Dyn Games Appl (2017) 7:609–633
DOI 10.1007/s13235-016-0209-z

Zero-Sum Pursuit-Evasion Differential Games with Many Objects: Survey of Publications

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Published online: 16 November 2016

Abstract If a pursuit game with many persons can be formalized in the framework of zero-sum differential games, then general methods can be applied to solve it. But difficulties arise connected with very high dimension of the phase vector when there are too many objects. Just due to this problem, special formulations and methods have been elaborated for conflict interaction of groups of objects. This paper is a survey of publications and results on group pursuit games.

Keywords Differential games · Conflict interacting groups of objects · Group pursuit problems · Constant-bearing method · Maximal stable sets

1 Introduction

In the theory of zero-sum differential games, time-optimal games are most typical, but simultaneously most difficult. For example, a number of pursuers chase one evader. The objective of the pursuers is to capture the evader as soon as possible. The pursuers can be joined into a group, which is controlled by the first player, and the evader is the second player. The

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condition of capture defines a set M in the joint phase space whereto the first player tries to guide the phase vector of the system; the second player counteracts this.

In a time-optimal problem, we are interested in finding the least guaranteed time of transfer the system to the set M or an upper estimate of this time for each initial position. Under some fixed control method of the first player (for example, under the optimal one), the transfer time depends on the behavior of the second player. Sometimes, it happens that there are some initial positions wherefrom the first player can provide reaching the target set M *exactly at* some prescribed time instant under *any* control of the second player. In this case, we can talk about passage to a problem with a fixed termination instant. Such games are simpler than the time-optimal ones.

The first publication, in which such an approach in a 1×1 game (a game with one pursuer and one evader) has been applied, is paper [51] by Pontryagin. There for the case of linear stationary dynamics, a method is suggested for description of dynamic advantage of the first player over the second one. Under assumption of such an advantage, there is a procedure for constructing some special set in the space *time* $\times$ *phase variable* wherefrom the first player can transfer the system to the set M at the given instant. But to produce the corresponding control, the first player should discriminate the second one. Namely, the second player announces its control at the current instant or for some small future period to the first one. In paper [52], Pontryagin has described a procedure for constructing the *maximal* set in the space *time* $\times$ *phase variable* wherefrom the transfer is possible. This procedure does not need any dynamic advantage of the first player, but still needs discrimination of the second player.

N.N. Krasovskii has started to call such sets the *stable bridges* and *maximal stable bridges*. It was proved [27,28] that if the transfer to the set M is possible under discrimination of the second player, then the positional strategy extremal to a corresponding bridge keeps the system motion in some small neighborhood of this stable bridge up to the termination instant. Therefore, such a strategy transfers the system into some small neighborhood of the set M. The size of this neighborhood decreases with decreasing the time step in the discrete control scheme. A time-optimal problem can be solved by means of solving a corresponding stationary problem with fixed termination instant. Namely, for a given initial point x_0, one should find an instant t_0 closest to the termination instant T such that the position (t_0, x_0) is inside the stable bridge. Then, the value $T - t_0$ is an upper estimate for the transfer time to the set M from the point x_0.

This was the state of art in the theory of differential games at the middle of 1970s, when group pursuit problems had been started to study intensively. Here, we should emphasize that by that time, the existence of the value function in time-optimal problems and problems with fixed termination instant was proved. Of course, an outstanding contribution in development of differential games has been made by Isaacs' book [26]. In this book, it had been discovered for a wide class of differential games that the value function obeys a first-order partial differential equation of the Hamilton–Jacobi type and a method had been suggested for constructing the solution. The method is based on finding singular surfaces in the game space where the equation is violated (or has no sense) and which separate domains where the equation is satisfied in the classical sense. But it was clear that the singular surfaces can be found more or less effectively only in problems of small dimension. In games with many objects, dimension of the phase vector is usually quite high, and one cannot practically apply the Isaacs' methodology. Numerical methods for differential games were invented at the beginning of 1980s and also embraced only problems of low dimension.

The aim of this paper is to give a short survey of the group pursuit problems. We start (Sect. 2) with publications dealing with certain pursuit problems 2×1 and 1×2. Also, works by L.A. Petrosyan are mentioned where the constant-bearing approach method (well known in engineering practice) was applied to studies of differential games for the first time. Further in our paper, the main attention is paid to the problems with many pursuers and one evader. In Sect. 3, a problem is described, which has been considered by Pschenichnyi in work [56]. This problem involves equal inertialess objects. B.N. Pschenichnyi has shown how to solve such a problem applying the constant-bearing approach method. In Sect. 4 on the basis of book [17], we give a generalization of this approach for the case of general linear dynamics of the objects. Except the general construction, we also give some important types of problems that can be solved analytically without numerical methods. Usually, analytical solution of capture problems of one evader by several pursuers can be obtained under assumption of dynamic advantage of any pursuer over the evader. In Sect. 5, a computational procedure for constructing maximal stable sets for games with linear dynamics and fixed termination instant is considered. In Sect. 6, the simplest (from the point of view of number of pursuers) problem with two pursuers and one evader is described and some examples of maximal stable sets obtained numerically are given. In Sect. 7, a number of difficult group pursuit problems are enlisted that are studied nowadays in various scientific teams.

Of course, problems with many participants include nonzero-sum formulations as well that are usual for economic situations. Also, problems of this type are formalized in the framework of mean-field games, evolutionary games, etc. We do not touch such formulations in this survey. Moreover, we consider problems only where the phase vector is finite-dimensional and the objects' controls are constrained geometrically.

2 First Works on Group Pursuit

An object is called having *dynamics of simple motion*, if it changes direction and/or magnitude of its velocity instantaneously. The maximal velocity magnitude is bounded a priori. This definition has been given by Isaacs in [26].

(1) In work [25], Breakwell and Hagedorn have considered a problem in the plane that involves two pursuers and one evader. All three objects have simple motion dynamics. Two pursuers have equal maximal velocities, which are less than the maximal velocity of the evader. The evader should cross the segment between the pursuers "from left to right" at some instant. The payoff is the minimal value of distance from the evader to the closest pursuer. The first player that joins the pursuers minimizes the resultant payoff; the second player (which is the evader) maximizes it. The authors have shown that in this problem there are initial locations of the objects that produce curvilinear optimal motions for the evader and one pursuer.

(2) Paper [11] by the same authors includes a similar fact about optimal curvilinear motions in games with simple motion that has been established for a problem with one pursuer and two evaders. In this game, three objects move in the plane and the pursuer minimizes total time of capture both of two evaders. The pursuer's velocity is now greater than the evaders' ones. It was justified that there are singular surfaces of dispersal and focal types in this game. When the system moves along the focal surface (in the 4-dimensional phase space of reduced coordinates), trajectories appear, which correspond to curvilinear motions of the objects in the original plane.

At the 8th International Symposium on Dynamic Games and Applications (took place at Chateau Vaalsbroek, Maastricht, NL in 1998), P. Bernhard made a talk "Isaacs, Breakwell, and their sons." Unfortunately, the text of the talk has not been published. In this talk, P. Bernhard remembered that at the First International Conference on the Theory and Applications of

Differential Games held in Amherst in 1969, R. Isaacs pointed importance of the problem "Obstacle Tag Chase game" formulated in his book [26]. Namely, R. Isaacs supposed that in the chase problem with a round obstacle (phase constraint that forces the objects to be outside a round) curvilinear optimal motions can arise despite of simple motion dynamics. P. Bernard noted also that J.V. Breakwell thought about this non-trivial idea during that Conference and after it.

These two works by J.V. Breakwell and P. Hagedorn can be regarded as some result of these reflections. So, maybe, the main outcome of these works is not a control method of players that can be generalized to other problems, but just the fact of existence of curvilinear optimal motions in zero-sum games with simple motion dynamics.

Computations in papers [11,25] are very complicated. It would be reasonable to check them and, possibly, to rewrite in an easier way. The principal point of paper [11] is classification of appearing singular surfaces. But it is not clear enough whether it is correct or not.

In work [2] among several formulations of pursuit problems 1×2, a time-optimal problem is considered similar to that from [11]. Unfortunately, in [2] there is no detailed comparison of obtained results with the ones of paper [11]. Also, there is no complete interpretation of appearing singular surfaces.

(3) In the second half of 1960s, L.A. Petrosyan has published several works (see, for example, [42,43,46]), in which he has studied the game with a life line formulated in the R. Isaacs' book. In this game, the evader tries to reach the boundary of a given set (the "life line"); the pursuer tries to intercept the evader before the reach. L.A. Petrosyan studied this problem assuming that the objects under consideration have simple motion dynamics. Also, problems have been investigated, which involve many pursuers or evaders. In particular, it was proved that if there are many pursuers and only one evader, then for many situations the constant-bearing approach is the optimal behavior for the pursuers. Majority of the obtained results has been included to books [44,45].

When the constant-bearing approach is used, the line of sight "pursuer – target" remains parallel to its original direction during the entire pursuit. To implement this method in a game, it is necessary for each pursuer to know the value $v(t)$ of the evader's control v at the current instant t.

3 Pursuit Problem with Equal Objects Having Simple Motion Dynamics

Now, we pass to work [56] by Pschenichnyi published in 1976. The formulation of the problem does not include anything connected with an optimality criterion. In a finite-dimensional space R^n, a differential game is considered that includes one evader and k pursuers. It is assumed that $k \geq n+1$. All objects have simple motion dynamics and are equal (that is, have equal maximal velocities):

$$\begin{aligned} &\dot{z}_{p_i} = u_i, \quad |u_i| \leq 1, \quad i = \overline{1,k}, \\ &\dot{z}_e = v, \qquad |v| \leq 1. \end{aligned} \tag{1}$$

A capture of the evader means coincidence $z_{p_i}(t) = z_e(t)$ for at least one index i and some instant $t \geq 0$. Assume that when producing a control action at an instant t, the pursuers know the control $v(t)$ of the evader at this instant. This is called the *discrimination of the evader*. One needs to give conditions for possibility of the capture and to estimate its time.

Let us describe the main facts connected with game (1) on the basis of books [17, Section 3] and [23, Section 3].

After passage to relative coordinates $z_i = z_{p_i} - z_e$, one gets

$$\dot{z}_i = u_i - v, \ |u_i| \le 1, \ |v| \le 1, \ i = \overline{1, k}. \tag{2}$$

The capture happens if $z_i(t) = 0$ for at least one i and some instant $t \ge 0$.

Let us fix the initial positions z_{i0}, $i = \overline{1, k}$. Suppose that $z_{i0} \ne 0$ for all i (that is, there is no capture at the initial time instant). Let

$$Z_0 = \mathrm{co}\left\{z_{i0} : \ i = \overline{1, k}\right\}$$

be the convex hull of points z_{i0}, $i = \overline{1, k}$. Below, the symbol int Z_0 denotes its interior.

It is not too difficult to prove that if $0 \notin \mathrm{int}\, Z_0$, then the evader can escape capture in the infinite time interval $[0, \infty)$. Indeed, let l^* be a unit vector such that $\langle l^*, z\rangle \le 0$ for all $z \in Z_0$. Then, the constant control $v^* = -l^*$ allows one to keep the relation $z_i(t) \ne 0$ for any open-loop control $u_i(\cdot)$.

The main difficulty of this problem is in investigation of the case when $0 \in \mathrm{int}\, Z_0$. In the original coordinates, this inclusion means

$$z_{e0} \in \mathrm{int\ co}\left\{z_{p_i 0} : \ i = \overline{1, k}\right\}.$$

B.N. Pschenichnyi does not seek for an optimal control of the first player (who controls the group of the pursuers) in a time-optimal problem. Instead, he assigns the constant-bearing approach method for all pursuers. He proves that this method provides the capture not later than some instant despite the behavior of the evader and gives an upper estimate for this instant.

Let the evader be initially at the point z_{e0} in the original coordinates and the ith pursuer be at the point $z_{p_i 0}$. Then the vector $z_{i0} = z_{p_i 0} - z_{e0}$ defines the direction of the initial line of sight from the evader to the ith pursuer.

(1) Denote by S the ball of radius 1 with the center at the origin of the space R^n. For each i and v obeying the inequality $|v| \le 1$, consider relation

$$z_{i0} + \chi\,(S - v) \ni 0. \tag{3}$$

Find the minimal positive value $\chi = \chi_i^*(v)$ such that relation (3) holds. The value $\chi_i^*(v)$ is the first instant of absorption of the origin by relative system $\dot{z}_i = u_i - v$, $|u_i| \le 1$, under a constant value v of the second player's control and with the initial position z_{i0}. The symbol z_{i0} is not included to arguments of χ_i^* because the point $z_{i0} \ne 0$ is regarded to be fixed. If for some i and v relation (3) is false for all $\chi > 0$, assume $\chi_i^*(v) = +\infty$.

Denote $\alpha_i^*(v) = 1/\chi_i^*(v)$. For $\chi_i^*(v) < +\infty$, the value $\alpha_i^*(v)$ is the maximal positive α such that the relation

$$(S - v) \ni -\alpha z_{i0}$$

is true. If $\chi_i^*(v) = +\infty$, then let $\alpha_i^*(v) = 0$.

From the condition $0 \in \mathrm{int}\, Z_0$, it follows that

$$\delta = \min_v \max_i \alpha_i^*(v) > 0. \tag{4}$$

Here, the minimum is taken over all controls v of the evader under constraint $|v| \le 1$, and the maximum is taken over all indices $i = \overline{1, k}$.

If at the current instant t one has $z_i(t) \ne 0$ and $\alpha_i^*\big(v(t)\big) \ne 0$, then let us define a vector $u_i^*\big(v(t)\big) \in \partial S$ in such a way that

$$u_i^*\big(v(t)\big) - v(t) = -\alpha_i^*(v(t))z_{i0}.$$

If $z_i(t) = 0$ or $\alpha_i^*\big(v(t)\big) = 0$, assume $u_i^*\big(v(t)\big) = v(t)$.

(2) Let a feasible control $v(t)$ be given (that is, a measurable function $t \mapsto v(t)$ obeying the constraint $\big|(v(t)\big| \leq 1)$. Show that for any $i = \overline{1,k}$ the control $t \to u_i^*\big(v(t)\big)$ provides the constant-bearing approach of the point $z_{p_i}(t)$ with the point $z_e(t)$ until they coincide for the first time.

One has

$$\begin{aligned} z_i(t) &= z_{i0} + \int_0^t u_i^*\big(v(s)\big)ds - \int_0^t v(s)ds \\ &= z_{i0} - \int_0^t \alpha_i^*\big(v(s)\big)ds \cdot z_{i0} = \left(1 - \int_0^t \alpha_i^*\big(v(s)\big)ds\right)z_{i0}. \end{aligned}$$

Then for any instant t such that

$$\int_0^t \alpha_i^*\big(v(s)\big)ds < 1,$$

the vector $z_i(t)$ has the same direction as the vector $z_{i0} = z_{p_i0} - z_{e0}$. Thus, the chosen control of the ith pursuer indeed provides the constant-bearing approach. At the first instant $\bar{t}$ when

$$\int_0^{\bar{t}} \alpha_i^*\big(v(s)\big)ds = 1,$$

one gets $z_i(\bar{t}\,) = 0$.

(3) Let z_0 be the vector with nonzero components $z_{10}, z_{20}, \dots, z_{k0}$. Following [17, Section 3], introduce the instant

$$T(z_0) = \min\left\{t > 0 : \inf_{v(\cdot)} \max_i \int_0^t \alpha_i^*\big(v(s)\big)ds = 1\right\}. \tag{5}$$

Consider an arbitrary feasible control $v(\cdot)$ of the evader. For the instant $T(z_0)$, one gets

$$\max_i \int_0^{T(z_0)} \alpha_i^*\big(v(s)\big)ds \geq 1. \tag{6}$$

Let us fix the index i (without introducing an additional denotation) that provides the maximum in the left-hand side of (6). Thus,

$$\int_0^{T(z_0)} \alpha_i^*\big(v(s)\big)ds \geq 1.$$

Suppose $\tilde{t} \leq T(z_0)$ to be such that

$$\int_0^{\tilde{t}} \alpha_i^*\big(v(s)\big)ds = 1.$$

Let write the component $z_i(\tilde{t})$ of system (2) at the instant $\tilde{t}$:

$$z_i(\tilde{t}) = z_{i0} + \int_0^{\tilde{t}} u_i^*\big(v(s)\big)ds - \int_0^{\tilde{t}} v(s)ds$$
$$= z_{i0} - \int_0^{\tilde{t}} \alpha_i^*\big(v(s)\big)ds \cdot z_{i0} = \Bigg(1 - \int_0^{\tilde{t}} \alpha_i^*\big(v(s)\big)ds\Bigg) z_{i0}.$$

Thus, we obtain $z_i(\tilde{t}) = 0$. For $t \geq \tilde{t}$ until $T(z_0)$, the control $u_i^*\big(t, v(t)\big) = v(t)$ is applied.

So, the suggested control of the pursuers based on the constant-bearing approach guarantees the capture at the instant $T(z_0)$ for any evader's control $v(\cdot)$.

(4) Let us prove the estimate

$$T(z_0) \leq \frac{k}{\delta}. \tag{7}$$

Take an $\varepsilon > 0$ and fix a measurable function $v(\cdot)$ that provides in (5) the value

$$\max_i \int_0^{T(z_0)} \alpha_i^*\big(v(s)\big)ds \leq 1 + \varepsilon.$$

For any $t \leq T(z_0)$, it is true that

$$\max_i \int_0^t \alpha_i^*\big(v(s)\big)ds \geq \frac{1}{k} \sum_{i=1}^k \int_0^t \alpha_i^*\big(v(s)\big)ds$$
$$= \frac{1}{k} \int_0^t \sum_{i=1}^k \alpha_i^*\big(v(s)\big)ds \geq \frac{1}{k} \int_0^t \min_{|v| \leq 1} \sum_{i=1}^k \alpha_i^*(v)ds \geq$$
$$\geq \frac{1}{k} \int_0^t \min_{|v| \leq 1} \max_i \alpha_i^*(v)ds = \frac{1}{k} \cdot t \cdot \delta.$$

Assuming $t = T(z_0)$, we get

$$1 + \varepsilon \geq \frac{1}{k} T(z_0)\delta.$$

Therefore,

$$T(z_0) \leq (1 + \varepsilon)\frac{k}{\delta}.$$

Now vanishing ε, one obtains estimate (5).

Let us draw some conclusions. The constant-bearing approach method is wide-used in engineering practice in guiding and intercept problems. It has turned out that in many person pursuit problems (in the case of equal objects with simple motion dynamics), this method gives a clear geometric picture of the pursuit and allows one to finish the pursuit not later the instant $T(z_0)$ despite of the evader's behavior. Moreover, the first player (that joins all pursuers) guarantees the pursuit termination exactly at the instant $T(z_0)$. This instant can be easily estimated from above if the initial positions of objects are known. Also, the conditions providing possibility of the capture are clear from the geometric point of view: the evader should be inside the convex hull of the initial positions of the pursuers.

Studying the problem, we assume that the initial instant t_0 is equal to 0. In this case, the instant $T(z_0)$ of the guaranteed capture depends on z_0. One can do in an other way: fix the termination instant T, and let the initial instant t_0 depend on z_0. With that, choose it in such a way that the difference $T - t_0(z_0)$ would be minimal. So, in this case, we could talk about capture of the evader in the framework of a differential game with a fixed termination instant.

4 Pursuit Problems with Objects Having General Linear Dynamics

Now, we are going to give some generalization of the facts from the previous section for the case of objects having general stationary linear dynamics.

Let there be k pursuers and one evader:

$$\begin{aligned} \dot{\mathbf{z}}_{p_i} &= A_{p_i}\mathbf{z}_{p_i} + B_{p_i}u_i, && u_i \in \mathcal{P}_i,\ i = \overline{1,k}, \\ \dot{\mathbf{z}}_e &= A_e\mathbf{z}_e + B_e v, && v \in \mathcal{Q}. \end{aligned} \tag{8}$$

Here, $\mathcal{P}_i$, $\mathcal{Q}$ are convex compact constraints on the controls of corresponding objects.

Take some initial positions $\mathbf{z}_{p_i0}$, $i = \overline{1,k}$, $\mathbf{z}_{e0}$, and an instant T that is considered as the instant of termination of the game. Let z_{p_i} and z_{ei}, $i = \overline{1,k}$, be vectors composed of some prescribed components of the vectors $\mathbf{z}_{p_i}$ and $\mathbf{z}_e$, respectively. For each i, assume that dimensions of the vectors z_{p_i} and z_{ei} are the same. The game is regarded to be finished successfully if at the instant T at least for one i the relation $z_{p_i}(T) = z_{ei}(T)$ is true. Emphasize that for each number i, the capture condition of the evader by the ith pursuer is defined by coincidence of some collections of components of the vectors $\mathbf{z}_{p_i}$ and $\mathbf{z}_e$. The collections of components of $\mathbf{z}_e$ can be different for different i.

Consider fundamental Cauchy matrices $\mathbf{Z}(T, t; A_{p_i})$, $\mathbf{Z}(T, t; A_e)$ that correspond to the matrices A_{p_i}, $i = \overline{1,k}$, A_e of system (8). Since system (8) is stationary, these fundamental matrices depend on the difference $T - t$ only.

Let $Z_i(T, t; A_{p_i})$, $Z_i(T, t; A_e)$ be submatrices of the matrices $\mathbf{Z}(T, t; A_{p_i})$, $i = \overline{1,k}$, $\mathbf{Z}(T, t; A_e)$ composed of the rows with the same numbers as the components of the vectors z_{p_i}, z_{ei} have in the vectors $\mathbf{z}_{p_i}$ and $\mathbf{z}_e$. Then

$$\begin{aligned} x_{p_i}(t) &= Z_i(T, t; A_{p_i})\mathbf{z}_{p_i}(t),\ i = \overline{1,k}, \\ x_{ei}(t) &= Z_i(T, t; A_e)\mathbf{z}_e(t) \end{aligned} \tag{9}$$

are the forecast values of the selected coordinates of the pursuers' and evader's positions to the instant T along a motion of system (8) under zero controls of the objects in the interval $[t, T]$. Note that

$$x_{p_i}(T) = z_{p_i}(T),\ x_{ei}(T) = z_{ei}(T).$$

Let

$$x_i(t) = x_{p_i}(t) - x_{ei}(t). \tag{10}$$

Evolution of the variables $x_i(t)$, $i = \overline{1,k}$, is described as

$$\dot{x}_i = Z_i(T, t; A_{p_i})B_{p_i}u_i - Z_i(T, t; A_e)B_e v, \quad u_i \in \mathcal{P}_i,\ v \in \mathcal{Q}. \tag{11}$$

Now, instead of system (8), system (11) can be considered and all reasonings can be made in the framework of this system.

Denote

$$D_i(t) = Z_i(T, t; A_{p_i})B_{p_i}, \ E_i(t) = Z_i(T, t; A_e)B_e, \ i = \overline{1, k}.$$

Then

$$\dot{x}_i = D_i(t)u_i - E_i(t)v, \ \ u_i \in \mathcal{P}_i, \ v \in \mathcal{Q}, \ i = \overline{1, k}. \tag{12}$$

The objective of the first player that joins the pursuers is to provide the equality $x_i(T) = 0$ for at least one index $i = \overline{1, k}$. The second player, which is the evader, tries to hinder this.

Below, it is assumed that for any $i = \overline{1, k}$ and for any $t \le T$ the following inclusions hold:

$$E_i(t)\mathcal{Q} \subset D_i(t)\mathcal{P}_i. \tag{13}$$

Presence of this condition is equivalent to the following requirement: for any $v \in \mathcal{Q}$, there is such an $u_i \in \mathcal{P}_i$ that

$$D_i(t)u_i - E_i(t)v = 0.$$

Thus, we introduce some dynamic advantage of any pursuer over the evader. In book [17], there are other, less tough variants of the advantage definition, but we limit us to condition (13).

For any $i = \overline{1, k}$, the initial location $x_{i0} = x_i(t_0)$ in system (12) depends on the initial locations $\mathbf{z}_{p_i0}$, $\mathbf{z}_{e0}$ in original system (8) and on the chosen instant t_0, namely, on the difference $T - t_0$. One has

$$x_{i0} = Z_i(T, t_0; A_{p_i})\mathbf{z}_{p_i0} - Z_i(T, t_0; A_e)\mathbf{z}_{e0}.$$

Our aim is to obtain conditions for choosing the instant t_0 that provide fulfillment of at least one of the equalities $x_i(T) = 0$, $i = \overline{1, k}$.

(1) Fix an arbitrary instant $t_0 < T$. At first, assume $x_{i0} \neq 0$, $i = \overline{1, k}$. Taking ideas of book [17] as the basis, let us do in the same way as in the previous section.

For each i, consider the relation

$$x_{i0} + \big(D_i(t)\mathcal{P}_i - E_i(t)v\big)\chi \ni 0, \ \ v \in \mathcal{Q}, \ t \in [t_0, T]. \tag{14}$$

Find the minimal $\chi = \chi_i^*(t, v) > 0$ that meets this requirement. If for some i, t, v relation (14) is not true for all $\chi > 0$, take formally $\chi_i^*(t, v) = +\infty$. In the case $\chi_i^*(t, v) < +\infty$, assume

$$\alpha_i^*(t, v) = 1/\chi_i^*(t, v).$$

If $\chi_i^*(t, v) = +\infty$, then $\alpha_i^*(t, v) = 0$.

If at the current instant t one has $x_i(t) \neq 0$ and $\alpha_i^*\big(t, v(t)\big) \neq 0$, then define a vector $u_i^*\big(t, v(t)\big) \in \partial\mathcal{P}_i$ in such a way that

$$D_i(t)u_i^*\big(t, v(t)\big) - E_i(t)v(t) = -\alpha_i^*\big(t, v(t)\big) \cdot x_{i0}. \tag{15}$$

If $x_i(t) = 0$ or $\alpha_i^*\big(t, v(t)\big) = 0$, assume $u_i^*\big(t, v(t)\big)$ such that

$$D_i(t)u_i^*\big(t, v(t)\big) = E_i(t)v(t). \tag{16}$$

Let $x_{i0} = 0$ for some i. Then, the control $u_i^*\big(t, v(t)\big)$ is chosen by relation (16) in the entire interval $[t_0, T]$.

We should provide that for any feasible control $v(\cdot)$ (that is, a measurable function $t \mapsto v(t)$ obeying the constraint $v(t) \in \mathcal{Q}$) the function $t \to u_i^*\big(t, v(t)\big)$ is measurable too. Corresponding explanations are given in [17, Section 3].

(2) Show that the control $t \to u_i^*(t, v(t))$ provides the constant-bearing approach control of the point $x_{p_i}(t)$ to the point $x_{ei}(t)$ up to the first instant of their coincidence, that is, when $x_i(t) = 0$. Let $t \to v(t)$ be a measurable function. One has

$$x_i(t) = x_{i0} + \int_{t_0}^{t} D_i(s) u_i^*\big(s, v(s)\big) ds - \int_{t_0}^{t} E_i(s) v(s) ds$$
$$= x_{i0} + \int_{t_0}^{t} \alpha_i^*\big(s, v(s)\big) ds \cdot x_{i0} = \left(1 - \int_{t_0}^{t} \alpha_i^*\big(s, v(s)\big) ds\right) x_{i0}.$$

Then for any instant t such that

$$\int_{t_0}^{t} \alpha_i^*\big(s, v(s)\big) ds < 1,$$

the vector $x_i(t) = x_{p_i}(t) - x_{ei}(t)$ has the same direction as the vector $x_{i0} = x_{p_i}(t_0) - x_{ei}(t_0)$. At the first instant $\bar{t}$, when

$$\int_{t_0}^{\bar{t}} \alpha_i^*\big(s, v(s)\big) ds = 1,$$

one obtains $x_i(\bar{t}\,) = 0$. After the instant $\bar{t}$, the control $u_i^*\big(t, v(t)\big)$ is chosen in such a way that relation (16) holds.

(3) For the given T, we are interested to choose t_0 as close to T as possible in such a way that to get $x_i(T) = 0$ for at least one $i = \overline{1, k}$ by means of the control $u_i^*\big(t, v(t)\big)$ constructed as it is described above.

Let $x_{i0} \neq 0$ for all $t_0 < T$ and for all $i = \overline{1, k}$. Consider the value

$$t_0^* = \sup\left\{ t_0 < T : \inf_{v(\cdot)} \max_i \int_{t_0}^{T} \alpha_i^*\big(s, v(s)\big) ds = 1 \right\}. \tag{17}$$

If $t_0^* > -\infty$, then in (17) instead of sup one can write max.

Suppose that $t_0^* > -\infty$. Take the instant t_0^* as the initial one in system (8). Using it, define the initial vectors x_{i0}. Let us show that at the instant T the relation $x_i(T) = 0$ holds for at least one i.

Consider a feasible control $v(\cdot)$ of the evader. One gets

$$\max_i \int_{t_0^*}^{T} \alpha_i^*\big(s, v(s)\big) ds \geq 1. \tag{18}$$

Let us fix (without introducing an additional denotation) the index i that provides the maximum in the right-hand side of (18). One has

$$\int_{t_0^*}^{T} \alpha_i^*\big(s, v(s)\big) ds \geq 1.$$

Let $\tilde{t} \le T$ be such that

$$\int\limits_{t_0^*}^{\tilde{t}} \alpha_i^*\big(s, v(s)\big) ds = 1.$$

One obtains

$$\begin{aligned} x_i(\tilde{t}) &= x_{i0} + \int\limits_{t_0^*}^{\tilde{t}} D_i(s) u_i^*\big(s, v(s)\big) ds - \int\limits_{t_0^*}^{\tilde{t}} E_i(s) v(s) ds \\ &= x_{i0} - \int\limits_{t_0^*}^{\tilde{t}} \alpha_i^*\big(s, v(s)\big) ds \cdot x_{i0} = \Bigg(1 - \int\limits_{t_0^*}^{\tilde{t}} \alpha_i^*\big(s, v(s)\big) ds\Bigg) x_{i0} = 0. \end{aligned}$$

After the instant $\tilde{t}$, the control $u_i^*\big(t, v(t)\big)$ is chosen in such a way that relation (16) holds. Hence, $x_i(T) = 0$.

So, if $t_0^* > -\infty$, then the suggested controls of the pursuers based on the constant-bearing approach (in the forecast coordinates) guarantee the capture at the instant T for any evader's control $v(\cdot)$. The value $T - t_0^*$ is taken as the upper estimate for the capture time.

Assume that $x_{i0} = 0$ for at least some $t_0 < T$ and some $i = \overline{1, k}$. Consider an instant

$$t_0^* = \max\{t_0 < T : x_{i0} = 0 \text{ for at least one } i = \overline{1, k}\}.$$

Take the instant t_0^* as the initial one in system (8). Fix the index i, for which $x_{i0} = 0$ at this instant. The control $u_i^*\big(t, v(t)\big)$ computed by (16) in $[t_0^*, T]$ provides the equality $x_i(t) = 0$ for $t \in [t_0^*, T]$.

Thus, the suggested method for the first player's control provides capture at the instant T. Until the first instant when the equality $x_i(t) = 0$ holds, the ith pursuer control is generated according to the constant-bearing method. With that, at any current instant t, it uses the value $v(t)$ [see (15)]. After the first instant when $x_i(t)$ reaches the origin until the instant T, the ith pursuer also discriminates the evader; namely, the pursuer takes $u_i^*(t)$ according to (16). The control law is simple, but it differs from (15). Thus, if we define formally the feasible strategies of the first player a priori, then we would use strategies with instantaneous discrimination of the second player of kind $u_i = u_i\big(t, x_{i0}, v\big)$ but with a capability to change the control law when the system reaches the manifold $x_i = 0$.

Of course, a crucial question arises about a possibility of pursuers' *positional* control construction that guarantees the capture of the evader at the given instant T. The word "positional" means that the control is constructed on the basis of the current phase state of the game without any discrimination of the evader. Taking into account the general statement "if the capture can be done exactly with a discrimination, then it is possible to provide an ε-capture without discrimination," one can hope to construct the corresponding positional control. Nevertheless, an accurate investigation of this topic is usually very complicated and is a problem itself.

Note that we do not speak here about conditions that guarantee evader's escape. Such conditions have been studied, but in contrast to the problem considered in Sect. 3, corresponding statements are not so simple.

Paper [56] by Pschenichnyi and further works by Chikrii, Grigorenko, Petrov, and their colleagues (see, for example, [8,15,21,22,47–49,57,59]) have improved significantly the theory of many person pursuit games.

At first, for games of this type, situations of capture have been started to study that imply not exact coincidence of positions of the pursuer and evader, but guiding the evader to some neighborhood of the pursuer. Also, problems with state constraints have been studied. Moreover, techniques have been elaborated that reduce problems with state constraints to ones without constraints. Sufficient conditions have been formulated that guarantee multiple capture. At the same time, problems with one pursuer and many evaders are studied too. Entire experience accumulated allowed one to explore problems with a group of pursuers and a group of evaders.

There are three important books [7,16,23] published in Russian. They contain a systematic description of theoretic results obtained on this topic during the last quarter of the previous century and at the beginning of the current one. Book [16] has been translated into English [17].

These books include a lot of examples that have been solved by the suggested methods. Among them let us note the problems with objects, whose dynamics is described by a differential relation of the following type:

$$z^{(p)} + a_1 z^{(p-1)} + \cdots + a_{p-1}\dot{z} + a_p z = u.$$

Here, z is a finite-dimensional phase vector of the object, $z^{(p)}$ is the pth time derivative, a_i, $i = 1, \ldots, p$, are numeric coefficients, and u is the vector control constrained by a convex compactum.

All three books suggest both own authors' investigations and ideas of other scientists. They can be used as teaching aids too. In book [23], an evasion differential game with a linear dynamics is considered, for which sufficient conditions are formulated that provide escape for the evader in the infinite time interval for any initial locations of objects. The basis of the method is the local evasion maneuver elaborated by Pontryagin and Mischenko in works [53,55]. This method is expanded to the situation of many pursuers in paper [38]. Also, in book [23] there is a section devoted to an evasion method worked out by Chernous'ko [14]. This method provides escape of an object with simple motion from a group of pursuers that have simple motion dynamics too. Under assumption of advantage in velocity of the evader over each pursuer, an algorithm has been suggested providing a certain distance evasion from all pursuers with keeping the motion inside a prescribed neighborhood of a given basic trajectory.

5 Procedure for Constructing Maximal Stable Sets

In Sect. 4, we have described an approach to solution of a stationary linear pursuit problem with k pursuers and one evader. Underline the following essential points of our description.

1. The original dynamics of all objects is supposed to be linear and stationary. It is agreed that the dynamic capabilities of each pursuer are not worse than the ones of the evader.
2. The initial locations of the objects are given.
3. An instant T is fixed that is regarded as the termination one.
4. A new phase vector is introduced, which is composed of vectors x_i, $i = \overline{1,k}$ [see (9) and (10)]. Each $x_i(t)$ is the difference of the forecasts under zero controls of the objects to the instant T of the selected subvectors of the ith pursuer's and evader's phase vectors

that are involved in the relations defining the rendezvous of this pursuer and the evader along the motion of system (8). System (12) obtained as a result does not contain the new phase variables in its right-hand part. The capture condition can now be written as $x_i(T) = 0$ for at least one $i = \overline{1, k}$.
5. At the current instant t, to generate their controls, the pursuers use the value $v(t)$ of the evader's control at this instant.
6. The initial instant t_0^* is chosen in a special way.
7. Each pursuer produces its control on the basis of the constant-bearing method that is implemented in the forecast coordinates. It is shown that for any behavior of the evader, it is captured at the instant T if the initial locations are related to the instant t_0^*.

Introduce the set $M = \{x : x_i = 0 \text{ for at least one } i = \overline{1, k}\}$. Let $x = \left(x_1, \ldots, x_k\right)^\top$ and $x(t_0^*)$ be the initial vector at the instant t_0^*. Consider the motion bundle appearing in the space t, x when the first player produces its control on the basis of the constant-bearing method. This bundle is emanated at the instant t_0^* from the initial location $x(t_0^*)$ and is generated by all feasible realizations of the second player's control. The bundle is stopped on the set M at the instant T. This bundle can be regarded as a stable set (a stable bridge).

(1) A general method for constructing a *maximal* stable set ending on a given arbitrary terminal set M at the fixed instant T has been described for linear systems in papers by Pontryagin [52,54]. Krasovskii [27,28] has formulated the concept of stable sets for a general case and put it on the basis of the differential game theory. Procedures for constructing maximal stable sets in differential games with a fixed termination instant have been considered in works [39,58,68,69]. Below, following [52,54], we give a schematic description of a method for constructing maximal stable sets in linear games.

We suppose that the original linear stationary game with a fixed termination instant T is already reduced to the type

$$\dot{x} = D(t)u - E(t)v, \quad u \in \mathcal{P},\ v \in \mathcal{Q}, \tag{19}$$

and a closed terminal set M is defined in the coordinates x. It is necessary to construct in the space $t, x, t \in [t_*, T]$, the maximal stable set W ending on the set M at the instant T. Here and below, $t_* < T$ is some instant that defines the interval of constructions. Further, the symbol $W(t)$ denotes a time section (t-section) of the set W at the instant t: $W(t) = \big\{x : (t, x) \in W\big\}$.

(2) A theoretic computational scheme of a backward construction can be described as follows. In the interval $[t_*, T]$, introduce a time grid $\Theta = \{t_1 = t_* < t_2 < \cdots < t_N = T\}$. In each semi-interval $[t_i, t_{i+1})$, dynamics (19) is changed by a constant dynamics

$$\dot{x} = D(t_i)u - E(t_i)v, \quad u \in \widetilde{\mathcal{P}},\ v \in \widetilde{\mathcal{Q}}. \tag{20}$$

Here, the sets $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{Q}}$ are some approximations of the sets $\mathcal{P}$ and $\mathcal{Q}$. Assume $\widetilde{W}(t_N) = \widetilde{M}$ where $\widetilde{M}$ is some approximation of the set M.

Suppose that the set $\widetilde{W}(t_{i+1})$ is built. Then, the set $\widetilde{W}(t_i)$ can be found by formula

$$\widetilde{W}(t_i) = \big(\widetilde{W}(t_{i+1}) + (-\Delta_i)D(t_i)\widetilde{\mathcal{P}}\big) \overset{*}{-} (-\Delta_i)E(t_i)\widetilde{\mathcal{Q}}. \tag{21}$$

In this formula, $\Delta_i = t_{i+1} - t_i$, the signs "$+$" and "$\overset{*}{-}$" mean Minkowski sum and difference (algebraic sum and geometric difference):

$$\mathcal{A} + \mathcal{B} = \{a + b : a \in \mathcal{A},\ b \in \mathcal{B}\}, \quad \mathcal{A} \overset{*}{-} \mathcal{B} = \bigcap_{b \in \mathcal{B}} (\mathcal{A} - b).$$

The operation of geometric difference has been introduced to the control theory and theory of differential games by Pontryagin in works [51,52] and nowadays is used widely both in differential games and in computational geometry.

It can happen that for some t_i the obtained set $\widetilde{W}(t_i)$ is empty. In this case, further backward constructions are ceased.

(3) Formula (21) has a very clear sense. The term $\widetilde{W}(t_{i+1}) + (-\Delta_i)D(t_i)\widetilde{\mathcal{P}}$ is a set composed of points such that from any of them taken at the instant t_i, the first player can guide a motion of system (20) to the set $\widetilde{W}(t_{i+1})$ at the instant t_{i+1} under some constant control from the set $\widetilde{\mathcal{P}}$ if the second player's control v equals zero. Then, the set

$$\big(\widetilde{W}(t_{i+1}) + (-\Delta_i)D(t_i)\widetilde{\mathcal{P}}\big) - (-\Delta_i)E(t_i)v$$

is the collection of points at the instant t_i wherefrom the first player can guide a motion of system (20) to the set $\widetilde{W}(t_{i+1})$ at the instant t_{i+1} under some constant control from the set $\widetilde{\mathcal{P}}$ if the second player's control $v \in \widetilde{\mathcal{Q}}$ is known. Intersection of these sets for all possible controls $v \in \widetilde{\mathcal{Q}}$ of the second player gives the set $\widetilde{W}(t_i)$ wherefrom the first player can guide the motion of system (20) to the set $\widetilde{W}(t_{i+1})$ if the second player reveals its constant control in the interval $[t_i, t_{i+1})$.

The recurrent procedure for constructing sets $\widetilde{W}(t_i)$ in the backward time is actually the dynamic programming procedure applied to a differential game of type (19). Vanishing diam Θ, one can speak about the limit set W in the space t, x. With appropriate approximations and quite small diameter of the grid Θ, the sets $\widetilde{W}(t_i)$ are close to the sets $W(t_i)$ in Hausdorff metrics.

(4) The theoretic scheme being clear ideologically meets significant algorithmic difficulties during practical implementation. They arise in subroutines for constructing Minkowski sum and difference, especially, if the sets are non-convex. In the case of convex polygons in the plane, these algorithms can be implemented very effectively and easy [24,31,67]. Computational problems grow catastrophically with increasing dimension of the phase vector of system (19). But despite this, there are some implementations of the sum and difference procedures for multi-dimensional convex polyhedra [10,50,70,71]. In the case of non-convex sets in the plane [in the case of two-dimensional phase vector of system (19)], there are some realizations (often, heuristic) made by mathematicians [18,41] and computer graphics specialists.

In problems of group pursuit, dimension of the phase vector of system (19) depends on the number of objects and conditions of capture. Often, dimension is extremely high. Due to this, usually, the group pursuit problems are not studied by means of construction of the maximal stable sets W. This concerns even well worked out grid methods for constructing the sets W (see, for example, [37]) or the value function [6,9].

Nevertheless, there are interesting problems with few objects where the maximal stable sets W can be built.

6 Game with Two Pursuers and One Evader

6.1 Problem Formulation

Let motions of the pursuers P_1, P_2, and the evader E be described in the vector form as follows:

$$\begin{aligned}\dot{\mathbf{z}}_{p_i} &= A_{p_i}\mathbf{z}_{p_i} + B_{p_i}u_i, \ |u_i| \le \mu_i, \ \mathbf{z}_{p_i} \in R^{n_i}, \ i = 1, 2, \\ \dot{\mathbf{z}}_e &= A_e\mathbf{z}_e + B_e v, \qquad |v| \le \nu, \ \mathbf{z}_e \in R^{n_e}.\end{aligned} \tag{22}$$

Here, u_1, u_2, and v are scalar controls; A_{p_1}, A_{p_2}, and A_e are square matrices; B_{p_1}, B_{p_2}, and B_e are column matrices.

Denote by z_{p_i}, $i = 1, 2$, and z_e the first components of the vectors $\mathbf{z}_{p_i}$, $i = 1, 2$, and $\mathbf{z}_e$, respectively. Assume that they are the geometric coordinates of the objects.

We fix an instant T. The payoff function is introduced as

$$\varphi = \min\bigl\{\bigl|z_{p_1}(T) - z_e(T)\bigr|, \bigl|z_{p_2}(T) - z_e(T)\bigr|\bigr\}. \tag{23}$$

Consider the following zero-sum differential game: the first player using controls u_1 and u_2 minimizes the payoff φ; the second one maximizes the payoff value by its control v.

6.2 Practical Motivation

The considered problem arises when studying a pursuit in upper atmosphere layers. The scheme of the pursuit is given in Fig. 1 and is taken from works by J. Shinar. We assume that both nominal trajectories are in the plane. The components of the nominal velocities along the horizontal axis are *quite large*, and the angles between the nominal velocities and the horizontal axis are *quite small*. Therefore, the longitudinal motion can be regarded as *uniform*, and we can consider *only the lateral motion* and measure the lateral misses between the pursuers and evader at the instants T_1 and T_2 of nominal collisions. Linearization of the original nonlinear dynamics along the nominal trajectories gives a linear differential game. Considering a particular case when $T_1 = T_2$, we obtain the formulation described above.

6.3 Zero-Effort Miss Coordinates

We denote by $x_i(t)$, $i = 1, 2$, the value of the difference $z_e - z_{p_i}$ that is predicted from the current instant t and the current positions $\mathbf{z}_e(t)$, $\mathbf{z}_{p_i}(t)$ to the instant T under the condition that zero controls act in system (22) in the interval $[t, T]$. (Here, we subtract the coordinates of the pursuers from the coordinates of the evader instead of the opposite subtraction that is used in the previous sections because this way is used in the works by J. Shinar and his collaborators.) We have

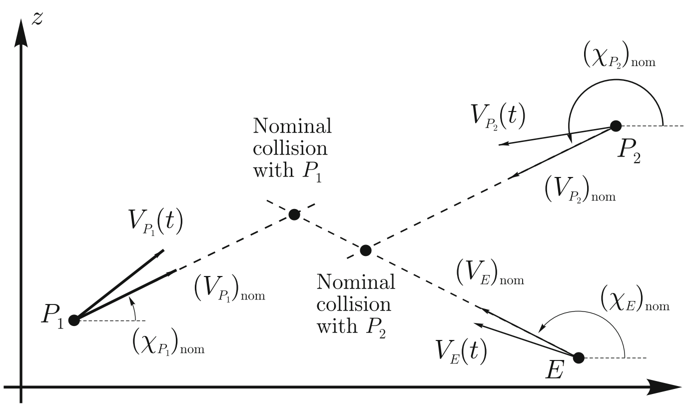


Fig. 1 Scheme of the interception with two pursuing objects

$$x_i(t) = Z^1(T, t; A_e)\mathbf{z}_e(t) - Z^1(T, t; A_{p_i})\mathbf{z}_{p_i}(t), \; i = 1, \, 2,$$

where the upper index 1 marks the first rows of the fundamental Cauchy matrices $\mathbf{Z}(T, t; A_{p_i})$ and $\mathbf{Z}(T, t; A_e)$ that correspond to the matrices A_{p_i} and A_e and are written for the instants T and t. Since the matrices A_{p_i}, A_e do not depend on the time t, the matrices $\mathbf{Z}(T, t; A_{p_i})$ and $\mathbf{Z}(T, t; A_e)$ depend on the difference $T - t$ only. Often, the values $x_i(t)$, $i = 1, 2$, are called the *zero-effort miss* coordinates [64]. Note that $x_i(T) = z_e(T) - z_{p_i}(T)$.

Differentiating the values $x_i(t)$ by t, we obtain

$$\begin{aligned} &\dot{x}_i(t) = Z^1(T, t; A_e)B_e v - Z^1(T, t; A_{p_i})B_{p_i}u_i, \\ &|u_i| \le \mu_i, \; |v| \le \nu, \; t \le T_i, \; i = 1, \, 2. \end{aligned} \tag{24}$$

From results of the differential game theory, it follows (see, for instance, [12,27,28]) that the differential game with dynamics (24) and the payoff function

$$\varphi = \min\bigl\{\bigl|x_1(T)\bigr|, \, \bigl|x_2(T)\bigr|\bigr\}$$

is equivalent (in the sense of magnitude of the value function) to the differential game with dynamics (22) and payoff function (23). Dimension of the phase vector $x = (x_1, x_2)^\top$ is equal to two and the phase vector x is absent in the right-hand part of system (24).

So, we have a standard differential game with a fixed termination instant T. However, the level sets (the Lebesgue sets) $\{x : \varphi(x) \le c\}$, $c \ge 0$, of the payoff function φ are not convex. This fact makes the problem interesting for a mathematical investigation.

6.4 Variants of Dynamics

In the literature devoted to 1×1 pursuit problems, the following variants of the objects' dynamics have been suggested.

1. **First-order link** The following dynamics gives the simplest description of inertiality of the servomechanisms that transforms the control signal u to the acceleration (see, for example, [62]):

$$\ddot{z} = a, \quad \dot{a} = (u - a)/\tau. \tag{25}$$

   The value τ is called the time constant and defines the time interval until the acceleration reaches the desired level.
2. **Oscillating link** Paper [63] investigates problems with the dynamics

$$\ddot{z} = a, \quad \ddot{a} = -\omega^2 a - \zeta\dot{a} + u. \tag{26}$$

   Here, in contrast to (25), the servomechanisms' dynamics is described by a second-order differential equation that corresponds to an oscillating contour with the own frequency ω and viscous friction with the factor ζ.
3. **Tail/canard control** Work [61] studies 1×1 games in the case when the control is created by deflection of aerodynamic rudders. The dynamics description is the following:

$$\ddot{z} = a + du, \quad \dot{a} = \bigl((1 - d)u - a\bigr)/\tau. \tag{27}$$

The parameter d is defined by disposition of the aerodynamic rudders. Its positive (negative) values correspond to the case when the rudders are placed in the head (tail) part of the object. As before, the symbol τ denotes the time constant.

Games 2×1 with dynamics of types (25) and (26) were studied in works [20,30,36]. Games 2×1 with dynamics (27) have not been studied earlier.

To make possible to take into account the dynamics variants (25)–(27), this section considers a more general formulation, in which the linear dynamics of each object is described by its own vector differential equation with a scalar control restricted on modulus. For each object, the first coordinate of the phase vector is regarded as the coordinate of the object position on the straight line.

6.5 Examples of Solvability Sets

Let for $c \geq 0$ the set W_c be the maximal stable set in the three-dimensional space t, x ending at the set $M_c = \{x : \varphi(x) \leq c\}$. In other words, W_c is the set where the magnitude of the value function is not greater than c. The set W_c is also called the *solvability set* corresponding to the miss c. If $(t_0, x_0) \in W_c$, then the first player guarantees the game termination with a miss $\varphi \leq c$. If $(t_0, x_0) \notin W_c$, there is no such a guarantee.

We construct the solvability sets W_c numerically. We cut off infinite strips of the cross-like terminal set M_c at some level in each coordinate x_1, x_2 and further use our own algorithms that are founded on theoretical constructions [27,28] developed in Ekaterinburg, Russia. The algorithms implement a procedure of dynamic programming in the backward time. With that, we use [20,29,30] the fact that the t-sections $W_c(t) = \{x : (t, x) \in W_c\}$ of the solvability sets W_c are sets in the plane R^2 (just these sets are produced by the algorithm).

When investigating the solvability sets, we try to detect their important structural peculiarities. Some of them have been outlined by J. Shinar in works dealing with linear pursuit problems of the type 1×1.

6.5.1 Example 1

Consider the case when the evader's behavior is described by system (25), and the behavior of each pursuer is described by system (27). Thus, if to use denotations of system (22), we have

$$A_{p_i} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/\tau_{p_i} \end{pmatrix}, \qquad B_{p_i} = \begin{pmatrix} 0 \\ d_{p_i} \\ (1 - d_{p_i})/\tau_{p_i} \end{pmatrix},$$
$$A_e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/\tau_e \end{pmatrix}, \qquad B_e = \begin{pmatrix} 0 \\ 0 \\ 1/\tau_e \end{pmatrix},$$
$$|u_i| \leq \mu_i, \; i = 1, 2, \quad |v| \leq \nu.$$

Let the pursuers P_1 and P_2 be equal. Choose the values of parameters as follows:

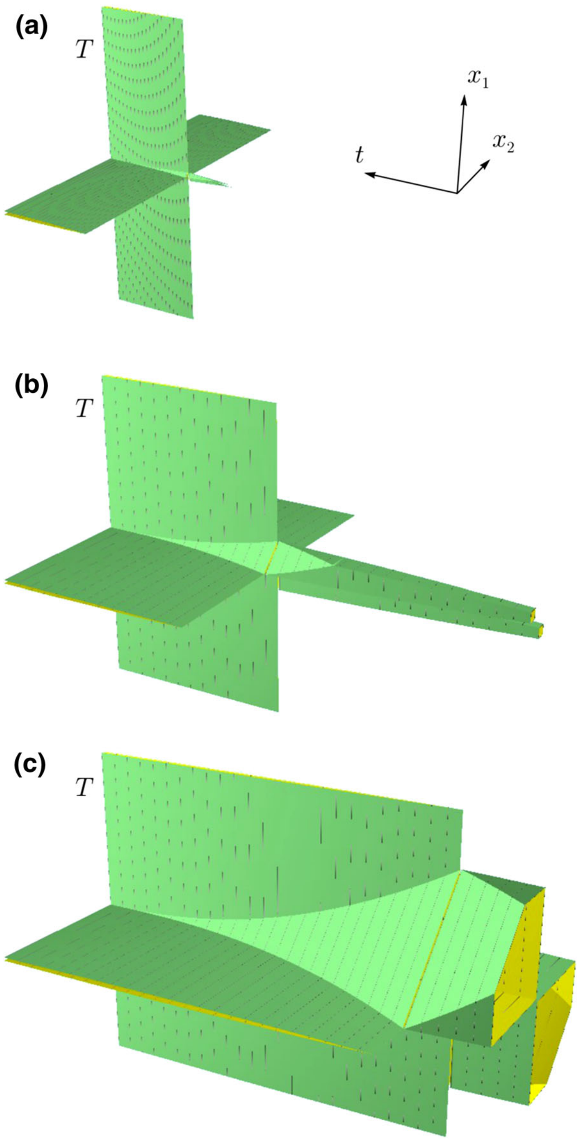
$$\mu_1 = \mu_2 = 0.9, \; \nu = 1, \; \tau_{p_1} = \tau_{p_2} = 1/0.9, \; \tau_e = 1, \; T = 15.$$

Firstly, let us suppose that $d_{p_1} = d_{p_2} = d = 0$. Then each of the objects has the dynamics of the first-order link for his control. Under this, for the chosen values of the parameters, the following relations hold:

$$\frac{\mu_i}{\nu} = \frac{0.9}{1} < 1, \;\; \frac{\mu_i}{\nu} \cdot \frac{\tau_e}{\tau_{p_i}} = \frac{0.9}{1} \cdot \frac{1}{1/0.9} = 0.81 < 1, \;\; i = 1, \; 2.$$

This corresponds [62] to the case of weak pursuers. The three-dimensional solvability set W_c for $c = 2.0$ in the 2×1 game is shown in Fig. 2b. Note that $W_0(t) = \varnothing$ for any $t < T$ in the

Fig. 2 Example 1: Solvability sets for three different variants of dynamics, $c = 2.0$: **a** $d < 0$, **b** $d = 0$, **c** $d > 0$



case of weak pursuers, that is, there are no initial positions, from which the first player can guarantee the exact encounter.

Let now $d > 0$. The three-dimensional solvability set for $c = 2.0$ is shown in Fig. 2c. It is seen that this set is significantly larger than one for $d = 0$. Figure 3 presents the set W_c corresponding to $c = 0$. From any initial position in this set, the first player guarantees the zero miss, that is, the exact encounter.

At last, let $d = -0.5 < 0$. In this case, again $W_0(t) = \varnothing$ for $t < T$. The solvability set for $c = 2.0$ is shown in Fig. 2a.

The sets in Figs. 2 and 3 are drawn in the same scale from the same point of view.

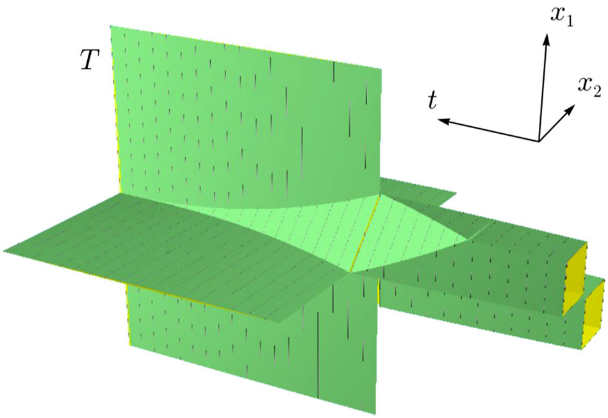


Fig. 3 Example 1: Solvability set for $c = 0$ in the case $d > 0$

6.5.2 Example 2

In the second example, all three objects have their dynamics of type (25) with the following parameters:

$$\mu_1 = 0.8,\ \mu_2 = 1.3,\ \nu = 1.0,\ \tau_{p_1} = 1/20.0,\ \tau_{p_2} = 1/0.5,\ \tau_e = 1.0,\ T = 15.0.$$

The peculiarity of this example is that the pursuers have qualitatively different dynamics capabilities. Both of them have varying dynamic advantage over the evader. The first pursuer is weaker than the evader in some period near the termination instant and stronger when there is a lot of time until the end of the pursuit. The second one is, vice versa, weaker than the evader when the time-to-go is large and stronger near the termination instant.

The solvability set corresponding to the miss level $c = 0.1$ is shown in Fig. 4.

The obtained solvability set is finite in time. From the theoretical point of view, this is due to the fact that the first pursuer, which is weaker finally, acts in the vertical direction and cannot prevent the contraction of the solvability set in this direction. The capabilities of the second pursuer are inessential in this situation. From the practical point of view, explanation

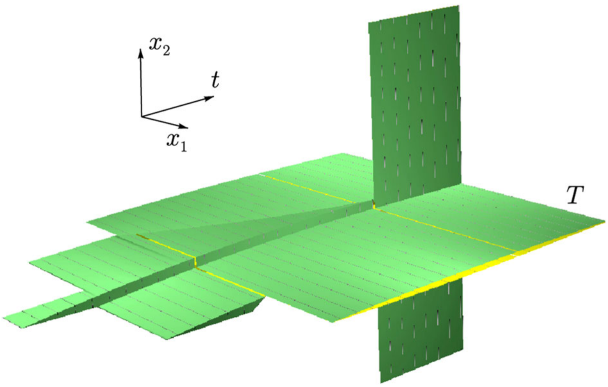


Fig. 4 Example 2: Solvability set for $c = 0.1$, situation of different pursuers

is the following: if the evader is stronger than a pursuer and has enough time, then he can escape through the influence zone of this pursuer, and the desirable level of the miss cannot be obtained.

6.5.3 Example 3

Now, let us take the dynamics of form (26) for the pursuers and the dynamics of type (25) for the evader. We have

$$
\begin{aligned}
A_{p_i} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_{p_i}^2 & -\zeta_{p_i} \end{pmatrix}, \qquad B_{p_i} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \\
A_e &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/\tau_e \end{pmatrix}, \qquad B_e = \begin{pmatrix} 0 \\ 0 \\ 1/\tau_e \end{pmatrix}, \\
|u_i| &\le \mu_i,\ i = 1,\ 2, \quad |v| \le \nu.
\end{aligned}
$$

Our aim is to show an example with "exotic" solvability set in a 2×1 game. Choose the following parameters:

$$
\mu_1 = \mu_2 = 0.3,\ \nu = 1.3,\ \omega_{p_1} = \omega_{p_2} = 0.5,
$$
$$
\zeta_{p_1} = \zeta_{p_2} = 0.0025,\ \tau_e = 1.0,\ T = 30.
$$

Figure 5 shows the solvability set W_c for $c = 1.6$. Its "peculiarity" is in the presence of two time periods with narrow "throats." Earlier, for three-dimensional solvability sets in model problems 1×1 of cosmic pursuit, examples with one narrow throat have been constructed [32]. For a problem 2×1 with dynamics of form (25) in work [29], an example is given where in some period of time, the solvability set disjoins into two parts. Each of them has a narrow throat. If in a problem 2×1 the pursuers have dynamics (26), then the number of throats can be even greater. In the case of a 1×1 game of form (26) for the pursuer and of form (25) for the evader, the possibility of situation with several narrow throats has been predicted in [63].

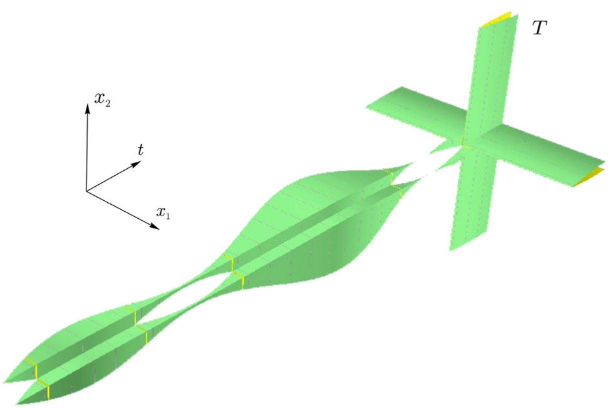


Fig. 5 Example 3: "Exotic" solvability set with two areas of disconnectivity of time sections

We have shown here results of numerical construction of solvability sets in a game 2×1 with fixed termination instant T. From the ideological point of view, each of them is the maximal sets in the space t, x, wherefrom the first player guides the system to the prescribed terminal set discriminating the second one; the terminal set is cross-like. We have said nothing about construction of the positional strategy of the first player that guarantees (with a sufficiently small time step of the discrete control scheme) reaching any close neighborhood of the terminal set at the termination instant T. As it is said in Sect. 4, construction of such a strategy is a separate problem. Theoretical investigations of this topic are set forth in books [27,28,66].

7 Other Pursuit Problems with Many Objects

1. Finalizing setting forth results of numerical study of the intercept problem at a fixed instant with two pursuers and one evader, note again that we deal with a problem where all objects move in a direct line. If a problem of planar intercept would be investigated in the same way, we would have a four-dimensional space of forecast coordinates. Such problems are very actual from the point of view of studying spatial intercept. In the literature on differential games, there are papers, whose results can be used as test examples for procedures solving problems of this type. Namely, paper [40] suggests an investigation of a problem with simple motion in the plane. The result is computed at some prescribed instant as the distance between the evader and closest pursuer. It is shown there that the value function is the program maximin function where the maximum is taken over open-loop controls of the evader and the minimum is taken over open-loop controls of the pursuers. In work [35], two inertial objects chase an inertialless one in the plane. The payoff at some fixed termination instant is the distance between the evader and closest pursuer. Some formulae are given for computation of the value function in different domains of the phase space.
2. Among pursuit problems with many persons, a very important class includes problems with false targets. The evader at some instant or instants shoots off a false target (or several targets) to put out the pursuer(s). The pursuer (or pursuers) can distinguish the false target only when being not far from it than some distance r. Moreover, the pursuer is capable of observing an object (the true or false target) if it is in its detection zone with radius $R > r$. The pursuer is interested to intercept the true target. How to formalize correctly and adequately problems of this type? E.P. Maslov and his collaborates have suggested some variants of formulations and studied the problem for objects with simple motion [1,3,60].
3. In works [4,5], problems with many pursuers and one evader in the plane are considered. Their peculiarity is in assumption that each pursuer starts to chase only if the evader is closer to it than to other pursuers. It is natural to use Voronoi diagrams during solving problems of this type.
4. In problems of group pursuit, it is desirable to exclude too tight approach of pursuers. Moreover, information interchange can be carried out only with objects that are not too far. From the mathematic point of view, these demands can be included to the formulation by means of some phase constraints. With that, of course, the aim of the group of pursuers has some global objective, for example, to destroy all evaders. In work [65], it is suggested to introduce some special objective and goal functions that allow one to take into account local and global aims of all objects. The authors of that paper try to use these functions to produce a Lyapunov function, whose change in time obeys to some inequality. Unfortunately, it is very difficult to construct explicitly the corresponding inequality.

5. If one can formulate a zero-sum differential game, which takes into account the phase constraints for the objects, adequately reflects objectives of pursuers and evaders, and allows one to prove existence of the value function, then the value function is a Lyapunov function that can be used for construction of optimal feedback controls. In works [13,19], a fabulous plan is declared about realization of numerical methods for solving extremely hard problems of group pursuit.
6. The problems are even more difficult when the objects are constrained by a "virtual container" that can change its shape and/or orientation during motion. Such problems are suggested in [33,34].

Finishing our survey, we should emphasize that the further development of methods for solving pursuit games *group* $\times$ *group*, doubtless, will be connected with improvement of existing numerical procedures and creating new ones for the case of multi-dimensional differential games and corresponding partial differential equations of Hamilton–Jacobi type. Nowadays, such algorithms are worked out in workgroups of A.B. Kurzhanski (Moscow State University, Moscow, Russia), V.N. Ushakov (Krasovskii Institute of Mathematics and Mechanics, Ekaterinburg, Russia), M. Falcone (Universitá degli Studi di Roma "La Sapienza," Rome, Italy), C. Tomlin, S. Sastry (Berkley University, Berkley, USA), N.L. Grigorenko (Moscow State University, Moscow, Russia), N.D. Botkin (TU München, Munich, Germany).

8 Conclusion

Pursuit problems with one pursuer and two evaders or with two pursuers and one evader can be formulated in such a way that their solution could be obtained by means of general methods of zero-sum differential games. Solvability sets corresponding to certain values of the payoff can be constructed, and singular surfaces can be found and classified. Both of them are theoretically interesting because some exotic examples can be discovered.

But if the total number of objects is large, investigations involving general methods with usual optimality criteria are hardly implementable due to extremely high dimension of the phase vector of obtained games. In this case, one way of simplification is to seek for some special formulations that are still adequate to original practical problems and provide some solution. Formulations of this type have been found in the class of linear differential games and games with simple motion. With that, the method of constant-bearing approach well known to engineers has been successfully applied in the original or forecast coordinates.

Works included in this survey have been selected just according to these two ideas. Also, a number of papers have been added where some formulations are discussed that reflect modern practical situations. Necessity in capability to solve such problems will stimulate development of theory and numerical methods for zero-sum differential games in the near future.

Acknowledgements This work has been partially supported by Russian Foundation for Basic Research, Project No. 15-01-07909.

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