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Chapter 13
Game with Two Pursuers and One Evader: Case of Weak Pursuers

Sergey Kumkov, Valerii Patsko, and Stéphane Le Ménec

Abstract This paper deals with a zero-sum differential game, in which the first player controls two pursuing objects, whose aim is to minimize the minimum of the misses between each of them and the evader at some given instant. The case is studied when the pursuers have equal dynamic capabilities, but are less powerful than the evader. The first player's control based on its switching lines is analyzed. Results of numeric application of this control are given.

Keywords Pursuit differential games • Fixed termination instant • Positional control • Switching lines

13.1 Introduction

In papers (Ganebny et al. 2012a,b; Le Ménec 2011), a model pursuit problem with two pursuers and one evader is considered. Three inertial objects move in a straight line. Control of each object is scalar and has a bounded value. At some prescribed instant T_1, the distance between the first pursuer and the evader is measured; also, at some instant T_2, the distance between the second pursuer and the evader is checked. The pursuers act together, and their aim is to minimize the payoff, which is the minimum of these two distances. The pursuers can be joined into one player, which will be called *the first player*. The evader is treated as *the second player*, who maximizes the payoff. The obtained problem can be considered as a pursuit

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game, because its practical source is a spacial pursuit, where the instant T_1 (T_2) is the instant of the rendezvous of the first (second) pursuing object with the evading object along the nominal trajectories.

From the point of view of the differential game theory, the model problem described above is interesting because the level sets of the payoff function are non-convex, and, therefore, the time sections of the level sets of the Value function are non-convex too. The authors in works (Ganebny et al. 2012a,b; Le Ménec 2011) distinguish variants of the problem parameters giving qualitatively different solutions of the problem and studied numerically corresponding level sets of the Value function.

The simplest case is the situation of "strong" pursuers when both of them have dynamic advantage over the evader. The most difficult case is when the dynamic advantage passes from a pursuer to the evader or back during the pursuit process. In particular, in this case, level sets appear, whose time sections lose connectedness during the process, and further get it back.

The main problem is to construct efficiently optimal (or quasioptimal) feedback controls of the player. The standard approaches from the differential game theory need either storing entire Value function, or fast computing its value in the neighborhood of the current point. With that, the optimal control is built using some variant of generalized gradient of the Value function (Bardi and Capuzzo-Dolcetta 1997; Isaacs 1965; Krasovskii 1985; Krasovskii and Subbotin 1974, 1988; Tarasyev et al. 1995).

The authors have experience (Botkin and Patsko 1983; Botkin et al. 1984; Patsko 2006; Patsko et al. 1994) of constructing optimal control in linear differential games with convex payoff function on the basis of switching lines and surfaces. Mentioning the switching lines, we mean some separation of the phase space at each instant into some domains, in which each component of the control keeps some of its extreme values. With that, we store the lines only without values of the Value function. In the problem with two pursuers and one evader, the payoff function is not convex, but the authors tried to extend (Ganebny et al. 2012a,b) their algorithms for constructing feedback control on the basis of switching lines for this situation too. For the case of "strong" pursuers, statements proving the optimality of corresponding controls are set forth in work (Ganebny et al. 2012a). For other variants of the game parameters, the switching lines are built also in papers (Ganebny et al. 2012a,b). But there was no strict proof of optimality of the corresponding feedback control methods.

In this paper, such a study is made for the case of equal "weak" pursuers. We assume that $T_1 = T_2$. Under these conditions, we formulate and prove statements about quasioptimality of the first player's control based on the switching lines. Also, we consider the question of stability of this control with respect to inaccuracies of numeric constructions and errors of measurements of the current position of the system.

Sections 13.2 and 13.3 of this paper deal with the formulation of the problem and passage to a two-dimensional equivalent differential game in coordinates of forecasted misses. These sections mostly repeat the corresponding text from paper (Ganebny et al. 2012a). The authors have not reduced this text to keep the readability. The remaining part of the paper is new. In Sect. 13.4, we introduce the concept of an approximating differential game, which is used to construct the switching lines. In the problem under consideration, the first player's control consists of two scalar components u_i, $i = 1, 2$, which are bounded by the constraints $|u_i| \leq \mu_i$. Each component has its own switching line depending on time. The algorithm for constructing the switching lines is described in detail in Sect. 13.5. On one side of the switching line, the corresponding control u_i takes value $u_i = +\mu_i$, on the other side, its value is $u_i = -\mu_i$. It is important that if the system has its current position just in the switching line, then the corresponding control u_i can be taken arbitrary from the interval $[-\mu_i, +\mu_i]$. Auxiliary statements concerning estimates of the Value function change along possible trajectories are proved in Sect. 13.6. Section 13.7 is devoted to the estimate of the guaranteed result, which is provided to the first player by the feedback control on the basis of the switching lines. Results of numeric simulations of the system with usage of the suggested control method are given in Sect. 13.8.

Note that there are a lot of publications dealing with group pursuit problems (multi-agent systems) (Abramyantz and Maslov 2004; Blagodatskih and Petrov 2009; Chikrii 1997; Grigorenko 1991; Hagedorn and Breakwell 1976; Levchenkov and Pashkov 1990; Petrosjan 1977; Petrosjan and Tomski 1983; Pschenichnyi 1976; Rikhsiev 1989; Stipanović et al. 2009, 2012). But these problems are difficult due to high dimension of the state vector and non-convexity of the payoff function. Therefore, usually, some strong conditions are assumed for the dynamics of the objects (for example, objects with simple motions are considered), of their initial positions, etc. In this work, where the number of objects is small, the authors obtain the solution without any essential simplifications of the problem.

13.2 Formulation of Problem

We consider a model differential game with two pursuers and one evader. All three objects move in a straight line. The dynamics descriptions for pursuers P_1 and P_2 are

$$\begin{aligned}
&\ddot{z}_{P_1} = a_{P_1}, && \ddot{z}_{P_2} = a_{P_2},\\
&\dot{a}_{P_1} = (u_1 - a_{P_1})/l_{P_1}, && \dot{a}_{P_2} = (u_2 - a_{P_2})/l_{P_2},\\
&|u_1| \leq \mu_1, && |u_2| \leq \mu_2,\\
&a_{P_1}(t_0) = 0, && a_{P_2}(t_0) = 0.
\end{aligned} \tag{13.1}$$

Here, z_{P_1} and z_{P_2} are the geometric coordinates of the pursuers; a_{P_1} and a_{P_2} are their accelerations generated by the controls u_1 and u_2. The time constants l_{P_1} and l_{P_2} define how fast the controls affect the systems.

The dynamics of the evader E is similar:

$$\ddot{z}_E = a_E, \quad \dot{a}_E = (v - a_E)/l_E, \quad |v| \le \nu, \quad a_E(t_0) = 0. \tag{13.2}$$

Unlike papers (Ganebny et al. 2012a,b; Le Ménec 2011), this work deals with the case of equal pursuers only, that is, we assume that $\mu_1 = \mu_2 = \mu$ and $l_{P_1} = l_{P_2} = l_P$.

Comparing dynamic capabilities of each pursuer P_1 and P_2 and the evader E, one can introduce the parameters (Le Ménec 2011; Shinar and Shima 2002) $\eta = \mu/\nu$, $\varepsilon = l_E/l_P$. We investigate the case of weak pursuers, that is, the situation when the inequalities

$$\eta \le 1, \quad \eta\varepsilon \le 1$$

are true and, at least, one of them is strict.

Let us fix some instant T. At this instant, the misses of the pursuers with respect to the evader are computed:

$$r_{P_1,E}(T) = \big|z_E(T) - z_{P_1}(T)\big|, \quad r_{P_2,E}(T) = \big|z_E(T) - z_{P_2}(T)\big|. \tag{13.3}$$

Assume that the pursuers act in coordination. This means that we can join them into one player P (which will be called the *first player*). This player governs the vector control $u = (u_1, u_2)$. The evader is regarded as the *second player*. The resultant miss is the following value:

$$\varphi = \min\{r_{P_1,E}(T),\ r_{P_2,E}(T)\}. \tag{13.4}$$

At any instant t, both players know exact values of all state coordinates z_{P_1}, $\dot{z}_{P_1}$, a_{P_1}, z_{P_2}, $\dot{z}_{P_2}$, a_{P_2}, z_E, $\dot{z}_E$, a_E. The vector composed of these components is denoted as z. The first player choosing its feedback control minimizes the miss φ, the second one maximizes it.

Let the game interval be $[\bar{t}, T]$, where $\bar{t} < T$.

Following Krasovskii and Subbotin (1974, 1988), feasible strategies of the first player are considered as arbitrary functions $(t, z) \mapsto U(t, z)$ with their values in the set $\big\{(u_1, u_2) : |u_1| \le \mu, |u_2| \le \mu\big\}$.

The symbol $z\big(\cdot; t_0, x_0, U, \Delta, v(\cdot)\big)$ denotes a stepwise motion of system (13.1), (13.2), which starts from the position (t_0, x_0) when the first player applies a strategy U in a discrete control scheme with the step $\Delta > 0$ and the second player uses a measurable control $v(\cdot)$ with values $\big|v(t)\big| \le \nu$. The term "discrete scheme of control" means the following. Some grid of instants t_s with the step Δ (starting at the instant t_0) is introduced. Having a position $z(t_s)$ at the instant t_s, the first player computes the vector control $u = U\big(t_s, z(t_s)\big)$. The first player's control chosen at the instant t_s is kept until the instant $t_{s+1} = t_s + \Delta$. At the position $\big(t_{s+1}, z(t_{s+1})\big)$, a new control value is chosen, etc.

Assume

$$\Gamma(t_0, z_0, U, \Delta) = \sup_{v(\cdot)} \varphi\big(z(T; t_0.z_0, U, \Delta, v(\cdot)\big).$$

Here, the supremum is taken over all measurable functions $t \mapsto v(t)$ bounded by inequality $\big|v(t)\big| \leq \nu$. The quantity $\varphi\big(z(T)\big)$ is the value of the payoff function (13.3), (13.4) at the termination instant T on the motion $z\big(\cdot; t_0, z_0, U, \Delta, v(\cdot)\big)$.

The quantity $\Gamma(t_0, z_0, U, \Delta)$ is the guarantee of the first player provided by the strategy U at the initial position (t_0, z_0) in a discrete scheme of control with the step Δ. The best guarantee of the first player for the initial position (t_0, z_0) is defined by the formula

$$\Gamma(t_0, z_0) = \min_U \overline{\lim_{\Delta \to 0}} \Gamma(t_0, z_0, U, \Delta),$$

where the symbol $\overline{\lim}$ denotes the upper limit. In Krasovskii and Subbotin (1974, 1988), it is shown that the minimum on feasible strategies U is reached.

It is known that the best guarantee $\Gamma(t_0, z_0)$ of the first player coincides with the best guarantee for the second player defined symmetrically. Thus, the quantity $\Gamma(t_0, z_0)$ is also called the value $\boldsymbol{V}(t_0, z_0)$ of the Value function at the point (t_0, z_0).

In this paper for the case of weak equal pursuers, it is shown how to find a quasioptimal strategy of the first player (that is, a strategy close to the one optimal on the guaranteed result), which is stable with respect to inaccuracies of its numeric construction and errors of measurement of the current phase state.

13.3 Passage to Two-Dimensional Differential Game

At first, we pass to relative geometric coordinates

$$y_1 = z_E - z_{P_1}, \quad y_2 = z_E - z_{P_2} \tag{13.5}$$

in dynamics (13.1), (13.2) and payoff function (13.4). After this, we have the following notations:

$$\begin{aligned}
\ddot{y}_1 &= a_E - a_{P_1}, & \ddot{y}_2 &= a_E - a_{P_2}, \\
\dot{a}_{P_1} &= (u_1 - a_{P_1})/l_P, & \dot{a}_{P_2} &= (u_2 - a_{P_2})/l_P, \\
\dot{a}_E &= (v - a_E)/l_E, & |u_2| &\leq \mu, \\
|u_1| &\leq \mu, \ |v| \leq \nu, & \varphi &= \min\{|y_1(T)|, \ |y_2(T)|\}.
\end{aligned} \tag{13.6}$$

State variables of system (13.6) are y_1, $\dot{y}_1$, a_{P_1}, y_2, $\dot{y}_2$, a_{P_2}, a_E; u_1 and u_2 are controls of the first player; v is the control of the second one. The payoff function φ depends on the coordinates y_1 and y_2 at the instant T.

Let us introduce *zero effort miss coordinates* (Shima and Shinar 2002; Shinar and Shima 2002) x_1 and x_2 computed by formula

$$x_i = y_i + \dot{y}_i \tau - a_{P_i} l_P^2 h(\tau/l_P) + a_E l_E^2 h(\tau/l_E), \ \ i = 1, 2. \tag{13.7}$$

Here, x_i, y_i, $\dot{y}_i$, a_{P_i}, and a_E depend on t; $\tau = T - t$. Function h is described by the relation $h(\alpha) = e^{-\alpha} + \alpha - 1$. It is very important that $x_i(T) = y_i(T)$. Let $X(t, z)$ be a two-dimensional vector composed of the variables x_1, x_2 defined by formulae (13.5), (13.7).

The dynamics in the new coordinates x_1, x_2 is the following (Le Ménec 2011):

$$\begin{aligned} \dot{x}_1 &= -l_P h(\tau/l_P) u_1 + l_E h(\tau/l_E) v, \quad |u_1| \le \mu, \ |u_2| \le \mu, \\ \dot{x}_2 &= -l_P h(\tau/l_P) u_2 + l_E h(\tau/l_E) v, \quad |v| \le \nu. \end{aligned} \tag{13.8}$$

The payoff function is

$$\varphi\big(x_1(T), x_2(T)\big) = \min\{|x_1(T)|, \ |x_2(T)|\}.$$

The first player governs the controls u_1, u_2 and minimizes the payoff φ; the second one has the control v and maximizes φ.

Let $x = (x_1, x_2)^{\mathrm{T}}$ and $V(t, x)$ be the value of the Value function of game (13.8) at the position (t, x). From general results of the differential game theory, it follows that $\boldsymbol{V}(t, z) = V\big(t, X(t, z)\big)$. This relation allows us to compute the Value function of the original game (13.1)–(13.4) using the Value function for game (13.8). The transformation $(t, z) \mapsto x = X(t, z)$ helps also to map the feedback controls in game (13.8) to corresponding controls in game (13.1)–(13.4).

For any $c \ge 0$, a level set (a Lebesgue set) $W_c = \big\{(t, x) : V(t, x) \le c\big\}$ of the Value function in game (13.8) can be treated as the solvability set for the considered game with the result not greater than c, that is, for a differential game with dynamics (13.8) and the terminal set

$$M_c = \big\{(T, x) : |x_1| \le c, \ |x_2| \le c\big\}.$$

Let $W_c(t) = \{x : (t, x) \in W_c\}$ be the time section (t-section) of the set W_c at the instant t.

Problem (13.8) is of the second order on the state variable and can be rewritten as follows:

$$\dot{x} = \mathscr{D}_1(t) u_1 + \mathscr{D}_2(t) u_2 + \mathscr{E}(t) v, \qquad |u_1| \le \mu, \ |u_2| \le \mu, \ |v| \le \nu. \tag{13.9}$$

Here, $x = (x_1, x_2)^{\mathrm{T}}$; vectors $\mathscr{D}_1(t)$, $\mathscr{D}_2(t)$, and $\mathscr{E}(t)$ look like

$$\begin{aligned} &\mathscr{D}_1(t) = \big(-l_P h((T-t)/l_P)^{\mathrm{T}}, \, 0\big), \qquad \mathscr{D}_2(t) = \big(0, \, -l_P h((T-t)/l_P)\big)^{\mathrm{T}}, \\ &\mathscr{E}(t) = \big(l_E h((T-t)/l_E), \, l_E h((T-t)/l_E)\big)^{\mathrm{T}}. \end{aligned}$$

The control of the first player has two independent components u_1 and u_2. The vector $\mathscr{D}_1(t)$ ($\mathscr{D}_2(t)$) is directed along the horizontal (vertical) axis. The second player's control v is scalar.

13.4 Approximating Differential Game

Together with system (13.9), we shall consider an approximating system

$$\dot{x} = D_1(t)u_1 + D_2(t)u_2 + E(t)v, \qquad |u_1| \leq \mu,\ |u_2| \leq \mu,\ |v| \leq \nu, \tag{13.10}$$

which will be used for numeric constructions.

As system (13.10), let us take a system with piecewise-constant functions

$$D_i(t) = \mathscr{D}_i(t_j), E(t) = \mathscr{E}(t_j), t \in [t_j, t_{j+1}), i = 1, 2,$$

which approximate the functions $\mathscr{D}_i(\cdot)$, $\mathscr{E}(\cdot)$, $i = 1, 2$, in some partition of the axis t by instants t_j.

The symbol $x^{(1)}\big(t; t_*, x_*, u(\cdot), v(\cdot)\big)$ (or, shortly, $x^{(1)}(t)$) denotes the position of system (13.9) at an instant t if its motion starts at the instant t_* from the point x_* and is generated by some feasible measurable controls $u(\cdot), v(\cdot)$. Let $x^{(2)}\big(t; t_*, x_*, u(\cdot), v(\cdot)\big)$ (or, shortly, $x^{(2)}(t)$) be the analogical denotation for system (13.10). The difference of the motions $x^{(1)}(\cdot)$ and $x^{(2)}(\cdot)$ at an instant t brought by difference of dynamics (13.9) and (13.10) can be bounded from above by the value

$$\chi(t_*, t) = \sum_{i=1}^{2} \mu \int_{t_*}^{t} \big\| D_i(s) - \mathscr{D}_i(s) \big\| ds + \nu \int_{t_*}^{t} \big\| E(s) - \mathscr{E}(s) \big\| ds.$$

The payoff function for the approximating game is the same as for game (13.9). Note that it obeys the Lipschitz condition with the constant $\lambda = 1$.

Let $V^{(2)}(t, x)$ be the value of the Value function of the approximating game at the position (t, x). Since the right-hand side of the dynamics (13.10) does not include the state variable, the Lipschitz constant of the function $x \mapsto V^{(2)}(t, x)$ for any $t \leq T$ coincides (Subbotin and Chentsov 1981, pp. 110–111) with the Lipschitz constant of the payoff function, that is, with the number $\lambda = 1$.

To find the function $V^{(2)}$, we apply a backward numeric procedure to construct t-sections

$$W_c^{(2)}(t) = \big\{x : V^{(2)}(t, x) \leq c\big\}$$

of its level sets. An algorithm developed by Ganebny uses the specificity of the plane and can be applied to problems with the dynamics piecewise-constant in time. Descriptions of the procedure can be found in works (Ganebny et al. 2012a,b).

For any $c \geq 0$ and $t \leq T$, the set $W_c^{(2)}(t)$ (if it is nonempty) is symmetric with respect to the origin of the plane x_1, x_2 because the same property is incident to dynamics (13.10) (together with the constraints onto the controls) and the payoff function. Moreover, there is the symmetry with respect to the bisectrix of the second and fourth quadrants. The latter is the consequence of the assumption about equality of the pursuers' dynamic capabilities in the game with the common terminal instant T and symmetry of the payoff function with respect to this bisectrix.

The numeric examples below are given for the game with the following parameters:

$$\eta = 0.9, \quad \varepsilon = 0.7, \quad T = 20. \tag{13.11}$$

In Fig. 13.1, the evolution in time of the set $W_c^{(2)}(t)$ for $c = 3.0$ is shown. The symbol $\tau = T - t$ in the marks denotes the backward time. The upper-left subfigure corresponds to the instant T when the game terminates. The section of the level set at this instant is a cross having infinite strips along the axes. The upper-right subfigure shows some intermediate instant when the infinite strips have not collapsed, but they become narrower. The middle-left subfigure shows the instant when the infinite strips disappear, and the t-section of the level set consists of two right triangles touching each other at the origin. Further, these triangles are compressing; also, horizontal and vertical rectilinear parts appear, which grow in the backward time (see the middle-right subfigure). At some instant, the parts parallel to the bisectrix of the first and third quadrants disappear (the lower-left subfigure), and the pentagons turn into squares. After that, the squares contract (the lower-right figure) until the t-section of the level set becomes empty.

In Fig. 13.2, one can see two three-dimensional views of the set $W_c^{(2)}$ in the space t, x_1, x_2 for $c = 3.0$. On the boundary of the set, there are contours of 30 time sections with the time step 0.55. The constructions are made with quite fine time step $\Delta t = 0.05$, so, the obtained set is a quite good approximation of the ideal level set of the Value function of game (13.9).

Figure 13.3 at the top gives the picture of sections $W_c^{(2)}(t)$ computed at the instant $t = 17.0$ ($\tau = 3.0$) for values c in the range from 0 to 50 with the step $\Delta c = 0.5$. It is important to emphasize that there are two points of minimum of the function $V^{(2)}(t, \cdot)$, which are located on the bisectrix of the second and fourth quadrants. The picture of sections at the instant $t = 8.0$ ($\tau = 12.0$) is shown in Fig. 13.2 at the bottom.

Denote by $Z(t)$, $t < T$, the set consisting of these two minimum points of the function $V^{(2)}(t, \cdot)$ in the plane x_1, x_2 at the instant t. With decreasing the direct time t, the points of the sets $Z(t)$ go away from the origin. With that, the global minimal value $c_{\min}(t)$ of the Value function grows.

An important property of dynamics (13.9) is that the directions of the vectors $\mathscr{D}_1(t)$ and $\mathscr{D}_2(t)$ do not change in time. The vectors $D_1(t)$ and $D_2(t)$ of approximating dynamics (13.10) possess the same property. Namely, the vectors $\mathscr{D}_1(t)$ and $D_1(t)$ ($\mathscr{D}_2(t)$ and $D_2(t)$) are directed horizontally (vertically) contrary to the positive direction of the axis x_1 (x_2). In particular, this property provides appearance

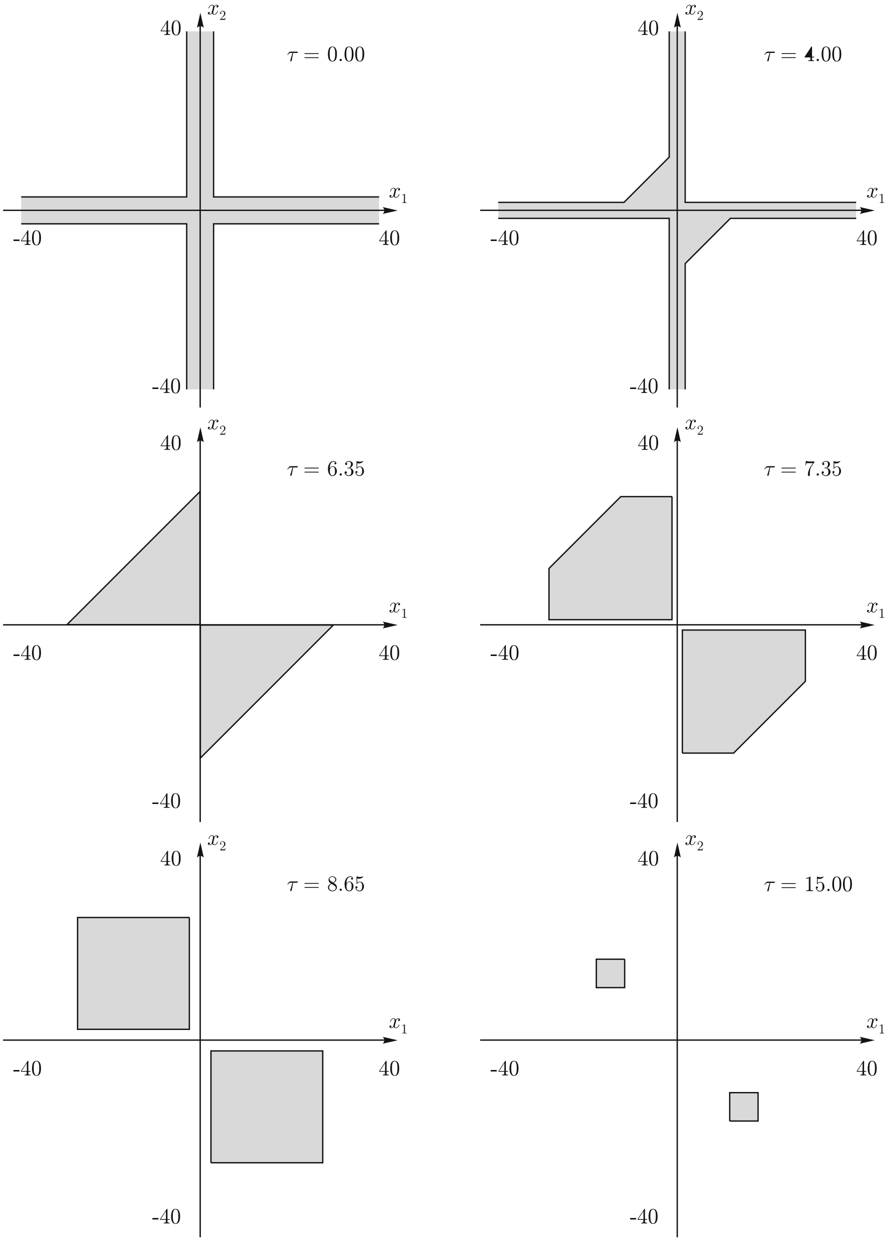

Fig. 13.1 Evolution of the set $W_{3.0}^{(2)}(t)$. The symbol $\tau = T - t$ denotes the backward time

of new horizontal and vertical parts of the boundary of the set $W_c^{(2)}(t)$ when it becomes disconnected.

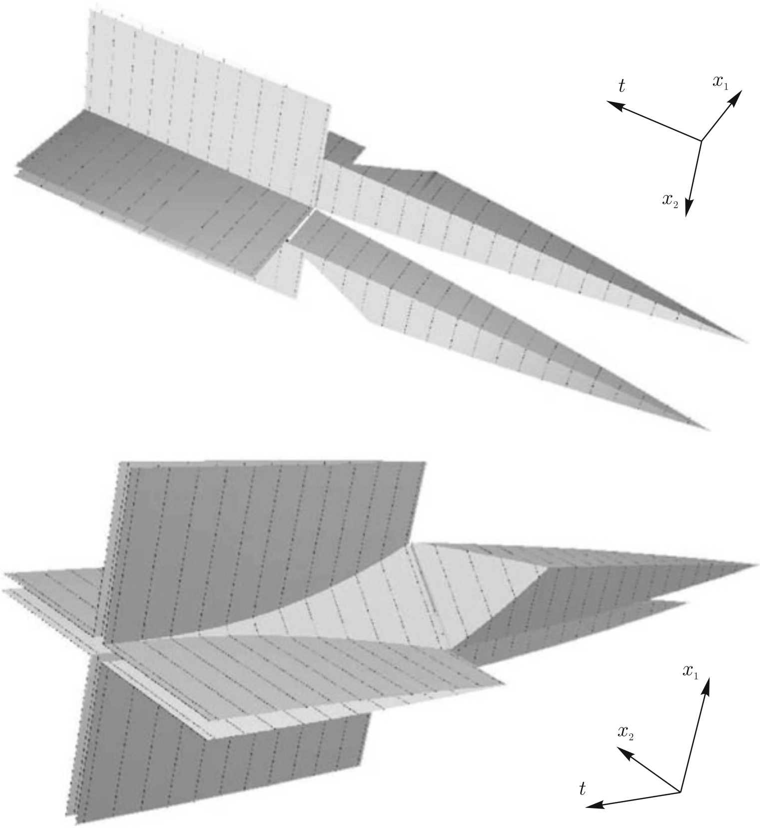

Fig. 13.2 Two three-dimensional views of the level set $W_{3.0}^{(2)}$

Denote by $\tilde{c}(t)$ the value of the function $V^{(2)}(t, \cdot)$ on the axis x_1 (which is the same for all points of the axis) at the instant t. Due to symmetry, the quantity $\tilde{c}(t)$ is also the value of the function $V^{(2)}(t, \cdot)$ on the axis x_2. The specific property of the case of weak pursuers is that the function $t \mapsto \tilde{c}(t)$ decreases with growth of t.

For $c \in [c_{\min}(t), \tilde{c}(t))$, the set $W_c^{(2)}(t)$ consists of two bounded non-intersecting parts.

13.5 Switching Lines

Taking into account that the vector $D_1(t)$ is directed horizontally, we shall study the restrictions of the function $V^{(2)}(t, \cdot)$ to horizontal lines.

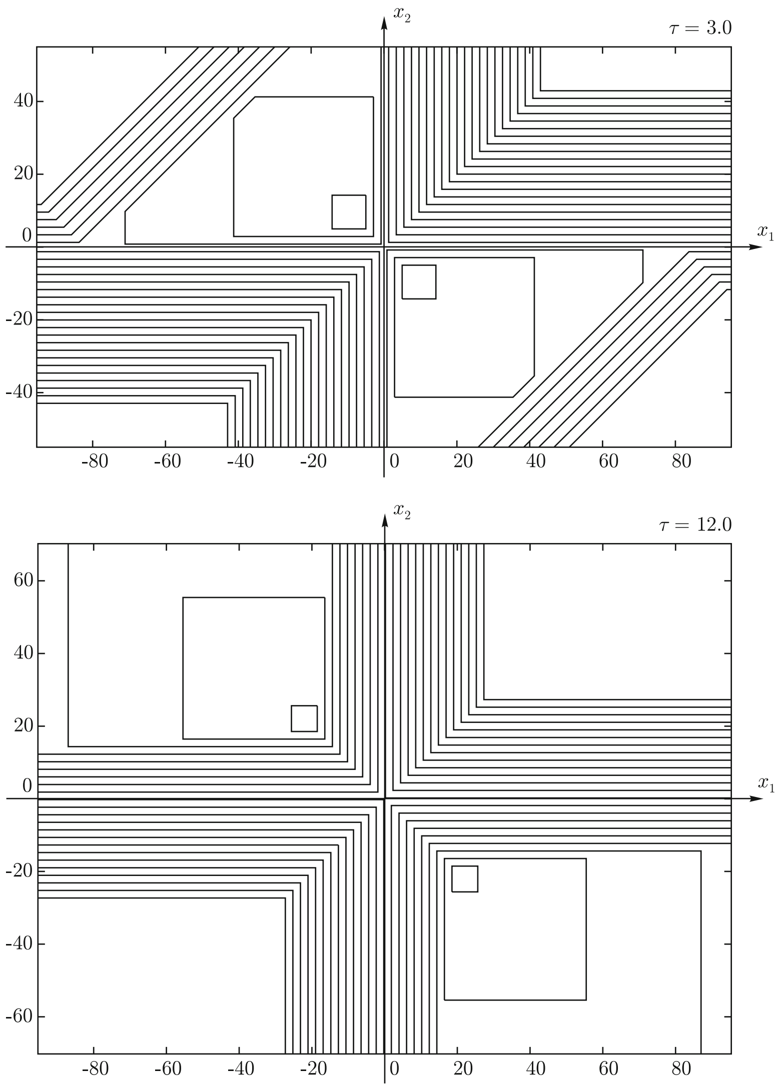

Fig. 13.3 Time sections of a system of level sets at two instants: *upper*: $t = 17.0$; *lower*: $t = 8.0$

In each horizontal line that does not cross the set $W^{(2)}_{\tilde{c}(t)}(t)$, there is only one point of minimum of the function $V^{(2)}(t, \cdot)$, and this points is located on the axis x_2. In points of the axis x_1, we have $V^{(2)}(t, x) = \tilde{c}(t)$. Therefore, the entire axis x_1 consists of points of minimum. For horizontal lines that cross the interior of the set $W^{(2)}_{\tilde{c}(t)}(t)$, there are a lot of points of minimum, and they are rectilinear part of the boundary of some set $W^{(2)}_c(t)$. Such a segment degenerates to a point for the line,

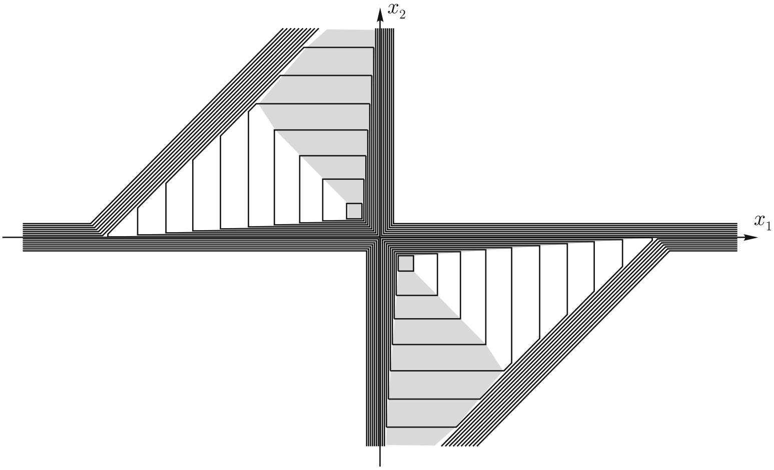

Fig. 13.4 Minima of restrictions of the Value function to horizontal lines

which passes through the point of the global minimum of the function $V^{(2)}(t,\cdot)$. In any horizontal line, the value $V^{(2)}(t,x)$ of the function $V^{(2)}(t,\cdot)$ grows if to move the point x along the line away from the segment of minima. For quite far points, the value is constant.

A set consisting of the segments of minimum is shown in Fig. 13.4 as a shadowed domain.

The same properties of the function $V^{(2)}(t,\cdot)$ take place for vertical lines too, which correspond to the vector $D_2(t)$.

Decomposing the plane x_1, x_2 into horizontal lines and considering restrictions of the function $V^{(2)}(t,\cdot)$, $t < T$, to each of them, one can take the middle point of the corresponding segment of minimum. In the horizontal axis, we take the origin as this middle point. The obtained collection of points can be seen in Fig. 13.5. Let us close this set adding two limit points at the horizontal axis. After that, let us add a horizontal segment that connects these two limit points. As a result, we get a line (Fig. 13.6), which will be denoted by $\Pi(1,t)$ and called the *switching line* for the control u_1 for system (13.10) at the instant t. On the right of this line, let $u_1 = +\mu$, on the left, $u_1 = -\mu$. On the switching line, the control u_1 can be taken arbitrary from the interval $[-\mu,+\mu]$.

The switching line $\Pi(2,t)$ (Fig. 13.7) for the control u_2 is symmetric to the line $\Pi(1,t)$ with respect to the bisectrix of the second and fourth quadrants. Above it, the control is $u_2 = +\mu$, below it, $u_2 = -\mu$. On the switching line, the control u_2 can be arbitrary from the interval $[-\mu,+\mu]$.

The lines $\Pi(1,t)$ and $\Pi(2,t)$ can be considered as exact ones for approximating system (13.10). From the further text, it will be clear that they define the optimal feedback control in system (13.10) and quasioptimal one (that is, close to the optimal one) in system (13.9). Generally speaking, we cannot compute exactly the

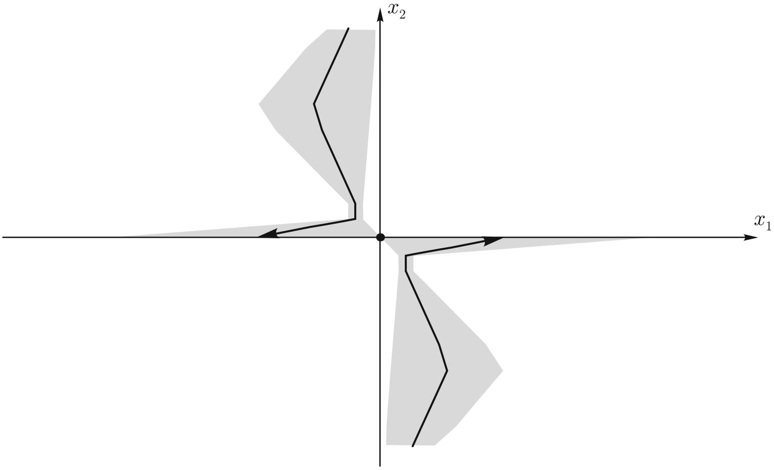

Fig. 13.5 Collection of middles of the minima intervals of the Value function restrictions to horizontal lines

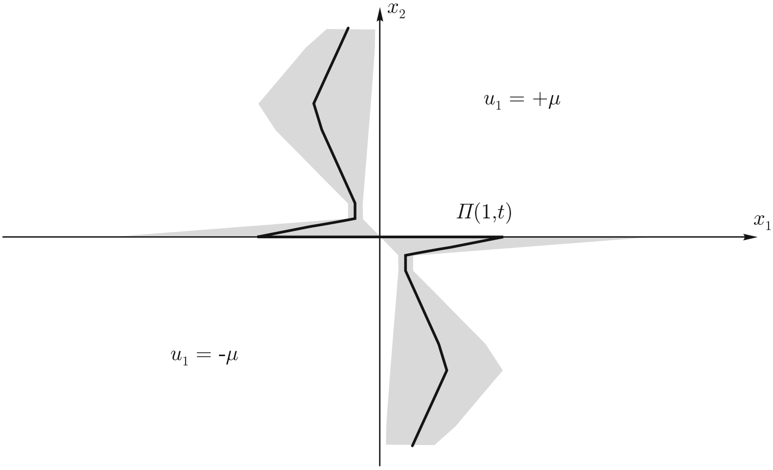

Fig. 13.6 Switching line for the control u_1

lines $\Pi(1,t)$ and $\Pi(2,t)$. For example, even if the sets $W_c^{(2)}(t)$ are constructed ideally, we work with some finite collection of them with some step on the parameter c. As a result, we get polygonal lines, which only approximate the ideal switching lines. Therefore, a very important question is what guarantee they provide for the first player.

For any $t < T$ and any horizontal (vertical) line passing through a point x, let us denote by $\mathscr{V}(1,t,x)$ (respectively, $\mathscr{V}(2,t,x)$) the minimal value of the Value function $V^{(2)}(t,\cdot)$ on this line. One has $\mathscr{V}(1,t,x) = V^{(2)}(t,x)$, when $x \in \Pi(1,t)$, and $\mathscr{V}(2,t,x) = V^{(2)}(t,x)$, when $x \in \Pi(2,t)$.

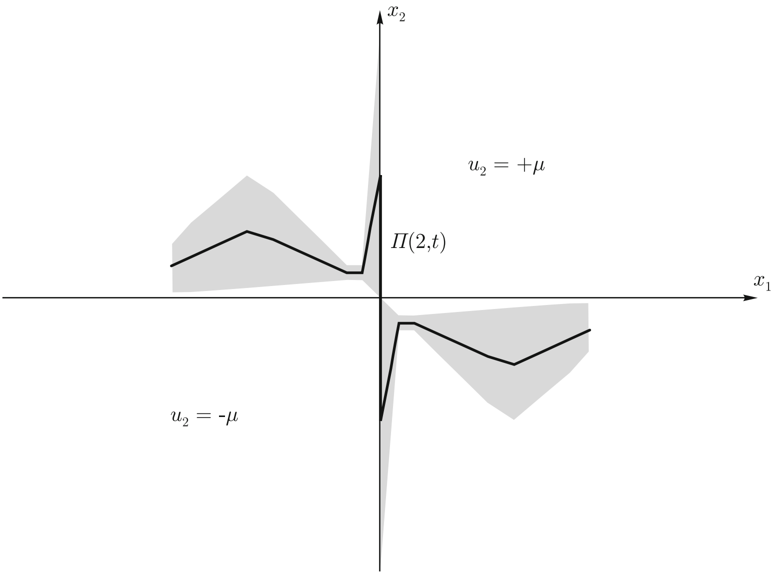

Fig. 13.7 Switching line for the control u_2

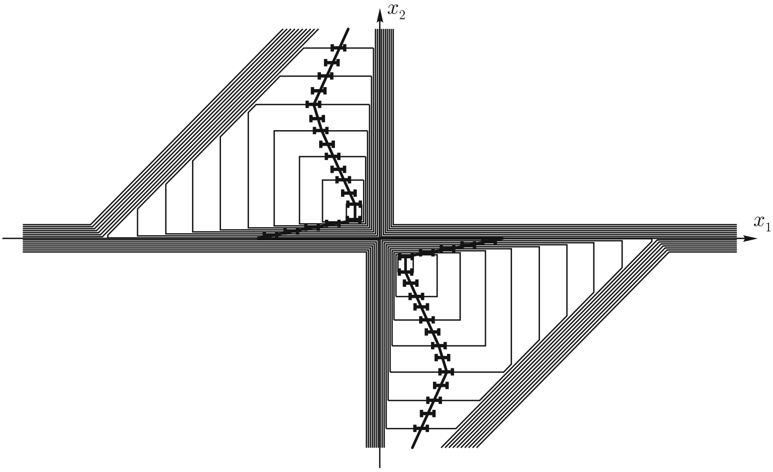

Fig. 13.8 r-extension of the switching line for the component u_1 of the first player's control

Take a number $r \geq 0$ and "expand" the line $\Pi(1, t)$ putting to it horizontal segments with the length $2r$. The obtained set (see Fig. 13.8) is denoted by $\Pi^r(1, t)$. In the same way, using vertical segments, one can construct the set $\Pi^r(2, t)$.

Geometric r-expansion of the ideal switching lines is introduced to deal with the case of inaccurate numeric construction of the switching lines. We would like to "enclose" the switching lines $\Pi(1, t)$ and $\Pi(2, t)$ by a domain, in which the

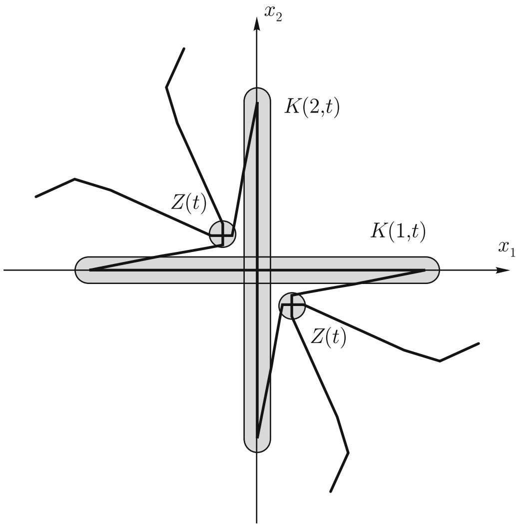

Fig. 13.9 α-neighborhoods of the set $Z(t)$ and the sets $K(1,t)$ and $K(2,t)$

inaccuracies of the construction or measurement errors can be concealed. With that, for the control u_1, it is convenient to use just the horizontal expansion, because in every horizontal line the value $\mathscr{V}(1,t,x)$ is the same for all points. Let after computations we have the following information: (1) we know the value $\mathscr{V}(1,t,x_*)$ in some point x_* at some instant t; (2) the distance from the point x_* to the switching line $\Pi(1,t)$ in the horizontal direction is not greater than r. Then we can obtain an upper estimate for $V^{(2)}(t,x_*)$:

$$V^{(2)}(t,x_*) \le \mathscr{V}(1,t,x_*) + \lambda r.$$

Due to similar reason, the vertical expansion is convenient for the control u_2.

But the expansions are inefficient at the horizontal part of the line $\Pi(1,t)$ and at the vertical part of the line $\Pi(2,t)$. Let us denote the horizontal (vertical) part of the line $\Pi(1,t)$ ($\Pi(2,t)$) by $K(1,t)$ ($K(2,t)$). Choose $\alpha > 0$ and consider closed α-neighborhoods $O\big(\alpha, K(i,t)\big)$, $i = 1, 2$, of these sets.

For further constructions, we need to "prohibit" a fast transfer from the r-expansion $\Pi^r(1,t)$ of the switching line $\Pi(1,t)$ to the r-expansion $\Pi^r(2,t)$ of the switching line $\Pi(2,t)$ and back. The lines $\Pi(1,t)$ and $\Pi(2,t)$ cross in the origin and in two points that constitute the set $Z(t)$. Let us introduce a closed α-neighborhood $O\big(\alpha, Z(t)\big)$ of the set $Z(t)$ (Fig. 13.9). Denote

$$\Pi^r_\alpha(i,t) = \mathrm{cl}\left[\Pi^r(i,t)\setminus\Big(O\big(\alpha, Z(t)\big)\bigcup O\big(\alpha, K(i,t)\big)\Big)\right], \quad i = 1,\ 2.$$

As it was said in Sect. 13.2, we assume that the initial instants in the considered games are in the interval $[\bar{t}, T]$. Let $Y = [\bar{t}, T] \times R^2$ be the space of the games.

The lines $\varPi(1,t)$ and $\varPi(2,t)$, $t < T$, depend continuously on the time. Thus, for any instant $\hat{t} \in [\bar{t}, T)$, one can find such quantities $\hat{\alpha} > 0$ and $\hat{r} > 0$ that

$$\varPi_\alpha^r(1,t) \bigcap \varPi_\alpha^r(2,t) = \varnothing, \quad t \in [\bar{t}, \hat{t}],\ \alpha \geq \hat{\alpha},\ r \in [0, \hat{r}]. \tag{13.12}$$

Moreover, for these values t, α, and r, there is an estimate $\vartheta(\hat{t}, \hat{\alpha}, \hat{r}) > 0$, which is less than the transfer time of systems (13.9) and (13.10) from one of the sets $\varPi_\alpha^r(1,\cdot)$ and $\varPi_\alpha^r(2,\cdot)$ to another.

13.6 Auxiliary Statements

Let us formulate a number of statements, which will be used during the proof of the main theorem about the guarantee when the first player applies in system (13.9) the control based on the switching lines constructed in system (13.10).

Denote by $\varPi_+(1,t)$ ($\varPi_-(1,t)$) the part of the plane, which is strictly on the right (strictly on the left) of the switching line $\varPi(1,t)$. If $x \in \varPi_+(1,t)$ ($x \in \varPi_-(1,t)$), then the control $u_1 = +\mu$ ($u_1 = -\mu$) directs the vector $D_1(t)u_1$ to the switching line, that is, to the area of less values of the Value function $V^{(2)}(t,\cdot)$. In the same way, the symbol $\varPi_+(2,t)$ ($\varPi_-(2,t)$) denotes the part of the plane above (below) the switching line $\varPi(2,t)$.

Let $\sigma = \max\{\|D_1(t)\| : t \in [\bar{t}, T]\}$. Since $\|D_1(t)\| = \|D_2(t)\|$ in the considered case of equal pursuers and equal termination instants $T_1 = T_2 = T$, the value σ is also an upper estimate for the norm of the vector $D_2(t)$ in the time interval $[\bar{t}, T]$.

Lemma 13.1. *Let us fix $i = 1, 2$. Let the position $(t_*, x_*) \in Y$ and the number $\delta > 0$, $t_* + \delta < T$, be such that $x_* \in \varPi_+(i, t_*)$ (or $x_* \in \varPi_-(i, t_*)$) and any motion of system (13.10) starting from the point x_* at the instant t_* stays in the set $\varPi_+(i,t)$ ($\varPi_-(i,t)$) for any instant $t \in [t_*, t_* + \delta]$. In the interval $[t_*, t_* + \delta]$, consider an arbitrary motion $x^{(1)}(\cdot)$ of system (13.9) starting from the point x_* at the instant t_* under some control $v(\cdot)$ of the second player and some control $u(\cdot)$ of the first player. The latter is such that $u_i \equiv +\mu$ ($u_i \equiv -\mu$) except, maybe, the interval $[t_*, t_* + \omega]$ with a length $\omega \leq \delta$.*

Then for any $t \in [t_*, t_* + \delta]$, the following estimate is true:

$$\mathscr{V}\big(\bar{i}, t, x^{(1)}(t)\big) \leq V^{(2)}(t_*, x_*) + 2\lambda\omega\sigma\mu + \lambda\chi(t_*, t_* + \delta). \tag{13.13}$$

Here, $\bar{i} = 2$ if $i = 1$, and $\bar{i} = 1$ if $i = 2$.

Remark 13.1. Let, for definiteness, $i = 2$ and from the variants $+$ and $-$ the sign $+$ be taken. Then $x_* \in \varPi_+(2,t)$, and the assumption about the "correct" control u_2 type is in agreement with this. The control can differ from $u_2 \equiv +\mu$ in some interval of the length ω only. A feasible control $u_1(\cdot)$ can be arbitrary. The value ω defines the second summand in the right-hand side of estimate (13.13). The third summand is standard addition, which estimates from above the increment of the Value function $V^{(2)}(t_*, t_*)$ caused by difference of the dynamics of systems (13.9) and (13.10).

Proof. Let us assume for definiteness that $i = 2$ and the sign $+$ is chosen.

Together with the motion $x^{(1)}\big(\cdot; t_*, x_*, u(\cdot), v(\cdot)\big)$ of system (13.9), which in the formulation of the lemma is denoted as $x^{(1)}(\cdot)$, let us consider a motion $x^{(2)}\big(\cdot; t_*, x_*, u(\cdot), v(\cdot)\big)$ (or, shortly, $x^{(2)}(\cdot)$) of system (13.10), which is emanated under the same controls $u(\cdot)$ and $v(\cdot)$.

Let $c_* = V^{(2)}(t_*, x_*)$.

Fix some arbitrary instant $t \in [t_*, t_* + \delta]$.

On the basis of the open-loop control $v(\cdot)$, which is considered in the interval $[t_*, t]$, choose an open-loop control $u_{\mathrm{st}}(\cdot)$ such that

$$x^{(2)}_{\mathrm{st}}(t) \in W^{(2)}_{c_*}(t), \tag{13.14}$$

where $x^{(2)}_{\mathrm{st}}(\cdot) = x^{(2)}_{\mathrm{st}}\big(\cdot; t_*, x_*, u_{\mathrm{st}}(\cdot), v(\cdot)\big)$ is a motion of system (13.10) starting from the point x_* at the instant t_* under controls $u_{\mathrm{st}}(\cdot)$ and $v(\cdot)$. Such a control can be chosen in any case on the basis of stability property (Krasovskii and Subbotin 1974, 1988) of the level set $W^{(2)}_{c_*}$ of the Value function $V^{(2)}$. Inclusion (13.14) means that

$$V^{(2)}\big(t, x^{(2)}_{\mathrm{st}}(t)\big) \le V^{(2)}(t_*, x_*). \tag{13.15}$$

Consider a new control $\hat{u}_{\mathrm{st}}(\cdot)$ with components $\hat{u}_{1\mathrm{st}}(\cdot) = u_{1\mathrm{st}}(\cdot)$ and $\hat{u}_{2\mathrm{st}}(\cdot) \equiv +\mu$. Let $\hat{x}^{(2)}_{\mathrm{st}}(\cdot)$ be a motion of system (13.10) starting from the point x_* at the instant t_* under controls $\hat{u}_{\mathrm{st}}(\cdot)$ and $v(\cdot)$.

The following component-wise relations are true:

$$\hat{x}^{(2)}_{1\mathrm{st}}(t) = x^{(2)}{}_{1\mathrm{st}}(t), \qquad \hat{x}^{(2)}_{2\mathrm{st}}(t) \le x^{(2)}{}_{2\mathrm{st}}(t). \tag{13.16}$$

Since the points $\hat{x}^{(2)}_{\mathrm{st}}(t)$ and $x^{(2)}{}_{\mathrm{st}}(t)$ are in the set $\Pi_+(2,t)$, it follows from (13.16) that

$$V^{(2)}\big(t, \hat{x}^{(2)}_{\mathrm{st}}(t)\big) \le V^{(2)}\big(t, x^{(2)}{}_{\mathrm{st}}(t)\big). \tag{13.17}$$

Due to the hypotheses of the lemma, the component $u_2(\cdot)$ of the vector control $u(\cdot)$ differs from the constant control $\hat{u}_{2\mathrm{st}}(t) \equiv +\mu$ in some interval of the length ω only. Therefore,

$$\left| x^{(2)}_2(t) - \hat{x}^{(2)}_{2\mathrm{st}}(t) \right| \le 2\omega\sigma\mu. \tag{13.18}$$

One has

$$\left| x^{(1)}_2(t) - x^{(2)}_2(t) \right| \le \chi(t_*, t). \tag{13.19}$$

From (13.18) and (13.19), it follows that

$$\left|x_2^{(1)}(t) - \hat{x}_{2\mathrm{st}}^{(2)}(t)\right| \le 2\omega\sigma\mu + \chi(t_*, t). \tag{13.20}$$

Consider a point z in the horizontal line passing through the point $x^{(1)}(t)$. The point z is the closest one to $\hat{x}_{\mathrm{st}}^{(2)}(t)$. Due to (13.20), one gets

$$\left\|z - \hat{x}_{\mathrm{st}}^{(2)}(t)\right\| \le 2\omega\sigma\mu + \chi(t_*, t).$$

Consequently,

$$V^{(2)}(t, z) \le V^{(2)}\big(t, \hat{x}_{\mathrm{st}}^{(2)}(t)\big) + 2\lambda\omega\sigma\mu + \lambda\chi(t_*, t).$$

Taking into account (13.15) and (13.17), this gives

$$V^{(2)}(t, z) \le V^{(2)}(t_*, x_*) + 2\lambda\omega\sigma\mu + \lambda\chi(t_*, t).$$

Thus,

$$\mathscr{V}(1, t, z) \le V^{(2)}(t_*, x_*) + 2\lambda\omega\sigma\mu + \lambda\chi(t_*, t). \tag{13.21}$$

Since $\mathscr{V}(1, t, z) = \mathscr{V}\big(1, t, x^{(1)}(t)\big)$, relation (13.21) gives (13.13). □

Lemma 13.2. *Let $(t_*, x_*) \in Y, \delta > 0,\ t_* + \delta \le T$. Assume that some arbitrary feasible controls $u(\cdot)$ and $v(\cdot)$ act in system (13.9) in the interval $[t_*, t_* + \delta]$. Then, along the corresponding motion $x^{(1)}(\cdot)$ starting from the point x_* at the instant t_*, the following estimate holds*

$$V^{(2)}\big(t, x^{(1)}(t)\big) \le V^{(2)}(t_*, x_*) + 4\lambda\sigma\mu(t - t_*) + \lambda\chi(t_*, t). \tag{13.22}$$

Proof. Fix some instant $t \in [t_*, t_* + \delta]$.

Consider a motion $x^{(2)}(\cdot)$ of system (13.10) starting from the point x_* at the instant t_* under the controls $u(\cdot)$ and $v(\cdot)$ from the lemma formulation. One has

$$\left\|x^{(1)}(t) - x^{(2)}(t)\right\| \le \chi(t_*, t). \tag{13.23}$$

Let $c_* = V^{(2)}(t_*, x_*)$. Use the stability property of the set $W_{c_*}^{(2)}$. Then on the basis of the open-loop control $v(\cdot)$, one can choose an open-loop control $u_{\mathrm{st}}(\cdot)$ such that

$$x_{\mathrm{st}}^{(2)}(t) = x^{(2)}\big(t; t_*, x_*, u_{\mathrm{st}}(\cdot), v(\cdot)\big) \in W_{c_*}^{(2)}(t).$$

This means that

$$V^{(2)}\big(t, x_{\mathrm{st}}^{(2)}(t)\big) \le V^{(2)}(t_*, x_*). \tag{13.24}$$

Taking into account the inequality

$$\left\|x^{(2)}(t) - x_{\mathrm{st}}^{(2)}(t)\right\| \le 4\sigma\mu(t - t_*)$$

and inequality (13.23), we obtain

$$\left\|x^{(1)}(t) - x_{\mathrm{st}}^{(2)}(t)\right\| \le 4\sigma\mu(t - t_*) + \chi(t_*, t).$$

Therefore,

$$V^{(2)}\big(t, x^{(1)}(t)\big) \le V^{(2)}\big(t, x_{\mathrm{st}}^{(2)}(t)\big) + 4\lambda\sigma\mu(t - t_*) + \lambda\chi(t_*, t).$$

From this due to (13.24), inequality (13.22) follows. □

Lemma 13.3. *Let* $(t_*, x_*) \in Y, \delta > 0, t_* + \delta < T$. *Assume that any motion of system* (13.10) *starting from the point* x_* *at the instant* t_* *does not reach the lines* $\Pi(i,t), i = 1, 2$, *for* $t \in [t_*, t_* + \delta)$.

Assume that along a motion $x^{(1)}(\cdot)$ *of system* (13.9) *starting from the point* x_* *at the instant* t_* *under some feasible controls* $u(\cdot)$ *and* $v(\cdot)$, *it is true that for any* $i = 1, 2$, *in the interval* $[t_*, t_* + \delta)$

- *either* $x^{(1)}(t) \in \Pi_+(i,t)$ *and* $u_i(t) = +\mu$;
- *or* $x^{(1)}(t) \in \Pi_-(i,t)$ *and* $u_i(t) = -\mu$.

Then the following estimate is true:

$$V^{(2)}\big(t_* + \delta, x^{(1)}(t_* + \delta)\big) \le V^{(2)}(t_*, x_*) + \lambda\chi(t_*, t_* + \delta).$$

The proof of Lemma 13.3 can be done in the same way as for Lemmas 13.1 and 13.2 using the stability property of the set $W_{c_*}^{(2)}$, where $c_* = V^{(2)}(t_*, x_*)$.

Lemma 13.4. *Let* $(t_*, x_*) \in Y, t^* \in (t_*, T)$. *Let* $0 \le \omega \le t^* - t_*$. *Assume that for a motion* $x^{(1)}(\cdot)$ *of system* (13.9) *starting from the point* x_* *at the instant* t_* *under some feasible controls* $u(\cdot)$ *and* $v(\cdot)$, *it be true that for any* $i = 1, 2$,

- *either* $x^{(1)}(t) \in \Pi_+(i,t)$ *in the interval* (t_*, t^*) *and* $u_i(t) = +\mu$ *in the interval* $[t_* + \omega, t^*]$;
- *or* $x^{(1)}(t) \in \Pi_-(i,t)$ *in the interval* (t_*, t^*) *and* $u_i(t) = -\mu$ *in the interval* $[t_* + \omega, t^*]$.

Then for any $t \in [t_*, t^*]$, *the following estimate is true:*

$$V^{(2)}\big(t, x^{(1)}(t)\big) \le V^{(2)}(t_*, x_*) + 4\lambda\omega\sigma\mu + \lambda\chi(t_*, t). \tag{13.25}$$

Proof. Divide the interval $[t_*, t^*]$ by instants $\{t_s\}$, $s = 1, 2, \dots, e$, $t_1 = t_*$, $t_e = t^*$, $t_{s+1} \le t_s + \delta$ in such a way that for any interval $[t_s, t_{s+1}]$, $s = 2, \dots, e - 1$ of the division, no motion of system (13.10) starting from the point $x^{(1)}(t_s)$ at the instant t_s

reaches the lines $\Pi(i,t)$, $i = 1, 2$, $t \in [t_s, t_{s+1}]$. This can be done due to continuity of the switching lines $\Pi(1,t)$ and $\Pi(2,t)$ in time and due to the assumption on the location of the points $x^{(1)}(t)$ with respect to the switching lines.

Due to Lemma 13.3 for any $s = 2, \ldots, e-1$ such that $t_s > t_* + \omega$, one has relation

$$V^{(2)}\big(t_{s+1}, x^{(1)}(t_{s+1})\big) \le V^{(2)}\big(t_s, x^{(1)}(t_s)\big) + \lambda\chi(t_s, t_{s+1}). \tag{13.26}$$

For s such that $t_s \in [t_*, t_* + \omega]$, from Lemma 13.2, it follows that

$$V^{(2)}\big(t_{s+1}, x^{(1)}(t_{s+1})\big) \le V^{(2)}\big(t_s, x^{(1)}(t_s)\big) + 4\lambda\delta\sigma\mu + \lambda\chi(t_s, t_{s+1}). \tag{13.27}$$

Fix $t \in [t_*, t^*]$. Using estimates (13.26) and (13.27) for $s = 1, 2, \ldots, e-1$ while $t_s < t$, we get the inequality

$$V^{(2)}\big(t, x^{(1)}(t)\big) \le V^{(2)}(t_*, x_*) + 4\lambda(\omega + \delta)\sigma\mu + \lambda\chi(t_*, t).$$

Passing to the limit as $\delta \to 0$, one obtains estimate (13.25). □

13.7 Theorem About Guarantee

13.7.1 Estimation of Inaccuracies for Multivalued Strategy of the First Player

Take an arbitrary instant $\hat{t} \in [\bar{t}, T)$. Using $\hat{t}$, choose $\hat{\alpha} > 0$ and $\hat{r} \in (0, \hat{\alpha})$ such that in the interval $[\bar{t}, \hat{t}]$, there is an estimate $\vartheta(\hat{t}, \hat{\alpha}, \hat{r}) > 0$, which is less than the time of the transfer of systems (13.9) and (13.10) from one of the sets $\Pi_{\hat{\alpha}}^{\hat{r}}(1,\cdot)$ and $\Pi_{\hat{\alpha}}^{\hat{r}}(2,\cdot)$ to another. Then, this estimate $\vartheta(\hat{t}, \hat{\alpha}, \hat{r})$ of the transfer time is held for $\alpha \ge \hat{\alpha}$, $r \in [0, \hat{r}]$ too. Note that $r < \alpha$. Instead of $\vartheta(\hat{t}, \hat{\alpha}, \hat{r})$, we write just ϑ.

Assume

$$S(i,\alpha,r,t) = O\big(\alpha, Z(t)\big) \bigcup O\big(\alpha, K(i,t)\big) \bigcup \Pi_\alpha^r(i,t),$$
$$i = 1,\ 2, \quad \alpha \ge \hat{\alpha},\ r \in [0, \hat{r}],\ t \in [\bar{t}, T).$$

Let us introduce a multivalued strategy $(t,x) \mapsto \mathbf{U}(t,x)$ of the first player. Define that

$$\mathbf{U}_i(t,x) = \big\{u_i : |u_i| \le \mu\big\}, \text{if } x \in S(i,\alpha,r,t), \quad i = 1,\ 2.$$

Outside the set $S(i,\alpha,r,t)$, $t < T$, the component $\mathbf{U}_i(t,x)$, $i = 1, 2$, of the strategy $\mathbf{U}$ is one-valued. Namely, in the position (t,x), the value u_i, $i = 1, 2$, equal either to $+\mu$, or $-\mu$ is taken in such a way that the vector $D_i(t)u_i$ is directed to the switching line $\Pi(i,t)$, which is ideal for system (13.10).

Let the first player apply in system (13.9) the strategy $\mathbf{U}$ in a discrete scheme of control with a step $\Delta \le \vartheta$. At each instant t_s of the discrete scheme, the first player computes the vector control $u \in \mathbf{U}^r_\alpha\big(t_s, x(t_s)\big)$.

We estimate increment of the function $V^{(2)}$ along a motion $x^{(1)}(\cdot)$ starting from the point x_0 at the instant $t_0 \in [\bar{t}, T]$ under the first player's strategy $\mathbf{U}$ in a discrete scheme with a step Δ and some feasible control $v(\cdot)$ of the second player.

Assume

$$\Pi^r_\alpha(t) = \Pi^r_\alpha(1,t) \bigcup \Pi^r_\alpha(2,t), \quad K(t) = K(1,t) \bigcup K(2,t).$$

A. Let us define the following time intervals.

1. The interval $\mathscr{T}_z = [t_z, t^z]$ from the instant t_z of the first entry of the point $x^{(1)}(t)$ in the set $O\big(\alpha, Z(t)\big)$ to the instant t^z of the last leaving the set. That is

$$t_z = \min\big\{t : x^{(1)}(t) \in O\big(\alpha, Z(t)\big)\big\}, \quad t^z = \max\big\{t : x^{(1)}(t) \in O\big(\alpha, Z(t)\big)\big\}.$$

If $\mathscr{T}_z = \varnothing$, then assume $t^z = t_0$.

2. The interval $\mathscr{T}_k = [t_k, t^k]$ from the instant t_k of the first entry of the point $x^{(1)}(t)$ in the set $O\big(\alpha, K(t)\big)$ to the instant t^k of the last leaving the set. This interval is considered only if $t_k \in [t^z, \hat{t})$.
3. The interval $\mathscr{T}_{\hat{c}} = [t_{\hat{c}}, t^{\hat{c}}]$ from the instant $t_{\hat{c}}$ of the first entry of the point $x^{(1)}(t)$ in the set $O\big(\alpha, W^{(2)}_{\hat{c}}(t)\big)$, where $\hat{c} = \tilde{c}(\hat{t})$, to the instant $t^{\hat{c}}$ of the last leaving the set. This interval is considered only
4. The interval $\mathscr{T}_\flat = [t_\flat, t^\flat]$ for $t^\flat \le \hat{t}$. Suppose that

$$x^{(1)}(t_\flat) \in \Pi^r_\alpha(t_\flat), \quad x^{(2)}(t^\flat) \in \Pi^r_\alpha(t^\flat).$$

Assume that the interval $\mathscr{T}_\flat$ is on the right of the instant t^z and beyond the interval $\mathscr{T}_k$. Moreover, let us agree that the interval $\mathscr{T}_\flat$ has the maximal possible length under these conditions.

From the properties conditioning the interval $\mathscr{T}_\flat$, it follows that only two cases of its location are possible: inside the interval $[t^z, t_k]$ or inside the interval $[t^k, \hat{t}]$. If the interval $\mathscr{T}_k$ is absent, then assume $t_k = \hat{t}$.

B. Compute estimates of changing the function $V^{(2)}$ along a motion $x^{(1)}(\cdot)$. The symbol $\mathrm{Var}\big(V^{(2)}, [t_*, t^*]\big)$ denotes the increment of the function $V^{(2)}$ on the interval $[t_*, t^*]$. At first, consider the intervals $\mathscr{T}_z$, $\mathscr{T}_k$, and $\mathscr{T}_{\hat{c}}$.

At the instant t^z, one has

$$V^{(2)}\big(t^z, x^{(1)}(t^z)\big) \le c_{\min}(t^z) + \lambda\alpha \le V^{(2)}(t_0, x_0) + \lambda\alpha. \tag{13.28}$$

At the instant t^k, the following estimate is true:

$$V^{(2)}\big(t^k, x^{(1)}(t^k)\big) \le \tilde{c}(t^k) + \lambda\alpha \le \tilde{c}(t_k) + \lambda\alpha.$$

Since

$$\tilde{c}(t_k) \le V^{(2)}\big(t_k, x^{(1)}(t_k)\big) + \lambda\alpha,$$

it holds

$$\mathrm{Var}\big(V^{(2)}, [t_k, t^k]\big) \le 2\lambda\alpha. \tag{13.29}$$

At the instant $t^{\hat{c}}$, it is true that

$$V^{(2)}\big(t^{\hat{c}}, x^{(1)}(t^{\hat{c}})\big) \le \hat{c} + \lambda\alpha. \tag{13.30}$$

C. The estimate for increment of $V^{(2)}$ along a motion $x^{(1)}(\cdot)$ on the interval $\mathscr{T}_\flat$ is not so easy. Assume for definiteness that $x^{(1)}(t_\flat) \in \Pi^r_\alpha(1, t_\flat)$.

Suppose $t_1 = t_\flat$. The symbol t_{1+} denotes the maximal instant belonging to the interval $[t_1, t_1 + \vartheta] \cap [t_1, t^b]$ such that $x^{(1)}(t) \in \Pi^r_\alpha(t)$. Since during a period of the length ϑ, the transfer from the set $\Pi^r_\alpha(1, \cdot)$ to the set $\Pi^r_\alpha(2, \cdot)$ is impossible, one has $x^{(1)}(t_{1+}) \in \Pi^r_\alpha(1, t_{1+})$. It can happen that $t_{1+} = t_1$. To estimate $V^{(2)}\big(t_{1+}, x^{(1)}(t_{1+})\big)$, we can involve Lemma 13.1.

Assume $t_{1+} < t^\flat$. Let t_2 be the minimal instant from the interval $[t_1 + \vartheta, t^\flat]$ such that $x^{(1)}(t) \in \Pi^r_\alpha(t)$. Both cases $x^{(1)}(t_2) \in \Pi^r_\alpha(1, t_2)$ and $x^{(1)}(t_2) \in \Pi^r_\alpha(2, t_2)$ are possible. In any case, the point $x^{(1)}(t)$ in the interval (t_{1+}, t_2) is outside the set $S(1, \alpha, r, t) \bigcup S(2, \alpha, r, t)$, and to estimate the quantity $\mathrm{Var}\big(V^{(2)}, [t_{1+}, t_2]\big)$ one can use Lemma 13.4. Note that $t_2 - t_1 \ge \vartheta$.

If $t_2 < t^\flat$, then introduce an instant t_{2+} defining it as the maximal one in the interval $[t_2, t_2 + \vartheta] \cap [t_2, t^\flat]$ such that $x^{(1)}(t) \in \Pi^r_\alpha(t)$. If $x^{(1)}(t_2) \in \Pi^r_\alpha(1, t_2)$, one has $x^{(1)}(t_{2+}) \in \Pi^r_\alpha(1, t_{2+})$. In the case $x^{(1)}(t_2) \in \Pi^r_\alpha(2, t_2)$, so one gets $x^{(1)}(t_{2+}) \in \Pi^r_\alpha(2, t_{2+})$. Assume that $t_{2+} < t^\flat$. Then introduce an instant t_3 defining it as the maximal one in the interval $[t_2 + \vartheta, t^\flat]$ such that $x^{(1)}(t) \in \Pi^r_\alpha(t)$, etc.

In the interval of the type $[t_j, t_{j+}]$ due to Lemma 13.1, one obtains

$$\mathscr{V}\big(i, t_{j+}, x^{(1)}(t_{j+})\big) \le V^{(2)}\big(t_j, x^{(1)}(t_j)\big) + 2\lambda\Delta\sigma\mu + \lambda\chi(t_j, t_{j+}). \tag{13.31}$$

Here, $i = 1$ if $x^{(1)}(t_j) \in \Pi^r_\alpha(1, t_j)$. The realization of the control u_2 under the strategy $\mathbf{U}$ can be "wrong" only in the interval $[t_j, t_j + \omega]$, where $\omega \le \Delta$. If $x^{(1)}(t_j) \in \Pi^r_\alpha(2, t_j)$, then assume $i = 2$ in the left-hand side of inequality (13.31).

Passing from $\mathscr{V}\big(i, t_{j+}, x^{(1)}(t_{j+})\big)$ to $V^{(2)}\big(t_{j+}, x^{(1)}(t_{j+})\big)$, we get

$$V^{(2)}\big(t_{j+}, x^{(1)}(t_{j+})\big) \le \mathscr{V}\big(i, t_{j+}, x^{(1)}(t_{j+})\big) + \lambda r.$$

Thus,

$$\mathrm{Var}\big(V^{(2)},[t_j,t_{j+}]\big)\le 2\lambda\varDelta\sigma\mu+\lambda r+\lambda\chi(t_j,t_{j+}). \tag{13.32}$$

For intervals of the type $[t_{j+},t_{j+1}]$, it follows that

$$\mathrm{Var}\big(V^{(2)},[t_{j+},t_{j+1}]\big)\le 4\lambda\varDelta\sigma\mu+\lambda\chi(t_{j+},t_{j+1}). \tag{13.33}$$

Due to relations (13.32) and (13.33),

$$\mathrm{Var}\big(V^{(2)},[t_j,t_{j+1}]\big)\le 6\lambda\varDelta\sigma\mu+\lambda r+\lambda\chi(t_j,t_{j+1}).$$

In the interval $[t_\flat,t^\flat]$, there are not more than $[(t^\flat-t_\flat)/\vartheta]$ intervals of the type $[t_j,t_{j+1}]$. (Here and below, $[\cdot]$ denotes the Entier operation.) The last interval terminating at the instant $t^\flat$ can be an interval of the type $[t_j,t_{j+}]$, where $t_{j+}-t_j\le\vartheta$. Gathering estimates for all intervals, we get

$$\mathrm{Var}\big(V^{(2)},[t_\flat,t^\flat]\big)\le\left(\left[\frac{t^\flat-t_\flat}{\vartheta}\right]+1\right)\cdot(6\lambda\varDelta\sigma\mu+\lambda r)+\lambda\chi(t_\flat,t^\flat). \tag{13.34}$$

D. As it was mentioned above, in the interval $[t^z,\hat t]$, not more than two intervals of the type $\mathscr{T}_\flat$ can be located. If there are two of them, then they are separated by an interval of the type $\mathscr{T}_k$. Denote the first of them by $[t_{\flat1},t^{\flat1}]$ and the second one by $[t_{\flat2},t^{\flat2}]$. In the intervals $(t^z,t_{\flat1})$, $(t^{\flat1},t_k)$, $(t^k,t_{\flat2})$, and $(t^{\flat2},\hat t)$, the point $x^{(1)}(t)$ is outside the set $S(1,\alpha,r,t)\bigcup S(2,\alpha,r,t)$. Therefore, in each interval, we can estimate the increment of the function $V^{(2)}$ using Lemma 13.4 assuming $\omega\le\varDelta$. Doing this and taking into account estimates (13.28), (13.29), and (13.34), we get

$$\begin{aligned}\mathrm{Var}\big(V^{(2)},[t_0,\hat t]\big)\le&\left(\left[\frac{\hat t-t_0}{\vartheta}\right]+2\right)\cdot(6\lambda\varDelta\sigma\mu+\lambda r)\\&+4\cdot4\lambda\varDelta\sigma\mu+3\lambda\alpha+\lambda\chi(t_0,\hat t).\end{aligned} \tag{13.35}$$

E. Consider the case when for some $t\ge\hat t$ the point $x^{(1)}(t)$ is inside the set $O\big(\alpha,W^{(2)}_{\hat c}(t)\big)$. Since $Z(t)\subset W^{(2)}_{\hat c}(t)$, this includes, in particular, the case when $t^z\ge\hat t$.

At the instant $t^{\hat c}$, one has estimate (13.30). For $t\ge t^{\hat c}$, the point $x^{(1)}(t)$ is outside the set $O\big(\alpha,W^{(2)}_{\hat c}(t)\big)$. Since

$$Z(t)\subset W^{(2)}_{\hat c}(t),\quad K(t)\subset W^{(2)}_{\hat c}(t),\quad r\le\alpha,$$

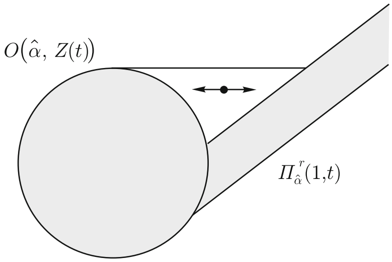

Fig. 13.10 Bad points near the switching set $S(1, \hat{\alpha}, r, t)$ at the place of conjunction of the sets $O\big(\hat{\alpha}, Z(t)\big)$ and $\Pi^r_{\hat{\alpha}}(1, t)$

we get that the motion $x^{(1)}(\cdot)$ is outside the sets $O\big(\alpha, Z(t)\big)$, $O\big(\alpha, K(t)\big)$, $\Pi^r_\alpha(t)$, and, therefore, along the motion $x^{(1)}(\cdot)$ for $t \geq t^{\hat{c}}$, the "correct" first player's control works except, maybe, an interval $[t^{\hat{c}}, t^{\hat{c}} + \omega]$, where $\omega \leq \Delta$. So, using Lemma 13.4, for $t \in [t^{\hat{c}}, T]$, one gets estimate

$$V^{(2)}\big(t, x^{(1)}(t)\big) \leq \hat{c} + \lambda\alpha + 4\lambda\Delta\sigma\mu + \lambda\chi(t^{\hat{c}}, t). \tag{13.36}$$

Let for $t \geq \hat{t}$ the point $x^{(1)}(t)$ be outside the set $O\big(\alpha, W^{(2)}_{\hat{c}}(t)\big)$. Then the motion is also outside the sets mentioned above, and in estimate of type (13.35) for $\mathrm{Var}\big(V^{(2)}, [t_0, t]\big)$, only the last summand grows.

F. Thus, the final estimate at the instant is the maximum of two values $F(T)$ and $L(T)$:

$$\begin{aligned} V^{(2)}\big(T, x^{(1)}(T)\big) &\leq \max\big\{F(t), L(t)\big\}, \\ F(T) &= V^{(2)}(t_0, x_0) + \left(\left[\frac{\hat{t} - t_0}{\vartheta}\right] + 2\right) \cdot (6\lambda\Delta\sigma\mu + \lambda r) \\ &\qquad + 16\lambda\Delta\sigma\mu + 3\lambda\alpha + \lambda\chi(t_0, T), \\ L(T) &= \hat{c} + \lambda\alpha + 4\lambda\Delta\sigma\mu + \lambda\chi(t_0, T). \end{aligned} \tag{13.37}$$

Recall that the quantity $\hat{c}$ depends on $\hat{t}$: $\hat{c} = \tilde{c}(\hat{t})$.

Since $V^{(2)}\big(T, x^{(1)}(T)\big) = \varphi\big(x^{(1)}_1(T), x^{(1)}_2(T)\big)$, inequality (13.37) is an estimate of the first player's guarantee when he uses the strategy **U** in a discrete scheme of control with a step Δ in system (13.9).

A problem with practical application of the strategy **U** is the following. At the instant $t \in [\bar{t}, T)$, there are "bad" points x located outside the set $S(1, \hat{\alpha}, r, t)$ ($S(2, \hat{\alpha}, r, t)$), for which the horizontal (vertical) direction to the line $\Pi(1, t)$ ($\Pi(2, t)$) cannot be determined as the horizontal (vertical) direction from the point x to the set $S(1, \hat{\alpha}, r, t)$ ($S(2, \hat{\alpha}, r, t)$) because the latter direction is not unique. This situation is shown schematically in Fig. 13.10. At the same time due to possible numeric inaccuracies, it is reasonable to think that the switching line for u_1 (u_2) obtained numerically is located in the set $S(1, \hat{\alpha}, r, t)$ ($S(2, \hat{\alpha}, r, t)$).

To exclude this problem for $t \in [\bar{t}, \hat{t}]$, one can do the following thing. Take into account that the line $\Pi(1,t)$ ($\Pi(2,t)$) for $t \in [\bar{t}, \hat{t}]$ crosses the horizontal lines outside the axis x_1 (x_2) with non-zero angle, and there is a lower estimate for this angle. Let us increase α up to some $\breve{\alpha} > \hat{\alpha}$ such that the set $O\big(\breve{\alpha}, Z(t)\big), t \in [\bar{t}, \hat{t}]$, covers that "bad" points in the horizontal (vertical) lines for the set $S(1, \hat{\alpha}, r, t)$ ($S(2, \hat{\alpha}, r, t)$), $r \in [0, \hat{r}]$. Then for each point $x \notin S(1, \breve{\alpha}, r, t)$ ($x \notin S(2, \breve{\alpha}, r, t)$), $t \in [\bar{t}, \hat{t}]$, there is no such a non-uniqueness of the direction to the set $S(1, \hat{\alpha}, r, t)$ ($S(2, \hat{\alpha}, r, t)$) for $r \in [0, \hat{r}]$. Estimate (13.37) holds, but we shall use it for $\alpha = \breve{\alpha}$ only.

For $t \in (\hat{t}, T)$, the choice of the control $u_i, i = 1, 2$, which takes into account the direction of the vector $D_i(t)u_i$ to the switching line $\Pi(i,t)$, is used for obtaining estimate (13.37) only for positions $x^{(1)}(t) \notin O\big(\alpha, W_{\hat{c}}^{(2)}(t)\big)$. If $\alpha \geq \hat{\alpha}, r \in [0, \hat{r}], r \leq \alpha$, and $t \in (\hat{t}, T)$, there is the inclusion $S(i, \alpha, r, t) \subset O\big(\alpha, W_{\hat{c}}^{(2)}(t)\big)$. Thus, the horizontal direction (for $i = 1$) from a point $x \notin O\big(\alpha, W_{\hat{c}}^{(2)}(t)\big)$ to the line $\Pi(1,t)$ coincides with the horizontal direction from this point to the set $S(1, \alpha, r, t)$. In the same way, the vertical direction from a point $x \notin O\big(\alpha, W_{\hat{c}}^{(2)}(t)\big)$ to the line $\Pi(2,t)$ coincides with the horizontal direction from this point to the set $S(2, \alpha, r, t)$.

Theorem 13.1. *Fix* $r \in [0, \hat{r}]$. *Let the multivalued strategy* $\mathbf{U}$ *defined in the interval* $[\bar{t}, T)$ *take the value* $\mathbf{U}_i(t,x) = \big\{u_i : |u_i| \leq \mu\big\}$ *in the set* $S(i, \breve{\alpha}, r, t), i = 1, 2$. *Let outside the set* $S(i, \breve{\alpha}, r, t)$ *the value* $\mathbf{U}_i(t,x)$ *equal either to* $+\mu$, *or to* $-\mu$ *be chosen in such a way that the vector* $D_i(t)\mathbf{U}_i(t,x)$ *is directed to the set* $S(i, \hat{\alpha}, r, t), i = 1, 2$. *Then for any initial position* $(t_0, x_0) \in Y$, *the strategy* $\mathbf{U}$ *in a discrete scheme of control with a step* $\Delta \leq \vartheta(\hat{t}, \hat{\alpha}, \hat{r})$ *guarantees in system* (13.9) *to the first player a result, which is described by formula* (13.37), *where* $\alpha = \breve{\alpha}$.

13.7.2 Stability of Suggested Control Method

Let $\hat{\xi}$ be the lower estimate for the angle between the line $\Pi(1,t)$ and horizontal lines outside the axis x_1, when $t \in [\bar{t}, \hat{t}]$. Define $\beta = r \sin \hat{\xi}$. Consider a neighborhood $O\big(\beta, \Pi(1,t)\big), t \in [\bar{t}, T)$. Take an arbitrary continuous line $\pi(1,t)$ in this neighborhood that will be used for constructing the component U_1^* of the strategy U^*. Let x be an arbitrary phase state at some instant t. Consider a ray with the beginning at this point and directing vector $D_1(t)$. If the ray crosses the line $\pi(1,t)$, then define $U_1^*(t,x) = +\mu$, otherwise $U_1^*(t,x) = -\mu$. In the same way, we can introduce a line $\pi(2,t)$ for constructing the component U_2^*. It is clear that the strategy U^* is a one-valued selector from the multivalued strategy $\mathbf{U}$.

Fix an arbitrary $\varepsilon > 0$. Choose an instant $\hat{t}$ such that $\hat{c} = \varepsilon/4$. Let the number $\breve{\alpha}$ obey the relation $3\lambda\breve{\alpha} = \varepsilon/2$. Choose numbers $\hat{\alpha} \in (0, \breve{\alpha})$ and $\hat{r} \in (0, \hat{\alpha}]$ such that there is an estimate $\vartheta(\hat{t}, \hat{\alpha}, \hat{r}) > 0$, which is less than the time of transfer of systems (13.9) and (13.10) from the set $\Pi_{\hat{\alpha}}^{\hat{r}}(1, \cdot)$ to the set $\Pi_{\hat{\alpha}}^{\hat{r}}(2, \cdot)$ and back in the interval $[\bar{t}, \hat{t}]$. Also, we demand that for chosen $\hat{\alpha}$, $\hat{r}$, and $\breve{\alpha}$, the property of absence of "bad" points x outside the sets

$$O\big(\breve{\alpha}, Z(t)\big) \bigcup O\big(\breve{\alpha}, K(i,t)\big) \bigcup \Pi^r_{\breve{\alpha}}(i,t), \quad i = 1,\ 2,$$

holds. Then the quantity

$$A(\hat{t}, \hat{\alpha}, \hat{r}) = \left[\frac{T - \bar{t}}{\vartheta(\hat{t}, \hat{\alpha}, \hat{r})} \right] + 2$$

is a fixed number. We choose $r^* \in (0, \hat{r}]$ and $\Delta^* \le \vartheta(\hat{t}, \hat{\alpha}, \hat{r})$ such that

$$A(\hat{t}, \hat{\alpha}, \hat{r}) \cdot (6\lambda\Delta^*\sigma\mu + \lambda r^*) + 16\lambda\Delta^*\sigma\mu \le \frac{\varepsilon}{2}.$$

For $\beta = r \sin\hat{\xi}$, one has

$$O\big(\beta, \Pi(i,t)\big) \subset S(i, \hat{\alpha}, r, t), \quad i = 1,\ 2,\ t \in [\bar{t}, \hat{t}].$$

From this due to (13.37), where $\alpha = \breve{\alpha}$, it follows that there exist such $\beta^* > 0$ and $\Delta^* > 0$ that for any $\beta \in [0, \beta^*]$ and $\Delta \in (0, \Delta^*]$, the following estimate is true:

$$\varphi\big(x_1^{(1)}(T), x_2^{(1)}(T)\big) \le V^{(2)}(t_0, x_0) + \varepsilon + \lambda\chi(t_0, T). \tag{13.38}$$

Estimates (13.37) and (13.38) concern the case when at an instant t the first player knows the exact position $x^{(1)}(t)$ of system (13.9) while constructing its control. Now, let us consider the case of inexact measurements.

Assume that instead of the true position $x^{(1)}(t)$ at an instant t, the first player gets some measurement $\zeta(t)$ such that $\big\|\zeta(t) - x^{(1)}(t)\big\| \le h$. He uses this measurement to produce the control $U^*\big(t, \zeta(t)\big)$. As a consequence from estimates (13.37) and (13.38), the next statement follows.

Corollary 13.1. *For any $\varepsilon > 0$, one can choose numbers $\gamma^* > 0, h^* > 0$, and $\Delta^* > 0$ such that if the strategy U^* in system (13.9) is built on the basis of the switching lines $\pi(1,t)$ and $\pi(2,t)$ located for each $t \in [\bar{t}, T)$ in the sets $O\big(\gamma^*, \Pi(1,t)\big)$ and $O\big(\gamma^*, \Pi(2,t)\big)$, respectively, the measurement inaccuracy is not greater than h^*, and the step $\Delta > 0$ of the discrete scheme of control obeys the inequality $\Delta \le \Delta^*$, then for any initial position $(t_0, x_0) \in Y$ and for any realization $v(\cdot)$ of the second player's control, estimate (13.38) holds.*

To prove this statement, it is sufficient to take $\gamma^* \le \beta^*/2, h^* \le \beta^*/2$.

Remark 13.2. We talk about the strategy U^* as a quasioptimal one for system (13.9). The last summand in estimate (13.38) decreases as approximating system (13.10) gets closer to system (13.9). With that, the value $V^{(2)}(t_0, x_0)$ tends to the value $V(t_0, x_0)$ of the Value function for system (13.9). It is reasonable to investigate the limit of the switching lines $\Pi(1,t)$ and $\Pi(2,t)$. It is natural to try to prove that the limit lines define an optimal strategy of the first player in game (13.9). But in this work, we do not deal with such a study.

Remark 13.3. The ideal switching lines $\Pi(1,t)$ and $\Pi(2,t)$ for system (13.10) define an optimal strategy for all initial positions $(t_0, x_0) \in Y$ in this system. This strategy is stable with respect to small inaccuracies of numeric constructions and errors of measurement of the phase state of the system. This follows from estimate (13.38).

13.8 Simulation Results

Let the pursuers P_1, P_2, and the evader E move in the plane. This plane is called the *original geometric space*. At the initial instant t_0, velocities of all objects are parallel to the horizontal axis and sufficiently greater than the possible changes of the lateral velocity components. Velocity of each object has a constant component parallel to the horizontal axis. Magnitudes of these components are such that the horizontal crossings of the objects P_1 and E and the objects P_2 and E happen at the instant T. The dynamics of lateral motion is described by relations (13.1), (13.2); the resultant miss is given by formula (13.4).

Parameters of the game are taken as (13.11). The initial lateral velocities and accelerations are assumed to be zero:

$$\dot{z}^0_{P_1} = \dot{z}^0_{P_2} = \dot{z}^0_E = 0, \;\; a^0_{P_1} = a^0_{P_2} = a^0_E = 0.$$

The initial instant is $t_0 = 0$.

In the following figures, the horizontal axis is denoted by the symbol d. So, the coordinate d shows the longitudinal position of the objects. Controls of the objects are built on the basis of exact measurements of the players' positions and affect the vertical (lateral) coordinate.

In Fig. 13.11, trajectories of the objects are shown for the following values of the initial lateral deviations: $z_{\rho 1}(t_0) = -2, z_{\rho 2}(t_0) = 5$. The first player (who joins the pursuers) applies the quasioptimal control generated by the switching lines built in the framework of system (13.10) under quite fine grids on the parameter c and in time. The control of the second player (evader) is produced on the basis of its switching lines, which are also built in the framework of system (13.10). A typical picture of the switching lines for the second player's control is given in Fig. 13.12. There are six domains, in which the feedback control of the second player keeps one of the extreme values $+\nu$ and $-\nu$. The arrows show directions of the vector $E(t)\nu$ in different domains. The procedure for constructing switching lines for the second player is described in Ganebny et al. (2012a). The control of the second player defined by its switching lines is not justified theoretically yet. We consider it as an empirical quasioptimal one.

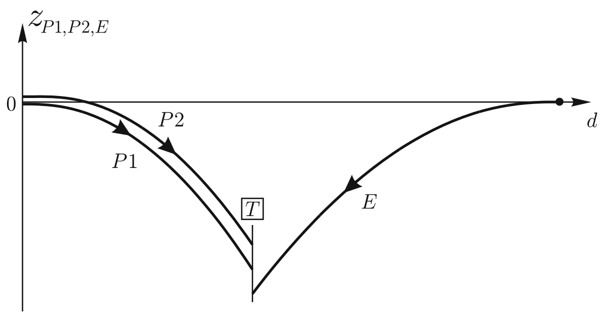

Fig. 13.11 Trajectories of the objects in the original geometric space for small initial deviations after application of the quasioptimal controls

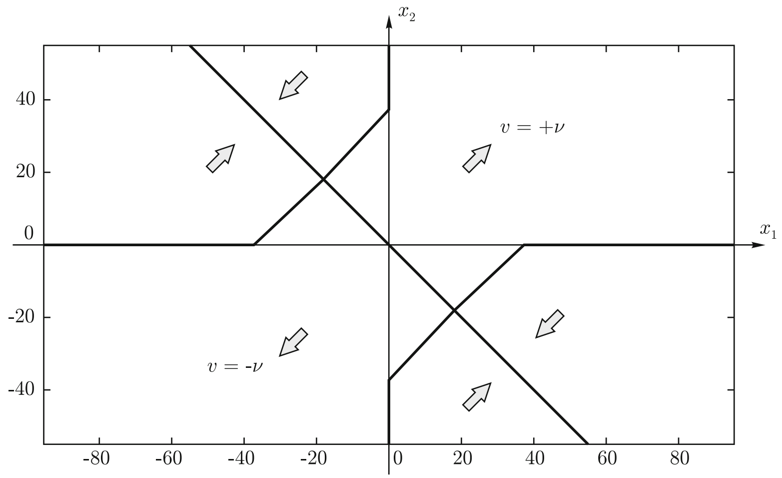

Fig. 13.12 Typical picture of the second player's switching lines

In Fig. 13.13, one can see the trajectories for the same initial lateral deviations, but under a random control of the second player (at each step of the discrete scheme a uniformly distributed value is taken from the interval $[-\nu, +\nu]$ and kept during this step). In comparison with the case of quasioptimal control of the second player, here, the situation of the exact capture is present.

Figure 13.14 shows trajectories for large initial lateral deviations: $z_{\rho_1}(t_0) = -120, z_{\rho_2}(t_0) = 150$. The first player uses its quasioptimal control based on the switching lines. The empirical quasioptimal control of the second one is produced by its switching lines.

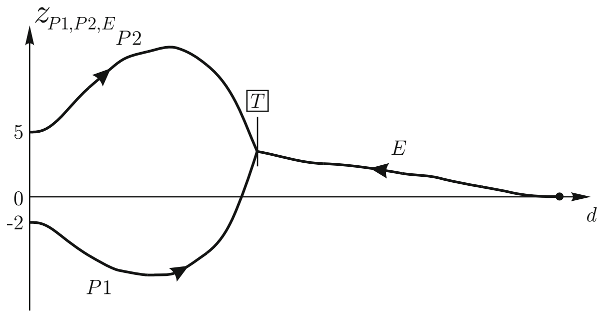

Fig. 13.13 Trajectories of the objects in the original geometric space for small initial deviations after application of the quasioptimal control of the first player and a random control of the second one

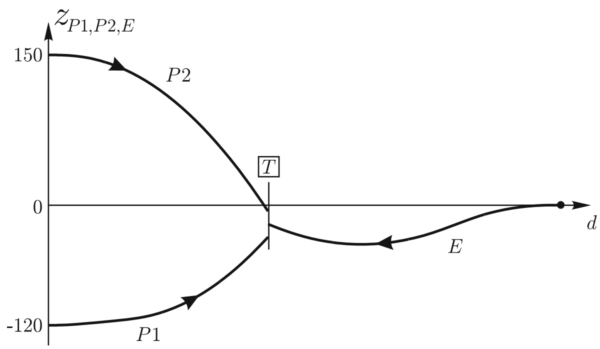

Fig. 13.14 Trajectories of the objects in the original geometric space for large initial deviations after application of the quasioptimal controls

13.9 Conclusion

The main result of the work is in description and justification of a quasioptimal feedback control of the first player in a zero-sum differential game with two “weak” equal pursuers and one evader. The approach is based on construction of two switching lines depending on time for two scalar controls of the first player in the approximating system. The control is stable with respect to inaccuracies of numeric constructions of the switching lines and errors of measurements of the current phase state of the system.

A specific property of the considered problem, which allowed to justify the suggested method of control, is that at the point of crossing the switching lines the value of the Value function of the approximating game decreases with decreasing time-to-go.

Estimates obtained during proof of the main theorem are quite simple, and the interest is in the general scheme of reasoning.

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