ISSN 1064-2307, Journal of Computer and Systems Sciences International, 2023, Vol. 62, No. 3, pp. 445–468. © Pleiades Publishing, Ltd., 2023.
ISSN 1064-2307, Journal of Computer and Systems Sciences International, 2023. © Pleiades Publishing, Ltd., 2023.
Russian Text © The Author(s), 2023, published in Izvestiya Akademii Nauk, Teoriya i Sistemy Upravleniya, 2023, No. 4, pp. 25–49.

Three-Dimensional Reachability Set For a Dubins Car: Reduction of the General Case of Rotation Constraints to the Canonical Case

V. S. Patsko[a,*] **and A. A. Fedotov**[a]

[a] *Institute of Mathematics and Mechanics, Ural Branch, Russian Academy of Sciences, Yekaterinburg, 620108 Russia*
** e-mail: patsko@imm.uran.ru*

Received November 6, 2022; revised December 12, 2022; accepted February 6, 2023

Abstract—In mathematical control theory, a Dubins car is a nonlinear motion model described by differential relations, in which the scalar control determines the instantaneous angular rate of rotation. The value of the linear velocity is assumed to be constant. The phase vector of the system is three-dimensional. It includes two coordinates of the geometric position and one coordinate having the meaning of the angle of inclination of the velocity vector. This model is popular and is used in various control tasks related to the motion of an aircraft in a horizontal plane, with a simplified description of the motion of a car, small surface and underwater vehicles, etc. Scalar control can be constrained either by a symmetric constraint (when the minimum rotation radii to the left and right are the same) or asymmetric constraint (when rotation is possible in both directions, but the minimum rotation radii are not the same). Usually, problems with symmetric and asymmetric constraints are considered separately. It is shown that when constructing the reachability set at the moment, the case of an asymmetric constraint can be reduced to a symmetric case.

DOI: 10.1134/S1064230723030115

INTRODUCTION

The term *Dubins car* is used to denote a controlled object that moves on a two-dimensional plane with a constant linear velocity and the set constraints on the instantaneous angular speed of rotation.

In addition to the article [1], the publications [2] and [3] should also be considered as classical works where such an object was considered. There are numerous works related to the Dubins car. Many of them are listed in the bibliography for the article [4]. In [5] and [6], substantial problems are described in which the Dubins car model and its various modifications/generalizations are applied.

We note some recent works. In [7], the problem of interception by a Dubins car in the shortest time of an object whose motion is predetermined is studied. In [8], a very general formulation of control problems with two geometric coordinates and two-dimensional control is studied. As one of the special cases of applying the obtained results, the authors consider the problem of the performance of the Dubins car. In [9], a variant of the algorithmic construction of the optimal open-loop trajectories of the Dubins car in the time-optimal problem was proposed. The numerical construction of a three-dimensional reachability set for the Dubins car in the presence of moving obstacles on the plane of geometric coordinates was considered in [10]. Some control problems for a hybrid system composed using the Dubins car were studied in [11].

This paper studies the 3D reachability set for the Dubins car. Without loss of generality, the value of the linear velocity of the object is assumed to be unity and the initial phase state is assumed to be zero. By the reachability set $G(t_f)$, we mean the totality of all three-dimensional phase states x, y and φ (x and y are the coordinates of the geometric position, while φ is the direction angle of the velocity vector), to each of which the object can be transferred at the given moment of time t_f. We denote by $G_\varphi(t_f)$ the two-dimensional section of the set $G(t_f)$ by the angular coordinate φ and call it the φ-section.

We name the canonical case when the scalar control u (meaning the instantaneous velocity of the rotation) is constrained by $|u| \le 1$. Problems with asymmetric constraints $u \in [u_1, u_2]$, in which $u_1 < 0$ and $u_2 > 0$, are also considered in the literature (see, for example, [12]).

As a practical example of asymmetric constraints, we can point to the coordinated motion (see, for example, [13, pp. 60, 61] and [14, p. 764]) of an aircraft in a horizontal plane at a constant linear velocity. Rotations in this case are carried out by changing the angle of the roll. For the model kinematic description, the kinematics of the Dubins car is used. Violation of the symmetry of the roll constraints (including due to incorrect operation of the actuators) leads to the consideration of an asymmetric case.

This study is a continuation of the article [15]. It is shown that the study of φ-sections for an arbitrary asymmetric case is reduced to a study of the φ-sections in the canonical case. In other words, an affine one-to-one correspondence has been established between the φ-sections of the reachability set for an arbitrary asymmetric case and φ-sections for the canonical case. An analytical description of the φ-sections for the canonical case is given in [4] and [16].

Now, we briefly describe the content of this article. In Section 1 we present the problem statement of finding the reachability set for a Dubins car with asymmetric control constraints $u_1 < 0$ and $u_2 > 0$. The reachability set $G(t_f)$ is represented as a set of its sections along the angular coordinate φ. The φ-sections for the general case need to be described by reducing them to consideration of φ-sections in the canonical case.

The study is based on the maximum principle of L.S. Pontryagin [17], which is satisfied by motions and controls leading to the boundary of the reachability set. As a simple consequence of the maximum principle in Section 2, when describing the boundary, we find that we can restrict ourselves to piecewise constant controls with a finite number of switchings and with values in the three-element set $\{u_1, 0, u_2\}$. In Section 3, statements are formulated and proved about the properties of motions and controls that satisfy the maximum principle. It is shown that when constructing the boundary of the reachability set, it suffices to take six types of controls with at most two switchings. This result can be extended to points from the interior of the reachable set under an appropriate restriction of the control constraints. In Section 4, at a fixed value of the angular coordinate $\varphi \geq 0$, the motions generated by the controls from Section 3 are described. The points obtained at the moment t_f form a continuous closed curve $\mathcal{A}_\varphi(t_f)$ on the geometric coordinate plane. This curve contains all points of the boundary of the considered φ-section, but part of it, generally speaking, also passes into the interior of the φ-section. An important property is that the curve $\mathcal{A}_\varphi(t_f)$ is symmetrical about the X axis of some auxiliary rectangular coordinate system X and Y (depending on φ).

Section 5 is central to the article. In it at $\varphi \geq 0$ a correspondence is established between the curve $\mathcal{A}_\varphi(t_f)$ and some curve $\mathcal{A}^c_\varphi(t^c_f)$ considered for the canonical case. The moment t^c_f is given by a certain formula and differs from the moment t_f. Based on this correspondence, a similar relationship is derived between the φ-sections $G_\varphi(t_f)$ and $G^c_\varphi(t^c_f)$ for the original and canonical cases in the auxiliary coordinate system. In Section 6, the φ-sections are described for values $\varphi < 0$, based on the case $\varphi > 0$.

The six types of control considered in this article coincide in the canonical case with the six variants indicated in [1] for the performance problem. In relation to this, we emphasize that in our article, we are talking about the description of the boundary of the reachability set at the given moment t_f. From a logical point of view, the set of open-loop controls that solve the problem of performance, generally speaking, is insufficient for the complete construction of the boundary of the reachable set at the moment.

1. PROBLEM STATEMENT

Let the motion of a controlled object on a plane be described by a system of differential equations

$$\dot{x} = \cos\varphi, \quad \dot{y} = \sin\varphi, \quad \dot{\varphi} = u; \quad u \in [u_1, u_2], \quad u_1 < 0 < u_2. \tag{1.1}$$

Here x and y are the coordinates of the geometric position, φ is the angle of inclination of the velocity vector (Fig. 1) counted counterclockwise from the positive direction of the x axis. The speed value is one. We consider measurable functions of time that satisfy the constraint $u_1 \leq u(t) \leq u_2$ as permissible controls $u(\cdot)$. The values of angle φ are considered in the interval $(-\infty, \infty)$. We denote the phase vector $(x, y, \varphi)^{\mathrm{T}}$ of system (1.1) by z.

We record z_0 the phase state of system (1.1) at the initial moment of time t_0. The reachability set $G(t_f)$ at the time $t_f > t_0$ is the collection of all points z of the three-dimensional phase space into each of which system (1.1) can be transferred at the moment t_f through some admissible control on the interval $[t_0, t_f]$ from the starting point z_0.

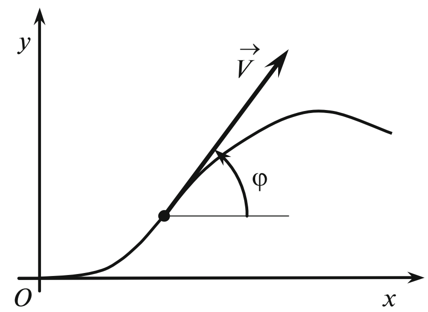

Fig. 1. Initial coordinate system.

We denote by $G_\varphi(t_f)$ the two-dimensional φ-section of the set $G(t_f)$. Note that if some point $(x, y)^T$ belongs to $\partial G_\varphi(t_f)$, then the point $(x, y, \varphi)^T$ belongs to $\partial G(t_f)$. The converse is generally not true. Here, symbol ∂ indicates the boundary of the set.

When studying reachable sets, without loss of generality, we set $t_0 = 0$ and $z_0 = (0, 0, 0)^T$. We will call system (1.1) in the particular case of symmetric constraints $u_1 = -1$ and $u_2 = 1$ *canonical*. The corresponding reachability set will be denoted by $G^c(t_f)$.

The aim of this article is to show and justify a method for obtaining φ-sections of the reachability set $G(t_f)$ of the original system (1.1) for arbitrary $\varphi \in [t_f u_1, t_f u_2]$ based of the φ-sections of the reachability set of the canonical system.

2. PONTRYAGIN'S MAXIMUM PRINCIPLE

It follows from the general results of mathematical control theory [18] that the reachable set $G(t_f)$ is closed and limited. It is also known that controls that lead to the boundary of the reachable set satisfy the Pontryagin maximum principle (PMP). We write down the relations of the maximum principle for system (1.1).

Assume $u^*(\cdot)$ is some admissible control and $(x^*(\cdot), y^*(\cdot), \varphi^*(\cdot))^T$ is the corresponding motion of system (1.1) on the interval $[t_0, t_f]$. The differential equations of the adjoint system are written as

$$\dot\psi_1 = 0, \quad \dot\psi_2 = 0, \quad \dot\psi_3 = \psi_1 \sin\varphi^*(t) - \psi_2 \cos\varphi^*(t). \tag{2.1}$$

The PMP means that there is a nonzero solution $(\psi_1^*(\cdot), \psi_2^*(\cdot), \psi_3^*(\cdot))^T$ of system (2.1) for which the following equality is satisfied almost everywhere (a.e.) on the interval $[t_0, t_f]$:

$$\psi_1^*(t)\cos\varphi^*(t) + \psi_2^*(t)\sin\varphi^*(t) + \psi_3^*(t)u^*(t)$$
$$= \max_{u\in[u_1,u_2]} [\psi_1^*(t)\cos\varphi^*(t) + \psi_2^*(t)\sin\varphi^*(t) + \psi_3^*(t)u].$$

Thus, the maximum condition has the form

$$\text{a.e.} \quad \psi_3^*(t)u^*(t) = \max_{u\in[u_1,u_2]} \psi_3^*(t)u, \quad t \in [t_0, t_f]. \tag{2.2}$$

Assuming that $u^*(\cdot)$ satisfies (2.2), we formulate some simple properties. Note that the functions $\psi_1^*(\cdot)$ and $\psi_2^*(\cdot)$ are constants. We denote them by ψ_1^* and ψ_2^*. If $\psi_1^* = 0$ and $\psi_2^* = 0$, then $\psi_3^*(t) = \text{const} \neq 0$ in between $[t_0, t_f]$. Consequently, in this case, either $u^*(t) = u_2$ or $u^*(t) = u_1$ a.e.

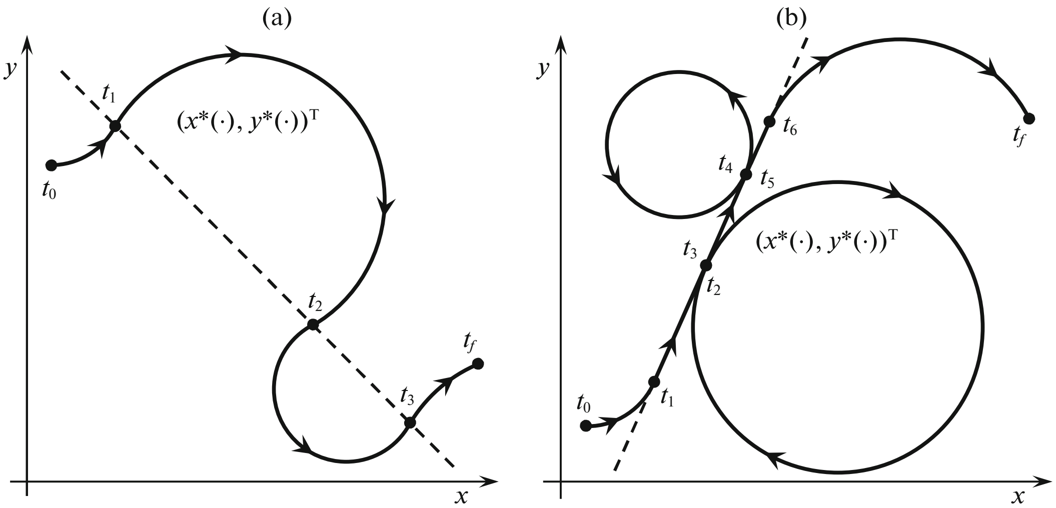

Fig. 2. Trajectories of the maximum principle and switching straight.

We now assume that at least one of the numbers ψ_1^* or ψ_2^* is not zero. Based on (1.1) and (2.1), we can write the expression $\psi_3^*(t) = \psi_1^* y^*(t) - \psi_2^* x^*(t) + C$. Hence it follows that $\psi_3^*(t) = 0$ if and only if the point $(x^*(t), y^*(t))^{\mathrm{T}}$ of the geometric position at moment t satisfies the equation of a straight line:

$$\psi_1^* y - \psi_2^* x + C = 0. \tag{2.3}$$

The straight line of switching (2.3) is not universal: when the motion that satisfies the PMP changes, generally speaking, the straight line of switching also changes.

By relation (2.2), if $\psi_3^*(t) > 0$ ($\psi_3^*(t) < 0$) over a certain period of time, then $u^*(t) = u_2$ ($u^*(t) = u_1$) a.e. on this interval. At the same time, the corresponding motion in the projection onto the plane x, y takes place along an arc of a circle of radius $1/u_2$ counterclockwise in the half plane $\psi_1^* y - \psi_2^* x + C > 0$ (along the arc of a circle of radius $-1/u_1$ clockwise in the half plane $\psi_1^* y - \psi_2^* x + C < 0$). We will call the left (right) cycle the section of motion with duration $2\pi/u_2$ (according to the duration $-2\pi/u_1$) on which a.e. $u^*(t) = u_2$ ($u^*(t) = u_1$). The trajectory of the motion in such a section in the projection onto the plane x, y represents a circle.

If $\psi_3^*(t) = 0$ on a certain interval of time, then on this interval the motion $(x^*(t), y^*(t))^{\mathrm{T}}$ goes in a straight line. In other words, $\varphi^*(t) = \mathrm{const}$. That is why $u^*(t) = 0$ a.e. on this interval.

Figure 2 shows the motions of system (1.1) that satisfy the PMP. Here and in the following explanatory figures, for definiteness, we assume $u_2 > -u_1$. The trajectory in Fig. 2a has three switching points and control sequence u_2, u_1, u_2, u_1. Figure 2b shows a trajectory with six switching points with the control sequence $u_2, 0, u_1, 0, u_2, 0, u_1$; there are left and right cycles.

Only the following three variants for the relative position of the trajectory of motion $(x^*(t), y^*(t))^{\mathrm{T}}$ and straight-line switching are possible.

1. The trajectory intersects the straight line (2.3) at some moment at a nonzero angle (Fig. 2a). Then the trajectory is a set of arcs of circles with the same angular opening between adjacent points of intersection of the switching of the straight line. Function $\psi_3^*(\cdot)$ changes sign on the interval $[t_0, t_f]$ a finite number of times.

2. The trajectory touches the straight line (2.3) at some moment (Fig. 2b). Then the trajectory is a set of arcs of circles and straight sections. The rectilinear segments lie on the straight line (2.3) and the arcs of the circles touch this line. In this case, any complete section in the form of an arc of a circle which is

not extreme is one or more consecutive cycles. Function $\psi_3^*(\cdot)$ in between $[t_0, t_f]$ either does not change sign or changes sign a finite number of times.

3. The trajectory does not intersect with the straight line (2.3). In this case, the function $\psi_3^*(\cdot)$ has the same sign throughout the interval $[t_0, t_f]$ and the trajectory is an arc of a circle.

Thus, if the maximum condition (2.2) is satisfied, then the function $\psi_3^*(\cdot)$ in between $[t_0, t_f]$ can only change sign a finite number of times. Therefore, as the control $u^*(\cdot)$, generating motion $z^*(\cdot)$ and satisfying the PMP, we can take a piecewise constant control with the values $u_1, 0, u_2$ and a finite number of switchings on the interval $[t_0, t_f]$. For definiteness, we assume that such a control is piecewise continuous on the right. Moment t_f is not included in the number of switching times.

The foregoing allows us to formulate the following two assertions.

Statement 1. We assume that the motion $z^*(\cdot)$ of system (1.1) on the interval $[t_0, t_f]$ is generated by an admissible measurable control and, in addition, the PMP is satisfied. Then the motion $z^*(\cdot)$ can be implemented using a piecewise constant control with a finite number of switchings and with values in the three-element set $\{u_1, 0, u_2\}$.

Statement 2. We assume that the motion $z^*(\cdot)$ of system (1.1) on the interval $[t_0, t_f]$ is generated by the piecewise constant control $u^*(\cdot)$ and at the same time, the PMP is fulfilled. Then, the following points are valid:

(a) points of the geometric position of system (1.1) on the plane x, y at the moment of switching of the control $u^*(\cdot)$ lie on the switching line (2.3);

(b) if in the motion $z^*(\cdot)$ there are no sections with zero control, no cycles, and the number of control switchings $u^*(\cdot)$ are more than two, then the increment of the angle between adjacent switching moments is the same in absolute value;

(c) if in the motion $z^*(\cdot)$ there are no sections with zero control, there is at least one cycle, and the number of control switchings $u^*(\cdot)$ is more than one, then all points of the geometric position at the moments of switching coincide;

(d) if in the motion $z^*(\cdot)$ there is a section with zero control, then any section of motion between adjacent switching moments in which constant control u_1 or u_2 is implemented represents one or more successive cycles.

3. PROPERTIES OF MOTIONS THAT SATISFY THE PMP

In the main part of this section, we will prove several lemmas and, based on them, Theorem 1 on the number and nature of control switching leading to the boundary of the set $G(t_f)$. In accordance with Statement 1, we consider piecewise constant controls $u(t) \in \{u_1, 0, u_2\}$ with a finite number of switching times. Such controls are sufficient to construct the boundary of the reachability set. Lemmas 1–3 study motions without segments with zero control. Lemma 4 analyzes the case with a segment of zero control. The final result is formulated in Theorem 1. Theorem 2 is an analog of Theorem 1, but, for controls leading to an arbitrary point of the set $G(t_f)$. We will use the symbol *int* to denote the interior of a set.

Lemma 1. *We assume that the motion $z(\cdot)$ in some interval $[\bar{t}, \hat{t}]$ is generated by the piecewise constant control $u(\cdot)$ with two switching points t_1 and t_2, where $\bar{t} < t_1 < t_2 < \hat{t}$. In this case, the control sequentially takes the values u_1, u_2, and u_1 (respectively, u_2, u_1, and u_2). We assume that $\varphi(\bar{t}) = \varphi(\hat{t})$. Then in the same time interval there is a control $\tilde{u}(\cdot)$ with two switching points $\tilde{t}_1$ and $\tilde{t}_2$, where $\bar{t} < \tilde{t}_1 < \tilde{t}_2 < \hat{t}$, which is transfers from the point $\tilde{z}(\bar{t}) = z(\bar{t})$ to the point $\tilde{z}(\hat{t}) = z(\hat{t})$ and sequentially takes the values u_2, u_1, and u_2 (respectively u_1, u_2, and u_1).*

Proof. Assume that control $u(\cdot)$ takes values u_1, u_2, and u_1 sequentially. We set $\tilde{t}_1 = \bar{t} - (\hat{t} - t_2)\, u_1/u_2$ and $\tilde{t}_2 = \hat{t} + (t_1 - \bar{t})u_1/u_2$. It is obvious that $\bar{t} < \tilde{t}_1$ and $\tilde{t}_2 < \hat{t}$. Furthermore,

$$\tilde{t}_2 - \tilde{t}_1 = \left(\hat{t} - \bar{t}\right)\left(1 + \frac{u_1}{u_2}\right) - \frac{u_1}{u_2}(t_2 - t_1). \tag{3.1}$$

We write equality $\varphi\left(\bar{t}\right) = \varphi\left(\hat{t}\right)$ in the form $u_1\left(t_1 - \bar{t}\right) + u_2\left(t_2 - t_1\right) + u_1\left(\hat{t} - t_2\right) = 0$. From here $\hat{t} - \bar{t} = \left(t_2 - t_1\right)\left(u_1 - u_2\right)/u_1$. Substituting the obtained expression for $\hat{t} - \bar{t}$ into (3.1), we have

$$\tilde{t}_2 - \tilde{t}_1 = -\frac{u_2}{u_1}(t_2 - t_1). \tag{3.2}$$

Taking into account (3.2), we obtain $\bar{t} < \tilde{t}_1 < \tilde{t}_2 < \hat{t}$.

We set the control $\tilde{u}(\cdot)$ in the interval $\left[\bar{t}, \hat{t}\right]$ between two switching points $\tilde{t}_1$ and $\tilde{t}_2$, as well as values u_2, u_1, and u_2 in the three areas of constancy. We consider the corresponding motion $\tilde{z}(\cdot) = \left(\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{\varphi}(\cdot)\right)^{\mathrm{T}}$ emerging at moment $\bar{t}$ from the point $\tilde{z}\left(\bar{t}\right) = z\left(\bar{t}\right)$. We show that $z\left(\hat{t}\right) = \tilde{z}\left(\hat{t}\right)$.

From the definition of moments $\tilde{t}_1$ and $\tilde{t}_2$, as well as relations (3.2), we have

$$-u_1\left(t_1 - \bar{t}\right) = u_2\left(\hat{t} - \tilde{t}_2\right), \quad u_2\left(t_2 - t_1\right) = -u_1\left(\tilde{t}_2 - \tilde{t}_1\right), \quad -u_1(\hat{t} - t_2) = u_2\left(\tilde{t}_1 - \bar{t}\right).$$

In other words,

$$\begin{aligned} \varphi(t_1) - \varphi(\bar{t}) &= \tilde{\varphi}(\tilde{t}_2) - \tilde{\varphi}(\hat{t}), \\ \varphi(t_2) - \varphi(t_1) &= \tilde{\varphi}(\tilde{t}_1) - \tilde{\varphi}(\tilde{t}_2), \\ \varphi(\hat{t}) - \varphi(t_2) &= \tilde{\varphi}(\bar{t}) - \tilde{\varphi}(\tilde{t}_1). \end{aligned} \tag{3.3}$$

Adding the left and right parts of these equalities, we get $\varphi\left(\hat{t}\right) - \varphi\left(\bar{t}\right) = \tilde{\varphi}\left(\hat{t}\right) - \tilde{\varphi}\left(\bar{t}\right)$. Hence, taking into account the condition $\tilde{\varphi}\left(\bar{t}\right) = \varphi\left(\bar{t}\right)$, we arrive at the equality $\tilde{\varphi}\left(\hat{t}\right) = \varphi\left(\hat{t}\right)$. Substituting this equality together with the equality $\varphi\left(\bar{t}\right) = \varphi\left(\hat{t}\right)$ into the first and third relations from (3.3), we have

$$\varphi(t_1) = \tilde{\varphi}\left(\tilde{t}_2\right), \quad \varphi(t_2) = \tilde{\varphi}\left(\tilde{t}_1\right). \tag{3.4}$$

We integrate the first equation of system (1.1) by the control $u(\cdot)$ on $\left[\bar{t}, \hat{t}\right]$:

$$\begin{aligned} x\left(\hat{t}\right) &= x\left(\bar{t}\right) + \frac{1}{u_1}\left(\sin\varphi(t_1) - \sin\varphi\left(\bar{t}\right)\right) + \frac{1}{u_2}\left(\sin\varphi(t_2) - \sin\varphi(t_1)\right) \\ &+ \frac{1}{u_1}\left(\sin\varphi\left(\hat{t}\right) - \sin\varphi(t_2)\right) = x\left(\bar{t}\right) + \left(\frac{1}{u_2} - \frac{1}{u_1}\right)\left(\sin\varphi(t_2) - \sin\varphi(t_1)\right). \end{aligned}$$

Integrating by the control $\tilde{u}(\cdot)$, we have

$$\begin{aligned} \tilde{x}\left(\hat{t}\right) &= x\left(\bar{t}\right) + \frac{1}{u_2}\left(\sin\tilde{\varphi}\left(\tilde{t}_1\right) - \sin\tilde{\varphi}\left(\bar{t}\right)\right) + \frac{1}{u_1}\left(\sin\tilde{\varphi}\left(\tilde{t}_2\right) - \sin\tilde{\varphi}\left(\tilde{t}_1\right)\right) \\ &+ \frac{1}{u_2}\left(\sin\tilde{\varphi}\left(\hat{t}\right) - \sin\tilde{\varphi}\left(\tilde{t}_2\right)\right) = x\left(\bar{t}\right) + \left(\frac{1}{u_2} - \frac{1}{u_1}\right)\left(\sin\tilde{\varphi}\left(\tilde{t}_1\right) - \sin\tilde{\varphi}\left(\tilde{t}_2\right)\right). \end{aligned}$$

Taking into account (3.4), we obtain $\tilde{x}\left(\hat{t}\right) = x\left(\hat{t}\right)$. Direct integration also establishes the equality $\tilde{y}\left(\hat{t}\right) = y\left(\hat{t}\right)$.

The case when the initial control sequentially takes the values u_2, u_1, and u_2 is treated in a similar way. We just need to swap the symbols u_1 and u_2. Lemma 1 is proved.

Lemma 2. *We assume that the motion* $z(\cdot)$ *in between* $[t_0, t_f]$ *is generated by the piecewise constant control* $u(\cdot)$ *with values* u_1, u_2 *and two switching points* t_1, t_2, *and* $t_0 < t_1 < t_2 < t_f$. *We assume that*

$$\left|(\varphi(t_1) - \varphi(t_0)) + (\varphi(t_f) - \varphi(t_2))\right| > |\varphi(t_2) - \varphi(t_1)|. \tag{3.5}$$

Then $z(t_f) \in \operatorname{int} G(t_f)$.

Proof. Condition (3.5) substantively means that the sum of the angle increments φ in the first and third sections of the constancy of the control, taken in absolute value, is greater than the modulus of the increment of the angle in the middle section. Without loss of generality, we accept the following sequence of control values $u(\cdot)$: u_1, u_2, u_1. Then condition (3.5) can be written as

$$-(\varphi(t_1) - \varphi(t_0)) - (\varphi(t_f) - \varphi(t_2)) > \varphi(t_2) - \varphi(t_1). \tag{3.6}$$

We assume on the contrary that $z(t_f) \in \partial G(t_f)$. Then any control leading to this point satisfies the PMP.

1. We consider the case when the points $(x(t_1), y(t_1))^{\mathrm{T}}$ and $(x(t_2), y(t_2))^{\mathrm{T}}$ of the geometric position on the plane x, y at the times of switching t_1 and t_2 do not match. By Statement 2c, we obtain that the motion $z(\cdot)$ in between $[t_0, t_f]$ has no cycles.

We choose the moments $\bar{t} \in (t_0,\ t_1)$ and $\hat{t} \in (t_2,\ t_f)$ so that the following equality is fulfilled:

$$-\left(\varphi(t_1) - \varphi(\bar{t})\right) - \left(\varphi(\hat{t}) - \varphi(t_2)\right) = \varphi(t_2) - \varphi(t_1). \tag{3.7}$$

The possibility of selecting $\bar{t}$ and $\hat{t}$ with provision (3.7) follows from the monotonic change of angle φ depending on the total length of the two extreme gaps with the control u_1. We can take, for example,

$$\bar{t} = t_1 + q(t_0 - t_1), \quad \hat{t} = t_2 + q(t_f - t_2), \quad q = \left(1 + \frac{\varphi(t_0) - \varphi(t_f)}{\varphi(t_2) - \varphi(t_1)}\right)^{-1}.$$

Indeed, the inequality $\varphi(t_0) - \varphi(t_f) > 0$ is valid due to condition (3.5), and the inequality $\varphi(t_2) - \varphi(t_1) > 0$ is satisfied because the control u_2 is valid in the interval $[t_1, t_2)$. That is why $q \in (0,1)$ and, therefore, $\bar{t} \in (t_0,\ t_1)$ and $\hat{t} \in (t_2,\ t_f)$. For moments $\bar{t}$ and $\hat{t}$ introduced in this way, we have

$$\varphi(\bar{t}) = (\varphi(t_1) - \varphi(t_0))q + \varphi(t_0), \quad \varphi(\hat{t}) = (\varphi(t_2) - \varphi(t_f))q + \varphi(t_f).$$

By substituting these equalities into relation (3.7), we verify that it holds. Relation (3.7) means that $\varphi(\bar{t}) = \varphi(\hat{t})$.

Based on Lemma 1 in between $[\bar{t},\ \hat{t}]$, we consider the motion leaving the point $z(\bar{t})$, arriving at the point $z(\hat{t})$, and using the sequential control u_2, u_1, and u_2 with two switching points $\tilde{t}_1$ and $\tilde{t}_2$.

Together with the original motion $z(\cdot)$ generated by the control $u(\cdot)$, we introduce an auxiliary motion $\tilde{z}(\cdot) = \left(\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{\varphi}(\cdot)\right)^{\mathrm{T}}$ leaving the starting point $\tilde{z}(t_0) = z(t_0)$ by the control

$$\tilde{u}(t) = \begin{cases} u_1, & t \in [t_0, \bar{t}), \\ u_2, & t \in [\bar{t}, \tilde{t}_1), \\ u_1, & t \in [\tilde{t}_1, \tilde{t}_2), \\ u_2, & t \in [\tilde{t}_2, \hat{t}), \\ u_1, & t \in [\hat{t}, t_f]. \end{cases}$$

Control $\tilde{u}(\cdot)$ is different from the original control $u(\cdot)$ only in between $[\bar{t}, \hat{t})$. In this case, $\tilde{z}(\hat{t}) = z(\hat{t})$. In other words, $\tilde{z}(t_f) = z(t_f)$. Hence, $\tilde{z}(t_f) \in \partial G(t_f)$. Therefore, control $\tilde{u}(\cdot)$ satisfies the PMP.

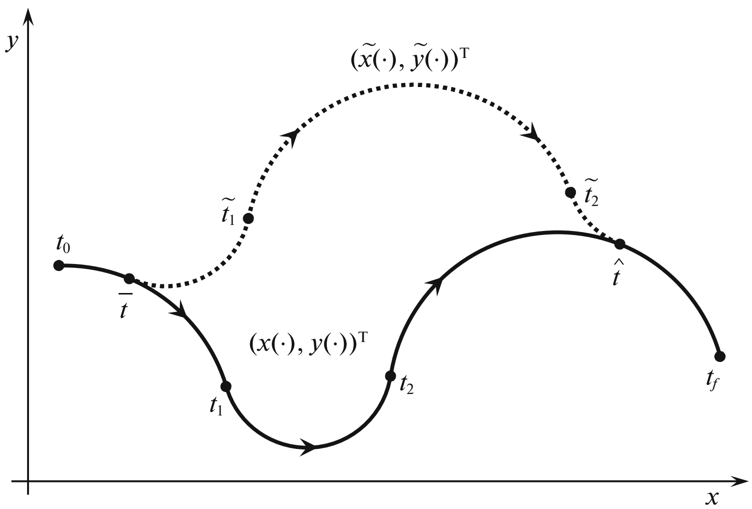

Fig. 3. Explanation of point 1 of the proof of Lemma 2.

Motion $\tilde{z}(\cdot)$ has no cycles in the interval $[t_0, t_f]$. This follows from the definition of control $\tilde{u}(\cdot)$ and the absence of cycles in motion $z(\cdot)$. By Statement 2b, we obtain that the increment of angle φ modulo any adjacent moments of control switching $\tilde{u}(\cdot)$ is constant along the given motion.

However, it is not. We take successive switching moments $\bar{t}$, $\tilde{t}_1$, $\tilde{t}_2$, and $\hat{t}$ in the auxiliary motion. For it, the increment of the angle between adjacent switching moments is also constant in absolute value. Let us denote this value by $\Delta\tilde{\varphi}$. Then $\tilde{\varphi}(\hat{t})$ can be expressed through $\tilde{\varphi}(\bar{t})$ taking into account the sequence of control values $\tilde{u}(\cdot)$ in between $[\bar{t}, \hat{t})$:

$$\tilde{\varphi}(\hat{t}) = \tilde{\varphi}(\bar{t}) + \Delta\tilde{\varphi} - \Delta\tilde{\varphi} + \Delta\tilde{\varphi} = \tilde{\varphi}(\bar{t}) + \Delta\tilde{\varphi}.$$

Since $\Delta\tilde{\varphi} \neq 0$, then $\tilde{\varphi}(\hat{t}) \neq \tilde{\varphi}(\bar{t})$. We obtain a contradiction.

In other words, $z(t_f) \in \operatorname{int} G(t_f)$ when $(x(t_1), y(t_1))^{\mathrm{T}} \neq (x(t_2), y(t_2))^{\mathrm{T}}$. The trajectories of the initial and auxiliary motions for this case are shown schematically in Fig. 3.

2. Now, we assume that $(x(t_1), y(t_1))^{\mathrm{T}} = (x(t_2), y(t_2))^{\mathrm{T}}$. In this case, the motion in the interval $[t_1, t_2]$ represents one or more consecutive cycles with control u_2. In this case, due to (3.6), the total accumulated angle in the first and third sections exceeds 2π in absolute value, i.e.,

$$-u_1\left((t_1 - t_0) + (t_f - t_2)\right) > 2\pi. \tag{3.8}$$

This allows us to set an auxiliary motion $\tilde{z}(\cdot)$ on the interval $[t_0, t_f]$ (leaving the point $\tilde{z}(t_0) = z(t_0)$ and arriving at the point $\tilde{z}(t_f) = z(t_f)$) through control $\tilde{u}(\cdot)$:

$$\tilde{u}(t) = \begin{cases} u_1, & t \in [t_0, \tilde{t}_1), \\ u_2, & t \in [\tilde{t}_1, \tilde{t}_2), \\ u_1, & t \in [\tilde{t}_2, \tilde{t}_3), \\ u_2, & t \in [\tilde{t}_3, t_f], \end{cases}$$

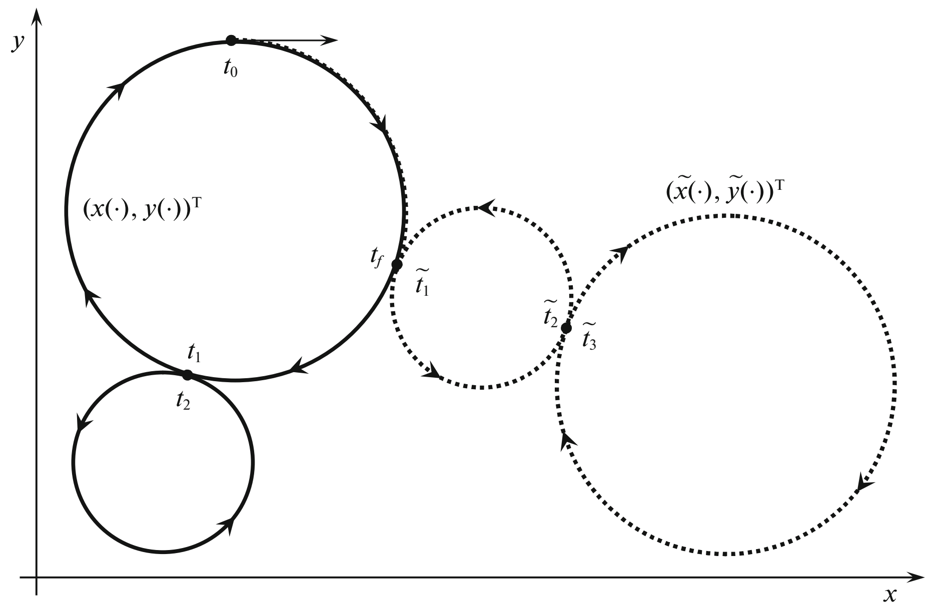

Fig. 4. Explanation of point 2 of the proof of Lemma 2.

with three switching points:

$$\tilde{t}_1 = t_0 - \left(-(t_1 - t_0)u_1 - (t_f - t_2)u_1 - 2\pi\right)/u_1 = t_f - (t_2 - t_1) + 2\pi/u_1,$$

$$\tilde{t}_2 = \tilde{t}_3 + 2\pi/u_1, \quad \tilde{t}_3 = t_f - \pi/u_2.$$

It can be verified using (3.8) that $t_0 < \tilde{t}_1 < \tilde{t}_2 < \tilde{t}_3 < t_f$ and $\tilde{z}(t_f) = z(t_f)$.

The formation of an auxiliary motion for the case $(x(t_1), y(t_1))^{\mathrm{T}} = (x(t_2), y(t_2))^{\mathrm{T}}$ is illustrated in Fig. 4. Here in between $[t_1, t_2]$ the initial motion $(x(\cdot), y(\cdot))^{\mathrm{T}}$ (solid line) has a cycle due to control u_2. The geometric positions at moments t_1 and t_2 correspond. The path of the auxiliary motion $(\tilde{x}(\cdot), \tilde{y}(\cdot))^{\mathrm{T}}$ is marked with a dotted line. The switching points $\tilde{t}_2$ and $\tilde{t}_3$ are taken so that the equalities $\tilde{t}_3 - \tilde{t}_2 = -2\pi/u_1$ and $t_f - \tilde{t}_3 = \pi/u_2$ are fulfilled. Therefore, the auxiliary motion on the section $[\tilde{t}_3, t_f]$ is a semicircle, and in the area $[\tilde{t}_2, \tilde{t}_3]$ it is an entire circle (i.e., forms a cycle). The geometric positions in the auxiliary motion at moments $\tilde{t}_1$ and t_f match. The positions at moments $\tilde{t}_2$ and $\tilde{t}_3$ also coincide.

For the auxiliary motion, we have $(\tilde{t}_2 - \tilde{t}_1)u_2 = (t_2 - t_1)u_2 - \pi$. The initial motion in the interval $[t_1, t_2]$ represents one or more consecutive cycles with control u_2. Therefore, the expression $(t_2 - t_1)u_2$ is a multiple of 2π. Therefore, the value $(\tilde{t}_2 - \tilde{t}_1)u_2$ is not a multiple of 2π. It follows from this that the geometric positions of the auxiliary motion at moments $\tilde{t}_1$ and $\tilde{t}_2$ do not match. Taking into account Statement 2c, we obtain that the auxiliary motion due to control $\tilde{u}(\cdot)$ does not satisfy the PMP. That is why $z(t_f) = \tilde{z}(t_f) \in \operatorname{int} G(t_f)$ in the case when the points of the geometric position of the initial motion at the moments of switching t_1 and t_2 match. Lemma 2 is proved.

Lemma 3. *Assume motion $z(\cdot)$ in the interval $[t_0, t_f]$ is generated by the piecewise constant control $u(\cdot)$ with values u_1, u_2 and three switching points t_1, t_2, t_3, with $t_0 < t_1 < t_2 < t_3 < t_f$. Then, $z(t_f) \in \operatorname{int} G(t_f)$.*

Proof. Of the two middle intervals $[t_1, t_2]$ and $[t_2, t_3]$ of the considered motion, we take the one in which the change in angle φ, taken modulo, is smaller. If the change in angles in modulo coincides, then we take any interval.

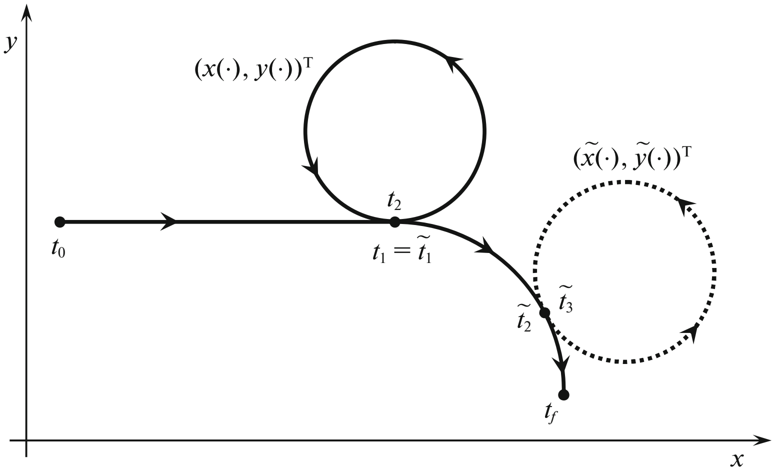

Fig. 5. Explanation of the proof of Lemma 4.

We assume that the interval $[t_2, t_3]$ is selected. Then $|\varphi(t_3) - \varphi(t_2)| \le |\varphi(t_2) - \varphi(t_1)|$. Additionally, considering that in the intervals $[t_1, t_2)$ and $[t_3, t_f]$ the controls are the same, we get

$$|\varphi(t_3) - \varphi(t_2)| < |\varphi(t_2) - \varphi(t_1)| + |\varphi(t_f) - \varphi(t_3)| = |\varphi(t_2) - \varphi(t_1) + \varphi(t_f) - \varphi(t_3)|.$$

From this it follows that motion $z(\cdot)$ satisfies the conditions of Lemma 2 in $[t_1, t_f]$ and, therefore, $z(t_f) \in \operatorname{int} G(t_f)$.

If the interval $[t_1, t_2]$ is selected, then the conditions of Lemma 2 are satisfied for motion $z(\cdot)$ in between $[t_0, t_3]$. That is why $z(t_3) \in \operatorname{int} G(t_3)$ and, therefore, $z(t_f) \in \operatorname{int} G(t_f)$. Lemma 3 is proved.

Lemma 4. *We assume motion $z(\cdot)$ in between $[t_0, t_f]$ is generated by the piecewise constant control $u(\cdot)$ with values in the set $\{u_1, 0, u_2\}$ and two switching points. We assume that there is only one interval with zero control and it is one of the two extreme intervals of control constancy. Then $z(t_f) \in \operatorname{int} G(t_f)$.*

Proof. We assume for definiteness that control $u(\cdot)$ takes the sequential values $0, u_2, u_1$. We assume on the contrary that $z(t_f) \in \partial G(t_f)$. Then control $u(\cdot)$ satisfies the PMP.

According to Statement 2d, motion $z(\cdot)$ in between $[t_1, t_2]$ represents one or several consecutive cycles (Fig. 5). That is why $(x(t_1), y(t_1))^{\mathrm{T}} = (x(t_2), y(t_2))^{\mathrm{T}}$.

We consider the auxiliary motion $\tilde{z}(\cdot)$ leaving point $z(t_0)$ at moment t_0 and which is given by the control

$$\tilde{u}(t) = \begin{cases} 0, & t \in [t_0, t_1), \\ -1, & t \in [t_1, \tilde{t}_1), \\ 1, & t \in [\tilde{t}_1, \tilde{t}_2), \\ -1, & t \in [\tilde{t}_2, t_f]. \end{cases}$$

Here $\tilde{t}_1 = t_1 + \varepsilon$ and $\tilde{t}_2 = t_2 + \varepsilon$, where value ε is taken from range $(0, \min\{t_f - t_2, 2\pi/u_2\})$.

Motions $\tilde{z}(\cdot)$ and $z(\cdot)$ match in between $[\tilde{t}_2, t_f]$. Hence, $\tilde{z}(t_f) = z(t_f)$. That is why $\tilde{z}(t_f) \in \partial G(t_f)$. Therefore, control $\tilde{u}(\cdot)$ satisfies the PMP. However, motion $\tilde{z}(\cdot)$ in between $[t_1, \tilde{t}_1]$ has no cycles, which contradicts Statement 2d.

Thus, $z(t_f) \in \operatorname{int} G(t_f)$. Lemma 4 is proved.

We formulate the main theorem on the nature of controls leading to the boundary of the reachable set.

Theorem 1. *Each point of the boundary of the reachability set $G(t_f)$ of system* (1.1) *can be reached at the moment t_f using a piecewise constant control with at most two switchings and with values in the three-element set $\{u_1, 0, u_2\}$. In this case, in the case of two switchings, we can limit ourselves to six variants for the control sequence*:

$$\begin{array}{lll} \text{U1) } u_2, 0, u_2; & \text{U2) } u_1, 0, u_2; & \text{U3) } u_2, 0, u_1; \\ \text{U4) } u_1, 0, u_1; & \text{U5) } u_2, u_1, u_2; & \text{U6) } u_1, u_2, u_1. \end{array} \tag{3.9}$$

Proof. Any point on the boundary of the reachability set is controlled by a control that satisfies the PMP. According to Statement 1, it can be considered piecewise constant with a finite number of switchings and with values in the three-element set $\{u_1, 0, u_2\}$.

We assume by contradiction that on the boundary of the reachability set $G(t_f)$ there is a point $\hat{z}$ that can be reached only through a control with three or more switchings. If there are several such controls, then we take the control $u^\diamond(\cdot)$ with the least number of switches. We denote the motion $z^\diamond(\cdot)$ generated by it.

We consider the motion $z^\diamond(\cdot)$ in the last four areas of the constancy of the control. There cannot be more than two sections with zero control among them. In this case, the following four cases are possible.

1. There are no areas with zero control. Then $z^\diamond(t_f) \in \operatorname{int} G(t_f)$ by Lemma 3. This is contrary to $z^\diamond(t_f) = \hat{z} \in \partial G(t_f)$.

2. There is only one area with zero control. In this case, three successive segments can be distinguished so that the segment with zero control is located at the beginning or at the end of such a triple. Based on Lemma 4, we have $z^\diamond(t_f) \in \operatorname{int} G(t_f)$, which contradicts the relation $z^\diamond(t_f) = \hat{z} \in \partial G(t_f)$.

3. There are two areas with zero control, and they are located at the edges. Here, similarly to the previous case, using Lemma 4, we establish that $z^\diamond(t_f) \in \operatorname{int} G(t_f)$, and we get a contradiction.

4. There are two (out of four) sections with zero control, and only one section with nonzero control is located between them. Control $u^\diamond(\cdot)$ satisfies the PMP. Therefore, according to Statement 2d, the middle segment with nonzero control, which lies between the segments with zero control, is one or several successive cycles. We transfer all cycles from the middle section to the starting point of the first straight section or to the end point of the second straight section. By gluing the rectilinear sections in time, we obtain an auxiliary motion $\tilde{z}(\cdot)$ leading to the same point at time t_f, which is the original motion $z^\diamond(\cdot)$. In this case, the auxiliary motion has one switching less than the original one. This contradicts the assumption made about the choice of control $u^\diamond(\cdot)$ with the least number of switchings.

As an explanation, Fig. 6 shows an example of motion with three switching points t_1, t_2, t_3 and control sequence $u_1, 0, u_2, 0$. In the third section (from moment t_2 until moment t_3), one or more cycles are implemented with control u_2. When forming an auxiliary motion, the cycles from the middle section of the initial motion are transferred to the starting point of the first straight section. We get two switching moments in the auxiliary motion $\tilde{t}_1 = t_1$ and $\tilde{t}_2 = t_1 + (t_3 - t_2)$ and the control sequence $u_1, u_2, 0$.

Thus, any point on the boundary of the reachability set $G(t_f)$ can be reached using a piecewise constant control with at most two switchings.

We turn to the question of the form of the sequence of controls. In addition to variants U1–U6 of the controls indicated in the formulation of Theorem 1 with two switchings, six more variants are logically possible:

$$7)\ 0, u_2, u_1;\ \ 8)\ 0, u_1, u_2;\ \ 9)\ u_2, u_1, 0;\ \ 10)\ u_1, u_2, 0;\ \ 11)\ 0, u_2, 0;\ \ 12)\ 0, u_1, 0.$$

Controls of form (7)–(10) cannot lead to the boundary of the reachable set due to Lemma 4.

We consider variants (11) and (12). Here, for each motion, the number of switchings can be reduced by one, analogously to how it was done for case 4. We obtain the control with one switching leading to the same point on the boundary. Theorem 1 is proved.

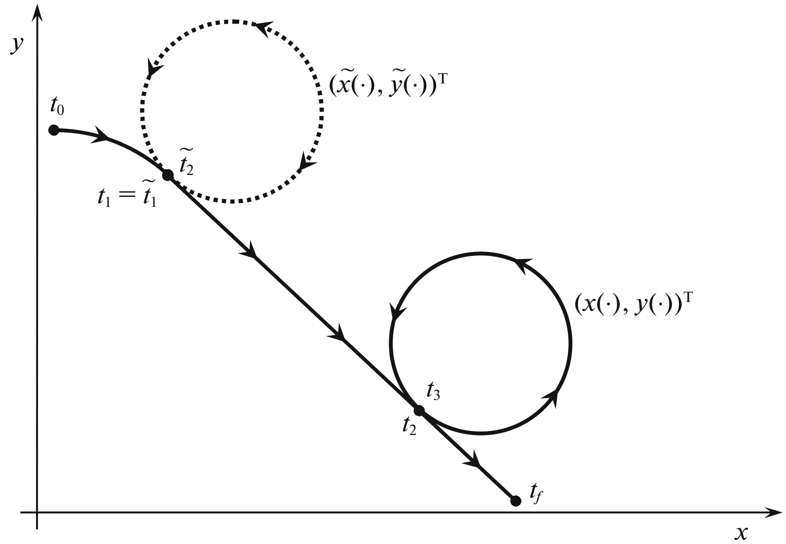

Fig. 6. Explanation of the proof of Theorem 1.

Taking into account Lemmas 1 and 2, we can refine the result of Theorem 1 depending on the sign of angle φ for the considered point $z = (x, y, \varphi)^T$ on the boundary of the set $G(t_f)$.

Note 1. If $\varphi > 0$, then only four of the six types of controls can be left in list (3.9): U1, U2, U3, and U6. When $\varphi < 0$, list (3.9) can be limited to four types of controls: U2, U3, U4, and U5. If $\varphi = 0$, then the following four types of controls can be left in list (3.9): U2, U3, U5, and U6; in this case, controls of types U5 and U6 generate the same set of points.

Proof. Let $\varphi = 0$. Any type of control U1 leads to the point $\varphi(t_f) > 0$. For controls such as U4 we have $\varphi(t_f) < 0$. Therefore, controls of types U1 and U4 are excluded. By Lemma 1 controls U5 and U6 generate the same set of points at $\bar{t} = t_0$ and $\hat{t} = t_f$.

Assume $\varphi > 0$. A control of type U4 is excluded by analogy with the case $\varphi = 0$. A control of type U5 is also ruled out, since by Lemma 2 such controls lead to the interior of the reachable set.

The case of $\varphi < 0$ is dealt with in a similar way. Here, we also get four control variants: U2, U3, U4, and U5. Note 1 is proved.

Based on Theorem 1, we formulate a theorem on controls leading to an arbitrary point of the reachability set $G(t_f)$.

Theorem 2. *We assume some continuous strictly increasing functions* $f_1(\alpha)$ *and* $f_2(\alpha)$, *defined on* [0, 1] *with values* $f_1(0) = f_2(0) = 0$ *and* $f_1(1) = f_2(1) = 1$ *at the extreme points, are given. Then for any point* $z = (x, y, \varphi)^T \in G(t_f)$, *there is such* $\alpha^\flat \in (0,1]$ *that it is possible to move to point z at time* t_f *using a piecewise constant control with at most two switchings and with values in the three-element set* $\{u_1^\flat, 0, u_2^\flat\}$, *where* $u_1^\flat = f_1(\alpha^\flat)u_1$ *and* $u_2^\flat = f_2(\alpha^\flat)u_2$. *In this case, in the case of two switchings, we can limit ourselves to six variants for the control sequence*:

$$\begin{array}{lll} \mathrm{U1}^\flat)\ u_2^\flat, 0,\ u_2^\flat; & \mathrm{U2}^\flat)\ u_1^\flat, 0,\ u_2^\flat; & \mathrm{U3}^\flat)\ u_2^\flat, 0,\ u_1^\flat; \\ \mathrm{U4}^\flat)\ u_1^\flat, 0,\ u_1^\flat; & \mathrm{U5}^\flat)\ u_2^\flat,\ u_1^\flat,\ u_2^\flat; & \mathrm{U6}^\flat)\ u_1^\flat,\ u_2^\flat,\ u_1^\flat. \end{array} \tag{3.10}$$

Proof. We denote by $G(t_f, \alpha)$ at $\alpha \in [0,1]$ a reachable set of a control system similar to system (1.1) with the only difference that control u is hampered by the constraint $u \in [u_1(\alpha), u_2(\alpha)]$, where $u_1(\alpha) = f_1(\alpha)u_1$ and $u_2(\alpha) = f_2(\alpha)u_2$. If $\alpha = 0$, then the only admissible control is the control $u(t) \equiv 0$ and the set $G(t_f, \alpha = 0)$ consists of one point. If $\alpha = 1$, then the set $G(t_f, \alpha = 1)$ coincides with $G(t_f)$.

Let $z = (x, y, \varphi)^{\mathrm{T}} \in \partial G(t_f)$. Then the desired result follows from Theorem 1 for $\alpha^{\flat} = 1$. In particular, a control that is identically equal to zero leads to the boundary of the set $G(t_f)$. Indeed, in this case the motion moves to the point $z(t_f) = (t_f, 0, 0)^{\mathrm{T}}$ with the maximum possible value of the first coordinate.

We now assume $z \in \operatorname{int} G(t_f)$. We denote by $\alpha^{\flat}$ the smallest $\alpha \in [0,1]$ at which $z \in G(t_f, \alpha)$. Such $\alpha^{\flat}$ exists due to the continuity of the functions f_1 and f_2, the continuous dependence of the set $G(t_f)$ in the Hausdorff metric of parameters u_1 and u_2, and also due to the closeness of the reachability set. The continuity of the reachability set with respect to parameters u_1 and u_2 follows from a certain very general property of differential inclusions described in [19] (Chapter 4, Theorem 5.4(b) and Corollary 5.5).

Note that $\alpha^{\flat} > 0$. Indeed, at $\alpha^{\flat} = 0$ we would have $z \in G(t_f, \alpha^{\flat} = 0)$. This is impossible because, as noted above, the one-point set $G(t_f, \alpha^{\flat} = 0)$ belongs $\partial G(t_f)$, while $z \in \operatorname{int} G(t_f)$.

We take the sequence $\{\alpha_i\}$ of positive numbers converging to $\alpha^{\flat}$ from below. We have $z \notin G(t_f, \alpha_i)$. We assume z_i is the point on the boundary of the set $G(t_f, \alpha_i)$ closest to point z. Let us show that the sequence $\{z_i\}$ converges to point z.

Taking into account the monotonic increase of the functions $f_1(\alpha)$ and $f_2(\alpha)$, we obtain a monotonic increase (by inclusion) of the sets $G(t_f, \alpha_i)$ as i increases. Due to the continuity of the change in the set $G(t_f, \alpha)$ by parameter α in the Hausdorff metric, we have $h\left(G(t_f, \alpha^{\flat}), G(t_f, \alpha_i)\right) \to 0$ as $i \to \infty$. Here h is the Hausdorff distance. Since the Euclidean distance $r(z, G(t_f, \alpha_i))$ from point z to the set $G(t_f, \alpha_i)$ does not exceed $h\left(G(t_f, \alpha_i), G(t_f, \alpha^{\flat})\right)$, we get $r(z, G(t_f, \alpha_i)) = r(z, z_i) \to 0$ as $i \to \infty$.

Since $z_i \in \partial G(t_f, \alpha_i)$, by Theorem 1 (considering the restrictions $u_1(\alpha_i)$ and $u_2(\alpha_i)$ for control u in it) point z_i can be reached using a control with at most two switchings with values in the three-element set $\{u_1(\alpha_i), 0, u_2(\alpha_i)\}$, where $u_1(\alpha_i) = f_1(\alpha_i)u_1$ and $u_2(\alpha_i) = f_2(\alpha_i)u_2$. In this case, in the case of two switchings, we can restrict ourselves to controls of the form

$$\begin{gathered} 1)\ u_2(\alpha_i), 0, u_2(\alpha_i); \quad 2)\ u_1(\alpha_i), 0, u_2(\alpha_i); \quad 3)\ u_2(\alpha_i), 0, u_1(\alpha_i); \\ 4)\ u_1(\alpha_i), 0, u_1(\alpha_i); \quad 5)\ u_2(\alpha_i), u_1(\alpha_i), u_2(\alpha_i); \quad 6)\ u_1(\alpha_i), u_2(\alpha_i), u_1(\alpha_i). \end{gathered} \tag{3.11}$$

From the sequence $\{\alpha_i\}$ we select the subsequence $\{\alpha_j\}$ with the same number of switchings for each j. In this case, in the same constant control interval in time order (there are not more than three of them), either the control with the same sign or the zero control is implemented.

The lengths of the control constancy sections are limited. Therefore, from the sequence $\{\alpha_j\}$, we can select the subsequence $\{\alpha_k\}$ for which the corresponding lengths of intervals of constancy of control have a limit (possibly equal to zero). The limit values of these lengths define some admissible control with at most two switchings that leads to the point z. The structure of this control satisfies the properties indicated in the statement of the theorem. Theorem 2 is proved.

Theorem 2 can be refined by analogy with Theorem 1, taking into account the sign of angle φ for the considered point $z = (x, y, \varphi)^{\mathrm{T}}$ in the reachability set $G(t_f)$.

Note 2. If $\varphi > 0$, then only four out of six types can be left in list (3.10) with two switchings: $U1^{\flat}$, $U2^{\flat}$, $U3^{\flat}$, and $U6^{\flat}$. When $\varphi < 0$, list (3.10) can be limited to four types: $U2^{\flat}$, $U3^{\flat}$, $U4^{\flat}$, and $U5^{\flat}$. If $\varphi = 0$, then list (3.10) can contain the following types of controls: $U2^{\flat}$, $U3^{\flat}$, $U5^{\flat}$, and $U6^{\flat}$; and controls of types $U5^{\flat}$ and $U6^{\flat}$ generate the same set of points.

Proof. If $z = (x, y, \varphi)^{\mathrm{T}} \in \partial G(t_f)$, then the proof of the remark above follows from Note 1 for $\alpha^{\flat} = 1$.

We assume that $z \in \operatorname{int} G(t_f)$.

Let $\varphi = 0$. This case is analyzed in the same way as in Note 1, with the controls of types U1, U4, U5, and U6 replaced by $U1^{\flat}$, $U4^{\flat}$, $U5^{\flat}$, and $U6^{\flat}$.

Let $\varphi > 0$. In proving Theorem 2, we considered the sequence $\{\alpha_i\}$ for which points z_i were chosen for $\partial G(t_f, \alpha_i)$. Since $\varphi > 0$, then starting from some number $\bar{i}$, we have $\varphi_i > 0$. For points z_i, where $i > \bar{i}$, we will consider the four variants indicated in Note 1 as the generating controls with two switchings, with the extreme values u_1 and u_2 being replaced by $u_1(\alpha_i)$ and $u_2(\alpha_i)$, respectively. When forming the subsequence $\{z_k\}$, we consider one of these variants to be implemented (in the case of two switchings). Then for the limit point z the corresponding limit control (if it also has two switchings) belongs to the same variant. Therefore, for point z, there are only four variants, $U1^\flat$, $U2^\flat$, $U3^\flat$, and $U6^\flat$ (the same as in Note 1), as possible generating controls with two switchings for $\varphi > 0$.

The case $\varphi < 0$ is dealt with in a similar way. We also get four control variants: $U2^\flat$, $U3^\flat$, $U4^\flat$, and $U5^\flat$. Note 2 is proved.

4. INTEGRATION FORMULAS FOR EXTREMAL MOTIONS

The set of possible values of φ of system (1.1) at the moment $t_f > 0$ is determined by the constraint $u \in [u_1, u_2]$ and is the segment $[t_f u_1, t_f u_2]$. In this and the next section, we assume that $\varphi \in [0, t_f u_2]$. The case $\varphi \in [t_f u_1, 0)$ will be discussed in the last section of the article.

According to Theorem 1, a piecewise constant control with at most two switchings leads to any point of the boundary $\partial G(t_f)$ of a 3D reachable set. At the same time, taking into account $\varphi \geq 0$, in the case of two switchings, we can restrict ourselves to four types of controls: U1, U2, U3, and U6 (Note 1). The same is true for points lying on the boundary of the φ-section $G_\varphi(t_f)$ at $\varphi \geq 0$.

In Sections 4.1–4.4 we will assume that $\varphi \in [0, t_f u_2)$. The extreme value $\varphi = t_f u_2$ corresponds to the one-point set $G_\varphi(t_f)$, which is described in Section 4.5.

4.1. One-Parameter Curves of Four Types in φ-Sections

We fix some $\varphi \in [0, t_f u_2)$. Using switching times t_1 and t_2, we introduce for each of the four types a one-parameter curve on the plane x, y. We rely on the fact that the switching times are related by the relation

$$\varphi = (\varphi(t_1) - \varphi(t_0)) + (\varphi(t_2) - \varphi(t_1)) + (\varphi(t_f) - \varphi(t_2)), \tag{4.1}$$

where $\varphi(t_0) = 0$. Expression (4.1) allows us for each type, using one parameter, to describe the controls leading to a fixed φ-section, i.e., build a one-parameter curve of end positions with the given φ at the moment t_f in the plane x, y.

1. We consider the sequence of controls $u_2, 0, u_2$ with two switching points t_1 and t_2 (type U1). In this case, $t_0 = 0 < t_1 < t_2 < t_f$.

Condition (4.1) for hitting the given φ-section takes the following form:

$$\varphi = (t_1 - t_0)u_2 + (t_f - t_2)u_2. \tag{4.2}$$

It follows from the formula that for a fixed φ the difference $t_2 - t_1$ (i.e., the duration of the average interval of the motion) is a constant: $t_2 - t_1 = t_f - t_0 - \varphi/u_2$. Considering $t_0 = 0$, we get

$$t_2 - t_1 = t_f - \frac{\varphi}{u_2}. \tag{4.3}$$

Taking t_1 as an independent variable, we obtain in $\varphi \in (0, t_f u_2)$ the range $(0, \varphi/u_2)$ of possible values of t_1. Integrating system (1.1) in the interval $[0, t_f]$ at fixed t_1, and then, taking into account (4.3), we arrive at the expression for the geometric position at moment t_f:

$$\begin{aligned}\begin{pmatrix} x(t_f) \\ y(t_f) \end{pmatrix} &= \frac{1}{u_2}\begin{pmatrix} \sin\varphi(t_1) \\ 1 - \cos\varphi(t_1) \end{pmatrix} + (t_2 - t_1)\begin{pmatrix} \cos\varphi(t_1) \\ \sin\varphi(t_1) \end{pmatrix} + \frac{1}{u_2}\begin{pmatrix} \sin\varphi - \sin\varphi(t_1) \\ \cos\varphi(t_1) - \cos\varphi \end{pmatrix} \\ &= \frac{1}{u_2}\begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix} + (t_2 - t_1)\begin{pmatrix} \cos\varphi(t_1) \\ \sin\varphi(t_1) \end{pmatrix} = \frac{1}{u_2}\begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix} + \left(t_f - \frac{\varphi}{u_2}\right)\begin{pmatrix} \cos\varphi(t_1) \\ \sin\varphi(t_1) \end{pmatrix}.\end{aligned} \tag{4.4}$$

Formula (4.4) defines a continuous one-parameter curve (with respect to parameter t_1) on the surface x, y.

We take a new independent parameter $s_1 = 2t_1u_2 - \varphi$. For it, the range of possible values $s_1 \in (-\varphi, \varphi)$ becomes symmetrical.

Substituting the expression $\varphi(t_1) = t_1u_2 = (s_1 + \varphi)/2$ into (4.4), we get a one-parameter curve on the plane x, y for $\varphi \in (0, t_f u_2)$:

$$\begin{pmatrix} x_{U1}(s_1) \\ y_{U1}(s_1) \end{pmatrix} = \frac{1}{u_2}\begin{pmatrix} \sin\varphi \\ 1-\cos\varphi \end{pmatrix} + \left(t_f - \frac{\varphi}{u_2}\right)\begin{pmatrix} \cos\left(\frac{s_1+\varphi}{2}\right) \\ \sin\left(\frac{s_1+\varphi}{2}\right) \end{pmatrix}. \tag{4.5}$$

2. Proceeding similarly, we establish that the motions corresponding to the types of controls U2, U3, and U6 also generate one-parameter curves on the plane x, y (with fixed φ). To describe them, we introduce the notation

$$\theta = \frac{u_1(\varphi - t_f u_2)}{u_2 - u_1}. \tag{4.6}$$

It is obvious that at $\varphi \in [0, t_f u_2)$ the inequality $\theta > 0$ is fulfilled. We define variants s_2, s_3, and s_6 through t_1 and θ:

$$s_2 = t_1u_1, \quad s_3 = t_1u_2 - \varphi, \quad s_6 = -2t_1u_1 - \theta. \tag{4.7}$$

Integrating system (1.1), we obtain the following formulas:

$$\begin{pmatrix} x_{U2}(s_2) \\ y_{U2}(s_2) \end{pmatrix} = \frac{1}{u_2}\begin{pmatrix} \sin\varphi \\ 1-\cos\varphi \end{pmatrix} \tag{4.8}$$

$$+\left(\frac{1}{u_2} - \frac{1}{u_1}\right)\left((\theta + s_2)\begin{pmatrix} \cos s_2 \\ \sin s_2 \end{pmatrix} - \begin{pmatrix} \sin s_2 \\ 1 - \cos s_2 \end{pmatrix}\right),$$

$$\begin{pmatrix} x_{U3}(s_3) \\ y_{U3}(s_3) \end{pmatrix} = \frac{1}{u_1}\begin{pmatrix} \sin\varphi \\ 1-\cos\varphi \end{pmatrix} \tag{4.9}$$

$$+\left(\frac{1}{u_2} - \frac{1}{u_1}\right)\left((\theta - s_3)\begin{pmatrix} \cos(s_3+\varphi) \\ \sin(s_3+\varphi) \end{pmatrix} + \begin{pmatrix} \sin(s_3+\varphi) \\ 1 - \cos(s_3+\varphi) \end{pmatrix}\right),$$

$$\begin{pmatrix} x_{U6}(s_6) \\ y_{U6}(s_6) \end{pmatrix} = \frac{1}{u_1}\begin{pmatrix} \sin\varphi \\ 1-\cos\varphi \end{pmatrix} + 2\sin\left(\frac{\varphi+\theta}{2}\right)\left(\frac{1}{u_2} - \frac{1}{u_1}\right)\begin{pmatrix} \cos\left(\frac{\varphi - s_6}{2}\right) \\ \sin\left(\frac{\varphi - s_6}{2}\right) \end{pmatrix}. \tag{4.10}$$

Formulas (4.8)–(4.10) define one-parameter curves on the plane x, y, corresponding to controls of types U2, U3, and U6 with two switchings. Variants s_2, s_3 and s_6 change in the following ranges:

$$s_2 \in (-\theta, 0), \quad s_3 \in (0, \theta), \quad s_6 \in (-\theta, \theta). \tag{4.11}$$

3. For $\varphi \in [0, t_f u_2)$ controls $u(\cdot)$ with one switching that satisfy the PMP can only have the following structure: $0, u_2$; $u_2, 0$; u_1, u_2, and u_2, u_1. It is easy to verify that the motion generated by each of these variants leads to the extreme point of at least one of curves (4.5) and (4.8)–(4.10). The control without switching (satisfying the PMP) is identically equal to zero. The corresponding motion gives $\varphi(t_f) = 0$, $x(t_f) = t_f$, $y(t_f) = 0$. This means that the point $(x(t_f), y(t_f))^{\mathrm{T}}$, corresponding to a zero control, can be in φ-section of the set $G(t_f)$ only when $\varphi = 0$. The obtained values $x(t_f)$ and $y(t_f)$ coincide with the values calculated by formula (4.5) for $\varphi = 0$ and $s_1 = 0$.

Thus, by Theorem 1, when describing the boundary $\partial G_\varphi(t_f)$ for $\varphi \in [0, t_f u_2)$, it suffices to use curves (4.5) and (4.8)–(4.10) with closed domains:

$$s_1 \in [-\varphi, \varphi], \quad s_2 \in [-\theta, 0], \quad s_3 \in [0, \theta], \quad s_6 \in [-\theta, \theta]. \tag{4.12}$$

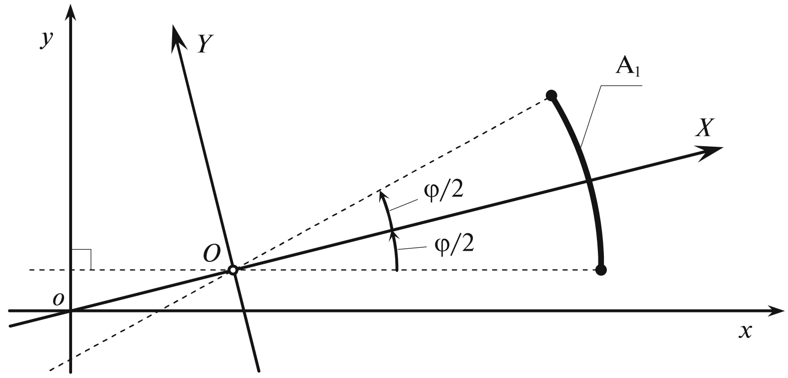

Fig. 7. Auxiliary coordinate system X, Y.

4.2. Auxiliary Coordinate System

In addition to the original coordinate system, we will use an auxiliary orthogonal system X, Y. Axis X of the auxiliary system passes through the origin of the original coordinate system x, y and is rotated at angle $\varphi/2$ about the x axis (Fig. 7). The origin of the auxiliary coordinate system is located at point $O = (\sin\varphi, 1-\cos\varphi)^{\mathrm{T}}$. Recalculation into the auxiliary system X, Y from the original system x, y is carried out according to the formula

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos(\varphi/2) & \sin(\varphi/2) \\ -\sin(\varphi/2) & \cos(\varphi/2) \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \sin\varphi \\ 1-\cos\varphi \end{pmatrix} \right). \tag{4.13}$$

The auxiliary coordinate system depends only on angle φ. Formula (4.13) defines a *one-to-one* affine correspondence (for fixed φ) between vectors $(x, y)^{\mathrm{T}}$ and $(X, Y)^{\mathrm{T}}$.

We recalculate curves (4.5) and (4.8)–(4.10), defined for $\varphi \in [0, t_f u_2)$ by parameters s_1, s_2, s_3, and s_6 in the closed intervals (4.12), into the auxiliary coordinate system X, Y and denote them by A_1, A_2, A_3, and A_6, respectively. To shorten the notation, we put

$$\xi_\varphi(u_2) = 2\sin\left(\frac{\varphi}{2}\right) \begin{pmatrix} \dfrac{1}{u_2} - 1 \\ 0 \end{pmatrix}. \tag{4.14}$$

We have

$$\mathrm{A}_1(s_1) = \left(\frac{1}{u_2} - \frac{1}{u_1}\right) \left[\theta \begin{pmatrix} \cos\left(\dfrac{s_1}{2}\right) \\ \sin\left(\dfrac{s_1}{2}\right) \end{pmatrix} \right] + \xi_\varphi(u_2), \tag{4.15}$$

$$\mathrm{A}_2(s_2) = \left(\frac{1}{u_2} - \frac{1}{u_1}\right) \left[(\theta + s_2) \begin{pmatrix} \cos\left(s_2 - \dfrac{\varphi}{2}\right) \\ \sin\left(s_2 - \dfrac{\varphi}{2}\right) \end{pmatrix} - 2\sin\left(\frac{s_2}{2}\right) \begin{pmatrix} \cos\left(\dfrac{s_2}{2} - \dfrac{\varphi}{2}\right) \\ \sin\left(\dfrac{s_2}{2} - \dfrac{\varphi}{2}\right) \end{pmatrix} \right] + \xi_\varphi(u_2), \tag{4.16}$$

$$\mathrm{A}_3(s_3) = \left(\frac{1}{u_2} - \frac{1}{u_1}\right) \left[(\theta - s_3) \begin{pmatrix} \cos\left(s_3 + \dfrac{\varphi}{2}\right) \\ \sin\left(s_3 + \dfrac{\varphi}{2}\right) \end{pmatrix} + 2\sin\left(\frac{s_3}{2}\right) \begin{pmatrix} \cos\left(\dfrac{s_3}{2} + \dfrac{\varphi}{2}\right) \\ \sin\left(\dfrac{s_3}{2} + \dfrac{\varphi}{2}\right) \end{pmatrix} \right] + \xi_\varphi(u_2), \tag{4.17}$$

$$\mathrm{A}_6(s_6) = \left(\frac{1}{u_2} - \frac{1}{u_1}\right) \left[-\begin{pmatrix} 2\sin\left(\dfrac{\varphi}{2}\right) \\ 0 \end{pmatrix} + 2\sin\left(\frac{\varphi + \theta}{2}\right) \begin{pmatrix} \cos\left(\dfrac{-s_6}{2}\right) \\ \sin\left(\dfrac{-s_6}{2}\right) \end{pmatrix} \right] + \xi_\varphi(u_2). \tag{4.18}$$

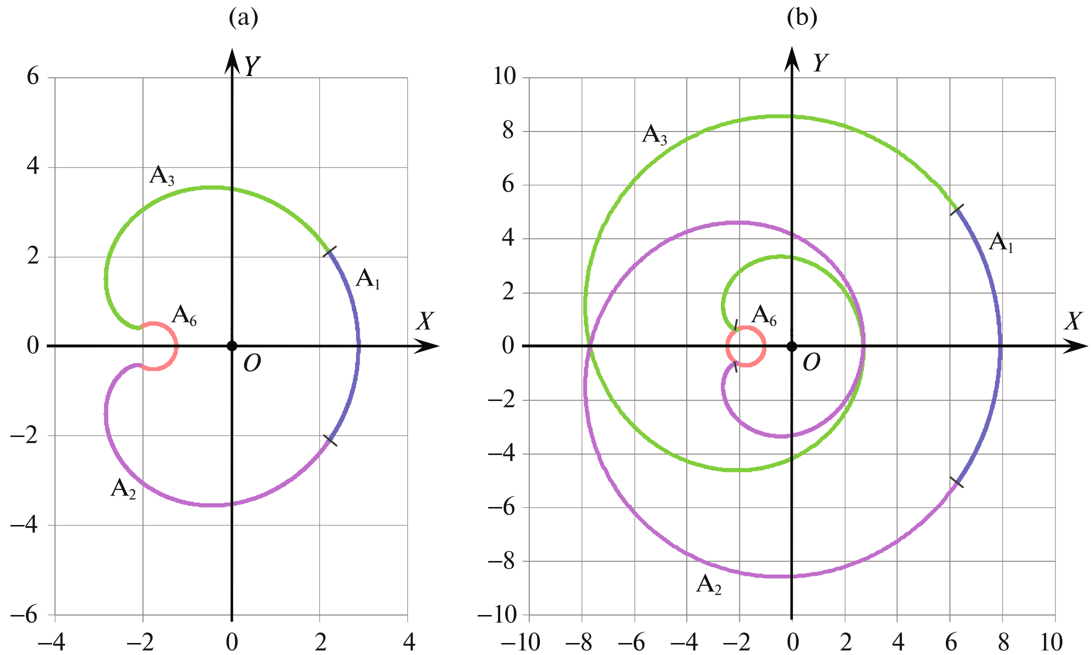

Fig. 8. Examples of the curve $\mathcal{A}_\varphi(u_1, u_2, t_f)$ for the asymmetric case: $u_1 = -2$, $u_2 = 3$, $\varphi = 0.4\pi$; values $t_f = 1.3\pi$ (a) and $t_f = 2.9\pi$ (b).

We note the structural analogy of formulas (4.15)–(4.18): a square bracket, the same coefficient in front of it, and the same last term. Variants s_1, s_2, s_3 and s_6 included in square brackets, change in ranges (4.12).

Curves A_1 and A_6 are arcs of circles. Each of them is symmetrical about the X axis due to the symmetry with respect to zero of the ranges of the variation in parameters s_1 and s_6. Curves A_2 and A_3 are mutually symmetrical about the axis X. This follows from the fact that the X components of points on curves A_2 and A_3 at $s_2 = -s_3$ match and the Y components differ only in sign.

4.3. Compound Closed Curve

With fixed $\varphi \in [0, t_f u_2)$, the set of curves A_1, A_2, A_3, and A_6 contains the boundary of the set $G_\varphi(t_f)$. We consider these curves in the sequence A_1, A_3, A_6, and A_2 bypassing them in ascending order of parameters s_1, s_2, s_3, and s_6. For the extreme values (4.12) of the parameters, we have

$$A_1(\varphi) = A_3(0), \quad A_3(\theta) = A_6(-\theta), \quad A_6(\theta) = A_2(-\theta), \quad A_2(0) = A_1(-\varphi).$$

As a result of gluing, we obtain a continuous piecewise-smooth closed curve on the plane X, Y. We denote it with the symbol $\mathcal{A}_\varphi(u_1, u_2, t_f)$ and we will call it a compound curve. This curve is symmetrical about the X axis and continuously depends on u_1, u_2, and t_f. The curve structure at fixed φ becomes more difficult with the growth of t_f. We emphasize that the curve $\mathcal{A}_\varphi(u_1, u_2, t_f)$ is composed of curves of four types and contains all points of the boundary of the set $G_\varphi(t_f)$.

Figure 8 gives two examples of the curve $\mathcal{A}_\varphi(u_1, u_2, t_f)$ for the asymmetric case $u_1 = -2$ and $u_2 = 3$ at $\varphi = 0.4\pi$. As noted above, the symmetry of the curve about the X axis is preserved by the auxiliary coordinate system. The articulation points of arcs A_1, A_3, A_6, and A_2 are marked with risks. For the value $t_f = 1.3\pi$ (Fig. 8a), curve $\mathcal{A}_\varphi(u_1, u_2, t_f)$ has no self-intersections. For the value $t_f = 2.9\pi$ (Fig. 8b), the curve $\mathcal{A}_\varphi(u_1, u_2, t_f)$ has self-intersection points.

Curve $\mathcal{A}_\varphi(u_1, u_2, t_f)$ will play a major role in solving the main task of this article, which is to establish a correspondence between φ-sections of $G_\varphi(t_f)$ reachable sets for the canonical and asymmetric cases.

4.4. A Family of Compound Curves

Assume $\varphi \in [0, t_f u_2)$, while $f_1(\alpha)$ and $f_2(\alpha)$ are some functions that satisfy the condition of Theorem 2. We consider an arbitrary point in the set $G_\varphi(t_f)$. By Theorem 2 and Note 2, there exists $\alpha^\flat \in (0,1]$ such that it is possible to move to this point at the moment t_f using a piecewise constant control with at most two switchings and with values in the three-element set $\{u_1^\flat, 0, u_2^\flat\}$, where $u_1^\flat = f_1(\alpha^\flat)u_1$ and $u_2^\flat = f_2(\alpha^\flat)u_2$. In this case, in the case of two switchings, we can limit ourselves to four types of controls: $\mathrm{U1}^\flat$, $\mathrm{U2}^\flat$, $\mathrm{U3}^\flat$, and $\mathrm{U6}^\flat$.

We denote by the symbol $\mathcal{A}_\varphi(u_1(\alpha), u_2(\alpha), t_f)$ a glued curve similar to the curve $\mathcal{A}_\varphi(u_1, u_2, t_f)$ and constructed using formulas (4.6) and (4.14)–(4.18) by substituting in them the values $u_1(\alpha) = f_1(\alpha)u_1$ and $u_2(\alpha) = f_2(\alpha)u_2$ instead of values u_1 and u_2. We find that the family of curves $\mathcal{A}_\varphi(u_1(\alpha), u_2(\alpha), t_f)$, where $\alpha \in (0,1]$, fills the whole set $G_\varphi(t_f)$, i.e.,

$$G_\varphi(t_f) = \bigcup_{\alpha \in (0,1]} \{\mathcal{A}_\varphi(u_1(\alpha), u_2(\alpha), t_f)\}. \tag{4.19}$$

Each curve $\mathcal{A}_\varphi(u_1(\alpha), u_2(\alpha), t_f)$ (by analogy with the curve $\mathcal{A}_\varphi(u_1, u_2, t_f)$) is symmetrical about the X axis of the auxiliary coordinate system. Therefore, due to (4.19), we obtain the symmetry of the set $G_\varphi(t_f)$ about the X axis.

4.5. Single Point φ-Section in the Special Case $\varphi = t_f u_2$

The extreme value $\varphi = t_f u_2$ is realized on the control $u(t) \equiv u_2$. By integrating system (1.1), we obtain a point with coordinates $x(t_f) = \sin(t_f u_2)/u_2$ and $y(t_f) = (1 - \cos(t_f u_2))/u_2$. We recalculate it into an auxiliary coordinate system and denote it by the symbol $e(t_f)$. We have

$$e(t_f) = \left(\frac{1}{u_2} - 1\right)\begin{pmatrix} 2\sin\left(t_f \dfrac{u_2}{2}\right) \\ 0 \end{pmatrix}. \tag{4.20}$$

Thus, the φ-section of $G_\varphi(t_f)$ at $\varphi = t_f u_2$ consists of one point, $e(t_f)$.

5. RELATIONSHIP OF φ-SECTIONS FOR THE ASYMMETRIC AND CANONICAL CASES

This section will show that any φ-section for the general asymmetric case is related by some one-to-one affine correspondence with an φ-section for the canonical case. This will mean that φ-sections for the asymmetric case can be obtained based on the description of φ-sections for the canonical case.

In system (1.1), we fix some values $u_1 < 0$, $u_2 > 0$, $t_f > 0$, $\varphi \in [0, t_f u_2]$. As before, we consider $t_0 = 0$ and $z(t_0) = (x_0, y_0, \varphi_0)^{\mathrm{T}} = (0,0,0)^{\mathrm{T}}$.

Together with the general case, we consider the canonical case, for which we set

$$u_1^{\mathrm{c}} = -1, \quad u_2^{\mathrm{c}} = 1, \quad t_f^{\mathrm{c}} = \frac{2u_1(\varphi - t_f u_2)}{u_2 - u_1} + \varphi, \quad \varphi^{\mathrm{c}} = \varphi. \tag{5.1}$$

We note that the input time t_f^{c} depends on values u_1, u_2, t_f, and φ. It is clear that $0 \le \varphi^{\mathrm{c}} \le t_f^{\mathrm{c}}$. By the symbol $G_\varphi^{\mathrm{c}}(t_f^{\mathrm{c}})$, we denote the φ-section of the reachability set at the moment t_f^{c} for the canonical case. By analogy with (4.6), we set

$$\theta^{\mathrm{c}} = \frac{u_1^{\mathrm{c}}(\varphi^{\mathrm{c}} - t_f^{\mathrm{c}} u_2^{\mathrm{c}})}{u_2^{\mathrm{c}} - u_1^{\mathrm{c}}}.$$

Taking into account (5.1), we have

$$\theta^c = \frac{t_f^c - \varphi}{2}. \quad (5.2)$$

Before establishing a correspondence formula between the φ-sections of $G_\varphi(t_f)$ and $G_\varphi^c\left(t_f^c\right)$, at $\varphi \in [0, t_f u_2)$ we obtain a similar formula for curves $\mathcal{A}_\varphi(u_1, u_2, t_f)$ and $\mathcal{A}_\varphi\left(u_1^c, u_2^c, t_f^c\right)$. Here $\mathcal{A}_\varphi\left(u_1^c, u_2^c, t_f^c\right)$ is a closed curve in the form of a glue of curves A_1^c, A_2^c, A_3^c, and A_6^c given by formulas (4.15)–(4.18) by substituting in them the values u_1^c, u_2^c, t_f^c, and θ^c instead of u_1, u_2, t_f, and θ. In other words, we will show that the curves $\mathcal{A}_\varphi(u_1, u_2, t_f)$ and $\mathcal{A}_\varphi\left(u_1^c,\ u_2^c,\ t_f^c\right)$, written in the auxiliary coordinate system (it depends only on φ), are related by the relation

$$\mathcal{A}_\varphi(u_1, u_2, t_f) = \frac{1}{2}\left(\frac{1}{u_2} - \frac{1}{u_1}\right)\mathcal{A}_\varphi\left(u_1^c,\ u_2^c,\ t_f^c\right) + \xi_\varphi(u_2). \quad (5.3)$$

The value $\xi_\varphi(u_2)$ is defined by formula (4.14). The proof of relation (5.3) will suggest a way of reasoning when establishing a correspondence between the φ-sections of $G_\varphi(t_f)$ and $G_\varphi^c\left(t_f^c\right)$.

To verify the validity of relation (5.3), it suffices to check it for the corresponding pairs of curves: (A_1, A_1^c), (A_2, A_2^c), (A_3, A_3^c), and (A_6, A_6^c). By substituting the value t_f^c from (5.1) into (5.2), we obtain the equality $\theta^c = \theta$, where θ is defined by (4.6). Therefore, given the equality $\varphi^c = \varphi$, we conclude that ranges (4.12) of the change in parameters s_1, s_2, s_3, and s_6 when curves A_1, A_2, A_3, and A_6 are given coincide with the corresponding ranges for curves A_1^c, A_2^c, A_3^c, and A_6^c.

We consider curves A_1 and A_1^c. Curve A_1 is given by formula (4.15) and corresponds to the moment t_f with control constraints u_1 and u_2. Curve A_1^c corresponds to the moment t_f^c with the control constraints u_1^c and u_2^c. Taking into account the fact that $u_1^c = -1$, $u_2^c = 1$, and $\theta^c = \theta$, it has the form

$$A_1^c(s_1) = \theta^c\left(\frac{1}{u_2^c} - \frac{1}{u_1^c}\right)\begin{pmatrix}\cos\left(\frac{s_1}{2}\right)\\ \sin\left(\frac{s_1}{2}\right)\end{pmatrix} + 2\sin\left(\frac{\varphi}{2}\right)\begin{pmatrix}\frac{1}{u_2^c} - 1\\ 0\end{pmatrix} = 2\theta\begin{pmatrix}\cos\left(\frac{s_1}{2}\right)\\ \sin\left(\frac{s_1}{2}\right)\end{pmatrix}. \quad (5.4)$$

Using formulas (4.15) and (5.4), we verify the validity of the equality

$$A_1(s_1) = \frac{1}{2}\left(\frac{1}{u_2} - \frac{1}{u_1}\right)A_2^c(s_1) + \xi_\varphi(u_2).$$

Similar equalities (in the auxiliary coordinate system) are also valid for other pairs of curves: (A_2, A_2^c), (A_3, A_3^c), and (A_6, A_6^c). Therefore, relation (5.3) holds.

Theorem 3. *Assume* $\varphi \in [0, t_f u_2]$. *Sections* $G_\varphi(t_f)$ *of the reachability set of the original system* (1.1) *are related in the auxiliary coordinate system with sections* $G_\varphi^c\left(t_f^c\right)$ *of the reachable set of the canonical system* (5.1) *by the relation*

$$G_\varphi(t_f) = \frac{1}{2}\left(\frac{1}{u_2} - \frac{1}{u_1}\right)G_\varphi^c\left(t_f^c\right) + \xi_\varphi(u_2). \quad (5.5)$$

Proof. At first, we will assume that $\varphi \in [0, t_f u_2)$. Let us use the scheme used to establish equality (5.3). Then we assume that $\varphi = t_f u_2$.

1. Let $\varphi \in [0, t_f u_2)$. For the asymmetric case, we consider the functions

$$f_1^{(1)}(\alpha) = \frac{2\alpha u_2}{\alpha(u_2 + u_1) + (u_2 - u_1)}, \quad f_2^{(1)}(\alpha) = \frac{2\alpha u_1}{\alpha(u_2 + u_1) - (u_2 - u_1)}. \quad (5.6)$$

These functions satisfy the conditions stipulated in Theorem 2.

We put $u_1^{(1)}(\alpha) = f_1^{(1)}(\alpha)u_1$, $u_2^{(1)}(\alpha) = f_2^{(1)}(\alpha)u_2$. Curves $\mathcal{A}_\varphi(u_1^{(1)}(\alpha), u_2^{(1)}(\alpha), t_f)$ while iterating $\alpha \in (0, 1]$ fill the whole set $G_\varphi(t_f)$ (Section 4.4).

For the canonical case defined in (5.1), we consider the functions

$$f_1^{(2)}(\alpha) = \alpha, \quad f_2^{(2)}(\alpha) = \alpha. \tag{5.7}$$

We put $u_1^{(2)}(\alpha) = f_1^{(2)}(\alpha)u_1^{\mathrm{c}}$ and $u_2^{(2)}(\alpha) = f_2^{(2)}(\alpha)u_2^{\mathrm{c}}$. Curves $\mathcal{A}_\varphi\left(u_1^{(2)}(\alpha), u_2^{(2)}(\alpha), t_f^{\mathrm{c}}\right)$ while iterating $\alpha \in (0, 1]$ fill the whole set $G_\varphi^{\mathrm{c}}\left(t_f^{\mathrm{c}}\right)$.

Let us establish a correspondence between the introduced families of curves by the formula

$$\mathcal{A}_\varphi\left(u_1^{(1)}(\alpha), u_2^{(1)}(\alpha), t_f\right) = \frac{1}{2}\left(\frac{1}{u_2} - \frac{1}{u_1}\right)\mathcal{A}_\varphi\left(u_1^{(2)}(\alpha), u_2^{(2)}(\alpha), t_f^{\mathrm{c}}\right) + \xi_\varphi(u_2), \quad \alpha \in (0,1]. \tag{5.8}$$

We fix some $\alpha \in (0, 1]$. Due to (5.6) and (5.7), we obtain

$$u_1^{(1)}(\alpha) = \frac{2\alpha u_1 u_2}{\alpha(u_2 + u_1) + (u_2 - u_1)}, \quad u_2^{(1)}(\alpha) = \frac{2\alpha u_1 u_2}{\alpha(u_2 + u_1) - (u_2 - u_1)}, \tag{5.9}$$

$$u_1^{(2)}(\alpha) = -\alpha, \quad u_2^{(2)}(\alpha) = -u_1^{(2)}(\alpha) = \alpha. \tag{5.10}$$

To simplify the notation, we set $u_1^{(1)} = u_1^{(1)}(\alpha)$, $u_2^{(1)} = u_2^{(1)}(\alpha)$, $u_1^{(2)} = u_1^{(2)}(\alpha)$, and $u_2^{(2)} = u_2^{(2)}(\alpha)$.

Using (5.6) and the inequality $(u_2 + u_1) < (u_2 - u_1)$, as well as taking into account that $\alpha \neq 0$, we have $u_1 \le u_1^{(1)} < 0$ and $0 < u_2^{(1)} \le u_2$. It is clear that $u_1^{\mathrm{c}} \le u_1^{(2)} < 0$ and $0 < u_2^{(2)} \le u_2^{\mathrm{c}}$.

With fixed values of φ, t_f, and t_f^{c}, we introduce the values

$$\theta^{(1)} = \frac{u_1^{(1)}\left(\varphi - t_f u_2^{(1)}\right)}{u_2^{(1)} - u_1^{(1)}}, \tag{5.11}$$

$$\theta^{(2)} = \frac{u_1^{(2)}\left(\varphi - t_f^{\mathrm{c}} u_2^{(2)}\right)}{u_2^{(2)} - u_1^{(2)}} = \frac{t_f^{\mathrm{c}} u_2^{(2)} - \varphi}{2}. \tag{5.12}$$

Formulas (5.11) and (5.12) are similar to relations (4.6) and (5.2), in which the controls u_1, u_2 and u_1^{c}, u_2^{c} were used. Using (5.11) and (5.12), we introduce the curves $\mathcal{A}_\varphi\left(u_1^{(1)}, u_2^{(1)}, t_f\right)$ and $\mathcal{A}_\varphi\left(u_1^{(2)}, u_2^{(2)}, t_f^{\mathrm{c}}\right)$. We calculate their constituent parts using formulas (4.15)–(4.18), substituting in them $u_1^{(1)}$, $u_2^{(1)}$, $\theta^{(1)}$ and $u_1^{(2)}$, $u_2^{(2)}$, $\theta^{(2)}$ instead of u_1, u_2, θ. The curve $\mathcal{A}_\varphi\left(u_1^{(1)}, u_2^{(1)}, t_f\right)$ (just as the curve $\mathcal{A}_\varphi(u_1, u_2, t_f)$) is in the set $G_\varphi(t_f)$ and the curve $\mathcal{A}_\varphi\left(u_1^{(2)}, u_2^{(2)}, t_f^{\mathrm{c}}\right)$ lies in the set $G_\varphi^{\mathrm{c}}\left(t_f^{\mathrm{c}}\right)$.

Let us show that $\theta^{(1)} = \theta^{(2)}$. Substituting into (5.11) expressions for $u_1^{(1)}$ and $u_2^{(1)}$ from (5.9), we get

$$\theta^{(1)} = \frac{\left(\dfrac{2\alpha u_1 u_2}{\alpha(u_2 + u_1) + (u_2 - u_1)}\right)\left(\varphi - t_f\left(\dfrac{2\alpha u_1 u_2}{\alpha(u_2 + u_1) - (u_2 - u_1)}\right)\right)}{\left(\dfrac{2\alpha u_1 u_2}{\alpha(u_2 + u_1) - (u_2 - u_1)}\right) - \left(\dfrac{2\alpha u_1 u_2}{\alpha(u_2 + u_1) + (u_2 - u_1)}\right)} \tag{5.13}$$

$$= \frac{\varphi(\alpha(u_2 + u_1) + (u_2 - u_1)) - 2\alpha u_1 u_2 t_f}{2(u_2 - u_1)}.$$

Formula (5.12) when substituting t_f^c from (5.1) and $u_2^{(2)}$ from (5.10) takes the form

$$\theta^{(2)} = \alpha\left(\frac{u_1(\varphi - t_f u_2)}{u_2 - u_1} + \frac{\varphi}{2}\right) - \frac{\varphi}{2}. \quad (5.14)$$

Comparing (5.13) and (5.14), we see that $\theta^{(1)} = \theta^{(2)}$.

We denote the composed curve $\mathcal{A}_\varphi(u_1^{(1)}, u_2^{(1)}, t_f)$ by $A_n^{(1)}$ and the composed curve $\mathcal{A}_\varphi\left(u_1^{(2)}, u_2^{(2)}, t_f^c\right)$ by $A_n^{(2)}$, $n = 1, 2, 3, 6$. Each component $A_n^{(1)}$ of the curve $\mathcal{A}_\varphi\left(u_1^{(1)}, u_2^{(1)}, t_f\right)$, in accordance with formulas (4.15)–(4.18), can be represented as

$$\left(\frac{1}{u_2^{(1)}} - \frac{1}{u_1^{(1)}}\right)\begin{bmatrix}\cdots\\ \cdots\end{bmatrix}_{A_n^{(1)}} + \xi_\varphi\left(u_2^{(1)}\right).$$

Here the square bracket $\begin{bmatrix}\cdots\\ \cdots\end{bmatrix}_{A_n^{(1)}}$ (matrix-column) fits the curve $A_n^{(1)}$ and is determined by the values of φ and $\theta^{(1)}$, as well as the parameter s_n. For the composed curve $\mathcal{A}_\varphi\left(u_1^{(2)}, u_2^{(2)}, t_f^c\right)$, the same formula is valid; however, in the notation, instead of superscript (1), we need to substitute index (2).

We take any curve $A_n^{(1)}$ of the four curves $\mathcal{A}_\varphi\left(u_1^{(1)}, u_2^{(1)}, t_f\right)$ included and the corresponding curve $A_n^{(2)}$ as part of $\mathcal{A}_\varphi\left(u_1^{(2)}, u_2^{(2)}, t_f^c\right)$. We write formula (5.8) to be proved for the chosen curves:

$$\left(\frac{1}{u_2^{(1)}} - \frac{1}{u_1^{(1)}}\right)\begin{bmatrix}\cdots\\ \cdots\end{bmatrix}_{A_n^{(1)}} + \xi_\varphi\left(u_2^{(1)}\right) = \quad (5.15)$$

$$= \frac{u_1 - u_2}{2u_1u_2}\left(\left(\frac{1}{u_2^{(2)}} - \frac{1}{u_1^{(2)}}\right)\begin{bmatrix}\cdots\\ \cdots\end{bmatrix}_{A_n^{(2)}} + \xi_\varphi(u_2^{(2)})\right) + \xi_\varphi(u_2).$$

In this formula, the square brackets match due to the equality $\theta^{(1)} = \theta^{(2)}$. Using the definitions of quantities $u_1^{(1)}$, $u_2^{(1)}$, $u_1^{(2)}$, and $u_2^{(2)}$, due to formulas (5.9) and (5.10), we see that the coefficients

$$\left(\frac{1}{u_2^{(1)}} - \frac{1}{u_1^{(1)}}\right), \quad \frac{u_1 - u_2}{2u_1u_2}\left(\frac{1}{u_2^{(2)}} - \frac{1}{u_1^{(2)}}\right)$$

with square brackets also match. To establish equality between the remaining terms on the left and right in formula (5.15), (after substituting the expressions for ξ_φ) we should check the relation

$$2\sin\left(\frac{\varphi}{2}\right)\left(\frac{1}{u_2^{(1)}} - 1\right) = \left(\frac{u_1 - u_2}{2u_1u_2}\right)2\sin\left(\frac{\varphi}{2}\right)\left(\frac{1}{u_2^{(2)}} - 1\right) + 2\sin\left(\frac{\varphi}{2}\right)\left(\frac{1}{u_2} - 1\right).$$

This relation is valid due to the definition of values $u_2^{(1)}$ and $u_2^{(2)}$ using formulas (5.9) and (5.10).

Thus, equality (5.15) and, therefore, equality (5.8) are proved. Taking (4.19) into account, we have

$$G_\varphi(t_f) = \bigcup_{\alpha\in(0,1]}\left\{\mathcal{A}_\varphi(u_1^{(1)}(\alpha), u_2^{(1)}(\alpha), t_f)\right\}, \quad G_\varphi^c\left(t_f^c\right) = \bigcup_{\alpha\in(0,1]}\left\{\mathcal{A}_\varphi(u_1^{(2)}(\alpha), u_2^{(2)}(\alpha), t_f^c)\right\}.$$

We obtain the required formula (5.5) for the correspondence between the φ-sections of $G_\varphi(t_f)$ and $G_\varphi^c\left(t_f^c\right)$ for the original and canonical variants of the control constraints.

2. Let $\varphi = t_f u_2$. In this case, the set $G_\varphi(t_f)$ consists of one point, which in the auxiliary coordinate system is written in form (4.20). For the canonical system, the quantity t_f^c due to (5.1) takes the value $t_f^c = \varphi$.

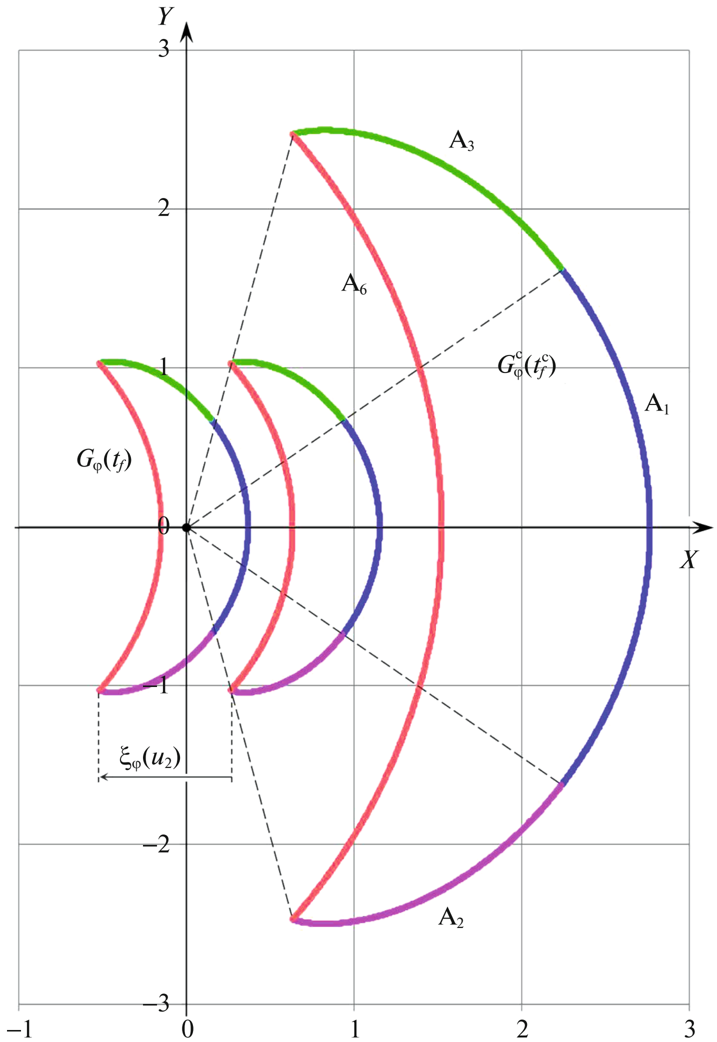

Fig. 9. Illustration of mutual correspondence of φ-sections in the nonsymmetric and canonical cases.

Then the maximum possible value of the angular coordinate is $t_f^c u_2^c = t_f^c = \varphi$. Therefore, the set $G_\varphi^c\left(t_f^c\right)$ also consists of one point, which we denote by $e^c\left(t_f^c\right)$. It is calculated by analogy with formula (4.20):

$$e^c\left(t_f^c\right) = \left(\frac{1}{u_2^c} - 1\right)\begin{pmatrix} 2\sin\left(t_f^c \dfrac{u_2^c}{2}\right) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Substituting $e(t_f)$ and $e^c\left(t_f^c\right)$ instead of $G_\varphi(t_f)$ and $G_\varphi^c\left(t_f^c\right)$ in (5.5), we obtain the equality

$$\left(\frac{1}{u_2} - 1\right)\begin{pmatrix} 2\sin\left(t_f \dfrac{u_2}{2}\right) \\ 0 \end{pmatrix} = 2\sin\left(\frac{\varphi}{2}\right)\begin{pmatrix} \dfrac{1}{u_2} - 1 \\ 0 \end{pmatrix},$$

which is valid because $\varphi = t_f u_2$.

Thus, relation (5.5) is also satisfied in the case when $\varphi = t_f u_2$. Theorem 3 is proved.

Figure 9 explains the application of formula (5.5). The initial data for the asymmetric case are taken in the form $u_1 = -2$, $u_2 = 3$, $\varphi = 0.4\pi$, and $t_f = 0.5\pi$. The moment t_f^c calculated from these data, to be used in the canonical system, is 1.28π. The set $G_\varphi^c\left(t_f^c\right)$ corresponding to this moment in the auxiliary coordinate system is shown in the figure on the right. In the multiplication of the set $G_\varphi^c\left(t_f^c\right)$ by the coefficient $(1/u_2 - 1/u_1)/2 \approx 0.417$, it is compressed relative to the zero of the auxiliary system (shown second from the right). The shift of the resulting set along the X axis by $\xi_\varphi(u_2) \approx -0.784$ gives the desired set $G_\varphi(t_f)$ (first from the left) for the asymmetric case.

Note 3. Formula (5.5) can be rewritten in the original coordinates x, y. The new formula will have the same form, but only the shift $\xi_\varphi(u_2)$ must be replaced in it by

$$\xi_\varphi(u_1, u_2) = \frac{1}{2}\left(\frac{1}{u_2} + \frac{1}{u_1}\right)\begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix}.$$

Note 4. In [20], the case of a one-sided rotation was considered, when $0 = u_1 < u_2$. Here each φ-section $G_\varphi(t_f)$ of the three-dimensional reachable set is either a segment of a circle (for $\varphi < 2\pi$) or a whole circle (with $\varphi \geq 2\pi$). Due to the continuous dependence of the reachability set on parameter u_1 (we assume that value u_2 is fixed), this φ-section represents the limit in the Hausdorff metric of the φ-sections of $G_\varphi(t_f)$ at $u_1 \to -0$. We emphasize that in this case the sets $G_\varphi(t_f)$ for $u_1 < 0$ are not convex. Therefore, the correspondence formula (5.5) is applicable only for the case $u_1 < 0 < u_2$. It cannot be used to obtain the limit set for $u_1 = 0$.

6. THE CASE $\varphi < 0$

We consider some $\varphi \in [u_1 t_f, 0)$ at $t_f > 0$. It is required to obtain a description of the set $G_\varphi(t_f)$. We indicate in the initial coordinates the symmetry property that allows us to do this.

We assume $u(\cdot)$ is an admissible control with values in the interval $[u_1, u_2]$, leading to the point $(x(t_f), y(t_f), \varphi(t_f))^{\mathrm{T}} \in G_\varphi(t_f)$, where $\varphi(t_f) = \varphi < 0$. Together with the original problem with constraints u_1 and u_2, we consider a new problem with constraints $\tilde{u}_1 = -u_2$ and $\tilde{u}_2 = -u_1$. We denote the three-dimensional reachable set obtained in it at the same value of t_f by $\widetilde{G(t_f)}$.

In the new task, we take the control $\tilde{u}(t) = -u(t)$, $t \in [0, t_f]$. We have $\tilde{u}_1 \leq \tilde{u}(t) \leq \tilde{u}_2$. The specificity of the equations of motion of the Dubins car is manifested in the fact that the current values $\tilde{\varphi}(t)$, $\tilde{x}(t)$ and $\tilde{y}(t)$ of the phase variables obtained at time t satisfy the relations $\widetilde{\varphi}(t) = -\varphi(t)$, $\widetilde{x}(t) = x(t)$, and $\widetilde{y}(t) = -y(t)$. In this case, $\widetilde{\varphi}(t_f) > 0$.

Thus, to find the set $G_\varphi(t_f)$ at $\varphi < 0$, we need to take the set $\widetilde{G_{\tilde{\varphi}}}(t_f)$ in a new task and reflect it about the x axis. In this way, the case $\varphi < 0$ in the original problem is reduced to the case $\widetilde{\varphi} > 0$ in the new task.

CONCLUSIONS

We study a three-dimensional reachability set for a nonlinear controllable object called the Dubins car. It is assumed that rotations are possible both to the left and to the right. The case with, generally speaking, asymmetric possibilities (tolerances) of such rotations is considered. The main results of the article are based on the PMP and are as follows.

The well-known theorem for the symmetric case on six types of piecewise constant controls with at most two switchings leading to the boundary of the reachable set can be extended to the nonsymmetric case. A similar theorem has also been proved for points from the interior of the reachability set (with decreasing tolerances for the left and right rotations).

The types of controls used at a fixed value of $\varphi \geq 0$ of the angular coordinate generate a continuous closed curve on the plane of geometric coordinates. The set of such curves, obtained by reducing the tolerances for rotations to the left and right, completely fills the considered φ-section. This makes it possible to justify $\varphi \geq 0$ for the formula of the calculation of the φ-section of the reachability set in the nonsymmetric case through the corresponding section in the canonical symmetric case. Herewith, in the canonical case, we take the same value of φ of the angular coordinate, but some other point in time for which the φ-section is calculated. The formula has the form of an affine transformation and defines a one-to-one correspondence of such φ-sections. The φ-sections at $\varphi < 0$ are analyzed based on the φ-sections for $\varphi > 0$.

As a result, the construction of the reachability set for the Dubins car in the asymmetric case is reduced to the canonical case. Due to the one-to-one correspondence obtained in this paper, this means that the presence of a description of the three-dimensional reachability set for some specific tolerances for the left and right rotations allows us to obtain a description of the reachability set for an arbitrary case with two-sided rotations.

The study of such a question in the case of a strictly unilateral rotation [21] (rotation is possible only in one direction with constraints from above and below on the instantaneous rotation radius) is also likely to have a positive answer. The authors will try to substantiate this in one of their following papers.

REFERENCES

1. L. E. Dubins, "On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents," Am. J. Math. **79** (3), 497–516 (1957).
2. A. A. Markov, "A few examples of the solution of a special type of problems on maximum and minimum values," Soobshch. Khar'kov. Mat. O-va, Vtoraya Ser. **1** (2), 250–276 (1889).
3. R. Isaacs, *Differential Games* (John Wiley and Sons, New York, 1965; Mir, Moscow, 1967).
4. V. S. Patsko and A. A. Fedotov, "Three-dimensional reachable set for the Dubins car: Foundation of analytical description," Commun. Optim. Theory **2022**, 23 (2022).
5. *Robot Motion Planning and Control*, Ed. by J.-P. Laumond (Springer, Berlin–Heidelberg, 1998).
6. S. M. LaValle, *Planning Algorithms* (Cambridge University Press, Cambridge, 2006).
7. M. E. Buzikov and A. A. Galyaev, "Time-optimal interception of a moving target by a Dubins car," Autom. Remote Control (Engl. Transl.) **82** (5) 745–758 (2021).
8. A. A. Ardentov, L. V. Lokutsievskii, and Yu. L. Sachkov, "Explicit solutions for a series of optimization problems with 2-dimensional control via convex trigonometry," Dokl. Math. **102** (1), 427–432 (2020).
9. S. P. Khabarov and M. L. Shilkina, "Geometric approach to solving the Dubins car problem with the formation of program motion trajectories," Nauchno-Tekh. Vestn. Inf. Tekhnol. Mekh. Optim. **21** (5), 653–663 (2021).
10. A. A. Zimovets, A. R. Matviichuk, A. V. Ushakov, and V. N. Ushakov, "Stability property in the convergence game problem in the presence of phase constraints," J. Comput. Syst. Sci. Int. **60** (4), 530–548 (2021).
11. A. S. Bortakovskii, "Fastest planar motion trajectories with unlimited curvature," J. Comput. Syst. Sci. Int. **61** (4) 512–522 (2022).
12. E. Bakolas and P. Tsiotras, "Optimal synthesis of the asymmetric sinistral/dextral Markov–Dubins problem," J. Optim. Theory Appl. **150** (2), 233–250 (2011).
13. A. Miele, *Flight Mechanics* (Dover, Mineola, N.Y., 1962; Nauka, Moscow, 1965).
14. T. Pecsvaradi, "Optimal horizontal guidance law for aircraft in the terminal area," IEEE Trans. Autom. Control **17** (6), 763–772 (1972).
15. V. S. Patsko, S. G. Pyatko, and A. A. Fedotov, "Three-dimensional reachability set for a nonlinear control system," J. Comput. Syst. Sci. Int. **42** (3), 320–328 (2003).
16. V. S. Patsko and A. A. Fedotov, "Analytical description of the reachability set for a Dubins car," Tr. Inst Mat. Mekh. Ural. Otd. Ross. Akad. Nauk **26** (1), 182–197 (2020).
17. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *Mathematical Theory of Optimal Processes* (Fizmatgiz, Moscow, 1983) [in Russian].
18. E. B. Lee and L. Markus, *Foundations of Optimal Control Theory* (Wiley, New York, 1967; Nauka, Moscow, 1972).
19. A. A. Tolstonogov, *Differential Inclusions in a Banach Space* (Kluwer, Dordrecht, 2000).
20. V. S. Patsko and A. A. Fedotov, "Reachability set at a certain time for a Dubins car in the case of a one-sided turn," Tr. Inst Mat. Mekh. Ural. Otd. Ross. Akad. Nauk **24** (1), 143–155 (2018).
21. V. S. Patsko and A. A. Fedotov, "The structure of the reachability set for the Dubins car with a strictly one-sided turn," Tr. Inst Mat. Mekh. Ural. Otd. Ross. Akad. Nauk **25** (3), 171–187 (2019).