LEVEL SETS OF VALUE FUNCTION AND SINGULAR SURFACES IN LINEAR DIFFERENTIAL GAMES

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Abstract: The paper deals with an algorithm of construction of level sets in linear differential games with fixed terminal time and convex payoff function depending on two components of the phase vector. Into this algorithm, the block for detection of singular points on the border of level set is included. Singular points give singular lines on the border of level set. Singular surfaces in the game space are collected using singular lines as skeleton. Examples of numerically calculated level sets and singular surfaces are represented. *Copyright © 1998 IFAC*

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1. INTRODUCTION

The concepts of the alternating integral (Pontryagin, 1967) and maximal stable bridge (Krasovskii and Subbotin, 1988) are the main ones in differential game theory. By means of these terms, the solvability set in a game of approach is usually described. For games with terminal payoff function, these terms define a level set of the value function. A system of level sets on a grid of values gives a representation of the value function in general. Information, obtained during construction of level sets, can be used for singularities analysis. Algorithmic description of this analysis is the main topic of the paper.

A linear antagonistic differential game

$$\begin{gathered} \dot{x} = A(t)x + B(t)u + C(t)v \\ x \in R^n,\ u \in P,\ v \in Q,\ T,\ \varphi(x(T)) \end{gathered} \qquad (1)$$

with fixed terminal time T and convex payoff function φ, which depends on two coordinates x_i, x_j of the phase vector, is considered. The first (second) player governs the control u (v) choosing it from the convex compact P (Q) and minimizes (maximizes) the value of the function φ.

It is known that the substitution $y(t) = X_{i,j}(T,t)x(t)$, where $X_{i,j}(T,t)$ is a matrix combined of two rows of the fundamental Cauchy matrix, provides the transformation to the equivalent differential game of the second order on phase variable.

At the beginning of the 80's, the backward constructions were elaborated (Subbotin and Patsko, Eds., 1984; Taras'ev and Ushakov, 1985) for building level sets of the value function in linear differential game (1). Software for interactive investigation of level sets was created recently in cooperation with V.L.Averbukh, D.A.Yurtaev, E.A.Shilov, A.I.Zenkov from the Department of System Support of the Institute of Mathematics and Mechanics.

As singular surfaces in theory of differential games, such sets in the game space are named where the optimal motions have some peculiarities (dispersion, refraction, junction, etc.). The classification of the singular surfaces was suggested by Isaacs (1965).

Necessary conditions, which characterize different types of singularity, were studied by Bernhard (1977) and Melikyan (1998).

In the paper, an attempt of elaboration of algorithm for global construction of complete system of singular surfaces is made. With that, level set of the value function is the base object. On the border of a level set, singular lines are detected. They are determined by various peculiarities of optimal motions coming along the border surface of the level set. Singular surfaces are built on the base of singular lines taken from a system of level sets. Similar idea of constructing singular surfaces was realized for a concrete problem by Shinar and Zarkh (1996).

Discussing algorithm for detection and classification of singularities is imbedded into the algorithm (Isakova, *et al.*, 1984) for backward construction of level sets. The level sets can be built for arbitrary polyhedra P and Q. The algorithm for detection of singularities has been elaborated now only for the case of scalar controls of the first and second players (i.e., the sets P and Q are segments).

2. CONSTRUCTING LEVEL SETS OF THE VALUE FUNCTION

Here, the algorithm (Isakova, *et al.*, 1984) for constructing level sets of the value function is described. It is needful for understanding further section.

2.1 Backward procedure

Assume that the transfer from the game (1) with the payoff function φ depending on two coordinates of the phase vector to the equivalent game

$$\begin{gathered} \dot{y} = D(t)u + E(t)v \\ y \in R^2,\ u \in P,\ v \in Q,\ T,\ \varphi(y_1(T), y_2(T)) \qquad (2) \\ D(t) = X_{i,j}(T,t)B(t),\ E(t) = X_{i,j}(T,t)C(t) \end{gathered}$$

is already done.

Let on the interval $[0,T]$ the sequence of instants $t_i:\ t_N = T,\ \dots,\ t_i = t_{i+1} - \Delta,\ \dots,\ t_0 = 0$ dividing the interval with a step Δ is given. The interest is to find the time sections $W_c(t_i) = \{y \in R^2 : V(t_i, y) \le c\}$ of level set $W_c = \{(t,y) \in [0,T] \times R^2 : V(t,y) \le c\}$ of the value function V for the given value of parameter c.

Replace the dynamics (2) by the piecewise-constant dynamics

$$\begin{gathered} \dot{y} = \mathbf{D}(t)u + \mathbf{E}(t)v \\ \mathbf{D}(t) = D(t_i),\ \mathbf{E}(t) = E(t_i),\ t \in [t_i, t_{i+1}) \end{gathered} \qquad (3)$$

Instead of the sets P and Q, let us consider their polyhedral approximations $\mathbf{P}$, $\mathbf{Q}$. Let $\hat{\varphi}$ be the approximating payoff function. For any c, its level set $\mathbf{M}_c = \{y \in R^2 : \hat{\varphi}(y) \le c\}$ is a convex polygon.

The approximating game (3) is taken so that a game with simple motion dynamics (Isaacs, 1965), polyhedral convex control constraints and the convex polygonal target set appears for each step $[t_i, t_{i+1}]$ of the backward procedure. On the base of $\boldsymbol{W}_c(t_N) = \mathbf{M}_c$, the game solvability set $\boldsymbol{W}_c(t_{N-1})$ can be computed. Further, starting from $\boldsymbol{W}_c(t_{N-1})$, set $\boldsymbol{W}_c(t_{N-2})$ can be built, and so on. As a result, the collection of convex polygons is obtained, which approximate sections $W_c(t_i)$ of level set W_c of the value function in the game (2).

Let $\mathcal{P}(t_i) = -D(t_i)\mathbf{P}$, $\mathcal{Q}(t_i) = E(t_i)\mathbf{Q}$. The support function $l \to \rho(l, \boldsymbol{W}_c(t_i))$ of the polygon $\boldsymbol{W}_c(t_i)$ is the convex hull (Pschenichnyi and Sagaidak, 1970) of the function

$$\gamma(l,t_i) = \rho(l, \boldsymbol{W}_c(t_i)) + \Delta\rho(l, \mathcal{P}(t_i)) - \Delta\rho(l, \mathcal{Q}(t_i)).$$

The function $\gamma(\cdot, t_i)$ is positively-homogeneous and piecewise-linear. The property of local convexity of this function can be violated only at the boundary of linearity cones of the function $\rho(\cdot, \mathcal{Q})$, i.e. at the boundary of cones generated by the normals to neighbor edges of the polygon $\mathcal{Q}$.

2.2 Algorithm of convex hull construction

Let us agree to omit the argument t_i in the notation of the function γ.

The linearity cones of γ are determined by the outer normals to the convex polygons $\boldsymbol{W}_c(t_i)$, $\mathcal{P}(t_i)$, $\mathcal{Q}(t_i)$. Gathering the normals of these sets and ordering them clockwise, the collection L of the vectors is obtained. The collection of values $\gamma(l)$ on the vectors $l \in L$ is denoted by Φ. The collections L, Φ describe completely the function γ.

The collection of the normals to $\mathcal{Q}(t_i)$ ordered clockwise is denoted by S. Vectors from S are called "suspicious". This name is connected with the fact that the function γ is locally convex on the cones, which interior does not contain the normals of the set $\mathcal{Q}(t_i)$. The violation of the local convexity can appear only on the cones which interior contains at least one normal of the polygon $\mathcal{Q}(t_i)$.

Let $L^{(1)} = L$, $\Phi^{(1)} = \Phi$, $S^{(1)} = S$. The k+1 step of the multistep convexing process consists in replacing the collections $L^{(k)}$, $\Phi^{(k)}$ by the collections $L^{(k+1)} \subset L^{(k)}$, $\Phi^{(k)} \subset \Phi^{(k+1)}$. The collection $S^{(k)}$ is also replaced by the new one $S^{(k+1)}$.

Describe now one step of the convexing process. Suppose that the angle between two neighbor vectors from the collection $L^{(k)}$ counted clockwise is less than π. Let $l \to \gamma^{(k)}(l)$ be the piecewise-linear function determined by the collections $L^{(k)}$, $\Phi^{(k)}$. Since $L^{(k)} \subset L^{(k-1)} \subset \ldots \subset L^{(1)}$, $\Phi^{(k)} \subset \Phi^{(k-1)} \subset \ldots \ldots \subset \Phi^{(1)}$, then for any vector $\bar{l} \in L^{(k)}$ the value $\gamma^{(k)}(\bar{l})$ is equal to $\gamma(\bar{l})$.

Take some vector $l_* \in S^{(k)}$ and check the local convexity of the function $\gamma^{(k)}$ on the cone generated by the vector l_* and two its neighbor vectors l_- and l_+ selected counterclockwise and clockwise from the collection $L^{(k)}$. In other words, check whether the inequality $l_*' y \le \gamma(l_*)$ is active in the triple of the inequalities $l_-' y \le \gamma(l_-)$, $l_*' y \le \gamma(l_*)$, $l_+' y \le \gamma(l_+)$. If the system of three inequalities is compatible, then (by virtue of the ordering the vectors l_-, l_*, l_+) only the middle one can be inactive.

The algorithm of verification: find the intersection point y_* of the straight lines $l_-' y = \gamma(l_-)$, $l_*' y = \gamma(l_*)$, and then check the inequality $l_+' y_* < \gamma(l_+)$. If it holds, the local convexity takes place. Otherwise, the local convexity is absent.

In the first case, the vector l_* is taken away from the collection $S^{(k)}$, and the remained set is denoted by $S^{(k+1)}$. Let $L^{(k+1)} = L^{(k)}$, $\Phi^{(k+1)} = \Phi^{(k)}$.

In the second case, two situations are distinguished. Let α be the angle counted clockwise from l_- to l_+.

1. $\alpha < \pi$. The vector l_* is taken away from the collection $S^{(k)}$, and, simultaneously, the vectors l_- and l_+ are included into this collection (one of them or even both can be there already). Denote the new collection of the "suspicious" vectors by $S^{(k+1)}$. The difference of the new collection $L^{(k+1)}$ from the collection $L^{(k)}$ is that the vector l_* is absent in $L^{(k+1)}$. When processing $\Phi^{(k)}$ to $\Phi^{(k+1)}$, the value $\gamma^{(k)}(l_*) = \gamma(l_*)$ is taken away.

2. $\alpha \ge \pi$. The constructing is ceased.

One step of the convexing algorithm has been described.

The algorithm finishes at the step with the number j when for the first time $S^{(j)} = \varnothing$, i.e. when the collection of the "suspicious" vectors becomes empty. It means that the function $\gamma^{(j)}$, which corresponds to the collections $L^{(j)}$ and $\Phi^{(j)}$, is locally convex everywhere. Thus, the function $\gamma^{(j)}$ is the convex hull of the function γ. In this case, let us denote the final collections $L^{(j)}$ and $\Phi^{(j)}$ as $\overline{L}$ and $\overline{\Phi}$, respectively.

The second variant of the termination is following: the angle α between the vectors l_- and l_+ becomes greater or equal to π after elimination the checked vector l_* from the collection of the "suspicious" vectors at some step. It means that the convex hull of the function γ does not exist, i.e. $\boldsymbol{W}_c(t_i) = \varnothing$. (If $\alpha = \pi$, it is possible that $\boldsymbol{W}_c(t_i)$ is a degenerate polygon, i.e. $\boldsymbol{W}_c(t_i)$ is a point or a segment. Further constructions are ceased in this case also.)

3. SINGULAR SURFACES

In this section, it is supposed that the sets *P* and *Q* are segments. So, $\mathcal{P}(t_i)$, $\mathcal{Q}(t_i)$ are also segments in the space y_1, y_2 for any time instant t_i.

Let assume in addition that the segments $\mathcal{P}(t_i)$, $\mathcal{Q}(t_i)$ are not parallel, and more than that, each of them is not parallel to neither one of edges of the polygon $\boldsymbol{W}_c(t_{i+1})$. Let call this assumption "the non-parallelism condition".

3.1 Optimal motions and singular points

On each time interval $[t_i, t_{i+1}]$, the approximating game (3) is the game with simple motions. The first player tries to transfer the system from the polygon $\boldsymbol{W}_c(t_i)$ onto the polygon $\boldsymbol{W}_c(t_{i+1})$, the second player tries to prevent it. Entering the discrimination of the second player, determine the optimal motions.

Let us fix some arbitrary point y_0 on the boundary of the polygon $\boldsymbol{W}_c(t_i)$. The second player control constant on the interval $[t_i, t_{i+1}]$ is called the optimal one if the first player can not direct the resulting motion from the point y_0 into the interior of the polygon $\boldsymbol{W}_c(t_{i+1})$. For fixed optimal control of the second player, the first player control (also constant on the same interval) is called the optimal parrying one if the corresponding motion comes onto the boundary of the set $\boldsymbol{W}_c(t_{i+1})$. Motion generated by optimal players' controls is called the optimal one.

The control v^* satisfying the condition of maximum $v^* = \arg\max\{l'v : v \in \mathcal{Q}(t_i)\}$ is called the extremal control of the second player on the vector *l*. Similarly, the extremal control of the first player on the vector *l* is the control u^*, which satisfies the condition of minimum $u^* = \arg\min\{l'u : \; u \in \mathcal{P}(t_i)\}$.

It is easy to see that if y_0 is an internal point of some edge of the polygon $\boldsymbol{W}_c(t_i)$, then the constant control of the second player is optimal if and only if it is extremal on the normal vector to $\boldsymbol{W}_c(t_i)$ at the given point. The optimal parrying control of the first player is extremal on the same vector.

If y_0 is a vertex of $\boldsymbol{W}_c(t_i)$, then two normals correspond to the vertex. The following statements describe the structure of the optimal controls.

Proposition 1. For any vertex of the polygon $\boldsymbol{W}_c(t_i)$, the second player control, being extremal at least on one of two normal vectors at this vertex, is optimal. With that, the optimal parrying control of the first player is extremal on the same vector.

A proof of this statement does not use a supposition of a scalar character of players' controls. Hence, the condition of non-parallelism is not used also. In the following statement, these assumptions are essential.

Proposition 2. Let the players' controls be scalar and the condition of non-parallelism is satisfied. Then for any vertex of the polygon $\boldsymbol{W}_c(t_i)$, the totality of optimal controls of the second player consists only of the extremal controls on two normal vectors of the polygon $\boldsymbol{W}_c(t_i)$ at this vertex.

The point y_0 is called regular if the optimal motion emanating from this point is unique and generated by the extreme players' controls. A point, which is not regular, is called the singular one. Here, "extreme" means that the constant control value is a boundary point of the segment $\mathcal{P}(t_i)$ (or $\mathcal{Q}(t_i)$).

3.2 Classification of singularities

Below, the classification of singular points of the polygon $\boldsymbol{W}_c(t_i)$ is described. It is based on the analysis of character of the optimal motions emanating from these points. For the classification, two marks are used. These marks are attached to normals participated in the process of convex hull construction of the function $\gamma(\cdot, t_i)$. The mark *FS* ("former suspicious") is added to normals which, during the convexing process, were denoted as "suspicious" ones, but, after the process, were remained in the final collection $\overline{L}$ (this collection determines the polygon $\boldsymbol{W}_c(t_i)$). The mark $N\mathcal{P}$ is added to normals, which were taken from the set $\mathcal{P}(t_i)$. The classification is represented in the Table.

The Table deals with the normals from the final collection $\overline{L}$. Individual normals or pairs of neighbor ones are analyzed. Quantity of considered normals is shown in the first column. In the second column, the marks are represented whose presence is checked out on the considered normals. With that, if only one mark is shown for a concrete vector, then the second mark is supposed to be absent. The third column contains the additional condition. Marks in the second column, together with the satisfaction of the additional condition, determine the singularity type (which is shown in the sixth column) of the object from the forth column. Such object can be the polygon $\boldsymbol{W}_c(t_i)$ vertex, which is incident for edges determined by normals (row 1), either the edge correspondent to the considered normal (row 3) or one of two shown normals (row 2). Inside the case of the row 1, a subdivision exists which is determined by the condition from the fifth column. Type of singularity is determined by the character of the optimal motions emanating from the marked point or points of the marked edge. The names of singularities are coordinated with ones used in the R.Isaacs' book.

The term "*P*-normal" ("*Q*-normal") means a normal taken from the set $\mathcal{P}(t_i)$ ($\mathcal{Q}(t_i)$).

Table. Singularities classification

	Flags	Additional condition	Object for marking	Secondary condition	Type of singularity
1	*FS*, *FS*	*Q*-normal is strictly between	Vertex	*P*-normal is not between	Dispersal for 2nd
				P-normal is between	Dispersal
1	$N\mathcal{P}$ +*FS*	*Q*-normal is strictly between	*P*-normal edge		Equivocal
2	$N\mathcal{P}$		Edge		Switching for 1st

The term "dispersal" corresponds to the situation when two optimal motions emanate from the given point. One of these motions is generated by players' controls, which are extremal on one of the regarded vectors with the mark *FS*. The second motion corresponds to the players' controls extremal on the second vector with the same mark. With that, all controls are extreme in its segments.

The term "equivocal" corresponds to the situation when from the vertex, determined by the normals with marks *FS* and NP +*FS*, two optimal motions go out. One motion is generated by players' controls, which are extremal on the vector with the mark *FS*. The second one is generated by controls extremal on the vector with the mark NP +*FS*. With that, on the first motion, both controls get the extreme values. But on the second one, the control of the first player is not extreme. (It is namely the difference of the equivocal case from the dispersal one.) The latter motion comes to the vertex of the polygon $\boldsymbol{W}_c(t_{i+1})$ where the cone generated by two normals contains the considered normal with the mark NP +*FS*. Each other point of the marked edge emanates only one motion, which comes to the mentioned vertex.

The term "switching for the first player" corresponds to the case when only one optimal motion emanates from each internal point of the marked edge. An extreme control of the second player and non-extreme control of the first player generate it. The difference from the equivocal case is in the fact that only one motion emanates from both vertices of the edge. With that, the first player control gets one extreme value at one vertex of the edge, and the opposite value at another vertex.

On different sides of the equivocal edge, both players' controls change. In dispersal situation, either only the second player changes its control or both players change their controls together. So, a special subdivision appears. In the switching situation, only the first player changes its control.

It can be proved that any point on the boundary of the polygon $\boldsymbol{W}_c(t_i)$ is regular if it is not a marked vertex or it is not included to any marked edge. And vice versa, any point, which is a marked vertex or included to a marked edge, is singular, except maybe the case when it is an endpoint of a marked edge.

3.3 Constructing singular surfaces

The data for the algorithm of singularity classification are accumulated during the process of the convex hull construction, and after that the algorithm of classification begins to work. Really, its work is reduced to the verification of the presence of situations described by rows of the Table. As a result, the collection of the points and intervals with description of type of their singularity is obtained for the current backward time section of the level set.

For graphical presentation of the singular surfaces, a collection of level sets is calculated on the given grid of magnitudes of the value function, and, simultaneously, the singular points and edges are detected on each section of each level set. Constructing the singular surfaces on the base of the singular points and intervals is carried out in the program of visualization.

Validity of the algorithm elaborated was verified on the test example from (Patsko and Tarasova, 1985). In that work, the singular surfaces appeared had been investigated analytically in detail. Results of our calculations coincide well with the results of the mentioned paper.

4. EXAMPLE

In this section, the examples of constructing level sets of the value function and singular surfaces are represented. The Gouraud shading is used for visualization of level set surface ("tube") built from separate sections by triangulation. The surface is illuminated by dot radiant, which position can be changed by user. Now, visualization of singular surfaces is implemented by the simplest methods.

The system is a conflict-controlled oscillator

$$\begin{aligned}\dot{x}_1 &= x_2 + v \\ \dot{x}_2 &= -x_1 + u\end{aligned} \qquad |u| \le 1,\ |v| \le 0.9,\ \varphi(x_1, x_2) = x_1^2 + x_2^2.$$

Three level sets shown in Fig. 1 were calculated on the interval $[0, 8]$ of the backward time τ with the step $\Delta = 0.05$. The sets are shown for values $c = 1.05,\ 1.4,\ 2.7$. The internal tube terminates. The outer tubes are visualized non-transparent.

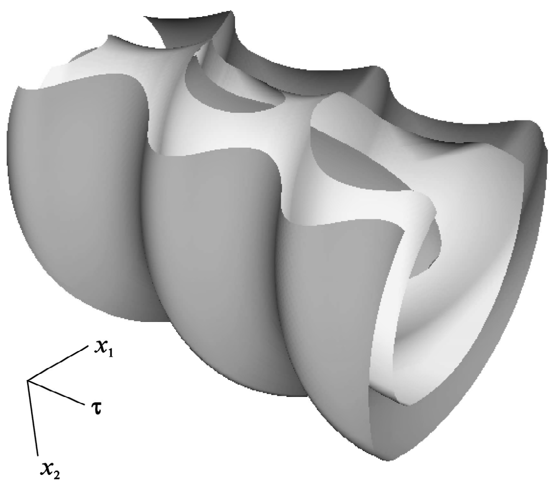

Fig. 1. Three level sets for "oscillator" system.

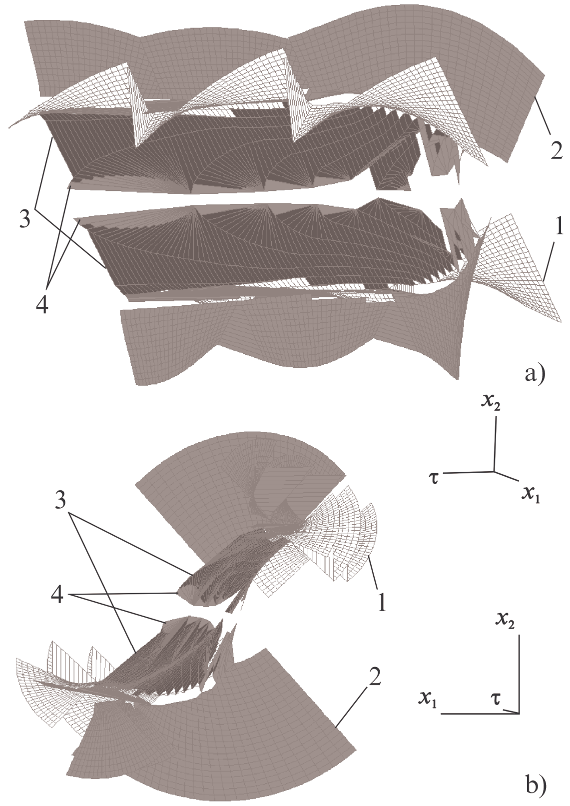

Fig. 2. Singular surfaces for "oscillator" system. a) View from the axis x_1. b) View from the axis τ. *Notations*: 1 – dispersal surface for the second player, 2 – switching surface for the first player, 3 – equivocal surface, 4 – dispersal surface.

To see the internal arrangement of the collection, two outer tubes are dissected by a plane parallel to the coordinate axes x_1, τ.

In Fig. 2, the singular surfaces are shown for two view-points: Fig. 2a corresponds to a view from the axis x_1, Fig. 2b is a view from the axis τ.

The surface closed to the axis τ is dispersal. On some distance from the axis τ, the equivocal surface is situated. It comes into the switching surface for the first player and dispersal one for the second player. The "empty" parts round the time axis can be filled by calculations with smaller time step Δ of the backward procedure.

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