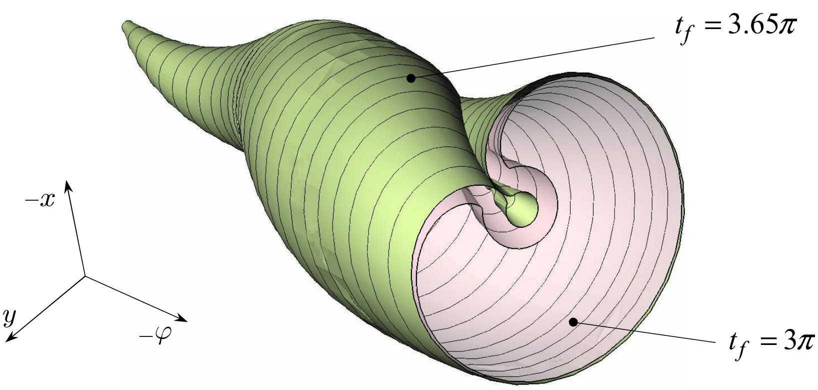

Fig. 10: The violation of the simple connectedness in the symmetric case. The reachable set cut off is made at $\varphi = 0$

A. Extremal motions

For the symmetric case, Fig. 11 shows a trajectory (in the plane x, y) satisfying the PMP. On this trajectory, the control takes two values 1 и -1, but has more than two switchings. In paper [21], it was proved that for such motions the three-dimensional phase state $(x(t_f), y(t_f), \varphi(t_f))^{\mathsf{T}}$ is strictly inside the reachable set. Therefore, in general, the PMP is not a sufficient condition for getting onto the boundary in the symmetric case. The same is true for the asymmetric case.

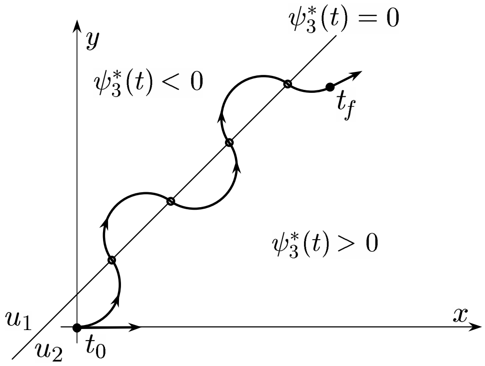

Fig. 11: The motion in the symmetric case satisfies the PMP but leads into the interior of the reachable set

Figure 12 presents another example when the PMP is not a sufficient condition. Here, for small values of $|\varphi|$ and $t_f = 2\pi$, the boundary fragment of the reachable set is shown. Simultaneously, parts of two surfaces lying strictly inside the reachable set are shown. The control $-1, +1, -1$ leads to the piece of the red color; the length of the middle part (under the constant control $u = +1$) is smaller than the sum length of the first and third parts. In [21], it was proved that such a motion comes strictly inside the reachable set at the instant t_f. In Fig. 12, the green part (symmetric to the red one) corresponds to the control $+1, -1, +1$.

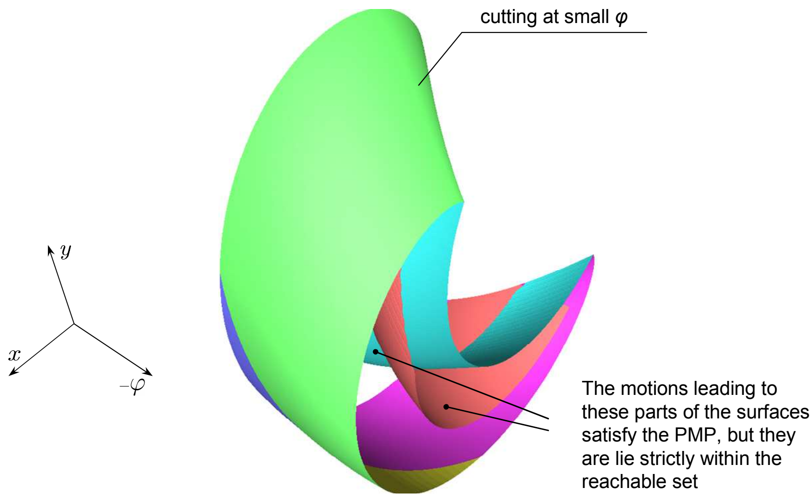

Fig. 12: PMP is only a necessary condition for controls leading to the boundary

By our opinion, a very interesting question arises about a complete description of the surfaces, the transfer onto which satisfies the PMP and which lie strictly inside the set $G(t_f)$. Whether the presence of such surfaces is due to some "stratification" of the reachable set into separate specific parts?

B. Uniqueness of extremal motions

In the symmetric and asymmetric cases, the uniqueness can be absent for motions getting onto the reachable set boundary.

Consider the reachable set section for $\varphi = 0$ and not large value t_f. In the symmetric case (Fig. 8), two types of the control $+1, -1, +1$ and $-1, +1, -1$ lead to each point on the "rear" part of such a section. In the plane x, y, the trajectories leading to the same phase point are shown in Fig. 13. Thus, here, we have exactly two separated motions that lead to the same point on the boundary of the set $G(t_f)$.

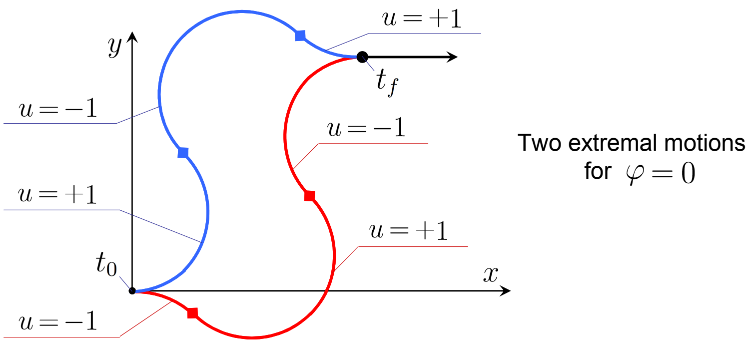

Fig. 13: Nonuniqueness of extremal motions

The analogous trajectories can be also built in the asymmetric case. The similar "dispersal" situation is often met in nonlinear controllable systems when constructing the motions from the terminal phase point to the initial one in the backward time.

Now we give an example with nonuniqueness of the extremal motion that leads onto the reachable set boundary and is connected with the situation of the cycle "transfer". Let us consider (Fig. 14) the symmetric case, in which the control $u = -1$ acts on the first time interval of the length 2π, and the control $u = +1$ operates on the last time interval (up to the instant t_f). This control generates the motion satisfying the PMP and getting onto reachable set boundary (the proof is not trivial). But the same point is reachable by *any* motion such that the control $u = +1$ acts on some initial interval, and the control $u = -1$ operates on the second interval of the length 2π, and the control $u = +1$ is applied on the last interval (up to the instant t_f). As a result, we obtain the "indistinct" totality of extremal motions leading into the same point on the boundary of the set $G(t_f)$.

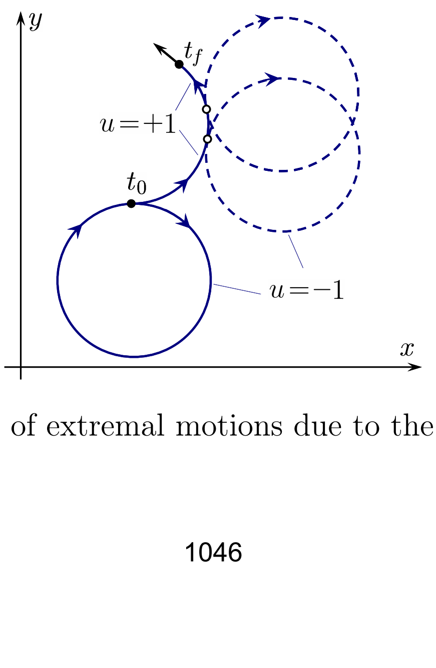

Fig. 14: The non-uniqueness of extremal motions due to the cycle "transfer" for $t_f > 2\pi$

VI. Conclusion

The paper deals with the Pontryagin Maximum Principle sufficiency for the controls leading onto the reachable set boundary *at the instant* for the object called the Dubins car.

It is shown that in the case of a strictly one-side turn and in the case of the one-side turn, the PMP is a sufficient condition for getting the boundary. In the symmetric and asymmetric cases, in general, the sufficiency property is not fulfilled. Corresponding examples are given.

The question on uniqueness of the extremal motions leading onto the boundary is also considered. It was found out that such a uniqueness exists only in the case of the strictly one-side turn.

The property of PMP sufficiency and the uniqueness of the extremal motions leading onto the boundary are closely connected with the convexity and strong convexity of sections (along the angular coordinate) of the three-dimensional reachable set.

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