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In the present publication the plenary and poster papers of the 26th Saint Petersburg International Conference on Integrated Navigation Systems (27 – 29 May 2019) are presented.

*The poster papers are marked with *.*

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Linear Bayesian Estimate for Multilateration Problem in the Presence of Outliers*

Dmitrii Bedin
Ural Federal University named after the First President of Russia B.N. Yeltsin
N.N. Krasovskii Institute of Mathematics and Mechanics of UrB RAS
Ekaterinburg, Russia
bedin@imm.uran.ru

***Abstract*—For a problem of multilateration in the presense of outliers, a Bayesian procedure is proposed for obtaining an estimate of the aircraft unknown position and the signal broadcast time. In the procedure, the estimate is obtained as a sum of various options at which receivers the outliers were occurred.**

***Index Terms*—multilateratioin (MLAT), outliers, Bayes rule**

I. Introduction

Multilateration is a type of a positioning system where several receivers record times of arrival (TOA) of the signal which is broadcasted by the transmitter. Knowning times of arrival, the position of the transmitter and the broadcast time can be estimated. The observation equations of this problem are similar to those used for GPS. The main differences of the multilateration problem in comparison with GPS are worse geometrical observation conditions and greater number of outliers in data. The main causes of the outliers are the multipath of signal propagation and confusions of the signals from different transmitters or broadcasting times.

The usual strategy against the outliers consists of determination of the outlier event and the "outlier" receivers using the techniques of the statistical hypotheses testing (so called the "fault detection and isolation"), then the exclusion of the determined outliers [1]–[3]. However, in the case when the number of observations is small, it is hardly possible to determine the outlier event and the corresponding receiver with a good confidence. There are four variables in the multilateration problem; so, roughly speaking, the "small" number of observations is when this number is less than eight.

Another approach is suggested wherein there is no exclusion of measurements of any receiver. The work of the algorithm was verified in the problem of determination of the aircraft position. The simulation with real geometrical locations of the receivers was performed.

II. Multilateration Problem

A task of multilateration is to determine the location $r \in \mathbb{R}^3$ of an aircraft by measuring the times t_i of arrival (TOA) of the aircraft signal to the receivers located at different geometric positions $r_i \in \mathbb{R}^3$. The number of receivers is equal to m. It is essential that the signal broadcasting time t is unknown and the measurements have random additive errors w_i. For convenience, choose the time scale such that the speed of light is equal to one. Using the vector of unknown parameters

$$\theta = \begin{bmatrix} t \\ r \end{bmatrix} \in \mathbb{R}^4$$

and the function of the model $g(\cdot)$, write the observation equations for the measurements t_i in the vector form

$$T = g(\theta) + w\,, \tag{1}$$

$$T = \begin{bmatrix} t_1 \\ \vdots \\ t_m \end{bmatrix}, \quad g(\theta) = \begin{bmatrix} t + \|r - r_1\| \\ \vdots \\ t + \|r - r_m\| \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}.$$

Assume that there are outliers that additionally affect the measurements. We will model them using random variables. Namely, we introduce two random variables: ω is the set of "erroneous" receivers where the outliers occurred (the number of such receivers will be $|\omega|$); μ is one-dimensional blunder value. Outliers from different receivers will be considered independent implementations of μ. Their joint vector by ω receivers will be μ_ω. Taking into account the outliers, the observation equation takes the following form:

$$T = g(\theta) + H_\omega \mu_\omega + w\,, \quad H_\omega = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ & \vdots & \\ 0 & \cdots & 1 \end{bmatrix}. \tag{2}$$

In the matrix $H_\omega \in \mathbb{R}_{m \times |\omega|}$, the number of columns coincides with the number of the "erroneous" receivers, each column has only one value "1" at place corresponding to one of the indices included in ω.

The purpose of the multilateration problem is to obtain an estimate $\hat{\theta} = \hat{\theta}(T)$ of the vector θ using all the measurements T. The estimate should work well in presence of outliers. Minimization of the standard deviation $\mathbf{E}\left\{\|\hat{\theta} - \theta\|^2\right\}$ is achieved in the case when the estimate is the conditional expectation $\hat{\theta} = \mathbf{E}\{\theta\,|T\}$, which in our case should take into account outlier influence on the model.

III. Bayesian Algorithm

Let's describe the expression for conditional expectation in more detail considering possible combinations of receivers and events "at the receivers ω blunders occurred, while the remaining receivers have not blunders"

$$\mathbf{E}\{\theta\,|T\} = \sum_{\omega\in\Omega} \mathbf{E}\{\theta\,|\omega, T\}\,\mathbf{P}(\omega\,|T)\;. \tag{3}$$

Here, Ω is the set of all possible variants of ω: $\Omega = \{\varnothing, \{1\}, \ldots, \{m\}, \{1,2\}, \ldots, \{m-1, m\}, \{1,2,3\}, \ldots\}$.

The conditional probability that the receivers of the set ω are outliers is calculated by the Bayes rule

$$\mathbf{P}(\omega\,|T) = \frac{p\,(T\,|\omega)\,\mathbf{P}(\omega)}{\sum\limits_{\omega'\in\Omega} p\,(T\,|\omega')\,\mathbf{P}(\omega')}\,, \tag{4}$$

where $\mathbf{P}(\omega)$ is a priori probability of the event "outliers occurred on set ω", and $p\,(T\,|\omega)$ is a conditional density for T given ω (likelihood). The formula has the form

$$\begin{aligned} &p\,(T\,|\omega) \\ &= \int\limits_{\theta\in\mathbb{R}^n}\int\limits_{\mu_\omega\in\mathbb{R}^{|\omega|}} p\,(T\,|\mu_\omega, \theta, \omega)\,p\,(\mu_\omega\,|\omega)\,p\,(\theta)\,d\theta d\mu_\omega \\ &= \int\limits_{\theta\in\mathbb{R}^n}\int\limits_{\mu_\omega\in\mathbb{R}^{|\omega|}} p_w\,(T - g(\theta) - H_\omega\mu_\omega) \\ &\qquad \times \prod_{\omega_i\in\omega} p_\mu\,(\mu_{\omega_i})\,p_\theta(\theta)\,d\theta d\mu_\omega\,. \end{aligned} \tag{5}$$

Here, $p_w(\cdot)$ is the density of w; $p_\mu(\cdot)$, $p_\theta(\theta)$ are a priori densities of μ and θ. Similarly, the conditional expectation in (3) can be detailed as

$$\begin{aligned} \mathbf{E}\{\theta\,|\omega, T\} &= \int\limits_{\theta\in\mathbb{R}^n} \theta\,\frac{p\,(T\,|\theta, \omega)\,p(\theta)}{\int\limits_{\theta'\in\mathbb{R}^n} p\,(T\,|\theta', \omega)\,p(\theta')d\theta'}d\theta \\ &= \frac{1}{p\,(T\,|\omega)}\int\limits_{\theta\in\mathbb{R}^n}\int\limits_{\mu_\omega\in\mathbb{R}^{|\omega|}} \theta\,p\,(T\,|\mu_\omega, \theta, \omega)\,p\,(\mu_\omega\,|\omega)\,p(\theta)\,d\theta d\mu_\omega \\ &= \frac{1}{p\,(T\,|\omega)}\int\limits_{\theta\in\mathbb{R}^n}\int\limits_{\mu_\omega\in\mathbb{R}^{|\omega|}} \theta\,p_w\,(T - g(\theta) - H_\omega\mu_\omega) \\ &\qquad \times \prod_{\omega_i\in\omega} p_\mu\,(\mu_{\omega_i})\,p_\theta(\theta)\,d\theta d\mu_\omega\,. \end{aligned} \tag{6}$$

Further calculations will be carried out under strong assumptions about the nature of outliers and random errors. We assume that the random errors are normally distributed with zero expectation of $w_i \sim \mathcal{N}(0, \sigma^2)$ and the standard deviation σ the same for all receivers. We also consider $\mu \sim \mathcal{N}(0, \sigma_\mu^2)$. The standard deviation σ_μ is assumed much larger than σ: $\sigma \ll \sigma_\mu$.

The distribution of θ is assumed to be degenerate. Technically, this is expressed in the substitution $p_\theta(\theta) = 1$ in integrals (5), (6). Although the function $p_\theta(\theta) \equiv 1$ is not a density function, and integral (5) is not a likelihood expression of $p\,(T\,|\omega)$ after substitution, this method corresponds to of limit in expressions (4), (6) on the scale parameter $\varepsilon \to 0$ in the case of the density of the form $p_\theta(\theta) = \varepsilon\rho(\varepsilon\theta)$ (for any given in advance "basic" density function $\rho(\theta)$).

Under these assumptions, integrands in (5), (6) have the form

$$\begin{aligned} p\,(T\,|\mu_\omega, \theta, \omega)\,p\,(\mu_\omega\,|\omega)\,p(\theta) &= \frac{1}{(2\pi\sigma^2)^{m/2}}\frac{1}{(2\pi\sigma_\mu^2)^{|\omega|/2}} \\ \times \exp\left\{-\frac{1}{2\sigma^2}\,(T - g(\theta) - H_\omega\mu_\omega)^2 - \frac{1}{2\sigma_\mu^2}\mu_\omega^2\right\}. \end{aligned} \tag{7}$$

Integral (5) of expression (7) cannot be calculated analytically due to the nonlinear character of g. To overcome this difficulty, the easiest way is the following. The initial estimate θ^* of θ can be calculated by minimizing the least-squares functional [4], [5]

$$J(\theta) = \sum_{i=1}^m (t_i - g_i(\theta))^2\;, \quad \theta^* = \underset{\theta}{\operatorname{argmin}}\,J(\theta)\,. \tag{8}$$

Further, a linear approximation to $g(\theta)$ in a neighbourhood of θ^* is used

$$g(\theta) = \theta^* + \frac{d}{d\theta}\,g(\theta^*)(\theta - \theta^*)\,.$$

In terms of linear approximation, the integration becomes easy. Introduce the notation

$$H = \frac{d}{d\theta}\,g(\theta), \quad D = \left(H^\mathsf{T}H\right)^{-1}, \quad R = I - HDH^\mathsf{T}\,.$$

In (5), it is convenient to integrate first θ and then μ. After integration we obtain

$$\begin{aligned} p\,(T\,|\omega) &= \frac{(2\pi\sigma^2)^{n/2}(2\pi\sigma^2)^{|\omega|/2}}{(2\pi\sigma^2)^{m/2}(2\pi\sigma_\mu^2)^{|\omega|/2}}\,\frac{|\det D|^{1/2}}{|\det R_\omega|^{1/2}}\times \\ \exp&\left\{-\frac{1}{2\sigma^2}(T - g(\theta^*))^\mathsf{T}\left(I - H_\omega R_\omega^{-1}{H_\omega}^\mathsf{T}\right)(T - g(\theta^*))\right\}, \end{aligned} \tag{9}$$

$$\tilde{\mu}_\omega = \mathbf{E}\{\mu_\omega|\omega, T\} = R_\omega^{-1}{H_\omega}^\mathsf{T}(T - g(\theta^*))\,, \tag{10}$$

$$\tilde{\theta} = \mathbf{E}\{\theta|\omega, T\} = \theta^* - DH^\mathsf{T}H_\omega R_\omega^{-1}{H_\omega}^\mathsf{T}(T - g(\theta^*))\,.$$

Here, $R_\omega = {H_\omega}^\mathsf{T}RH_\omega + \sigma^2/\sigma_\mu^2 I_{|\omega|\times|\omega|}$ is a block of the matrix R, which corresponds to the index set ω with a small regularization summand $\sigma^2/\sigma_\mu^2 I_{|\omega|\times|\omega|}$.

The final estimate is constructed as follows:

1) For each possible ω, expression (9) determines the likelihood of $p\,(T\,|\omega)$.
2) Expression (4) determines the conditional probabilities $p\,(\omega\,|T)$.
3) Expression (10) determines the conditional expectation of θ for each variant of ω.
4) The final estimate is calculated by formula (3)

$$\begin{aligned} &\mathbf{E}\{\theta\,|T\} = \theta^* - \\ &\quad DH^\mathsf{T}\left(\sum_{\omega\in\Omega}\mathbf{P}(\omega\,|T)\,H_\omega R_\omega^{-1}{H_\omega}^\mathsf{T}\right)(T - g(\theta^*))\,. \end{aligned} \tag{11}$$

IV. A Comparison with Standard Methods Against Outliers

In processing the GPS measurements, the common method against outliers [2], [3] is as follows:

1) detection of the outlier event;

2) isolation, i.e. the determination the receiver where the outlier was;
3) exclusion of the outlier and parameter estimation without it.

Since the mathematical formulation of the multilateration problem (1), (2) basically repeats the formulation of the GPS navigation problem, it would be natural to use the same methods. However, due to the worst geometric conditions of observation, and probably larger number of outliers, these methods often do not work correctly.

Thus, the most difficult cases in outlier control are those where the blunder is not so large to be noticeable but its impact on the estimate is sufficiently significant. In the problem of multilateration, a fairly small number m of measurements is typical, and with a small number of measurements, outliers are difficult to be detected.

Detection of an outlier event is usually carried out by analyzing the value $J(\theta^*)$ of the functional J at the optimal point [1]–[3]. The value $J(\theta^*)$ is random due to presence of the random errors. If there are no outliers and equation (1) is fulfilled, the value $J(\theta^*)$ has a chi-square distribution χ^2_{m-4} with $m-4$ degrees of freedom. In the case of the outliers presence and equation (2), with fixed ω, μ_ω, the value $J(\theta^*)$ have a noncentral chi-square distribution $\chi^2_{\hat{\mu},m-4}$ with $m-4$ degrees of freedom and the noncentrality parameter $\hat{\mu} = (H_\omega \mu_\omega)^{\mathsf{T}} R H_\omega \mu_\omega$.

Fig. 1 shows the empirical cumulative distribution functions of $J(\theta^*)$ for a small number of receivers $m = 5$. The blue line corresponds to the empirical cumulative distribution function $J(\theta^*)$ for the case without outliers. The red one is the same for the outliers at one receiver, the blunder has fixed value of 300 m. The vertical green line marks the threshold above which only 5% of the $J(\theta^*)$ values are found. Detection of outliers usually is made by the threshold rules. For this situation, it can be seen that the proposed threshold rule at the significance level of 5% has a large number of missed events about 23%.

Detection of where exactly the outlier occurred is usually made in two ways. The first option is to analyze the residuals of $T - g(\theta^*)$. The second one is connected with handling subsets α of the receivers. For each subset, the least-squares problem have to be solved again

$$J_\alpha(\theta) = \sum_{i\in\alpha} (t_i - g_i(\theta))^2 \ , \quad \theta^*_\alpha = \operatorname*{argmin}_\theta J_\alpha(\theta)\,,$$

until the subset α^* will be found satisfying the statistical "no outliers" test (or, in the simple case, the subset α with minimal $J_\alpha(\theta^*_\alpha)$ can be taken).

In the case of a small number of receivers m, the statistical test on the subset has less power at a given significance level than the test on the full sample. As a result, the number of errors increases when receivers with outliers are not actually excluded from the solution, but the receivers that do not have blunders are excluded.

The Bayesian method described above, instead of determination of the "erroneous" receivers, assigns to each receiver the conditional probability $\mathbf{P}(\omega\,|T)$ of the outlier event. By this, it is protected from such extreme events when the "erroneous" receivers are determined incorrectly.

A test was conducted to simulate the accuracy of the algorithms discussed above. A system of six receivers was considered, the aircraft was at an altitude of 2000 m. For the grid of specified horizontal positions of the aircraft, the times of arrival was calculated with outliers in them. For each six measurements $\{t_i\}_{i=1}^6$ one has a blunder necessarily (for a randomly selected receiver). The magnitude of the blunders was equal to 300 m (or 1 μs) with a random sign.

The figures below show the lines of the level of horizontal accuracy, i.e. the standard deviation of determining the horizontal position of the aircraft in meters. Fig. 2 shows the accuracy of the initial least-squares solution of (8). Fig. 3

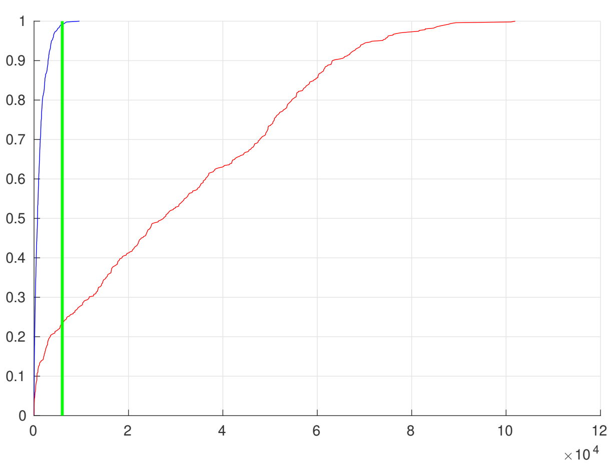

Fig. 1. Empirical cumulative distribution functions of $J(\theta^*)$ without outliers (blue) and with outliers (red)

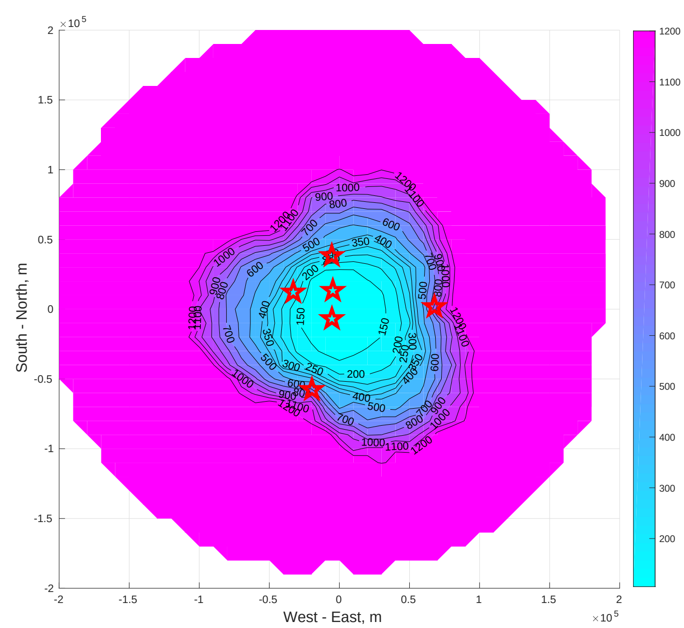

Fig. 2. Horizontal accuracy of the original least-squares solution

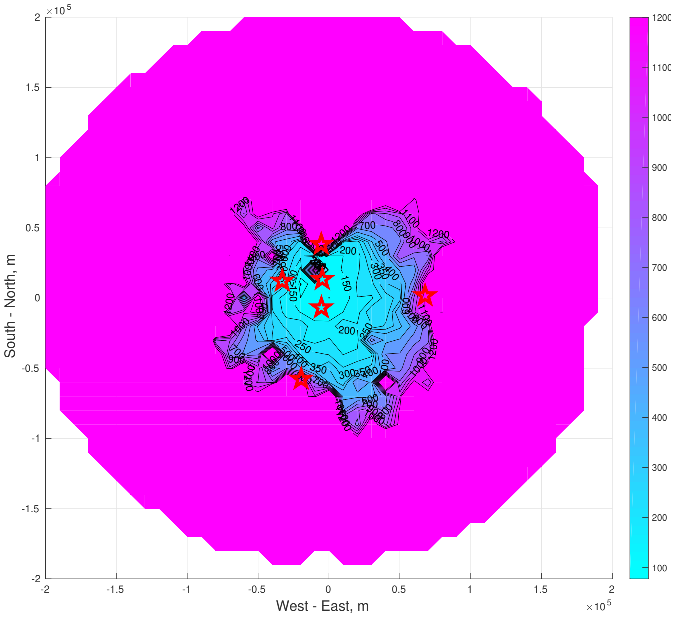

Fig. 3. Horizontal accuracy of the solution based on exclusion of the suggested outliers

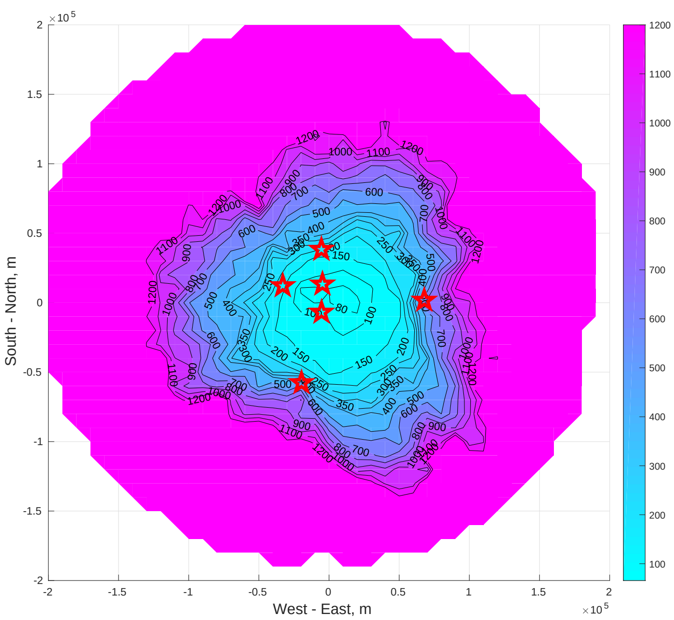

Fig. 4. Horizontal accuracy of the Bayesian algorithm

shows the accuracy of the algorithm based on [1], [3], which works with subsets of receivers. As can be seen, the accuracy of such an algorithm is even worse than the accuracy of the original least-squares solution, where there is not any operations against outliers. In Fig. 4, the accuracy of the proposed Bayesian algorithm is shown. For this it is supposed that $\mathbf{P}(\omega = \varnothing) = 0.61$, $\mathbf{P}(\omega = \{1\}) = \ldots = \mathbf{P}(\omega = \{6\}) = 0.065$, $\sigma = 30$ m, $\sigma_\mu = 300$ m. It should be noted that the assumptions about the probability law of outliers in the algorithm do not coincide with the assumptions used in the data simulation. Nevertheless, a good accuracy of determining the position of the aircraft has been obtained.

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