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Chapter 12
Open-Loop Solvability Operator in Differential Games with Simple Motions in the Plane

Liudmila Kamneva and Valerii Patsko

Abstract The paper deals with an open-loop solvability operator in two-person zero-sum differential games with simple motions. This operator takes a given terminal set to the set defined at the initial instant whence the first player can bring the control system to the terminal set if the player is informed about the open-loop control of the second player. It is known that the operator possesses the semigroup property in the case of a convex terminal set. In the paper, sufficient conditions ensuring the semigroup property in the non-convex case are formulated and proved for problems in the plane. Examples are constructed to illustrate the relevance of the formulated conditions. The connection with the Hopf formula is analysed.

Keywords Planar differential games • Semigroup property • Simple motions • Open-loop solvability operator

12.1 Introduction

The paper concerns the simplest model description of dynamics in the differential game theory:

$$\dot{x} = p + q, \quad p \in P, \quad q \in Q.$$

The system has no state variable x at the right-hand side, and the state velocity $\dot{x}$ is defined only by controls $p \in P$ and $q \in Q$ of the first and second players, where

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the constraints P and Q do not depend on the time. In Isaacs (1965), games with such dynamics are called games with simple motions.

In numerical methods of the differential game theory, the simple motion dynamics arises absolutely naturally under a local approximation of linear or nonlinear dynamics when the capabilities of the players are "frozen" in the time and state variables. In the framework of the simple motion dynamics, one calculates the next step of an iterative procedure to construct the value function of the game.

For example, one important class of differential games consists of the games with linear dynamics, a fixed terminal time, and a continuous terminal payoff function. For these games, the transfer to new variables is known (Krasovskii and Subbotin 1974, pp. 159–161; Krasovskii and Subbotin 1988, pp. 89–91). The new variables can be regarded as forecasting the state variables to the terminal instant by the "free" motion of the system under zero controls of the players. The transfer is performed by the Cauchy matrix of the original problem. The new dynamic system has no state variables at the right-hand side, but the controls of the players are multiplied by coefficients depending on time.

Under the numerical construction of level sets of the value function, the time interval is divided by a step, and the coefficients of the dynamics are frozen on each small time interval (Botkin 1984; Isakova et al. 1984). So, at each step we get some dynamics of simple motions. Being given a level set of the payoff function as the terminal set and going backward from the terminal set, one recalculates the level set at each time step using the dynamics of simple motions. Then one passes to the limit as the step of the partition goes to zero. If the operator of recalculation is properly chosen (at one step), then the limit set coincides with the level set (Lebesgue set) of the value function.

This is the scheme. To perform it effectively, it is very important to choose an operator to use at each step of the backward iterative procedure. It is most desirable that the operator possesses the semigroup property: if the dynamics is frozen on some time interval, then the use of any additional points of the partition does not change the result of the iterative procedure.

We investigate the operator known as the programmed absorption operator in Russian literature on the differential game theory (Krasovskii and Subbotin 1974, p. 122). It can be called the open-loop solvability operator as well. For the simple motion games, the semigroup property was established earlier (Pshenichnyy and Sagaydak 1971) in the case of the operator dealing with convex sets. In this paper, sufficient conditions providing the semigroup property in the non-convex case are formulated and proved for the problems in the plane. Examples are constructed to illustrate the relevance of the formulated conditions. In the appendix, we describe the connection between the question under investigation and the Hopf formula known in the differential game theory and the theory of partial-differential equations.

The results obtained in the paper can be useful for developments and justifications of numerical methods in the differential game theory.

12.2 Problem Statement: Open-Loop Solvability Operator

Consider a conflict-control dynamic system with simple motions (Isaacs 1965):

$$\frac{dx}{dt} = p + q, \quad p \in P, \quad q \in Q, \quad x \in \mathbb{R}^n. \tag{12.1}$$

Here, $t \in [0, \vartheta]$; p, q are controls of the first and second players; P, Q are convex compact sets in $\mathbb{R}^n$. Let M be a compact terminal set in $\mathbb{R}^n$.

For a differential game, the notion of the maximal stable bridge $W_0 \subset [0, \vartheta] \times \mathbb{R}^n$ terminating at the instant ϑ on the set M (i.e., $W_0(\vartheta) = M$) was introduced in Krasovskii and Subbotin (1974, p. 67), Krasovskii and Subbotin (1988, p. 61). Here, the notation $W_0(t)$ for a t-section of the set W_0 is used:

$$W_0(t) = \{x \in \mathbb{R}^n : (t, x) \in W_0\}, \quad t \in [0, \vartheta].$$

To guarantee the inclusion $x(\vartheta) \in M$, the positional strategy of the first player can be constructed (Krasovskii and Subbotin 1974, 1988) by the procedure of extremal aiming to the maximal stable bridge W_0. The set W_0 coincides with the solvability set in the problem of guidance over non-anticipating strategies (Bardi and Capuzzo-Dolcetta 1997; Subbotin 1995). The notion of the maximal stable bridge W_0 is very close to the notion of the viability kernel (Aubin 1991; Cardaliaguet et al. 1999), and its t-section $W_0(t)$ is well known as the alternating Pontryagin integral (Pontryagin 1967, 1981).

In the case of a convex set M, the Pshenichnyi formula describing constructively the sections $W_0(t)$, $t \in [0, \vartheta]$, is known (Pshenichnyy and Sagaydak 1971):

$$W_0(t) = (M - (\vartheta - t)P) \overset{*}{-} (\vartheta - t)Q. \tag{12.2}$$

Here, operations of the algebraic sum (the Minkowski sum) $A + B$ and the geometric difference (the Minkowski difference) $A \overset{*}{-} B$ of the sets $A, B \subset \mathbb{R}^n$ are used (see, for example, Hadwiger (1957); Polovinkin and Balashov (2004); Pontryagin (1967, 1981)):

$$A + B := \{d \in \mathbb{R}^n : \ d = a + b, \ a \in A, \ b \in B\},$$

$$A \overset{*}{-} B := \{d \in \mathbb{R}^n : \ d + B \subseteq A\} = \bigcap_{b \in B} (A - b).$$

The set $A + B$ is convex if the both sets A and B are convex. The set $A \overset{*}{-} B$ is convex in the case of a convex set A.

Define the open-loop solvability operator (the programmed absorption operator):

$$M \mapsto T_\tau(M) := (M - \tau P) \overset{*}{-} \tau Q, \quad \tau = \vartheta - t.$$

By (12.2), for a convex set M, we have

$$W_0(t) = T_{\vartheta - t}(M). \tag{12.3}$$

It is of interest to try to establish some conditions providing equality (12.3) in the case of a non-convex set M.

For any compact (generally speaking, non-convex) set M, the representation

$$W_0(t) = \bigcap_{\tau_1+\tau_2+\cdots+\tau_m=\vartheta-t,\ m\in\mathbb{N}} T_{\tau_1}(T_{\tau_2}(\ldots T_{\tau_m}(M)\ldots)) =: \tilde{T}_{\vartheta-t}(M)$$

is true (Pshenichnyy and Sagaydak 1971). Its right-hand side defines the operator with multiple recomputations:

$$M \mapsto \tilde{T}_\tau(M), \quad \tau = \vartheta - t.$$

Therefore, the operators T_τ and $\tilde{T}_\tau$ are equal (i.e., $T_\tau(M) = \tilde{T}_\tau(M)$ for all $\tau \in [0, \vartheta]$) if, for any $\tau_1, \tau_2 > 0$ such that $\tau_1 + \tau_2 \le \vartheta$, the following relation holds:

$$T_{\tau_1+\tau_2}(M) = T_{\tau_1}(T_{\tau_2}(M)). \tag{12.4}$$

Equality (12.4) is known as the semigroup property of the operator T_τ. In Pshenichnyy and Sagaydak (1971), the semigroup property was proved for any convex set M. This implies (12.2).

Thus, the question on the validity of (12.3) in the case of a non-convex set M is reduced to the formulation of conditions on the sets M, P, Q, and on the range of τ_1, τ_2 to provide equality (12.4).

12.3 Auxiliary Results

Let us remark two obvious properties:

$$T_\tau(M) = \bigcap_{q\in Q}(M - \tau(P + q)); \tag{12.5}$$

$$x \in T_\tau(M) \quad \Leftrightarrow \quad \forall q \in Q \quad (x + \tau(P + q)) \cap M \neq \varnothing. \tag{12.6}$$

The following two results are known (Pshenichnyy and Sagaydak 1971).

Lemma 12.1.

$$T_{\tau_1}(T_{\tau_2}(M)) \subseteq T_{\tau_1+\tau_2}(M). \tag{12.7}$$

Proof. Fix $x \in T_{\tau_1}(T_{\tau_2}(M))$. By (12.6), for any $q \in Q$ there exists $p_1 \in P$ such that

$$x + \tau_1 q + \tau_1 p_1 \in T_{\tau_2}(M),$$

and there exists $p_2 \in P$ such that

$$z := (x + \tau_1 q + \tau_1 p_1) + \tau_2 q + \tau_2 p_2 \in M.$$

Since the set P is convex, the following inclusion holds:

$$p_* := \frac{\tau_1 p_1 + \tau_2 p_2}{\tau_1 + \tau_2} \in P.$$

We have

$$x + (\tau_1 + \tau_2)(p_* + q) = z \in M.$$

Thus, for any $q \in Q$ there exists $p_* \in P$ such that

$$x + (\tau_1 + \tau_2)(p_* + q) \in M.$$

Then by (12.6), we obtain $x \in T_{\tau_1+\tau_2}(M)$. □

Lemma 12.2. *Assume the set M is convex. Then*

$$T_{\tau_1+\tau_2}(M) = T_{\tau_1}(T_{\tau_2}(M)), \quad \tau_1, \tau_2 > 0.$$

Proof. By Lemma 12.1, it remains to prove that

$$T_{\tau_1+\tau_2}(M) \subseteq T_{\tau_1}(T_{\tau_2}(M)).$$

Let $x \in T_{\tau_1+\tau_2}(M)$. Then, because of the property (12.6), for any $q_1 \in Q$ there exists $p_1 \in P$ such that

$$z_1 := x + (\tau_1 + \tau_2)(p_1 + q_1) \in M. \tag{12.8}$$

Let us prove the inclusion

$$x + \tau_1(p_1 + q_1) \in T_{\tau_2}(M). \tag{12.9}$$

Fix $q_2 \in Q$. By the same arguments as in (12.8), we find $p_2 \in P$ such that

$$z_2 := x + (\tau_1 + \tau_2)(p_2 + q_2) \in M.$$

Considering the convexity of the set M, we have

$$x + \tau_1(p_1 + q_1) + \tau_2(p_2 + q_2) = \frac{\tau_1 z_1 + \tau_2 z_2}{\tau_1 + \tau_2} \in M.$$

Thus, inclusion (12.9) holds. Therefore, by (12.6), we get $x \in T_{\tau_1}(T_{\tau_2}(M))$. □

In addition, three lemmas formulated and proved below are required. Lemma 12.3 claims inequality (12.11), which, in particular, is necessarily true if $T_{\tau_1+\tau_2}(M) = T_{\tau_1}(T_{\tau_2}(M))$. Further, a similar condition is used in the main Theorem 12.1. In Lemma 12.4, the case of a convex set M is considered, and thus, by Lemma 12.2, the semigroup property is necessarily true. The proof is based on Lemma 12.3. The lemma is also used in the proof of Lemma 12.5, which is in its turn necessary for our proof of Theorem 12.2.

Let $\rho(\,\cdot\,, A)$ be a support function of a compact set $A \subset \mathbb{R}^n$, i.e.,

$$\rho(\eta\,, A) = \max\{\langle x, \eta\rangle \,:\, x \in A\}, \quad \eta \in \mathbb{R}^n.$$

Write

$$H(s) = \max_{q\in Q}\langle q, s\rangle + \min_{p\in P}\langle p, s\rangle, \quad s \in \mathbb{R}^n.$$

Lemma 12.3. *Fix* $\tau_1, \tau_2 > 0$*, and assume that the sets* $T_{\tau_2}(M)$*,* $T_{\tau_1}(T_{\tau_2}(M))$ *are nonempty,* $\eta \in \mathbb{R}^n$*, and*

$$\rho(\eta, T_{\tau_1+\tau_2}(M)) = \rho(\eta, T_{\tau_1}(T_{\tau_2}(M))). \tag{12.10}$$

Then

$$\rho(\eta, T_{\tau_1+\tau_2}(M)) + \tau_1 H(\eta) \le \rho(\eta, T_{\tau_2}(M)). \tag{12.11}$$

Proof. Since $T_{\tau_1}(T_{\tau_2}(M)) \subseteq T_{\tau_1+\tau_2}(M)$, we have $T_{\tau_1+\tau_2}(M) \neq \varnothing$.

The definition of the set $T_{\tau_1}(T_{\tau_2}(M))$ implies the inclusion

$$T_{\tau_1}(T_{\tau_2}(M)) + \tau_1 Q \subseteq T_{\tau_2}(M) - \tau_1 P.$$

Then

$$\rho(\eta, T_{\tau_1}(T_{\tau_2}(M))) + \tau_1 \max_{q\in Q}\langle q, \eta\rangle \le \rho(\eta, T_{\tau_2}(M)) + \tau_1 \max_{p\in P}\langle -p, \eta\rangle.$$

So, we get (12.11) by (12.10) and taking into account the equality

$$\max_{p\in P}\langle -p, \eta\rangle = -\min_{p\in P}\langle p, \eta\rangle.$$

□

Lemma 12.4. *Let M be a convex set. Assume that $\tau_1, \tau_2 > 0$ and the sets $T_{\tau_2}(M)$, $T_{\tau_1+\tau_2}(M)$ are nonempty. Then, for any $\eta \in \mathbb{R}^n$, inequality (12.11) holds.*

Proof. Since the set M is convex, Lemma 12.2 implies

$$T_{\tau_1+\tau_2}(M) = T_{\tau_1}(T_{\tau_2}(M)).$$

Then $\rho(\eta, T_{\tau_1}(T_{\tau_2}(M))) = \rho(\eta, T_{\tau_1+\tau_2}(M))$. Using Lemma 12.3, we have (12.11). □

Lemma 12.5. *Let $\eta \in \mathbb{R}^n$ and $\eta \neq 0$. Assume that there exists $z_* \in M$ such that the intersection $M \cap \Pi_*$ of the set M and the half-space*

$$\Pi_* = \{x \in \mathbb{R}^n : \langle x - z_*, \eta \rangle \leq 0\}$$

is convex and its interior is nonempty.

Then there exists $\vartheta > 0$ such that, for any $\tau \in [0, \vartheta]$, the set $T_\tau(M)$ is nonempty and the function

$$\tau \mapsto \delta_\eta(\tau) := \rho(-\eta, M) - \tau H(-\eta) - \rho(-\eta, T_\tau(M))$$

increases on $[0, \vartheta]$.

Proof.

1) Define

$$\mu_* = \langle z_*, -\eta \rangle.$$

Observe that $\mu_* < \rho(-\eta, M)$. Choose any $\mu \in (\mu_*, \rho(-\eta, M))$, and write

$$\Pi_\mu = \Pi_* - (\mu - \mu_*)\eta/\|\eta\|.$$

Since the interior of the intersection $M \cap \Pi_\mu$ is nonempty (in view of the fact that the set $M \cap \Pi_*$ is convex and its interior is nonempty), for rather small $\tau > 0$, we get

$$T_\tau(M) \cap \Pi_\mu \neq \varnothing. \tag{12.12}$$

Thus, there exists $\tau_1^* > 0$ such that, for any $\tau \in [0, \tau_1^*]$, we have (12.12).

Set

$$\alpha = \min_{p \in P} \min_{q \in Q} \langle p + q, -\eta \rangle.$$

We are able to choose a value $\tau_2^* > 0$ such that

$$\tau\alpha \geq \mu_* - \mu, \quad \tau \in (0, \tau_2^*].$$

Indeed, since $\mu_* - \mu < 0$, any $\tau_2^* > 0$ can be taken if $\alpha \geq 0$; otherwise, we choose any sufficiently small $\tau_2^* > 0$.

Set $\vartheta = \min\{\tau_1^*, \tau_2^*\}$. (Thus, the value ϑ depends on the choice of μ.)

2) Fix $\tau \in [0, \vartheta]$. Let us show that

$$T_\tau(M) \cap \varPi_\mu \subseteq T_\tau(M \cap \varPi_*). \tag{12.13}$$

Choose $x \in T_\tau(M) \cap \varPi_\mu$. By (12.6), for any $q \in Q$ there exists $p_* \in P$ such that $x + \tau q + \tau p_* \in M$. Since $x \in \varPi_\mu$, we obtain $\langle x, -\eta\rangle \geq \mu$. Therefore, considering the choice of the value μ and the definition of the value α, we deduce

$$-\langle x, -\eta\rangle + \mu_* \leq -\mu + \mu_* \leq \tau\alpha \leq \tau\langle q + p_*, -\eta\rangle.$$

Hence $\langle x + \tau q + \tau p_*, -\eta\rangle \geq \mu_*$, i.e., $x + \tau q + \tau p_* \in \varPi_*$. Thus,

$$x + \tau q + \tau p_* \in M \cap \varPi_*,$$

and, consequently, $x + \tau q \in (M \cap \varPi_*) - \tau P$. Since $q \in Q$ is chosen arbitrarily, in view of (12.5), we get

$$x \in \bigcap_{q \in Q} \big((M \cap \varPi_*) - \tau(P + q)\big) = T_\tau(M \cap \varPi_*).$$

3) Fix $\tau_2, \tau_1 + \tau_2 \in [0, \vartheta]$. By (12.12), we have

$$T_{\tau_2}(M) \neq \varnothing, \quad T_{\tau_1+\tau_2}(M) \cap \varPi_\mu \neq \varnothing.$$

Considering (12.13), the convexity of the set $M \cap \varPi_*$, Lemma 12.2, and the monotonicity of T_τ, we calculate

$$T_{\tau_1+\tau_2}(M) \cap \varPi_\mu \subseteq T_{\tau_1+\tau_2}(M \cap \varPi_*) = T_{\tau_1}(T_{\tau_2}(M \cap \varPi_*)) \subseteq T_{\tau_1}(T_{\tau_2}(M)).$$

This implies that $T_{\tau_1}(T_{\tau_2}(M)) \neq \varnothing$ and

$$\rho(-\eta, T_{\tau_1+\tau_2}(M) \cap \varPi_\mu) \leq \rho(-\eta, T_{\tau_1}(T_{\tau_2}(M))). \tag{12.14}$$

Since $\varPi_\mu$ is a half-space with an outward normal vector η and the intersection $T_{\tau_1+\tau_2}(M) \cap \varPi_\mu$ is nonempty, we find

$$\rho(-\eta, T_{\tau_1+\tau_2}(M)) = \rho(-\eta, T_{\tau_1+\tau_2}(M) \cap \varPi_\mu).$$

We employ this identity in (12.14) to obtain the inequality

$$\rho(-\eta, T_{\tau_1+\tau_2}(M)) \leq \rho(-\eta, T_{\tau_1}(T_{\tau_2}(M))).$$

On the other hand, Lemma 12.1 implies the opposite inequality. So, equation (12.10) holds.

Hence, considering Lemma 12.3, we get inequality (12.11), which is equivalent to the inequality $\delta_\eta(\tau_1 + \tau_2) \geq \delta_\eta(\tau_2)$. Therefore, the function $\delta_\eta(\cdot)$ increases on the segment $[0, \vartheta]$. □

12.4 The Main Theorem

Now, we deal with the case of $\mathbb{R}^2$.

A set A is called *arcwise connected* (Schwartz 1967) (in the sequel, *connected* for brevity) if any two distinct points of the set A can be joined by a simple curve (arc) which lies in the set.

A set $A \subset \mathbb{R}^2$ is called *simply connected* (Schwartz 1967) if any simple closed curve can be shrunk to a point continuously in the set, i.e., the set consists of one piece and does not have any "holes."

A polygon is defined as a plane figure that is bounded by a closed path composed of a finite sequence of straight line segments (edges of the polygon).

Let us denote by $\mathscr{V}_A$ the set of all outward normal unit vectors to the edges of the polygon A. If A is a segment, we suppose that the set $\mathscr{V}_A$ consists of two opposite directed vectors that are normal to the segment A.

Let us formulate the main theorem.

Theorem 12.1. *Assume that*

(A1) $M \subset \mathbb{R}^2$ *is a simply connected compact set;*
(A2) $P \subset \mathbb{R}^2$ *is either a non-degenerate segment, or a convex polygon ;* $Q \subset \mathbb{R}^2$ *is a convex compact set;*
(A3) *for any* $x \in \mathbb{R}^2$ *and* $\nu \in \mathscr{V}_P$*, the set*

$$\Pi_M(x, \nu) = M \cap \{z \in \mathbb{R}^2 : \langle z, \nu \rangle \leqslant \langle x, \nu \rangle\}$$

is connected;
(A4) *for any* $\tau \in [0, \vartheta]$*, the set* $T_\tau(M)$ *is nonempty and connected;*
(A5) *for any* $\nu \in \mathscr{V}_P$*, the function*

$$\tau \mapsto \delta_\nu(\tau) := \rho(-\nu, M) - \tau H(-\nu) - \rho(-\nu, T_\tau(M))$$

increases on the segment $[0, \vartheta]$.

Then the operator T_τ *possesses the semigroup property on the segment* $[0, \vartheta]$. *(And, consequently,* $W_0(t) = T_{\vartheta - t}(M)$*,* $t \in [0, \vartheta]$.*)*

Our proof of Theorem 12.1 is based on Lemma 12.6. To formulate the lemma, let us introduce the following notations.

For the set $T_\tau(M) \neq \varnothing$, we define "an envelope set"

$$\mathrm{env}\,(T_\tau(M)) = \bigcap_{\nu \in \mathcal{V}_P} \{x \in \mathbb{R}^2 : \langle x, -\nu \rangle \leq \rho(-\nu, T_\tau(M))\}.$$

Note that $T_\tau(M) \subset \mathrm{env}\,(T_\tau(M))$. If P is a segment, then $\mathrm{env}\,(T_\tau(M))$ is a closed strip; if P is a polygon, then $\mathrm{env}\,(T_\tau(M))$ is a convex polygon.

Let $\mathscr{P}$ be the set of vertices of the segment or polygon P. For a vertex $p \in \mathscr{P}$, define a bundle of unit vectors

$$\mathscr{N}(p) = \big\{(p - x)/\|p - x\| : \ x \in P \setminus \{p\}\big\}.$$

If P is a segment, then the set $\mathscr{P}$ consists of two vertices, and the set $\mathscr{N}(p)$ consists of a unique vector for $p \in \mathscr{P}$.

Let $l(a, \eta)$ be a ray with the initial point $a \in \mathbb{R}^2$ and the direction along the vector $\eta \in \mathbb{R}^2$:

$$l(a, \eta) = \{a + \alpha\eta : \ \alpha \geq 0\}.$$

Lemma 12.6. *Assume that $\tau_1, \tau_2 > 0$, the sets $T_{\tau_2}(M)$ and $T_{\tau_1+\tau_2}(M)$ are nonempty, and the following conditions hold:*

(L1) *if $y \in \mathbb{R}^2$, $q_1 \in Q$, and*

$$(y + \tau_1(P + q_1)) \cap T_{\tau_2}(M) = \varnothing, \quad (y + \tau_1(P + q_1)) \cap \mathrm{env}\,(T_{\tau_2}(M)) \neq \varnothing,$$

then there exist $p_* \in \mathscr{P}$ and $q_2 \in Q$ such that

$$\forall \eta \in \mathscr{N}(p_*) \quad l(y + \tau_1(p_* + q_1) + \tau_2(p_* + q_2), -\eta) \cap M = \varnothing;$$

(L2) *for any $\nu \in \mathcal{V}_P$, we have*

$$\rho(-\nu, T_{\tau_1+\tau_2}(M)) + \tau_1 H(-\nu) \leq \rho(-\nu, T_{\tau_2}(M)).$$

Then

$$T_{\tau_1+\tau_2}(M) = T_{\tau_1}(T_{\tau_2}(M)). \tag{12.15}$$

Proof. Suppose that (12.15) is false. Then, by Lemma 12.1, we can find

$$y \in T_{\tau_1+\tau_2}(M) \setminus T_{\tau_1}(T_{\tau_2}(M)) \neq \varnothing.$$

Since $y \notin T_{\tau_1}(T_{\tau_2}(M))$, by (12.6), we find $q_1 \in Q$ such that

$$G_1 \cap T_{\tau_2}(M) = \varnothing, \quad G_1 := y + \tau_1(P + q_1). \tag{12.16}$$

1) Assume

$$G_1 \cap \operatorname{env}(T_{\tau_2}(M)) \neq \varnothing. \tag{12.17}$$

a) We now assert that there exists $q_2 \in Q$ such that

$$(G_1 + \tau_2(P + q_2)) \cap M = \varnothing. \tag{12.18}$$

Indeed, using (12.16), (12.17), and condition (L1), we find $p_* \in \mathscr{P}$ and $q_2 \in Q$ such that

$$\forall \eta \in \mathscr{N}(p_*) \quad l(b, -\eta) \cap M = \varnothing, \quad b := y + \tau_1(p_* + q_1) + \tau_2(p_* + q_2). \tag{12.19}$$

For any $z \in G_1$ we have the representation

$$z = y + \tau_1(\bar{p} + q_1), \quad \bar{p} \in P.$$

In addition, for any $p \in P$, we can write

$$z + \tau_2(p + q_2) = b - \eta_*, \quad \eta_* := \tau_1(p_* - \bar{p}) + \tau_2(p_* - p).$$

We have

$$\frac{p_* - \bar{p}}{\|p_* - \bar{p}\|} \in \mathscr{N}(p_*) \quad (p_* \neq \bar{p}), \qquad \frac{p_* - p}{\|p_* - p\|} \in \mathscr{N}(p_*) \quad (p \neq p_*).$$

Therefore, if $\eta_* \neq 0$, then $\eta_*/\|\eta_*\| \in \mathscr{N}(p_*)$. Considering (12.19), we get

$$z + \tau_2(p + q_2) \notin M = \varnothing.$$

Hence (12.18) holds.

b) Set $\tilde{q} = (\tau_1 q_1 + \tau_2 q_2)/(\tau_1 + \tau_2)$. Then

$$y + (\tau_1 + \tau_2)(P + \tilde{q}) = y + \tau_1(P + q_1) + \tau_2(P + q_2) = G_1 + \tau_2(P + q_2).$$

Using (12.18), we get $(y + (\tau_1 + \tau_2)(P + \tilde{q})) \cap M = \varnothing$. By (12.6), we conclude $y \notin T_{\tau_1+\tau_2}(M)$, that contradicts to our choice of y.

2) Assume $G_1 \cap \operatorname{env}(T_{\tau_2}(M)) = \varnothing$. Then, using the definition of the operator env, we write

$$G_1 \subseteq \bigcup_{\nu \in \mathscr{V}_P} \{x \in \mathbb{R}^2 : \langle x, -\nu \rangle > \rho(-\nu, T_{\tau_2}(M))\}.$$

Since G_1 is either a non-degenerate segment, or a convex polygon, and $\mathscr{V}_P$ is the set of outward normals to G_1, we deduce that there exists $\nu_0 \in \mathscr{V}_P$ such that

$$\forall z \in G_1 \quad \langle z, \ -\nu_0 \rangle > \rho(-\nu_0, T_{\tau_2}(M)). \tag{12.20}$$

Remark also that the inclusion $y \in T_{\tau_1+\tau_2}(M)$ implies the inequality

$$\langle y, -\nu_0 \rangle \le \rho(-\nu_0, T_{\tau_1+\tau_2}(M)). \tag{12.21}$$

Suppose

$$p_0 \in \operatorname{Arg} \max_{p \in P} \langle p, \nu_0 \rangle, \quad z_0 := y + \tau_1(p_0 + q_1).$$

Since $z_0 \in G_1$, using (12.20), (12.21), and the relations

$$\langle p_0, -\nu_0 \rangle = \min_{p \in P} \langle p, -\nu_0 \rangle, \quad \langle q_1, -\nu_0 \rangle \le \max_{q \in Q} \langle q, -\nu_0 \rangle,$$

we calculate

$$\begin{aligned} \rho(-\nu_0, T_{\tau_2}(M)) < \langle z_0, -\nu_0 \rangle &= \langle y, -\nu_0 \rangle + \tau_1 \langle p_0 + q_1, -\nu_0 \rangle \\ &\le \rho(-\nu_0, T_{\tau_1+\tau_2}(M)) + \tau_1 H(-\nu_0), \end{aligned}$$

that contradicts to condition (L2).

So, assuming the violation of (12.15), we obtain the contradictions in the both cases 1) and 2). □

Proof (of Theorem 12.1). Choose $\tau_2, \tau_1 + \tau_2 \in (0, \vartheta]$. To prove the equality

$$T_{\tau_1+\tau_2}(M) = T_{\tau_1}(T_{\tau_2}(M)),$$

check conditions (L1) and (L2) of Lemma 12.6.

For any $\nu \in \mathscr{V}_P$, the increase of the function $\delta_\nu(\cdot)$ on the segment $[0, \vartheta]$ implies the inequality $\delta_\nu(\tau_1 + \tau_2) \ge \delta_\nu(\tau_2)$, which is equivalent to the inequality in (L2).

Let us verify condition (L1). Fix $y \in \mathbb{R}^2$ and $q_1 \in Q$. Set $G_1 = y + \tau_1(P + q_1)$, and assume the following conditions hold:

$$G_1 \cap T_{\tau_2}(M) = \varnothing, \tag{12.22}$$

$$G_1 \cap \operatorname{env}(T_{\tau_2}(M)) \ne \varnothing. \tag{12.23}$$

Consider the following two cases: P is a non-degenerate segment and P is a convex polygon.

I. Let P be a non-degenerate segment.

Note that the set $\mathscr{P}$ is two-element (vertices of the segment P), and for any $p \in \mathscr{P}$, the set $\mathscr{N}(p)$ consists of a unique vector.

Since G_1 is a segment, which is parallel to P, and the set env $(T_{\tau_2}(M))$ is a strip, which is parallel to P, inequality (12.23) implies

$$G_1 \subset \text{ env}\,(T_{\tau_2}(M)).$$

Let us remark that the boundary of the set env $(T_{\tau_2}(M))$ is formed by two supporting lines of the connected set $T_{\tau_2}(M)$. Consequently, in view of (12.22), we can find a vertex $a_* := y + \tau_1(p_* + q_1)$, $p_* \in \mathscr{P}$, of the segment G_1 such that

$$l(a_*, \eta_*) \cap T_{\tau_2}(M) \neq \varnothing, \quad \mathscr{N}(p_*) = \{\eta_*\}. \tag{12.24}$$

Since $a_* \notin T_{\tau_2}(M)$, remembering (12.6), we find $q_2 \in Q$ such that

$$(a_* + \tau_2(P + q_2)) \cap M = \varnothing. \tag{12.25}$$

Besides, (12.24) implies that

$$\exists \alpha > 0 : \;\; a_* + \alpha\eta_* \in T_{\tau_2}(M).$$

Thus, due to (12.6), we have

$$(a_* + \alpha\eta_* + \tau_2(P + q_2)) \cap M \neq \varnothing. \tag{12.26}$$

Eqs. Define $b_* := a_* + \tau_2(p_* + q_2)$. In view of (12.25), (12.26), and $b_* \in a_* + \tau_2(P + q_2)$, we find

$$l(b_*, \eta_*) \cap M \neq \varnothing.$$

To get property (L1), it remains to prove that

$$l(b_*, -\eta_*) \cap M = \varnothing.$$

Assume the converse. Then we can find

$$x^* \in l(b_*, -\eta_*) \cap M.$$

Choose also $x_* \in l(b_*, \eta_*) \cap M$. Due to

$$l(b_*, \pm\eta_*) \subset \Pi_M(b_*, \nu) \cap \Pi_M(b_*, -\nu), \quad \nu \in \mathscr{V}_P,$$

the assumption (A3) implies that there exist continuous arcs $\gamma_+ \subset \Pi_M(b_*, \nu)$ and $\gamma_- \subset \Pi_M(b_*, -\nu)$ connecting the points x_* and x^*. As a result, the complex arc

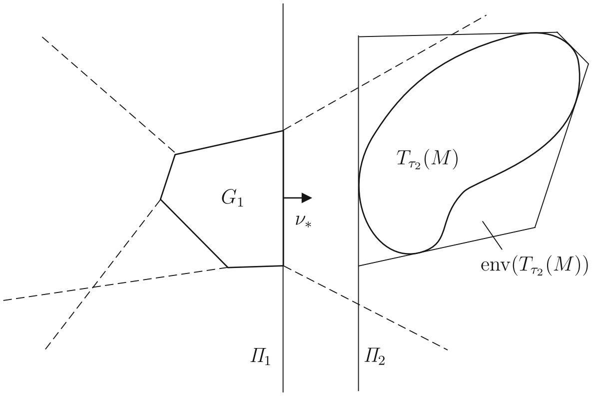


Fig. 12.1 Illustration to the proof of (12.29)

$\gamma_-\gamma_+ \subset M$ encircles the segment $a_* + \tau_2(P + q_2)$. Remembering (12.25), we get the contradiction with the simple connectedness of the set M. This contradiction completes the proof of property (L1).

II. Let P be a convex polygon.

1) We next show that there exists $p_* \in \mathscr{P}$ such that

$$\forall \eta \in \mathscr{N}(p_*) \quad l(y + \tau_1(p_* + q_1), \eta) \cap T_{\tau_2}(M) \neq \varnothing. \tag{12.27}$$

Assume the converse, i.e.,

$$\forall\, p \in \mathscr{P} \quad \exists\, \eta \in \mathscr{N}(p): \quad l(y + \tau_1(p + q_1), \eta) \cap T_{\tau_2}(M) = \varnothing. \tag{12.28}$$

Since G_1 is a convex polygon, we write

$$G_1 = \bigcap_{\nu \in \mathscr{V}_P} \{z \in \mathbb{R}^2 \colon \langle z, \nu \rangle \leq \rho(\nu, G_1)\}.$$

In view of (12.22), we have

$$T_{\tau_2}(M) \subset \mathbb{R}^2 \setminus G_1 = \bigcup_{\nu \in \mathscr{V}_P} \{z \in \mathbb{R}^2 \colon \langle z, \nu \rangle > \rho(\nu, G_1)\}.$$

Using (12.28) and the connectedness of the set $T_{\tau_2}(M)$, we find $\nu_* \in \mathscr{V}_P$ (Fig. 12.1) such that

$$T_{\tau_2}(M) \subset \{z \in \mathbb{R}^2 \colon \langle z, \nu_* \rangle > \rho(\nu_*, G_1)\}. \tag{12.29}$$

Define

$$\Pi_1 := \{z \in \mathbb{R}^2 : \langle z, \nu_* \rangle \le \rho(\nu_*, G_1)\}.$$

Therefore

$$\mathbb{R}^2 \setminus \Pi_1 = \{z \in \mathbb{R}^2 : \langle z, -\nu_* \rangle < -\rho(\nu_*, G_1)\}.$$

Because of (12.29), we have

$$\Pi_2 := \{z \in \mathbb{R}^2 : \langle z, -\nu_* \rangle \le \rho(-\nu_*, T_{\tau_2}(M))\} \subset \mathbb{R}^2 \setminus \Pi_1.$$

The last formula and the definition of the operator env imply

$$\text{env}\,(T_{\tau_2}(M)) \subset \Pi_2 \subset \mathbb{R}^2 \setminus \Pi_1.$$

Remembering that $G_1 \subset \Pi_1$, we have

$$\text{env}\,(T_{\tau_2}(M)) \cap G_1 = \varnothing,$$

that contradicts to (12.23). Thus, (12.27) holds.

2) Set $a_* = y + \tau_1(p_* + q_1)$. In view of (12.22), we get $a_* \notin T_{\tau_2}(M)$. Using (12.6), we find $q_2 \in Q$ such that

$$G_2 \cap M = \varnothing, \quad G_2 := a_* + \tau_2(P + q_2). \tag{12.30}$$

Besides, (12.27) implies that

$$\forall \eta \in \mathcal{N}(p_*) \quad \exists \alpha_\eta > 0 : \quad a_* + \alpha_\eta \eta \in T_{\tau_2}(M).$$

So, considering (12.6), we have

$$\forall \eta \in \mathcal{N}(p_*) \quad \exists \alpha_\eta > 0 : \quad (G_2 + \alpha_\eta \eta) \cap M \ne \varnothing. \tag{12.31}$$

For the chosen p_* and q_2, let us prove that

$$\forall \eta \in \mathcal{N}(p_*) \quad l(a_* + \tau_2(p_* + q_2), -\eta) \cap M = \varnothing. \tag{12.32}$$

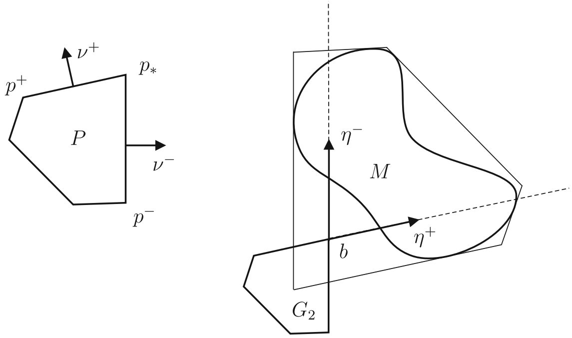
Let p^+ and p^- be the vertices of the polygon P adjoining the vertex p_* such that going around the vertices p^-, p_*, p^+ is counterclockwise (Fig. 12.2).

We have $p^+ \ne p^-$. Set

$$b = a_* + \tau_2(p_* + q_2), \quad \eta^\pm = p_* - p^\pm.$$

By ν^+ (ν^-) denote the outward normal vector to the edge of P that has its vertices at the points p_* and p^+ (respectively, p_* and p^-).

Fig. 12.2 Illustration to the verification of condition (L1)



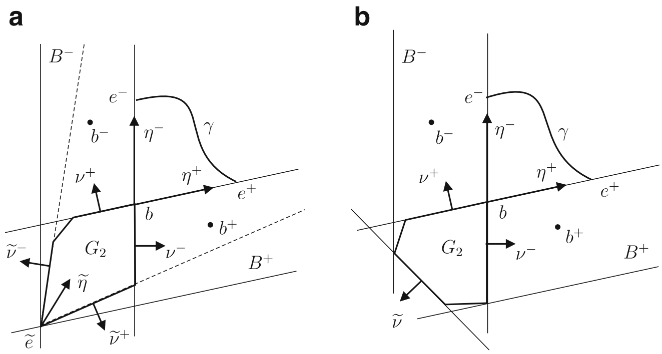


Fig. 12.3 Step 1 of the proof of (12.32)

The proof of (12.32) is divided into two steps. In the first step, we assert an auxiliary statement. At the second step, assuming that (12.32) does not hold, we obtain a contradiction with the simple connectedness of the set M.

Step 1. We claim that there exists a continuous arc γ (Fig. 12.3), which connects some points $e^+ \in l(b, \eta^+)$ and $e^- \in l(b, \eta^-)$, and the inclusion

$$\gamma \subset (b + K) \cap M \tag{12.33}$$

holds, where $K := \{\alpha\eta\colon\ \eta \in \mathcal{N}(p_*),\ \alpha \geq 0\}$.

Define

$$B^\pm = \{z + \alpha\eta^\pm :\ \alpha > 0,\ z \in G_2\} \setminus G_2.$$

Since $G_2 \cap M = \varnothing$ and (12.31) holds for $\eta = \eta^\pm/\|\eta^\pm\|$, we have $B^\pm \cap M \neq \varnothing$. Fix any $b^\pm \in B^\pm \cap M$. As $B^+ \cap B^- = \varnothing$, we conclude that $b^+ \neq b^-$.

a) Suppose

$$\operatorname{Arg}\min_{z\in G_2}\langle z, \nu^+\rangle = \operatorname{Arg}\min_{z\in G_2}\langle z, \nu^-\rangle =: E.$$

In this case, the convexity of G_2 implies that the set E consists of a unique vector, i.e., $E = \{\tilde{e}\}$, and $\tilde{e}$ is a vertex of the polygon G_2 (Fig. 12.3a).

By $\tilde{\nu}^+$ and $\tilde{\nu}^-$ denote the outward normals to the edges adjoining the vertex $\tilde{e}$. Assume that the normals are chosen in such a way that the counterclockwise angle from $\tilde{\nu}^-$ to $\tilde{\nu}^+$ is less than π. Set

$$\tilde{K} := \{\alpha(z - \tilde{e}) :\ \alpha \geq 0,\ z \in G_2\}.$$

Note that $\tilde{K} \subset K$, $\partial\tilde{K} \cap \partial K = \{0\}$, and

$$b^{\pm} \in B^{\pm} \subset \Pi_M(\tilde{e}, \tilde{\nu}^{\mp}). \tag{12.34}$$

Besides, assumption (A3) implies that the set $\Pi_M(\tilde{e}, \tilde{\nu}^{\pm})$ is connected.

Fix $\tilde{\eta} \in \tilde{K}$. Applying (12.31) for $\eta = \tilde{\eta}/\|\tilde{\eta}\|$, we can find $\tilde{\alpha} > 0$ such that

$$(G_2 + \tilde{\alpha}\tilde{\eta}) \cap M \neq \varnothing.$$

Choose $c \in (G_2 + \tilde{\alpha}\tilde{\eta}) \cap M$. Note that

$$c \in \Pi_M(\tilde{e}, \tilde{\nu}^{\pm}). \tag{12.35}$$

Consider the two possible cases: (i) $c \in (B^+ \cup B^-)$; (ii) $c \notin (B^+ \cup B^-)$.

(i) Suppose $c \in B^{\pm}$. Using (12.34), (12.35), and the connectedness of the set $\Pi_M(\tilde{e}, \tilde{\nu}^{\pm})$, we conclude that there exists a continuous arc $\gamma_1 \subset \Pi_M(\tilde{e}, \tilde{\nu}^{\pm})$ connecting the points c and $b^{\mp}$. In the set $\Pi_M(\tilde{e}, \tilde{\nu}^{\pm})$, the points c and $b^{\mp}$ are separated by the set $G_2 \cup (b + K)$. Applying (12.30), from the arc γ_1 we can single out the required continuous arc γ without self-intersections which lies in the set $b + K$ and connects some points $e^+ \in l(b, \eta^+)$ and $e^- \in l(b, \eta^-)$.

(ii) Suppose $c \notin (B^+ \cup B^-)$. In this case, the point c belongs to the interior of the set $b + K$. We have

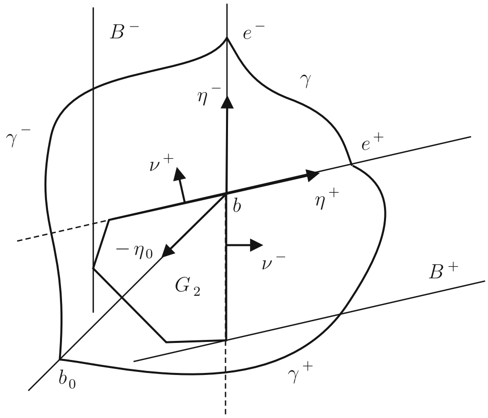
$$c,\ b^+ \in \Pi_M(\tilde{e}, \tilde{\nu}^-), \quad c,\ b^- \in \Pi_M(\tilde{e}, \tilde{\nu}^+).$$

Using the connectedness of the sets $\Pi_M(\tilde{e}, \tilde{\nu}^+)$ and $\Pi_M(\tilde{e}, \tilde{\nu}^-)$, we get that there exists a continuous arc $\gamma_1 \subset \Pi_M(\tilde{e}, \tilde{\nu}^+)$ connecting the points c and b^+, and there exists a continuous arc $\gamma_2 \subset \Pi_M(\tilde{e}, \tilde{\nu}^-)$ connecting the points b^- and c. By (12.30), from the complex arc $\gamma_1\gamma_2$ we can single out the required continuous arc γ without self-intersections which lies in the set $b + K$ and connects some points $e^+ \in l(b, \eta^+)$ and $e^- \in l(b, \eta^-)$.

b) It remains to consider the case (Fig. 12.3b)

$$\operatorname{Arg}\min_{z \in G_2}\langle z, \nu^+\rangle \neq \operatorname{Arg}\min_{z \in G_2}\langle z, \nu^-\rangle.$$

Fig. 12.4 Step 2 of the proof of (12.32)



In this case, we can find $\tilde{\nu} \in \mathscr{V}_P$ such that

$$B^{\pm} \subset \Pi_3 := \{z \in \mathbb{R}^2 : \langle z, \tilde{\nu} \rangle \leq \rho(\tilde{\nu}, G_2)\}.$$

Since $b^+, b^- \in \Pi_3$, assumption (A3) implies that there exists a continuous arc $\gamma_1 \subset \Pi_3 \cap M$ connecting the points b^+ and b^-. In the half-plain Π_3, the points b^+ and b^- are separated by the set $G_2 \cup (b + K)$. By (12.30), from the arc γ_1 we can single out the required continuous arc γ without self-intersections which lies in the set $b + K$ and connects some points $e^+ \in l(b, \eta^+)$ and $e^- \in l(b, \eta^-)$.

Thus, there exists the arc γ with the required properties.

Step 2. Suppose that (12.32) is false, i.e.,

$$\exists\ \eta_0 \in \mathscr{N}(p_*) : \quad l(b, -\eta_0) \cap M \neq \varnothing.$$

Choose $b_0 \in l(b, -\eta_0) \cap M$ (Fig. 12.4).

Let us construct a continuous closed arc without self-intersections which lies in the set M and encircles the set G_2. We have

$$e^+, b_0 \in \Pi_M(b, \nu^+), \quad e^-, b_0 \in \Pi_M(b, \nu^-).$$

By assumption (A3), there exists a continuous arc $\gamma^+ \subset \Pi_M(b, \nu^+)$ connecting the points b_0 and e^+, and there exists a continuous arc $\gamma^- \subset \Pi_M(b, \nu^-)$ connecting the points e^- and b_0. Without loss of generality, we can suppose that the arcs γ^+ and γ^- have no self-intersections.

The complex arc $\gamma^+\gamma\gamma^- \subset M$ is continuous and closed; it has no self-intersections and encircles the set G_2. Using (12.30), we obtain a contradiction with the connectedness of the set M. Thus, relation (12.32) is proved, i.e., condition (L1) holds. □

12.5 The Case of Polygon M

Assumptions (A1)–(A3) of Theorem 12.1 concern only the sets M and P; they are geometric and easy to verify. Assumptions (A4)–(A5) deal with the segment $[0, \vartheta]$.

Theorem 12.2 formulated below (based on Lemma 12.5) asserts that if M is a polygon with some geometric property with respect to P, then there exists some interval of τ where assumptions (A4)–(A5) hold.

Theorem 12.2. *Assume that*

(A1)* $M \subset \mathbb{R}^2$ *is a polygon;*
(A2) $P \subset \mathbb{R}^2$ *is either a non-degenerate segment or a convex polygon;*
$Q \subset \mathbb{R}^2$ *is a convex compact set;*
(A3) *for any* $x \in \mathbb{R}^2$ *and* $\nu \in \mathscr{V}_P$, *the set*

$$\Pi_M(x, \nu) = M \cap \{z \in \mathbb{R}^2 : \langle z, \nu \rangle \leqslant \langle x, \nu \rangle\}$$

is connected.

Then there exists $\vartheta > 0$ *such that the operator* T_τ *possesses the semigroup property on the segment* $[0, \vartheta]$. *(And, consequently, we get* $W_0(t) = T_{\vartheta - t}(M)$, $t \in [0, \vartheta]$.*)*

Proof. The assumptions of the theorem contain assumptions (A1)–(A3) of Theorem 12.1.

Note that the set $T_\tau(M)$ is connected for rather small $\tau > 0$, i.e., assumption (A4) holds for rather small ϑ.

Let $\nu \in \mathscr{V}_P$. By assumption (A3) of the theorem, for any $x \in \mathbb{R}^2$ the set $\Pi_M(x, \nu)$ is connected. Since M is a polygon, we choose $z_* \in M$ such that the set $\Pi_M(z_*, \nu)$ is either a triangle or a trapezium. We obtain that the assumptions of Lemma 12.5 hold for $\eta = \nu$. Consequently, there exists $\vartheta > 0$ such that assumption (A5) of Theorem 12.1 is true.

By assumptions (A1)–(A5) of Theorem 12.1, we get that the operator T_τ possesses the semigroup property on the segment $[0, \vartheta]$. □

12.6 Examples on Violation of Assumptions (A3)–(A5)

Let us show that no one assumption from (A3)–(A5) of Theorem 12.1 is excessive, i.e., violation of only one assumption of (A3)–(A5) allows one to find sets M, P, and Q, and instants τ_1 and τ_2 such that equality (12.4) is false.

Below, we consider that P and Q are the segments and $\tau_1 = \tau_2$ for all three examples. The sets M, $\tau_1 P$, $\tau_1 Q$ (thick solid line), $T_{\tau_2}(M)$ (dash line), $T_{\tau_1}(T_{\tau_2}(M))$ (hair line), and $T_{\tau_1+\tau_2}(M)$ (dotted line) are represented in Figs. 12.5–12.7.

Figure 12.5 shows an example such that only the geometric assumption (A3) of Theorem 12.1 is violated: as the set M has a triangle excision at the right, the set

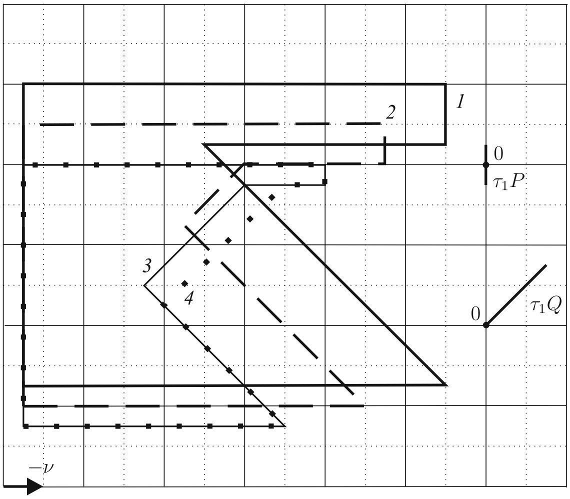


Fig. 12.5 Example 1. Assumption (A3) of Theorem 12.1 is violated; *1* is the boundary of the set M, *2* is the boundary of the set $T_{\tau_2}(M)$, *3* is the boundary of the set $T_{\tau_1}(T_{\tau_2}(M))$, *4* is the boundary of the set $T_{\tau_1+\tau_2}(M)$

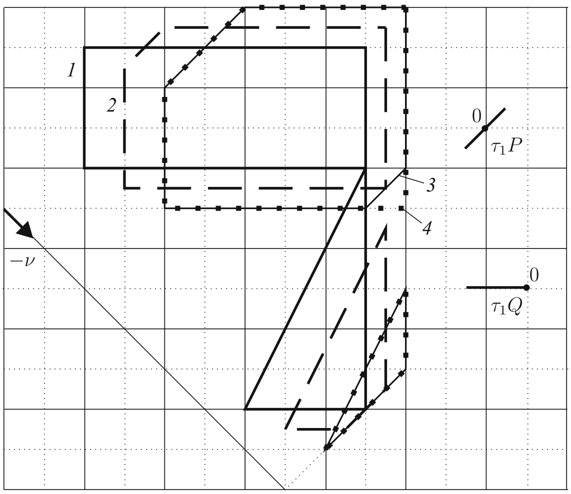


Fig. 12.6 Example 2. Assumption (A4) of Theorem 12.1 is violated; *1* is the boundary of the set M, *2* is the boundary of the set $T_{\tau_2}(M)$, *3* is the boundary of the set $T_{\tau_1}(T_{\tau_2}(M))$, *4* is the boundary of the set $T_{\tau_1+\tau_2}(M)$

$\Pi_M(x,\nu)$ is not connected for some points x (here, $\nu = (-1,0)^{\mathrm{T}}$). The difference between the boundaries of the sets $T_{\tau_1}(T_{\tau_2}(M))$ and $T_{\tau_1+\tau_2}(M)$ takes place in its middle part at the right.

The example in Fig. 12.6 is found in such a way that assumption (A4) concerning the connectedness of the set $T_{\tau_2}(M)$ is violated. The set P is a segment with the slope 45°. Each of the sets $T_{\tau_1}(T_{\tau_2}(M))$ and $T_{\tau_1+\tau_2}(M)$ consists of two disjoint parts. The underparts coincide (the triangle); the upsides are different.

Figure 12.7 gives an example such that the inequality $\delta_\nu(\tau_1 + \tau_2) < \delta_\nu(\tau_2)$ holds for $\nu = (-1,0)^{\mathrm{T}}$, i.e., assumption (A5) is violated. The sets $T_{\tau_1}(T_{\tau_2}(M))$ and $T_{\tau_1+\tau_2}(M)$ are different by small triangle in its upper part at the right.

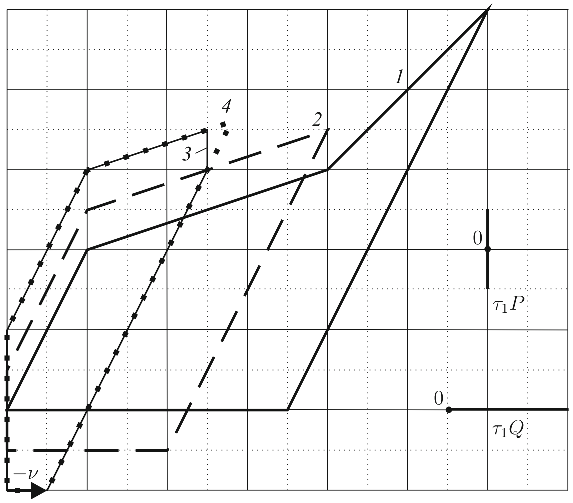


Fig. 12.7 Example 3. Assumption (A5) of Theorem 12.1 is violated: $\delta_\nu(\tau_1 + \tau_2) < \delta_\nu(\tau_2)$; *1* is the boundary of the set M, *2* is the boundary of the set $T_{\tau_2}(M)$, *3* is the boundary of the set $T_{\tau_1}(T_{\tau_2}(M))$, *4* is the boundary of the set $T_{\tau_1+\tau_2}(M)$

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Appendix

Let us consider a connection between the question investigated in the present work and the well-known Hopf formula (Alvarez et al. 1999; Bardi and Evans 1984; Hopf 1965).

1) For an arbitrary proper (i.e., not identically equal to $+\infty$) function $g : \mathbb{R}^n \to (-\infty, +\infty]$, we define the Legendre transform

$$g^*(s) = \sup_{x \in \mathbb{R}^n} [\langle x, s \rangle - g(x)], \quad s \in \mathbb{R}^n.$$

By $\operatorname{co} g$ denote the convex hull of the function g. Properties of the function g^* (Rockafellar 1970, Chap. 3, §16; Polovinkin and Balashov 2004) imply that if the proper function $\operatorname{co} g$ is continuous in $\mathbb{R}^n$, then

$$(\operatorname{co} g)^* = g^*. \tag{12.36}$$

The support function $\rho(\cdot, A)$ of a compact set $A \subset \mathbb{R}^n$ is connected with the indicator function

$$\sigma_A(x) = \begin{cases} 0, & x \in A, \\ +\infty, & x \notin A \end{cases}$$

of the set A by the relation

$$\rho(\cdot, A) = \sigma_A^*(\cdot). \tag{12.37}$$

2) The Hopf formula

$$w(t,x) := \sup_{s\in\mathbb{R}^n} [\langle x, s\rangle - \varphi^*(s) + (\vartheta - t)H(s)], \quad s \in R^n, \;\; x \in R^n, \;\; t \le \vartheta \tag{12.38}$$

represents the generalized (viscosity (Bardi and Capuzzo-Dolcetta 1997) or minimax (Subbotin 1991, 1995)) continuous solution of the Cauchy problem for the Hamilton–Jacobi equation

$$\begin{gathered} w_t(t,x) + H(w_x(t,x)) = 0, \; t \in (0,\vartheta), \quad x \in \mathbb{R}^n; \\ w(\vartheta, x) = \varphi(x), \; x \in \mathbb{R}^n, \end{gathered} \tag{12.39}$$

with convex continuous terminal function φ (Bardi and Evans 1984).

3) Set

$$H(s) = \max_{q\in Q}\langle q, s\rangle + \min_{p\in P}\langle p, s\rangle, \quad \varphi(s) = \sigma_M(s),$$

where M is a convex compact set. Consider the function $\overline{w}$ defined formally by the Hopf formula (12.38) for these data.

Let us show that

$$T_{\vartheta-t}(M) = \{x \in \mathbb{R}^n : \; \overline{w}(t,x) \le 0\}. \tag{12.40}$$

As a preliminary, for the convex sets A and B, we establish the relation

$$\rho(\,\cdot\,,\; A \overset{*}{-} B) = \mathrm{co}\,\big(\rho(\,\cdot\,,\; A) - \rho(\,\cdot\,,\; B)\big). \tag{12.41}$$

We use the equality

$$A \overset{*}{-} B = \bigcap_{b\in B}(A - b).$$

It is known that the support function for the intersection of an arbitrary collection of compact convex sets coincides (Rockafellar 1970) with the convex hull of the function that is the minimum of support functions for the sets used in the intersection. In our case,

$$\min_{b\in B}\rho(s,\ A-b)=\rho(s,\ A)+\min_{b\in B}\langle -b,\ s\rangle=\rho(s,\ A)-\rho(s,\ B).$$

Consequently, (12.41) holds.

Applying (12.41), for the convex set M and $\tau=\vartheta-t$, we get

$$\rho(s,T_\tau(M))=\mathrm{co}\ h_\tau(s), \tag{12.42}$$

where

$$h_\tau(s):=\rho(s,M)+\rho(s,-\tau P)-\rho(s,\tau Q)=\rho(s,M)-\tau H(s). \tag{12.43}$$

The compactness of the sets P and Q implies that the function co h_τ is continuous. Using (12.36) and (12.42), we get

$$h^*_\tau(x)=(\mathrm{co}\ h_\tau)^*(x)=\sup_{s\in\mathbb{R}^n}[\langle x,s\rangle-(\mathrm{co}\ h_\tau)(s)]=\sup_{s\in\mathbb{R}^n}[\langle x,s\rangle-\rho(s,T_\tau(M))]. \tag{12.44}$$

On the other hand, by (12.43) and (12.37), we calculate

$$h^*_\tau(x)=\sup_{s\in\mathbb{R}^n}[\langle x,s\rangle-h_\tau(s)]=\sup_{s\in\mathbb{R}^n}[\langle x,s\rangle-\sigma^*_M(s)+\tau H(s)]. \tag{12.45}$$

Since

$$T_{\vartheta-t}(M)=\{x\in\mathbb{R}^n:\ \sup_{s\in\mathbb{R}^n}[\langle x,s\rangle-\rho(s,T_{\vartheta-t}(M))]\le 0\},$$

applying (12.45), we write

$$T_{\vartheta-t}(M)=\{x\in\mathbb{R}^n:\ h^*_\tau(x)\le 0\}.$$

Comparing (12.38), (12.44), and (12.45), we get (12.40).

4) Thus, if the compact set M is convex, then

$$W_0(t)=T_{\vartheta-t}(M)=\{x\in\mathbb{R}^n:\ \overline{w}(t,x)\le 0\}.$$

Now, in the case of a convex set M, we have two variants of useful description of the set $W_0(t)$, whence the guidance problem of the first player to the set M at the fixed instant ϑ is solvable, namely, by the Pshenichnyi formula and by the Hopf formula. The Pshenichnyi formula deals with sets, while the Hopf formula uses functions.

In the paper, for the problems in the plane, we obtain sufficient conditions to describe the set $W_0(t)$ by the Pshenichnyi formula for a non-convex set M. The Hopf formula does not work in the non-convex case.

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