

Two Pursuers and One Evader: Model Differential Game

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I. Problem formulation

Let the motion of the pursuers P_1 , P_2 and the evader E be described in the vector form as follows:

$$\begin{aligned} \dot{\mathbf{z}}_{P_i} &= A_{P_i} \mathbf{z}_{P_i} + B_{P_i} u_i, & |u_i| &\leq \mu_i, \mathbf{z}_{P_i} \in R^{n_i}, i = 1, 2, \\ \dot{\mathbf{z}}_E &= A_E \mathbf{z}_E + B_E v, & |v| &\leq \nu, \mathbf{z}_E \in R^{n_E}. \end{aligned} \quad (1)$$

Here, u_1 , u_2 , and v are scalar controls; A_{P_1} , A_{P_2} , and A_E are square matrices; B_{P_1} , B_{P_2} , and B_E are column-matrices.

Denote by z_{P_i} , $i = 1, 2$, and z_E the first components of the vectors \mathbf{z}_{P_i} , $i = 1, 2$, and \mathbf{z}_E , respectively. Assume that they are the geometric coordinates of the objects.

We fix two instants T_1 and T_2 . The payoff function is introduced as

$$\varphi = \min\{|z_{P_1}(T_1) - z_E(T_1)|, |z_{P_2}(T_2) - z_E(T_2)|\}. \quad (2)$$

Consider the following zero-sum differential game: the first player using controls u_1 and u_2 minimizes the payoff φ , the second one maximizes the payoff value by its control v . We assume that, during the game, both players know exact values of all phase coordinates. It is necessary to propose a method for constructing the level sets of the value function (the solvability sets) and to investigate possibility of building the players' optimal strategies.

II. Practical Motivation

The considered problem arises when studying a pursuit in upper atmosphere layers. The scheme of the pursuit is given in Fig. 1 and is taken from works by J.Shinar. We assume that both nominal trajectories are in the plane. The components of the nominal velocities along the horizontal axis are quite large, and the angles between the nominal velocities and the horizontal axis are quite small. Therefore, the longitudinal motion can be regarded as uniform, and we can consider only the lateral motion and measure the lateral misses between the pursuers and evader at the instants T_1 and T_2 of nominal collisions. Linearization of the original non-linear dynamics along the nominal trajectories gives the linear differential game described above.

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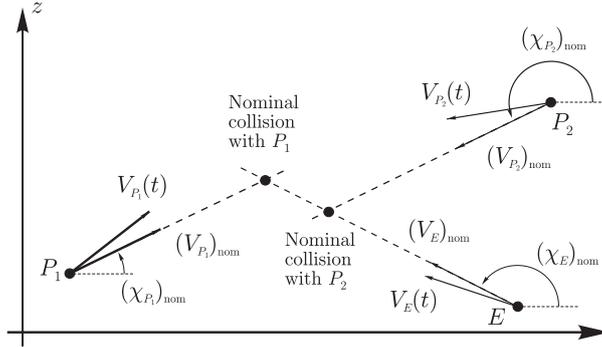


Figure 1. Scheme of the interception with two pursuing objects

III. Zero-Effort Miss Coordinates

We denote by $x_i(t)$, $i = 1, 2$, the value of the difference $z_E - z_{P_i}$ that is predicted from the current instant t and the current positions $\mathbf{z}_E(t)$, $\mathbf{z}_{P_i}(t)$ to the instant T_i under the condition that zero controls act in system (1) in the interval $[t, T_i]$. We have

$$x_i(t) = X_E^1(T_i, t)\mathbf{z}_E(t) - X_{P_i}^1(T_i, t)\mathbf{z}_{P_i}(t), \quad i = 1, 2,$$

where the upper index 1 marks the first rows of the fundamental Cauchy matrices $X_{P_i}(T_i, t)$ and $X_E(T_i, t)$ that correspond to the matrices A_{P_i} and A_E and are written for the instants T_i and t . Since the matrices A_{P_i} , A_E do not depend on the time t , the matrices $X_{P_i}(T_i, t)$ and $X_E(T_i, t)$ depend on the difference $T_i - t$ only. Often, the values $x_i(t)$, $i = 1, 2$, are called the *zero-effort miss* coordinates [1]. Note that $x_i(T_i) = z_E(T_i) - z_{P_i}(T_i)$.

Differentiating the values $x_i(t)$ by t , we obtain

$$\begin{aligned} \dot{x}_i(t) &= X_E^1(T_i, t)B_E v - X_{P_i}^1(T_i, t)B_{P_i} u_i, \\ |u_i| &\leq \mu_i, \quad |v| \leq \nu, \quad t \leq T_i, \quad i = 1, 2. \end{aligned} \quad (3)$$

From results of the differential game theory, it follows (see, for instance, [2–4]) that the differential game with dynamics (3) and the payoff function

$$\varphi = \min\{|x_1(T_1)|, |x_2(T_2)|\}$$

is equivalent (on value of the value function) to the differential game with dynamics (1) and payoff function (2). Computations with dynamics (3) are more convenient since the dimension of the phase vector $x = (x_1, x_2)^\top$ is equal two and the phase vector x is absent in the right part of system (3).

If $T_1 = T_2 = T$, then we have a standard differential game with a fixed termination instant T . However, even in this case, the level sets (the Lebesgue sets) $\{x : \varphi(x) \leq c\}$, $c \geq 0$, of the payoff function φ are not convex. Just this fact makes the problem interesting for a mathematical investigation. The case, when $T_1 \neq T_2$, does not differ substantially from the previous situation.

IV. Variants of Dynamics

In the literature devoted to one-to-one (in the sequel, 1×1) pursuit problems, the following variants of the objects' dynamics have been suggested.

1) **First order link.** The following dynamics gives the simplest description of inertiality of the servomechanisms that passes the control signal u to the acceleration (see, for example [5]):

$$\ddot{z}_P = a_P, \quad \dot{a}_P = (u - a_P)/l_P. \quad (4)$$

The value l_P is called the time constant and defines the time interval until the acceleration reaches the desired level.

2) **Oscillating link.** The paper [6] investigates problems with the dynamics

$$\ddot{z}_P = a_P, \quad \ddot{a}_P = -\omega^2 a_P - \zeta \dot{a}_P + u. \quad (5)$$

Here, in contrast to (4), the servomechanisms' dynamics is described by a second order differential equation that corresponds to an oscillating contour with the own frequency ω and viscous friction with the factor ζ .

3) **Tail/canard control.** The work [7] studied 1×1 games in the case when the control is created by deflection of aerodynamic rudders. The dynamics description is the following:

$$\ddot{z}_P = a_P + d_P u, \quad \dot{a}_P = ((1 - d_P)u - a_P)/l_P. \quad (6)$$

The parameter d_P is defined by the disposition of the aerodynamic rudders. Its positive (negative) values correspond to the case when the rudders are placed in the head (tail) part of the object. As before, the symbol l_P denotes the time constant.

Games 2×1 (two pursuers and one evader) with dynamics of type (4) were studied by the authors in works [8,9]. Games 2×1 with dynamics (5), (6) have not been studied earlier.

To make possible to take into account the dynamics variants (4)–(6), this paper considers a more general formulation, in which the linear dynamics of each object is described by its own vector differential equation with a scalar control restricted on modulus. For each object, the first coordinate of the phase vector is regarded as the coordinate of the object position on the straight line.

V. Examples of Solvability Sets

We construct the solvability sets W_c numerically. The set W_c is the collection of all initial positions (t_0, x_0) in the three-dimensional space t, x , for which the magnitude of the value function is not greater than $c \geq 0$. In other words, if $(t_0, x_0) \in W_c$, then the first player guarantees the game termination with a miss $\varphi \leq c$. If $(t_0, x_0) \notin W_c$, there is no such a guarantee. We use our own algorithms that are founded on theoretical constructions [2,3] developed in Ekaterinburg, Russia. The algorithms realize a procedure of dynamic programming in the backward time. With that, we use [9–11] the fact that the t -sections $W_c(t) = \{x : (t, x) \in W_c\}$ of the solvability sets W_c are sets in the plane R^2 (just these sets are produced by the algorithm). When investigating the solvability sets, we try to detect their important structural peculiarities. Some of them have been outlined by J. Shinar in works dealing with linear pursuit problems of the type 1×1 .

In the work, numerically obtained solvability sets will be shown for different variants of the dynamics of the objects.

A. Example 1

For example, consider the case when the evader's behavior is described by system (4), and the behavior of each pursuer is described by system (6). Thus, if to use denotations

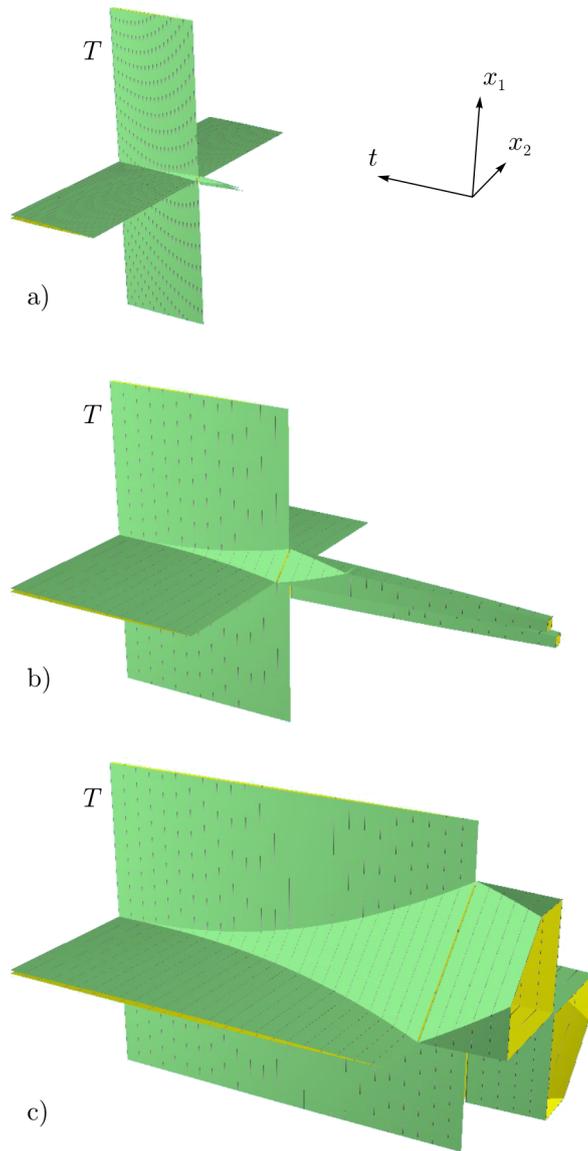


Figure 2. Solvability sets for three different variants of dynamics, $c = 2.0$: a) $d < 0$, b) $d = 0$, c) $d > 0$

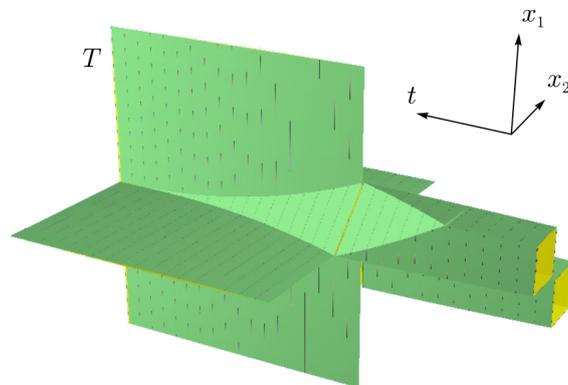


Figure 3. Capture zone (solvability set for $c = 0$) for the case $d > 0$

of system (1), we have

$$\begin{aligned}
A_{P_i} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/l_{P_i} \end{pmatrix}, & B_{P_i} &= \begin{pmatrix} 0 \\ d_{P_i} \\ (1-d_{P_i})/l_{P_i} \end{pmatrix}, \\
A_E &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/l_E \end{pmatrix}, & B_E &= \begin{pmatrix} 0 \\ 0 \\ 1/l_E \end{pmatrix}, \\
|u_i| &\leq \mu_i, \quad i = 1, 2, & |v| &\leq \nu.
\end{aligned}$$

Let the pursuers P_1 and P_2 be equal and $T_1 = T_2 = T$. Choose the values of parameters as follows:

$$\mu_1 = \mu_2 = 0.9, \quad \nu = 1, \quad l_{P_1} = l_{P_2} = 1/0.9, \quad l_E = 1, \quad T = 15.$$

Firstly, let us suppose that $d_{P_1} = d_{P_2} = d = 0$. Then each of the objects has the dynamics of the first order link for his control. Under this, for the chosen values of the parameters, the relations

$$\frac{\mu_i}{\nu} = \frac{0.9}{1} < 1, \quad \frac{\mu_i}{\nu} \cdot \frac{l_E}{l_{P_i}} = \frac{0.9}{1} \cdot \frac{1}{1/0.9} = 0.81 < 1, \quad i = 1, 2,$$

hold. This corresponds [5] to the case of weak pursuers. The three-dimensional solvability set W_c for $c = 2.0$ in the 2×1 game is shown in Fig. 2b. Note that $W_0(t) = \emptyset$ for any $t < T$ in the case of weak pursuers, *i.e.*, there are no initial positions, from which the first player can guarantee the exact encounter.

Let now $d > 0$. The three-dimensional solvability set for $c = 2.0$ is shown in Fig. 2c. It is seen that this set is significantly larger than one for $d = 0$. Figure 3 presents the three-dimensional set W_c corresponding to $c = 0$. From any initial position in this set, the first player guarantees the zero miss, *i.e.*, the exact encounter.

At last, let $d = -0.5 < 0$. In this case, again $W_0(t) = \emptyset$ for $t < T$. The three-dimensional solvability set for $c = 2.0$ is shown in Fig. 2a.

The sets in Fig. 2 and Fig. 3 are drawn in the same scale from the same point of view.

B. Example 2

Now, let us take the dynamics of form (5) for the pursuers and the dynamics of type (4) for the evader. We have

$$\begin{aligned}
A_{P_i} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_{P_i}^2 & -\zeta_{P_i} \end{pmatrix}, & B_{P_i} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \\
A_E &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/l_E \end{pmatrix}, & B_E &= \begin{pmatrix} 0 \\ 0 \\ 1/l_E \end{pmatrix}, \\
|u_i| &\leq \mu_i, \quad i = 1, 2, & |v| &\leq \nu.
\end{aligned}$$

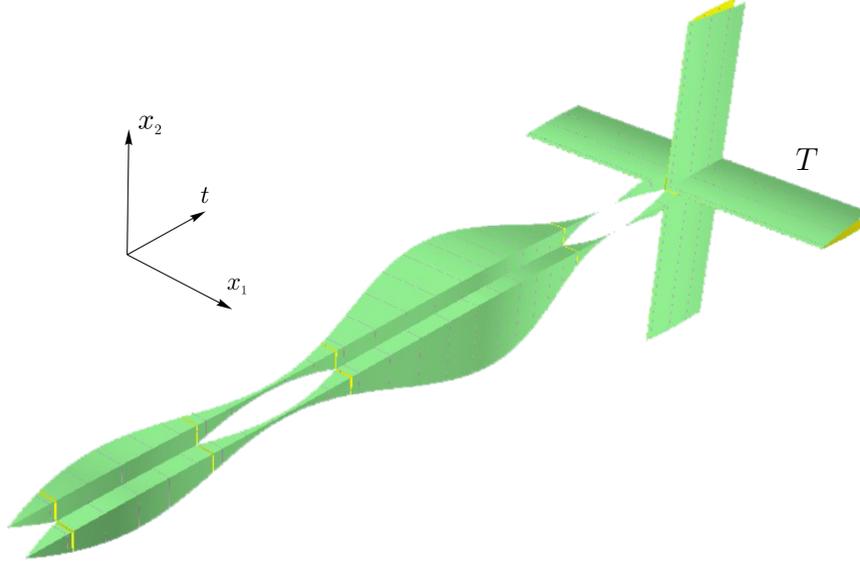


Figure 4. “Exotic” solvability set with two areas of disconnectivity of time sections

Our aim is to show an example with “exotic” solvability set in a 2×1 game. Choose the following parameters:

$$\mu_1 = \mu_2 = 0.3, \quad \nu = 1.3, \quad \omega_{P_1} = \omega_{P_2} = 0.5, \quad \zeta_{P_1} = \zeta_{P_2} = 0.0025, \quad l_E = 1.0, \quad T_1 = T_2 = 30.$$

Figure 4 shows the solvability set W_c for $c = 1.6$. Its “peculiarity” is in the presence of two time periods with narrow “throats”. Earlier, for three-dimensional solvability sets in model problems 1×1 of cosmic pursuit, examples with one narrow throat have been constructed [10]. For a problem 2×1 with dynamics of form (4) in work [11], an example is given, where in some period of time, the solvability set disjoins into two parts. Each of them has a narrow throat. If in a problem 2×1 the pursuers have dynamics (5), then the number of throats can be even greater. In the case of a 1×1 game of form (5) for the pursuer and of form (4) for the evader, a possibility of situation with several narrow throats have been predicted in [6].

VI. Optimal Feedback Controls

A problem of constructing feedback controls is a separate and very difficult problem. But in the case of “strong” pursuers, the optimal feedback objects’ controls can be constructed quite easily by means of switching lines. We call the pursuer P_i *strong* with respect to the evader E if the individual capture zone (the individual solvability set corresponding to $c = 0$) is not empty. In the talk, there will be corresponding mathematical statement and results of numerical computations.

Working with system (3), we assume

$$D_i(t) = X_{P_i}^1(T_i, t)B_{P_i}, \quad E_i(t) = X_{P_i}^1(T_i, t)B_E, \quad i = 1, 2.$$

A. Control of the First Player

Put $u_1^* = \mu_1 \text{sign } D_1(t)$ at the right of the vertical axis and $u_1^* = -\mu_1 \text{sign } D_1(t)$ at the left. Therefore, we take the axis x_2 as the switch line for the control u_1 . Similarly,

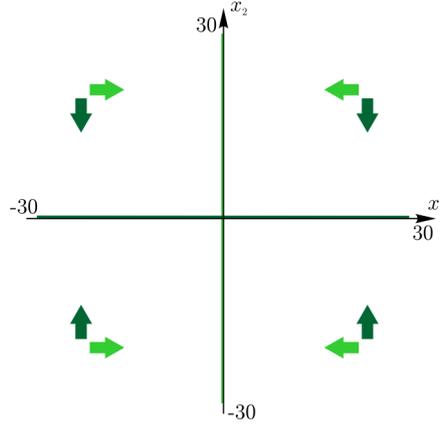


Figure 5. The optimal control synthesis of the first player in the case of strong pursuers

take the axis x_1 as the switch line for the control u_2 : over the axis, we suppose $u_2^* = \mu_2 \text{sign } D_2(t)$, and below the axis, let $u_2^* = -\mu_2 \text{sign } D_2(t)$. At the current point $x(t)$, this method of the first player's controls provides the direction of the vector $(-D_1(t), 0)^\top u_1^*$ $((0, -D_2(t))^\top u_2^*)$ to the point of minimum of the restriction of the value function $V(t, \cdot)$ onto the horizontal (vertical) line passing through the point $x(t)$. The minima of the restrictions equal zero and are reached on the axis x_2 (x_1). Taking into account this property, it is possible to prove that the introduced very simple method of control for the first player (by the feedback principle) is optimal in the case of strong pursuers. Moreover, it is important that this method is stable with respect to small informational errors of measurement of the current state $x(t)$. For dynamics of form (4), the proof of the corresponding statement is given in [9]. The optimal control synthesis of the first player is shown in Fig. 5. The light-green arrows show the direction of the vector $(-D_1(t), 0)^\top u_1^*$, and the dark-green ones correspond to the vector $(0, -D_2(t))^\top u_2^*$.

B. Control of the Second Player

In the case of strong pursuers under the additional assumption that $T_1 = T_2 = T$, it is easy to find also the optimal method v^* of control for the second player.

If $T_1 = T_2$, then $E_1(t) = E_2(t)$. Hence, the vector $E(t) = (E_1(t), E_2(t))^\top$ is directed

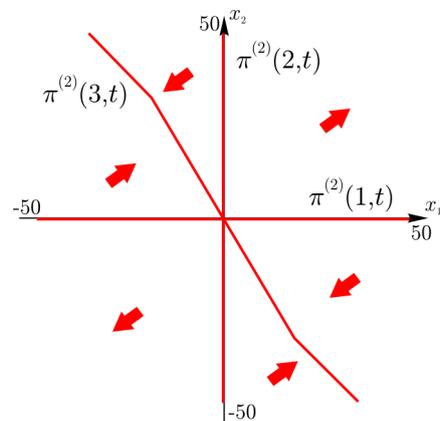


Figure 6. The optimal control synthesis of the second player in the case of strong pursuers

along the bisectrix of the first and third quadrants. From this, we obtain that in the first quadrant the optimal control v^* is calculated by formula $v^* = \nu \operatorname{sign} E_1(t)$. Such a control directs the vector $E(t)v^*$ applied to the current point $x(t)$ to the direction of increasing the value function. In the third quadrant, the optimal control v^* is given by the formula $v^* = -\nu \operatorname{sign} E_1(t)$. In the second quadrant for any fixed $t \in [\bar{t}, T]$, the sets

$$\mathcal{W}_c^{\text{II}}(t) = \operatorname{cl}(R^2 \setminus W_c(t)) \cap \{x : x_1 \leq 0, x_2 \geq 0\}$$

differ for any two values of c_1 and c_2 only by the shift along the bisectrix of the second quadrant. Each of the sets $\mathcal{W}_c^{\text{II}}(t)$ has at its boundary the half-infinite horizontal ray, the half-infinite vertical ray, and the segment parallel to the bisectrix of the first and the third quadrants (parallel to the vector $E(t)$, if $E(t) \neq 0$) that connects the beginnings of these rays. The length of such a slant segment is the same for all $c \geq 0$ for a certain instant t . Let us take a ray passing through the middle points of these slant segments. We connect the beginning of the ray with the origin. In the fourth quadrant, a similar ray is parallel to the bisectrix of this quadrant. The beginning point of this ray is also joined by a segment with the origin.

On the whole, we obtain a polygonal line, which consists of two rays and the segment passing through the origin and joining the beginning points of the rays. Denote this line by $\pi^{(2)}(3, t)$ and call it the switch line for the second player. Let us take the horizontal and vertical axes as the other switch lines $\pi^{(2)}(1, t)$ and $\pi^{(2)}(2, t)$. The optimal synthesis for the second player's control at some t is shown in Fig. 6. The arrows show the direction of the vector $E(t)v^*$ in six cells, into which the plane x_1, x_2 is divided by the three switch lines. In [9] for dynamics of form (4), the statement is given on optimality of the control v^* of the second player and its stability with respect to small errors of measurement of the position $x(t)$. The similar statement holds in the case of more general dynamics considered in this paper.

If the pursuers are strong and equal, then the switch line $\pi^{(2)}(3, t)$ does not depend on time and coincides with the bisectrix of the second and fourth quadrants.

VII. Motion Simulation

For representation of simulation results, consider motions of the pursuers P_1, P_2 and the evader E in the two-dimensional plane. We call this plane the *original geometric space*. Assume that during the motion the horizontal component of the velocity vector of each object remains constant. Let the values of these components be such that the instants of horizontal coincidence of P_1 with E and of P_2 with E are the same and equal to T . Thus, the controls affect only onto the vertical shift. The dynamics of the lateral motion is described by relations (1); the resultant miss is given by formula (2). In Figs. 7–9, the horizontal axis is denoted by the symbol r . So, the coordinate r shows the longitudinal positions of the objects.

To simulate the trajectories, let us take a system, which solvability sets are shown in Figs. 2c and 3. The evader has a dynamics of type (4), the pursuers are equal and have their dynamics as (6).

The parameters of the game:

$$\mu_1 = \mu_2 = 0.9, \quad \nu = 1, \quad l_{P_1} = l_{P_2} = 1/0.9, \quad d_{P_1} = d_{P_2} = 0.5, \quad l_E = 1, \quad T_1 = T_2 = T = 8.$$

The initial lateral velocities, accelerations and higher derivatives are zero:

$$\dot{z}_{P_1}^0 = \dot{z}_{P_2}^0 = \dot{z}_E^0 = 0, \quad \ddot{z}_{P_1}^0 = \ddot{z}_{P_2}^0 = 0, \quad a_{P_1}^0 = a_{P_2}^0 = a_E^0 = 0.$$

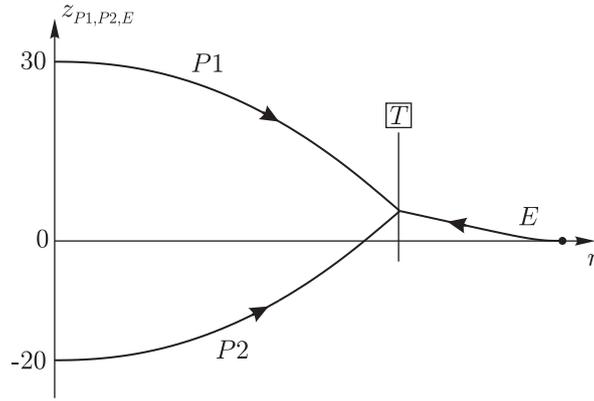


Figure 7. The trajectories of the objects in the geometric space; optimal controls of both players; the initial lateral deviations are not too large

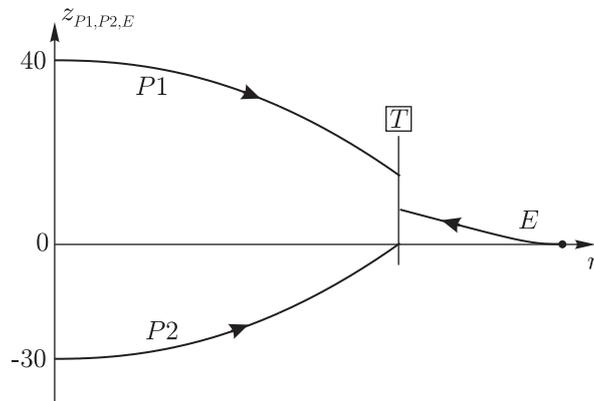


Figure 8. The trajectories of the objects in the geometric space; optimal controls of both players; the initial lateral deviations are large

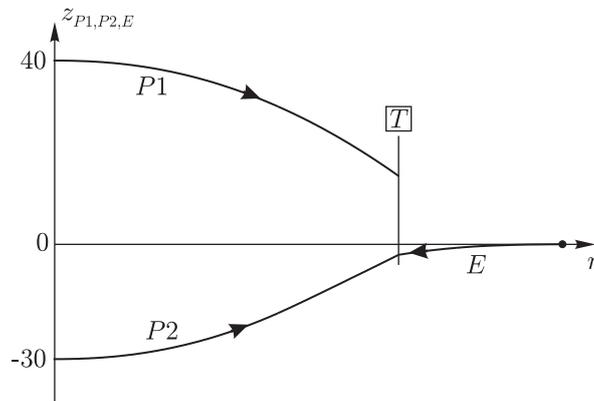


Figure 9. The trajectories of the objects in the geometric space; optimal controls of the pursuers, a random control of the evader; the initial lateral deviations are large

The initial instant is $t_0 = 0$.

Note that the termination instant $T = 8$ is taken such that both pursuers are stronger than the evader in the interval $[t_0, T]$.

Both players are controlled by the switch lines described in the previous section. Under this, they use the exact values of all phase coordinates of all objects.

In the first simulation, the initial lateral coordinates of the objects are $z_{P1} = 30$,

$z_{P_2} = -20$, $z_E = 0$. The resultant trajectories are given in Fig. 7. In this situation, there is the exact capture of the evader by both pursuers.

For the second variant, let $z_{P_1} = 40$, $z_{P_2} = -30$, $z_E = 0$ (Fig. 8). In this case, the initial lateral deviations of the pursuers from the evader are large: the pursuers are unable to provide zero payoff at the termination instant T .

For the third situation, we take the same initial lateral coordinates as in the second one. But now, the control of the evader is random. At the beginning of each step of the discrete scheme of control [2,3,9], it chooses randomly its control from the interval $[-\nu, \nu]$ and keeps it constant during the time step. The pursuers control optimally on the basis of the switching lines. The trajectories for some realization of the evader's control can be seen in Fig. 9. Here, the evader is captured by the second pursuer.

VIII. Conclusion

The results given in Sections V–VII are obtained by the authors during the last year and are new. Of course, they are connected with model mathematical formulations in the framework of the theory of differential games. But these formulations appear in real practical problems of space pursuit. The results show that the solvability sets in the 2×1 problem can rarely be constructed analytically. Therefore, to construct them one needs effective numerical algorithms and programs including programs for visualization. Solvability sets in quite typical situations can have a quite complicated structure. From the authors' point of view, its exotic peculiarities are important in actual engineering practice.

We emphasize that the problem of effective construction of optimal or quasioptimal feedback controls is extremely difficult for the formulations under consideration. But in the particular case of “strong” pursuers, it can be solved completely by means of methods that are convenient for engineers, namely, by means of switching lines. The authors have proved statements characterizing the optimality of the control on the basis of switching lines. Its stability with respect to measuring and computational inaccuracies is justified too.

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