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THREE-DIMENSIONAL REACHABLE SET FOR THE DUBINS CAR: FOUNDATION OF ANALYTICAL DESCRIPTION

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Dedicated to the memory of Professor Rafail Fedorovich Gabasov

Abstract. The Dubins car is a model of motion, in which the scalar control u determines the instantaneous angular speed of rotation. The paper considers the symmetric variant of the constraints $u\in[u_1,u_2]$, where $u_1=-u_2$ and $u_2=1$. We study the three-dimensional reachable set at a given instant $t_f>0$. An analytical description of two-dimensional sections of the set w.r.t. the angular coordinate φ is given. The boundary of each φ-section is formed with the help of a certain set of curves obtained using the Pontryagin maximum principle. This set includes circular arcs and circular involutes. The symmetry property of each φ-section w.r.t. a certain straight line is established. Classification of possible types of the φ-sections is proposed. The greatest difficulty is presented by analysis of the case with non-simply connected φ-sections. The range of values φ and t_f, at which the φ-sections are non-simply connected, is indicated.

Keywords. Dubins car; Extremal piecewise constant control; Pontryagin maximum principle; Sections of reachable set along the angle coordinate; Three-dimensional reachable set.

1. INTRODUCTION

The mathematical "Dubins car" is a model of controllable motion, in which two phase variables x,y are the coordinates of a point geometric position in the plane and the third variable φ is the angle formed by the velocity vector w.r.t. the positive direction of the axis x. The value of the linear velocity is considered to be constant and equal to 1. The scalar control u has the sense of the angular speed of rotation (or, equivalently, the instantaneous radius of turn) and is restricted by the constraint $u\in[-1,1]$.

The name of the model is related to work [1], in which L. Dubins established the properties of curves of the minimal length (with a radius of curvature bounded from below) that connect two points in the plane with the specified exit and entry directions. It corresponds to the time-optimal problem for an object moving at the constant speed and rotation radius restricted from

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below. The results obtained by L. Dubins have been reproven and supplemented with the help of the Pontryagin maximum principle in [2, 3].

Speaking about the previous history of the similar problems, we should note article [4] by A.A. Markov, in which he considered four mathematical problems associated with the design of railways. In 1951, R. Isaacs, while working for the Rand Corporation, submitted his first report [5] on the theory of differential games, in which he posed and outlined the solution to the "homicidal chauffeur" problem. It was R. Isaacs who first began to call the described controllable object with the word "car".

This model is often referred to as the "simplified unicycle". It is used when considering aircraft motions in the horizontal plane with the constant speed and small bank angles (see, e.g., [6, Chapter 4, Section 8.4], [7]). Such model is also applied to a simplified description of motion of the controllable wheeled "bogies" [8] and autonomous underwater vehicles [9]. The list of meaningful problems in the plane, the mathematical description of which (after some transformations) is reduced to the Dubins model, is available in book [10].

The literature devoted to solving specific control problems using the Dubins car is huge. The most popular are various variants of the time-optimal problem [11, 12, 13, 14, 15, 16] that includes problems with the phase constraints. The time-optimal problems with the requirement of passing through the given points are considered in works [17, 18, 19, 20]. Synthesis of the time-optimal control for the Dubins car under the standard three-dimensional termination condition is presented in paper [7]. The problems of possibility of elongation curves solving the time-optimal problem with the three-dimensional termination condition for a certain period of time are considered in [21]. In some works (e.g., [22, 23, 24]), solutions of the pursuit problems close to game problems are investigated where the pursuer has the dynamics of the Dubins car. Some papers study connection between the standard time-optimal problem of object transferring to a given set in the space of geometric coordinates with the solution of the "homicidal chauffeur" differential game [25, 26, 27].

Work [28] also deserves attention, in which the Dubins car found an unexpected use for soft manipulators.

The constraint on the control does not necessarily have the form $u \in [-1,1]$. A more general variant can be written as $u \in [u_1,u_2]$. For the case of $u_1 < 0$, $u_2 > 0$, synthesis of the time-optimal control is built in work [29]. Other models of motion are also studied; their description clearly comes from the Dubins car but is more complex [10, Chapter 13], [30, 31, 32, 33]. At the same time, the structure of the optimal controls often inherits the optimal structure of the similar problem with the dynamics of the Dubins car (see, e.g., work [34]).

The reachable set *at an instant* t_f under the given initial phase state at the instant t_0 is a set of all phase states, into each of which it is possible to transfer the system with the help of some permissible open-loop control exactly at the instant t_f. The reachable set at the instant t_f differs from the reachable set *up to the instant*. The latter consists of phase states, into each of which it is possible to transfer the system at some instant from the interval $[t_0, t_f]$.

For the numerical construction of the three-dimensional reachable sets, the grid methods (developed in the framework of the differential games theory and Hamilton-Jacobi-Bellman partial differential equations) can be used. Examples of such constructions are given in works [35, 36, 37, 38].

This paper examines the reachable set at an instant t_f. Let us denote it as $G(t_f)$. For the constraint $u\in[-1,1]$, an analytical description of boundary of the two-dimensional sections $G_\varphi(t_f)$ of the set $G(t_f)$ by the angular coordinate φ is given. In general, such φ-sections are non-convex and can be non-simply connected. Their structure depends on the instant t_f and the value φ. Description of the two-dimensional reachable set at an instant in geometric coordinates x,y (i.e., the projection of the three-dimensional set $G(t_f)$ into the plane x,y) was obtained in work [39].

The results presented in the paper are based on the statements from paper [40] where (with the help of the Pontryagin maximum principle [41, 42]) the statements about 6 types of piecewise constant open-loop control have been proven. One can limit by only these control types when constructing the boundary of the set $G(t_f)$. The selected types basically coincide with the variants obtained by L. Dubins in work [1].

Using these 6 types of the open-loop control, we analyze (with a fixed t_f) the ends of the corresponding motions that implement the specified value φ. Thus, in the φ-section of the set $G(t_f)$, a number of curves are selected and the boundary of the φ-section is constructed with the help of them. For $\varphi\geq 0$, we consider 5 different cases of the boundary formation of the φ-section. An analysis of some of these cases is very laborious.

The paper is organized as follows. In Section 2, the problem statement is given. In Section 3, information is given on 6 types of the extreme motions used to construct the boundary of the reachable set $G(t_f)$. For $\varphi\geq 0$, formulas for curves in the φ-sections generated by extreme motions are derived. An auxiliary coordinate system is considered, which is more convenient than the original one when analyzing the properties of curves in the φ-sections. Some curves are described, from the parts of which the boundary of the φ-sections is constructed. The properties of these curves are analyzed. In particular, it is established that two curves represent parts of the involutes of the circle. Section 4 is devoted to statements about the extreme motions leading to the interior of the set $G(t_f)$. Auxiliary statements based on the Jordan curve theorem are given in Section 5. In Section 6, classification of the φ-sections is given depending on the values t_f and φ. All possible variants are divided into five cases. In Sections 7–10, the detailed analysis of each of these cases is given. Of particular interest is Case 2, for which the φ-sections are not simply connected. Section 12 describes the symmetry property that allows one to find the φ-sections for $\varphi<0$ relying on the constructed φ-sections for $\varphi\geq 0$.

2. Problem statement

Consider the controllable system

$$\dot{x}=\cos\varphi,\quad \dot{y}=\sin\varphi,\quad \dot{\varphi}=u;\quad u\in[-1,1]. \tag{2.1}$$

Here, x and y are the coordinates of the geometric position in the plane, u is the scalar control. Let us agree that the positive value of the angle φ is counted from the positive direction of the axis x counterclockwise (Fig. 1). The phase vector $(x,y,\varphi)^{\mathsf{T}}$ of system (2.1) will be denoted as z.

As the initial state at the instant $t_0=0$, we set $x_0=y_0=\varphi_0=0$. The value of the angle φ at the instant t is calculated in the form of the integral

$$\int_0^t u(\tau)d\tau$$

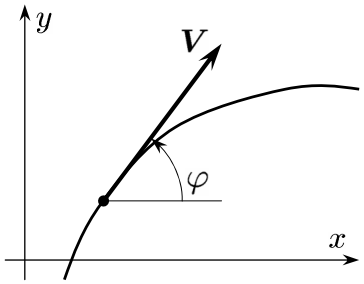

FIGURE 1. The coordinate system, $\boldsymbol{V} = (\dot{x},\dot{y})^{\mathsf{T}}$

of the open-loop control implemented on the interval $[0,t]$. Thus, $\varphi \in (-\infty, +\infty)$. As admissible open-loop controls, we take measurable functions of time that satisfy the constraint onto the control u.

The reachable set $G(t_f)$ is defined as the set of all those phase states of system (2.1) that can be obtained *at the instant* t_f using all the admissible measurable open-loop controls on the interval $[0,t_f]$. The choice of measurable controls as admissible is due to the desire to talk about the closedness of the reachable set $G(t_f)$ within the framework of the problem statement.

By the symbol $G_\varphi(t_f)$, we denote the φ-section of the set $G(t_f)$:

$$G_\varphi(t_f) = \{(x,y)^{\mathsf{T}} : (x,y,\varphi)^{\mathsf{T}} \in G(t_f)\}.$$

The purpose of this work is to obtain an analytical description of the φ-sections.

The symbol ∂ will mean the boundary of the set, the symbol "int" is its interior. Note that if some point P belongs to $\partial G_\varphi(t_f)$, then the point $(P,\varphi)^{\mathsf{T}}$ belongs to $\partial G(t_f)$. Generally speaking, the reverse is not true.

3. PROPERTIES OF CURVES GENERATED BY EXTREME MOTIONS

3.1. **Types of extreme motions.** We apply the Pontryagin maximum principle (PMP) to the open-loop controls transfering system (2.1) onto the boundary of the reachable set. Under this, we establish that for any point on the boundary there is some *piecewise constant* control leading to this point. The corresponding calculations are given in [40, 43]. It is shown that any point on the boundary of a three-dimensional set $G(t_f)$ can be reached using an open-loop control that takes values in the three-element set $\{-1,0,1\}$ and has no more than two switch instants. We identified 6 types of controls, by which one can limit to study the boundary.

Let us list these 6 types. The switch instants are denoted by t_1 and t_2. We suppose that $t_0 = 0$, $t_f > 0$ and $t_0 \le t_1 \le t_2 \le t_f$.

Any control of the type U1 takes the value $u = 1$ at some first time interval $[0,t_1)$, the value $u = 0$ is applied on some second interval $[t_1,t_2)$, and the value $u = 1$ is used on the third interval $[t_2,t_f]$. If one or two of the specified intervals are missing, then the resulting control is also assigned to the U1 type. The types U2 – U6 are defined similarly. Noting the control values on each of three intervals, we shall write the corresponding table in the form

$$\begin{array}{lll} \text{U1}: \ \ 1,\ 0,\ 1; & \text{U2}: -1,\ 0,\ 1; & \text{U3}: \ \ 1,\ 0,-1; \\ \text{U4}: -1,\ 0,-1; & \text{U5}: \ \ 1,-1,\ 1; & \text{U6}: -1,\ 1,-1. \end{array} \tag{3.1}$$

These types of controls coincide with those ones that were identified by L. Dubins in work [1] for solving the time-optimal problem. In relation to the controls of the types U5 and U6, additional conditions have been noted in the L. Dubins' work that are specific namely for the time-optimal problem. In the problem of constructing the boundary of the reachable set $G(t_f)$, the additional requirement for controls of the types U5 and U6 has the following form [40, p. 323]:

$$t_2 - t_1 \geq (t_1 - t_0) + (t_f - t_2). \tag{3.2}$$

In [40], it is shown that when inequality (3.2) is violated, the open-loop control of the types U5 or U6 leads to the interior of the reachable set $G(t_f)$.

Since the controls $\mathrm{U1}-\mathrm{U6}$ satisfy the PMP, we call them and corresponding motions extreme. For the controls U5 and U6, in the sequel, we shall assume that inequality (3.2) is fulfilled.

For each type of control, degeneration (reducing to zero) is possible of one or two intervals of control constancy. Formally, we shall refer such controls to more than one type of controls $\mathrm{U1}-\mathrm{U6}$ specified in (3.1).

The set of possible values φ of system (2.1) at the instant $t_f > 0$ is defined by the constraint $u \in [-1, 1]$ and represents the segment $[-t_f, t_f]$. The extreme values $\varphi = \mp t_f$ are provided on the controls $u(t) \equiv \mp 1$. We get the single-point φ-sections with the coordinates $x(t_f) = \sin t_f$, $y(t_f) = \mp(1 - \cos t_f)$.

For $\varphi > 0$, only four (namely, U1, U2, U3, and U6) from six types of control (specified in (3.1)) can lead to the boundary $\partial G_\varphi(t_f)$ of the corresponding φ-section. Indeed, any control of the type U4 gives the value $\varphi \leq 0$ at the instant t_f. For any control of the type U5 with $\varphi > 0$, condition (3.2) is violated. We emphasize that for any control of type U6 for $\varphi \geq 0$, the condition (3.2) is satisfied.

Let $\varphi = 0$ at the instant t_f. Then the control of the type U4 is identically equal to zero and can also be assigned to the type U1. Now we take an arbitrary control of the type U5 with the switch instants t_1 and t_2. We have $t_0 \leq t_1 \leq t_2 \leq t_f$ and $t_2 - t_1 = (t_f - t_2) + (t_1 - t_0)$. Consider the control of the type U6 with the switches at the instants $t_1^* = t_f - t_2$ and $t_2^* = t_1^* + (t_2 - t_1)$. Integrating system (2.1), it is easy to verify that the designed control of the type U6 leads to the same point $\big(x(t_f), y(t_f)\big)^{\mathsf{T}}$ as the initial control of the type U5. Therefore, for $\varphi = 0$, the controls of the type U5 generate the same set of points $(x, y)^{\mathsf{T}}$ at the instant t_f as the controls of the type U6.

Thus, the following lemma is valid.

Lemma 3.1. *For $\varphi \in [0, t_f)$, to construct the boundary of the φ-sections of the reachable set $G(t_f)$, one can restrict themselves to four types of controls* U1, U2, U3, *and* U6.

Further, when studying the φ-sections, detailed calculations are made under the assumption $0 \leq \varphi < t_f$. The constructed φ-sections (taking into account the symmetry of system (2.1)) determine the φ-sections for the condition $-t_f < \varphi < 0$ using some linear transformation.

3.2. **Formulas in the original coordinates for curves in φ-sections.** Fixing a certain value of φ at the instant $t_f > 0$, we get a connection between the switch instants t_1 and t_2 which provides this φ. Thus, for each control type providing the selected φ, we get corresponding one-parameter curve in the plane x, y.

We assume that $\varphi \geq 0$. In accordance with Lemma 3.1, we use controls of the types U1, U2, U3, and U6 to construct the boundary of the φ-sections.

Introduce the notation $\theta = (t_f - \varphi)/2$. Obviously, for $\varphi \in [0, t_f)$, the inequalities $\theta > 0$ and $\theta + \varphi > 0$ are fulfilled.

We give formulas describing the geometric positions in the plane x, y by virtue of controls of the types U1, U2, U3, and U6 for the selected values t_f and φ. The corresponding one-dimensional parameters s_1, s_2, s_3, and s_6 are determined using the formulas

$$s_1 = 2t_1 - \varphi, \quad s_2 = -t_1, \quad s_3 = t_1 - \varphi, \quad s_6 = 2t_1 - \theta. \tag{3.3}$$

We shall take the ranges of acceptable values of these parameters (marking the begin and end values) in the form

$$\begin{aligned} s_1 \in [s_1^b, s_1^e] = [-\varphi, \varphi], \quad & s_2 \in [s_2^b, s_2^e] = [-\theta, 0], \\ s_3 \in [s_3^b, s_3^e] = [0, \theta], \quad & s_6 \in [s_6^b, s_6^e] = [-\theta, \theta]. \end{aligned} \tag{3.4}$$

Integrating system (2.1) for four types of the controls under consideration, we get four curves in the plane x, y:

$$\begin{pmatrix} x_{\mathrm{U1}}(s_1) \\ y_{\mathrm{U1}}(s_1) \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix} + (t_f - \varphi) \begin{pmatrix} \cos\left(\dfrac{s_1 + \varphi}{2}\right) \\ \sin\left(\dfrac{s_1 + \varphi}{2}\right) \end{pmatrix}, \tag{3.5}$$

$$\begin{pmatrix} x_{\mathrm{U2}}(s_2) \\ y_{\mathrm{U2}}(s_2) \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix} + 2 \left((\theta + s_2) \begin{pmatrix} \cos s_2 \\ \sin s_2 \end{pmatrix} - \begin{pmatrix} \sin s_2 \\ 1 - \cos s_2 \end{pmatrix} \right), \tag{3.6}$$

$$\begin{pmatrix} x_{\mathrm{U3}}(s_3) \\ y_{\mathrm{U3}}(s_3) \end{pmatrix} = - \begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix} + 2 \left((\theta - s_3) \begin{pmatrix} \cos(s_3 + \varphi) \\ \sin(s_3 + \varphi) \end{pmatrix} + \begin{pmatrix} \sin(s_3 + \varphi) \\ 1 - \cos(s_3 + \varphi) \end{pmatrix} \right), \tag{3.7}$$

$$\begin{pmatrix} x_{\mathrm{U6}}(s_6) \\ y_{\mathrm{U6}}(s_6) \end{pmatrix} = - \begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix} + 4 \sin\left(\frac{t_f + \varphi}{4}\right) \begin{pmatrix} \cos\left(\dfrac{\varphi - s_6}{2}\right) \\ \sin\left(\dfrac{\varphi - s_6}{2}\right) \end{pmatrix}. \tag{3.8}$$

In work [44], other (but equivalent) formulas for the parametric representation of the indicated curves in the original coordinate system x, y were obtained. They were used there for the analytical description of the φ-sections in the assumption $t_f \le 2\pi$. The relations (3.5)–(3.8) are more compact and more effective for the analysis of the φ-sections under any value $t_f > 0$.

3.3. **Auxiliary coordinate system.** We shall use an auxiliary orthogonal coordinate system as in work [44]. It is convenient for revealing the symmetry properties of the φ-sections boundary. We define the auxiliary system X, Y through the original system x, y as follows:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos(\varphi/2) & \sin(\varphi/2) \\ -\sin(\varphi/2) & \cos(\varphi/2) \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix} \right). \tag{3.9}$$

For a fixed φ, this linear transformation consists of a rotation and a shift. It is one-to-one and preserves the distance between points.

The axis X of the auxiliary system passes through the origin of the initial system (point o) and is rotated by an angle $\varphi/2$ counterclockwise relative to the axis x (Fig. 2). The reference point of the auxiliary system (point O) coincides with the center of the circumference containing arc (3.5). The axis X divides this arc in half.

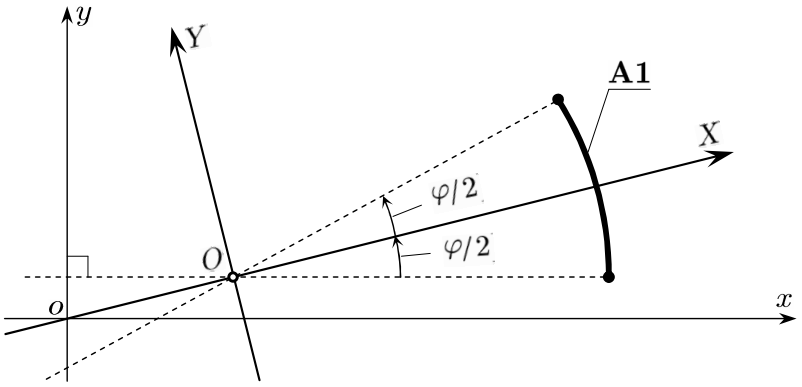

FIGURE 2. Auxiliary coordinate system X,Y

We denote by A1, A2, A3, and A6 curves (3.5), (3.6), (3.7), and (3.8), which are generated by the controls U1, U2, U3, and U6 in the auxiliary system. Their analytical representation has the form

$$\mathrm{A1}(s_1) = \begin{pmatrix} X_{\mathrm{U1}}(s_1) \\ Y_{\mathrm{U1}}(s_1) \end{pmatrix} = (t_f - \varphi) \begin{pmatrix} \cos\left(\frac{s_1}{2}\right) \\ \sin\left(\frac{s_1}{2}\right) \end{pmatrix} = 2\theta \begin{pmatrix} \cos\left(\frac{s_1}{2}\right) \\ \sin\left(\frac{s_1}{2}\right) \end{pmatrix}, \tag{3.10}$$

$$\mathrm{A2}(s_2) = \begin{pmatrix} X_{\mathrm{U2}}(s_2) \\ Y_{\mathrm{U2}}(s_2) \end{pmatrix} = 2(\theta + s_2) \begin{pmatrix} \cos\left(s_2 - \frac{\varphi}{2}\right) \\ \sin\left(s_2 - \frac{\varphi}{2}\right) \end{pmatrix} - 4\sin\left(\frac{s_2}{2}\right) \begin{pmatrix} \cos\left(\frac{s_2}{2} - \frac{\varphi}{2}\right) \\ \sin\left(\frac{s_2}{2} - \frac{\varphi}{2}\right) \end{pmatrix}, \tag{3.11}$$

$$\mathrm{A3}(s_3) = \begin{pmatrix} X_{\mathrm{U3}}(s_3) \\ Y_{\mathrm{U3}}(s_3) \end{pmatrix} = 2(\theta - s_3) \begin{pmatrix} \cos\left(s_3 + \frac{\varphi}{2}\right) \\ \sin\left(s_3 + \frac{\varphi}{2}\right) \end{pmatrix} + 4\sin\left(\frac{s_3}{2}\right) \begin{pmatrix} \cos\left(\frac{s_3}{2} + \frac{\varphi}{2}\right) \\ \sin\left(\frac{s_3}{2} + \frac{\varphi}{2}\right) \end{pmatrix}, \tag{3.12}$$

$$\mathrm{A6}(s_6) = \begin{pmatrix} X_{\mathrm{U6}}(s_6) \\ Y_{\mathrm{U6}}(s_6) \end{pmatrix} = -4 \begin{pmatrix} \sin\left(\frac{\varphi}{2}\right) \\ 0 \end{pmatrix} + 4\sin\left(\frac{t_f + \varphi}{4}\right) \begin{pmatrix} \cos\left(\frac{-s_6}{2}\right) \\ \sin\left(\frac{-s_6}{2}\right) \end{pmatrix}. \tag{3.13}$$

The parameters s_1, s_2, s_3, s_6 and the corresponding ranges of their changes are defined in (3.3) and (3.4).

Further analysis of the φ-sections of the reachable set $G(t_f)$ will be carried out in the auxiliary coordinate system.

3.4. **The simplest properties of curves A1, A2, A3, and A6.** The curves A1 and A6 are arcs of circumferences. Each of them is symmetric w.r.t. the axis X, since the following relations are valid:

$$X_{\mathrm{U1}}(s_1) = X_{\mathrm{U1}}(-s_1), \quad Y_{\mathrm{U1}}(s_1) = -Y_{\mathrm{U1}}(-s_1);$$

$$X_{\mathrm{U6}}(s_6) = X_{\mathrm{U6}}(-s_6), \quad Y_{\mathrm{U6}}(s_6) = -Y_{\mathrm{U6}}(-s_6).$$

The center of the circumference (the arc of which is the curve A1) coincides with the origin of the auxiliary system. The radius R_{A1} of the circumference is calculated using the formula

$$R_{A1} = 2\theta = t_f - \varphi. \tag{3.14}$$

The angular span of the arc A1 is equal to φ. Therefore, the curve A1 has no self-intersections for $\varphi \in [0, 2\pi)$. For $\varphi \geq 2\pi$, the curve A1 covers the entire circumference. The "overlap" points can be reached at different values of the parameter s_1, which differ by 4π. If $\varphi = 0$, then the curve A1 degenerates to the point $\left(t_f, 0\right)^{\mathsf{T}}$. Let us denote by the symbol C_{A1} the circle of the radius R_{A1} with the center at the origin of the auxiliary coordinate system.

The center of the circumference, the arc of which is the curve A6, is located at the point

$$H = -4\left(\sin\left(\varphi/2\right), 0\right)^{\mathsf{T}}, \tag{3.15}$$

on the axis X. The radius R_{A6} of the circumference is equal to

$$R_{A6} = 4\left|\sin\left((t_f + \varphi)/4\right)\right|. \tag{3.16}$$

Its value depends on t_f, φ and becomes zero when $(t_f + \varphi)$ is multiple of 4π. The angular span of the arc A6 is equal to θ. If $(t_f + \varphi) < 4\pi$, then $\theta < 2\pi$ and the curve A6 has no self-intersections. We denote by the symbol C_{A6} the circle of the radius R_{A6} with the center at the point H.

The curves A2 and A3 are mutually symmetric w.r.t. the axis X. This property is defined by equalities

$$X_{\mathrm{U2}}(s_2) = X_{\mathrm{U3}}(s_3), \quad Y_{\mathrm{U2}}(s_2) = -Y_{\mathrm{U3}}(s_3) \tag{3.17}$$

that are valid for any parameter values $s_2 = -s_3$ from ranges (3.4).

For controls of the type U3 forming the curve A3, let us analyze the degeneracy of the control constancy intervals. The first interval is non-degenerate for $s_3 > 0$, since we have $t_1 = \varphi + s_3 > 0$ by virtue of (3.3) and (3.4). Duration of the second interval is determined by the formula $t_2 - t_1 = t_f - \varphi - 2s_3$. It vanishes only when $s_3 = (t_1 - \varphi)/2 = \theta$, i.e., at the last point of the curve A3. Duration of the third interval is $t_f - t_2 = s_3$. Therefore, the third interval degenerates only at the initial point of the curve A3. Thus, none of the control constancy intervals degenerates at the internal points of the curve A3. The similar property is true for the curve A2.

In Fig. 3, the curve A3 corresponding to $\varphi = 0.3\pi$ and $t_f = 2.7\pi$ is shown. The geometric method of constructing points $\mathrm{A3}(s_3)$ by controls of the type U3 is explained for several values of the parameter s_3. The motions leave the starting point o with the direction indicated by the arrow. On the interval $[0, t_1)$, each motion goes with $u = +1$ along the dotted circumference counterclockwise. Then, it continues on the interval $[t_1, t_2)$ by a straight-line motion with $u = 0$. The third interval $[t_2, t_f]$ corresponds to the motion along the arc of the circumference with $u = -1$ clockwise.

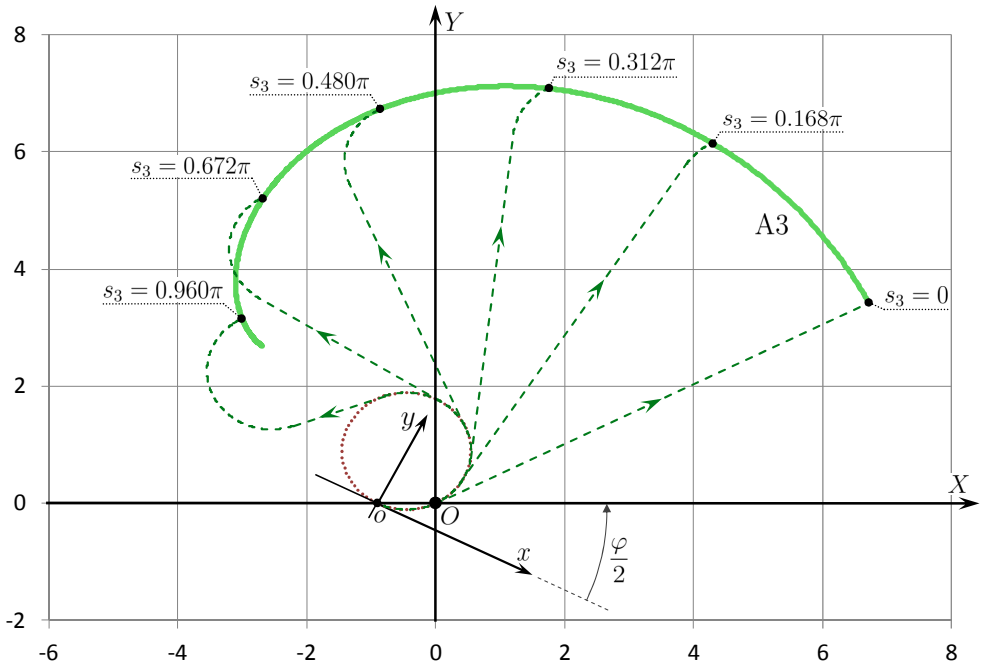

FIGURE 3. Formation of the curve A3 by means of controls of the type U3

Consider the curves A1, A2, A3, and A6 in the sequence A1, A3, A6, and A2 bypassing them in ascending parameters s_1, s_2, s_3, and s_6. For the extreme values of the parameters, we have

$$\mathrm{A1}(s_1^e) = \mathrm{A3}(s_3^b), \quad \mathrm{A3}(s_3^e) = \mathrm{A6}(s_6^b), \quad \mathrm{A6}(s_6^e) = \mathrm{A2}(s_2^b), \quad \mathrm{A2}(s_2^e) = \mathrm{A1}(s_1^b).$$

The listed docking points are denoted by $\mathcal{P}_{1,3}$, $\mathcal{P}_{3,6}$, $\mathcal{P}_{2,6}$, and $\mathcal{P}_{1,2}$. As a result of splicing, we get a continuous piecewise smooth closed curve, which we denote by the symbol $\mathrm{A}_\varphi(t_f)$.

By virtue of Lemma 3.1, the boundary of the φ-section for $\varphi \in [0, t_f)$ is a subset of the curve $\mathrm{A}_\varphi(t_f)$. When selecting parts of the curves A1, A3, A6, and A2 that form the boundary of the φ-section, significant difficulty is caused by presence of self-intersections of the curve $\mathrm{A}_\varphi(t_f)$.

In Fig. 4, two examples of the curve $\mathrm{A}_\varphi(t_f)$ are given. The docking points of the curves A1, A3, A6, and A2 are marked with risks. For the values $t_f = 3\pi$ and $\varphi = 0.4\pi$ (Fig. 4a), the curve $\mathrm{A}_\varphi(t_f)$ has no self-intersections. There are many self-intersection points for the values $t_f = 10\pi$ and $\varphi = 0.4\pi$ (Fig. 4b). When changing s_6 in the range $[-\theta, \theta]$, the points $\mathrm{A6}(s_6)$ go along the boundary of the circle C_{A6} clockwise from the point of docking with the curve A3 to the point of docking with the curve A2. Herewith, on Fig. 4b, the point $\mathrm{A6}(s_6)$ makes more than two turns.

Structure of the curve $\mathrm{A}_\varphi(t_f)$ (for a fixed φ) becomes more complicated with the growth of t_f.

Further in this section, location of the curves A1, A2, A3, and A6 in the plane X, Y will be investigated.

3.5. **Curves A2 and A3 are parts of circle's involutes.** Each of the curves A2 and A3 is some part of the involute of the circle. We show it for the curve A2. Let us rewrite equation (3.11)

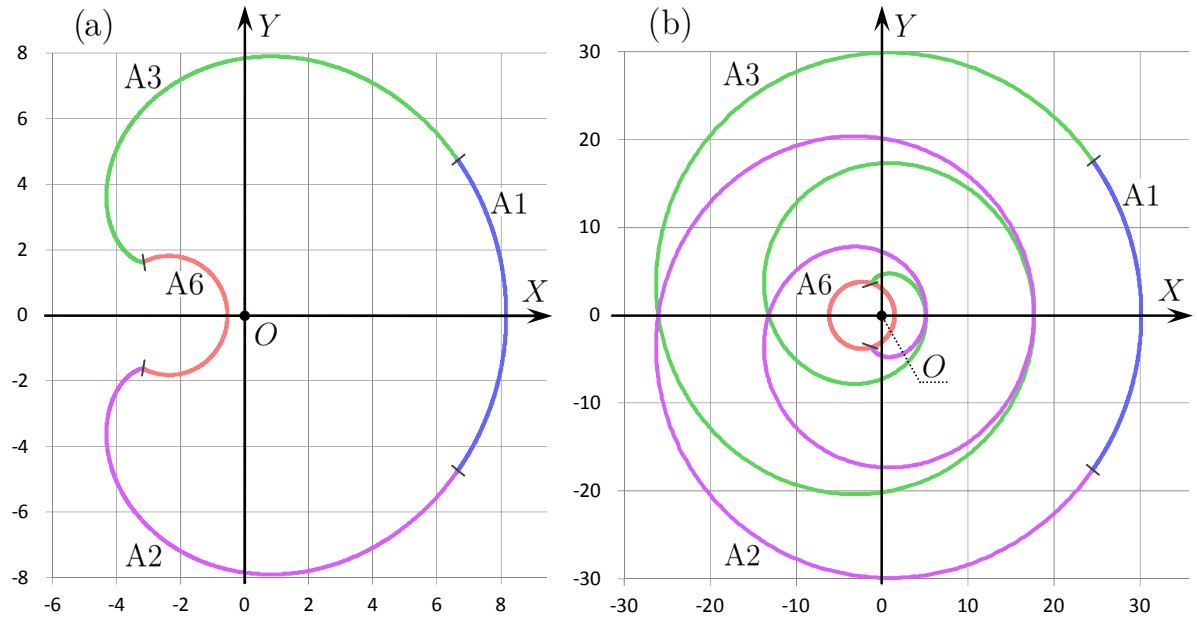

FIGURE 4. The curve $\mathrm{A}_\varphi(t_f)$ for $t_f = 3\pi$, $\varphi = 0.4\pi$ (a), and $t_f = 10\pi$, $\varphi = 0.4\pi$ (b)

for the curve A2 in the equivalent form using the trigonometric transformation of the last term:

$$\mathrm{A2}(s_2) = 2\,(\theta + s_2)\begin{pmatrix}\cos\left(s_2 - \dfrac{\varphi}{2}\right)\\ \sin\left(s_2 - \dfrac{\varphi}{2}\right)\end{pmatrix} + 2\begin{pmatrix}-\sin\left(s_2 - \dfrac{\varphi}{2}\right)\\ \cos\left(s_2 - \dfrac{\varphi}{2}\right)\end{pmatrix} - 2\begin{pmatrix}\sin\left(\dfrac{\varphi}{2}\right)\\ \cos\left(\dfrac{\varphi}{2}\right)\end{pmatrix}. \tag{3.18}$$

The canonical form of the parametric representation of the involute of a circle is taken in the following form (see [45, p. 252, formula (1)], [46, §11, p. 43]):

$$\begin{aligned} x_1 &= r\cos\tau + r\tau\sin\tau,\\ x_2 &= r\sin\tau - r\tau\cos\tau. \end{aligned} \tag{3.19}$$

Here, x_1 and x_2 are the rectangular coordinates, $\tau \geq 0$ is a parameter. The radius of the base circle is r, its center is located at the origin. The involute at $\tau = 0$ leaves the point $(0, r)^{\mathsf{T}}$.

Let us represent the curve A2, given by formula (3.18), in form (3.19) using the rotation and shift operations. The shift is provided by the third term in (3.18), it does not depend on s_2. Next, we shall replace the variables $\tau = \theta + s_2$, $\tau \in [0, \theta]$. After that, the first two terms in (3.18) (after rearranging them) will be written as follows:

$$2\begin{pmatrix}-\sin\left(\tau - \theta - \dfrac{\varphi}{2}\right)\\ \cos\left(\tau - \theta - \dfrac{\varphi}{2}\right)\end{pmatrix} + 2\tau\begin{pmatrix}\cos\left(\tau - \theta - \dfrac{\varphi}{2}\right)\\ \sin\left(\tau - \theta - \dfrac{\varphi}{2}\right)\end{pmatrix}. \tag{3.20}$$

Further, we introduce the rotation matrix

$$\begin{pmatrix}\cos\psi & \sin\psi\\ -\sin\psi & \cos\psi\end{pmatrix}$$

and multiply it by vector (3.20). Assuming $\psi = \frac{\pi}{2} - \theta - \frac{\varphi}{2}$, we get for (3.20) representation of form (3.19):

$$2\begin{pmatrix}\cos\tau\\ \sin\tau\end{pmatrix} + 2\tau\begin{pmatrix}\sin\tau\\ -\cos\tau\end{pmatrix}.$$

Thus, the curve A2 is the initial part of the circle's involute. As a result, it does not have self-intersections. The center of the base circle is located at the point $2\,(-\sin(\varphi/2), -\cos(\varphi/2))^{\mathsf{T}}$, its radius is 2. The starting point of the involute corresponds to the value $s_2 = -\theta$.

The curve A3 is symmetric to the curve A2 and, also, represents the initial part of the involute of the circle. For it, the radius of the base circle is the same as for the curve A2, and its center is located at the point $2\,(-\sin(\varphi/2), \cos(\varphi/2))^{\mathsf{T}}$ symmetrically w.r.t. the axis X.

Note that the point O of the auxiliary system X, Y origin always belongs to both base circles.

In Fig. 5 for $t_f = 2.5\pi$ and $\varphi = \pi/3$, the base circles corresponding to the involutes A3 and A2 are shown. Dotted straight lines generate points of the curve A3 (of the curve A2, respectively) as the points of the circle's involute. Curves A1 and A6 are also presented.

We emphasize that the dotted lines (coming to the curve A3 or the curve A2) are not the trajectories of the system (2.1) from the initial phase point $x_0 = y_0 = \varphi_0 = 0$ and do not satisfy the boundary condition for φ when hitting the curve A3 or the curve A2.

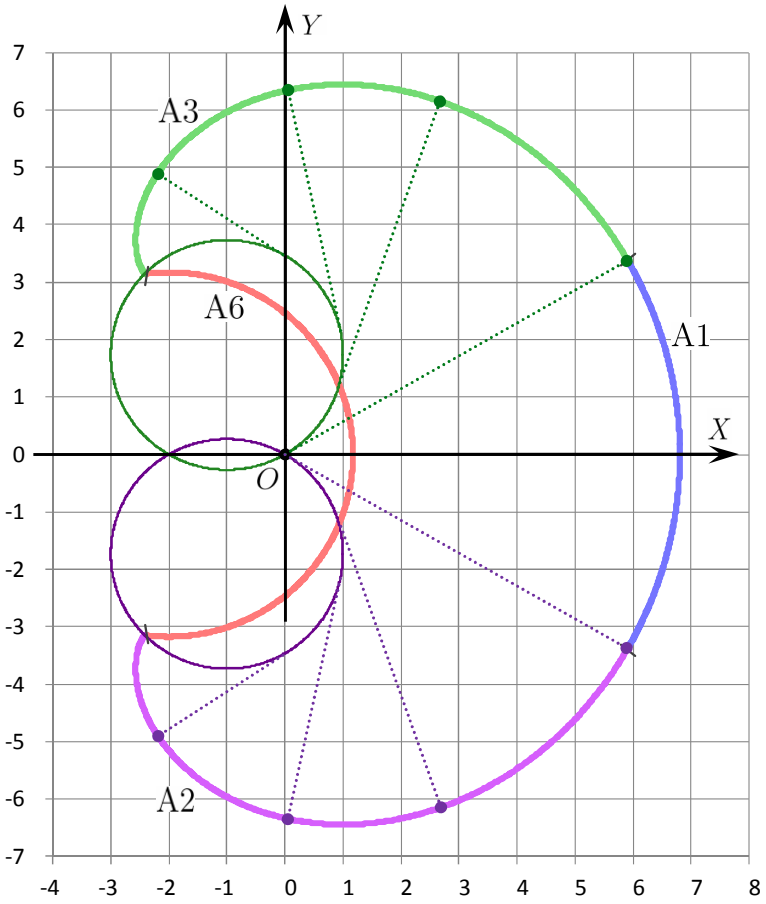

FIGURE 5. Representation of the curves A3 and A2 in the form of involutes

3.6. **Mutual arrangement of curves A1, A2, A3, and A6.** Denote by the symbol $R_{1,6}(s_6)$ the distance from the center of the curve A1 (it coincides with the origin O of the auxiliary system) to the points of the curve A6. Let $R_{6,1}(s_1)$, $R_{6,2}(s_2)$, and $R_{6,3}(s_3)$ be respectively distances from the center H of the curve A6 to the points of the curves A1, A2, and A3.

The denotations used here are illustrated in Fig. 6. Calculations are made for $t_f = 2.4\pi$ and $\varphi = 0.8\pi$.

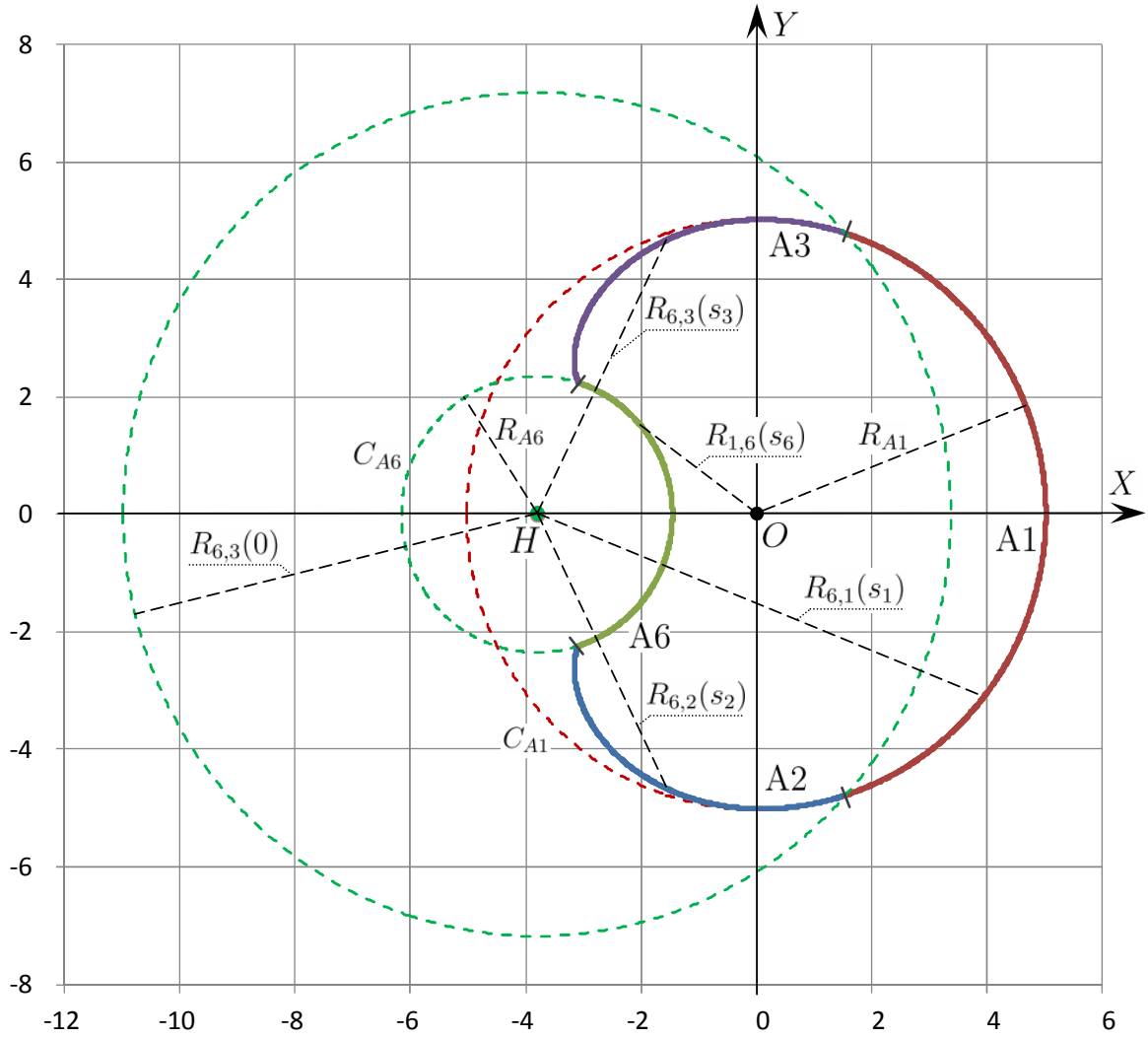

FIGURE 6. Explanation of denotations for the study of the mutual arrangement of the curves A1, A2, A3, and A6

1) Let us start with analysis of the relative arrangement of the curves A1 and A6. We show that the following relation

$$(R_{A1})^2 - \big(R_{1,6}(s_6)\big)^2 > 0, \qquad s_6 \in \left[s_6^b, s_6^e\right] = [-\theta, \theta] \tag{3.21}$$

is fulfilled. Therefore, the curve A6 belongs to intC_{A1}.

By virtue of formulas (3.10) and (3.13), we get $(R_{A1})^2 - \big(R_{1,6}(s_6)\big)^2 = 4D_{1,6}(s_6)$,

where

$$D_{1,6}(s_6) = \theta^2 - 4\sin^2\left(\frac{\varphi}{2}\right) + 8\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi+\theta}{2}\right)\cos\left(\frac{s_6}{2}\right) - 4\sin^2\left(\frac{\varphi+\theta}{2}\right).$$

The sign of inequality (3.21) is defined by the expression for $D_{1,6}(s_6)$.

It can be seen that $D_{1,6}(s_6) > 0$ if $\theta > 4$. Now let $0 < \theta \leq 4$. Given the range of the values s_6, we have $-2 \leq -\theta/2 \leq s_6/2 \leq \theta/2 \leq 2$. Therefore, $\cos(\theta/2) \leq \cos(s_6/2) \leq \cos(0) = 1$. The expression $\cos(s_6/2)$ is included linearly in the definition of the function $D_{1,6}(s_6)$. So, its minimum is reached either at $s_6 = 0$, or at $s_6 = \pm\theta$. Taking into account the parity of the function $D_{1,6}(s_6)$, the condition $D_{1,6}(s_6) > 0$ needs to be checked only for two values $s_6 = 0$ and $s_6 = \theta$.

Using trigonometric transformations, we obtain

$$\begin{aligned}
D_{1,6}(0) &= \theta^2 - 4\sin^2\left(\frac{\varphi}{2}\right) + 8\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi+\theta}{2}\right) - 4\sin^2\left(\frac{\varphi+\theta}{2}\right) \\
&= \theta^2 - 4\sin^2\left(\frac{\theta}{2}\right) = 4\left(\left(\frac{\theta}{2}\right)^2 - \sin^2\left(\frac{\theta}{2}\right)\right) > 0, \\
D_{1,6}(\theta) &= \theta^2 - 4\sin^2\left(\frac{\varphi}{2}\right) + 8\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi+\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right) - 4\sin^2\left(\frac{\varphi+\theta}{2}\right) \\
&= \theta^2 - 4\left(\sin\left(\frac{\varphi}{2}\right) - \sin\left(\frac{\varphi+\theta}{2}\right)\right)^2 \\
&= \theta^2 - 4\left(2\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}+\frac{\varphi}{2}\right)\right)^2 > \theta^2 - 16\sin^2\left(\frac{\theta}{4}\right) > 0.
\end{aligned}$$

Thus, for $\varphi \in [0, t_f)$, the inequality $D_{1,6}(s_6) > 0$ is satisfied for all $s_6 \in [s_6^b, s_6^e] = [-\theta, \theta]$. This means that A6 $\subset$ intC_{A1}. As a consequence, we see that the arcs A1 and A6 have no common points.

2) Let describe the docking of the curves A2 and A3 with the curve A1.

The radius of curvature of the involute at the current point is the distance from it along the generating straight line to the point of contact of the line with the base circle [45, p. 252].

For the involute A3 at the current value of the parameter s_3, the point of contact of the generating straight line with the base circle is a vector representing the second term in the right part of formula (3.12). The radius of curvature of the curve A3 at the current point is the modulus of the first term in (3.12), *i.e.*, $2(\theta - s_3)$. For the starting point of the curve A3 (where $s_3 = 0$), the point of tangency of the generating straight line with the base circle coincides with the origin of the coordinates of the plane X, Y, and the radius of curvature is 2θ. The circumpherence, on which the arc A1 lies, has the same center and the same radius. Consequently, the curves A1 and A3 are smoothly joined with the coincidence of curvature. Since the radius of curvature of the involute A3 monotonically decreases with the growth of s_3, the entire curve A3 (excepting the starting point) belongs to intC_{A1} [47, Theorem 2]. It follows that the curves A1 and A3 have only one common point (i.e., the junction point $\mathcal{P}_{1,3}$).

Similarly, due to symmetry, the curve A2 (excepting the point $\mathcal{P}_{1,2}$ that is the point of junction with the curve A1) belongs to intC_{A1}.

3) To estimate the relative location of the curves A2 and A6, consider the function $s_2 \to \left(R_{6,2}(s_2)\right)^2$. Based on (3.11) and (3.13), we have

$$\left(R_{6,2}(s_2)\right)^2 = 4\Bigg((\theta+s_2)^2+\left(2\sin\left(\frac{s_2}{2}\right)\right)^2+\left(2\sin\left(\frac{\varphi}{2}\right)\right)^2-4(\theta+s_2)\sin\left(\frac{s_2}{2}\right)\cos\left(\frac{s_2}{2}\right)$$
$$+4(\theta+s_2)\cos\left(s_2-\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right)-8\sin\left(\frac{s_2}{2}\right)\cos\left(s_2-\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right)\Bigg).$$

Substituting $s_2 = s_2^b = -\theta$, we get

$$\left(R_{6,2}(-\theta)\right)^2 = 4\left(2\sin\left(\frac{\varphi+\theta}{2}\right)\right)^2 = (R_{A6})^2.$$

Let us express the derivative of the function $\left(R_{6,2}(s_2)\right)^2$ with respect to the parameter s_2:

$$\left(\left(R_{6,2}(s_2)\right)^2\right)'_{s_2} = 8(\theta+s_2)\left(1-\cos(s_2)-2\sin\left(s_2-\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right)\right).$$

After trigonometric transformations, we have

$$\left(\left(R_{6,2}(s_2)\right)^2\right)'_{s_2} = 8(\theta+s_2)(1-\cos(s_2-\varphi)) \geq 0.$$

The resulting inequality turns into the equality only in a finite number of values of the parameter s_2. Therefore, the function $\left(R_{6,2}(s_2)\right)^2$ is strictly monotonically increasing in the parameter s_2. Thus, for $\varphi < t_f$, the inequality $R_{6,2}(s_2) > R_{A6}$ is valid for all $s_2 \in \left(s_2^b, s_2^e\right]$. The similar inequality $R_{6,3}(s_3) > R_{A6}$ is also fulfilled for the points of the curve A3.

4) Let $0 \leq \varphi < 2\pi$. Let us show that the curve A1 lies outside the interior of the circle, whose center is at the point H (Fig. 6) and the radius is the distance from the point H to the points $\mathcal{P}_{1,3}$ and $\mathcal{P}_{1,2}$.

The square of the distance from the point H to the junction point of the curves A1 and A3 can be calculated by the formula

$$\left(R_{6,3}(s_3=0)\right)^2 = 4\left(\theta^2+\left(2\sin\frac{\varphi}{2}\right)^2+2\theta\sin\varphi\right).$$

The square of the distance from the point H to the points of the curve A1 is found by the formula

$$\left(R_{6,1}(s_1)\right)^2 = \left(X_{U1}(s_1)+4\sin\left(\frac{\varphi}{2}\right)\right)^2+\left(Y_{U1}(s_1)\right)^2$$
$$=\left(2\theta\cos\frac{s_1}{2}\right)^2+16\theta\cos\frac{s_1}{2}\sin\frac{\varphi}{2}+\left(4\sin\frac{\varphi}{2}\right)^2+\left(2\theta\sin\frac{s_1}{2}\right)^2.$$

Consider the difference

$$\left(R_{6,1}(s_1)\right)^2-\left(R_{6,3}(0)\right)^2 = 16\theta\cos\frac{s_1}{2}\sin\frac{\varphi}{2}-8\theta\sin\varphi = 16\theta\sin\frac{\varphi}{2}\left(\cos\frac{s_1}{2}-\cos\frac{\varphi}{2}\right).$$

Taking into account the inequalities $-\pi < -\varphi/2 \leq s_1/2 \leq \varphi/2 < \pi$ and the cosine properties, we obtaine that the last factor of the written difference (like the difference itself) cannot be less than zero.

Therefore, the minimum distance from the point H to the points of the curve A1 is achieved at the points $\mathcal{P}_{1,3}$ and $\mathcal{P}_{1,2}$, i.e., $R_{6,1}(-\varphi) = R_{6,1}(\varphi) \leq R_{6,1}(s_1)$ for any $s_1 \in [-\varphi, \varphi]$.

5) Let us prove that the curves A2 and A3 can intersect only at points on the axis X.

Suppose the contrary, that there is a point of intersection of the curves A2 and A3 lying outside the axis X. Let this point correspond to some values of the parameters $\widetilde{s_2}$ and $\widetilde{s_3}$. The curves A2 and A3 are symmetric w.r.t. the axis X. Therefore, there is a point on the curve A3 corresponding to some parameter $\widehat{s_3}$, for which $X_{\mathrm{U3}}(\widehat{s_3}) = X_{\mathrm{U2}}(\widetilde{s_2})$, $Y_{\mathrm{U3}}(\widehat{s_3}) = -Y_{\mathrm{U2}}(\widetilde{s_2})$. Since $Y_{\mathrm{U3}}(\widehat{s_3}) = -Y_{\mathrm{U3}}(\widetilde{s_3}) \neq 0$, then $\widehat{s_3} \neq \widetilde{s_3}$. Thus, two different points were obtained on the curve A3 that are symmetric w.r.t. the axis X. Therefore, they are equally distanced from any point on the axis X, in particular, from the point H. Above, the strict monotony of changing the distance from the point H to the points of the curve A3 was established. Therefore, the points of the curve A3 specified by the parameters $\widetilde{s_3}$ and $\widehat{s_3}$ have different distances to the point H. We came to the contradiction.

3.7. **List of properties of curves A1, A2, A3, and A6.** Let us summarize the properties of the curves A1, A2, A3, and A6 in the plane X,Y at $0 \leq \varphi < t_f$.

1. Each of the curves A1 and A6 is symmetric w.r.t. the axis X and is an arc of the circumference, whose center lies on the axis X. The span of the arc can be greater than 2π. The curve A1 degenerates to a point only when $\varphi = 0$. The curve A6 degenerates to a point only when $(t_f + \varphi)$ is multiple to 4π.

2. Each of the curves A2 and A3 represents the initial part of the involute of the circle, therefore, it does not have self-intersections.

3. The curves A2 and A3 are mutually symmetric w.r.t. the axis X and can intersect only at points on this axis.

4. All internal points of the curve A2 (respectively, A3) are generated by controls of the type U2 (U3) with three non-degenerate constancy intervals.

5. The distance $R_{6,2}(s_2)$ from the center H of the circle C_{A6} to the point on the curve A2 (defined by the parameter s_2) increases monotonically as the parameter s_2 grows.

6. The distance $R_{6,3}(s_3)$ from the center H of the circle C_{A6} to the point on the curve A3 (defined by the parameter s_3) decreases monotonically as the parameter s_3 grows.

7. The curves A2 and A3 are smoothly mated with the curve A1. The radii of curvature at the docking points are the same.

8. The curves A2 and A3 (excepting the points of their junction with the curve A1) as well as the entire curve A6 lie in the interior of the circle C_{A1}.

9. The curves A2 and A3 (excepting the points of their junction with the curve A6) lie outside the circle C_{A6}.

10. For $\varphi \in [0, 2\pi)$, the distance from the center H of the circle C_{A6} to the curve A1 is acheived at the curve extreme points.

11. Connection of the curves A1, A2, A3, and A6 forms a piecewise smooth closed curve that is symmetric w.r.t. the axis X.

4. Auxiliary statements about extreme motions leading to the interior of the reachable set $G(t_f)$

Paper [40] contained statements characterizing the extreme motions leading to the interior of the reachable set $G(t_f)$. In this paper, additional statements on this topic will be required. When considering them, we shall use the following important property [40, 43]. Let some control $u^*(\cdot)$ and the corresponding motion $z^*(\cdot)$ of system (2.1) satisfy the PMP. Then, if the control $u^*(\cdot)$ has more than two switch instants, then the geometric positions of $(x^*(t), y^*(t))^{\mathsf{T}}$ at these instants

lie on a single straight line, which is called the switching one. Also, if the control $u^*(\cdot)$ has an interval with $u^*(t) \equiv 0$, then the corresponding part of the rectilinear motion in the plane x,y lies on the straight switch line.

Now we prove two statements about extreme motions with cycles. By a cycle, we mean a part of motion with the constant control $u = \pm 1$ and the length greater or equal to 2π.

Lemma 4.1. *Let a motion $z(\cdot)$ of system (2.1) with $\varphi(t_f) \geq 0$ be generated by a control of the type* U2 *or* U3 *with three non-degenerate control constancy intervals. If at least one of these two extreme intervals has duration of at least 2π, then $z(t_f) \in \operatorname{int} G(t_f)$.*

Proof. From the assumption of two extreme time intervals of control constancy, the presence of a "cycle" on at least one of them follows.

Consider a control $u(\cdot)$ of the type U3: $u(t) = 1$ on the interval $t \in [0, t_1)$, $u(t) = 0$ for $t \in [t_1, t_2)$, and $u(t) = -1$ on the interval $t \in [t_2, t_f)$. Since $\varphi(t_f) \geq 0$, the presence of a cycle on the third interval entails the presence of a cycle on the first interval. Therefore, it is sufficient to consider the presence of a cycle only on the first interval (Fig. 7a), i.e., $t_1 \geq 2\pi$. Take the instant $\hat{t} \in (t_2, t_f)$ so that $\hat{t} - t_2 < 2\pi$. Let us construct an additional motion $\tilde{z}(\cdot)$ with the control

$$\tilde{u}(t) = \begin{cases} 1, & t \in [t_0, t_1 - 2\pi), \\ 0, & t \in [t_1 - 2\pi, t_2 - 2\pi), \\ -1, & t \in [t_2 - 2\pi, \hat{t} - 2\pi), \\ 1, & t \in [\hat{t} - 2\pi, \hat{t}), \\ -1, & t \in [\hat{t}, t_f]. \end{cases}$$

Such a motion comes at the instant t_f to the same point and with the same value φ as the original motion (Fig. 7). At the same time, the additional motion at the switch instant $\hat{t}$ does not lie on the switch line (along which the motion goes at $u = 0$). Thus, PMP will not be performed for the additional motion. Therefore, $z(t_f) \in \operatorname{int} G(t_f)$.

For the curve A2, the statement is true due to the symmetry of the curves A2 and A3. □

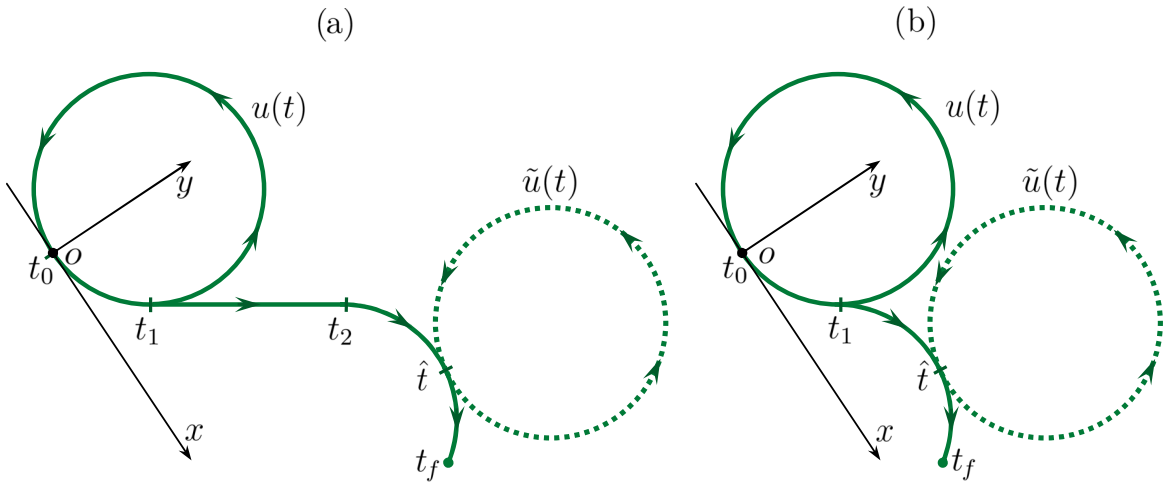

FIGURE 7. Illustration for the proofs of Lemmas 4.2 and 4.3

Lemma 4.2. *Let a motion $z(\cdot)$ of system (2.1) be generated by a control of the type* U6. *Wherein, some two adjacent intervals of the control constancy are non-degenerate. If at least one of them has duration greater than 2π, then $z(t_f) \in \operatorname{int} G(t_f)$.*

Proof. By assumption, we have a cyclic motion with a duration greater than 2π on one of two non-degenerate adjacent parts (with constant control) of the original motion $z(\cdot)$. We transfer a cycle of duration 2π from this part to any interior point of a non-degenerate neighboring part. We get an additional motion $\tilde{z}(\cdot)$ going to the same phase point at the time t_f as the original motion $z(\cdot)$. Additional motion has at least four consecutive non-degenerate parts of control constancy with values of ± 1, i.e., at least three switch instants. Due to Lemma 3 from [40], the additional motion leads to the point $\tilde{z}(t_f) \in \operatorname{int} G(t_f)$. Therefore, $z(t_f) \in \operatorname{int} G(t_f)$.

The proof is illustrated in Fig. 7b. Here, for the original motion, the first and second intervals of control constancy are non-degenerate. The cycle is located on the first interval. This cycle is carried over to the second motion part to the point $(x(\hat{t}), y(\hat{t}))^{\mathsf{T}}$ of the original motion. We get the additional motion with three switch instants. □

The proof of the next statement is the longest.

Lemma 4.3. *Let a motion $z(\cdot)$ be generated by a control of the type* U3 *and lead to the point $z(t_f) = \big(x(t_f), y(t_f), \varphi(t_f)\big)^{\mathsf{T}}$, for which $\varphi(t_f) \geq 0$. Suppose that for the point $\big(x(t_f), y(t_f)\big)^{\mathsf{T}}$ after transferring it to the auxiliary system X, Y, the inequality $Y(t_f) < 0$ is fulfilled. In addition, it is assumed that all three time intervals of constancy of the control are non-degenerate, and the duration of both the first and third intervals is less than 2π. Then $z(t_f) \in \operatorname{int} G(t_f)$.*

Proof. Let E be the geometric position of the motion at the instant t_f. The method of proof will be as follows. Together with the original motion leading to the point E for the given value φ at the instant t_f, we shall construct an additional motion leading from the same starting point to the point E for the same values φ and t_f. The additional motion could be built in such a way that the PMP will not be performed for it. Consequently, $z(t_f) \in \operatorname{int} G(t_f)$.

1) By assumption, the control of the type U3 has three non-degenerate time intervals with the values $1, 0, -1$. The points of the geometric position at the switch instants t_1 and t_2 are denoted by B and D (Figs. 8 – 10). The circumference along which the motion goes on the interval $[0, t_1]$ will be called the first (marked by the symbol C1), and the circumference with the motion on the interval $[t_2, t_f]$ is called the second (marked by the symbol C2). The original coordinate system is denoted by x, y, the auxiliary one is denoted by X, Y. The point O is the origin of the auxiliary system. The axis X of the auxiliary system passes through the starting point of the motion and constitutes an angle $\varphi/2$ with the direction of the axis x. The angle is counted counterclockwise from the axis x. The point E is assumed to be below the axis X. The direction vector of the motion at the points O and E makes the angle $\varphi/2$ with the axis X (counted from the axis X counterclockwise).

2) Since $\varphi \geq 0$ and the original motion is a motion of the type U3, then $t_1 \geq \varphi$. For the instant $t = \varphi$, the corresponding geometric position in the coordinates x, y is described by the relation $(x(t), y(t))^{\mathsf{T}} = (\sin\varphi, 1 - \cos\varphi)^{\mathsf{T}}$. From formula (3.9), it is seen that such a point coincides with the point O.

Consider the straight line EO with the direction from E to O. Let β be the angle between this straight line and the axis X. This angle is counted counterclockwise from the axis X. Since the

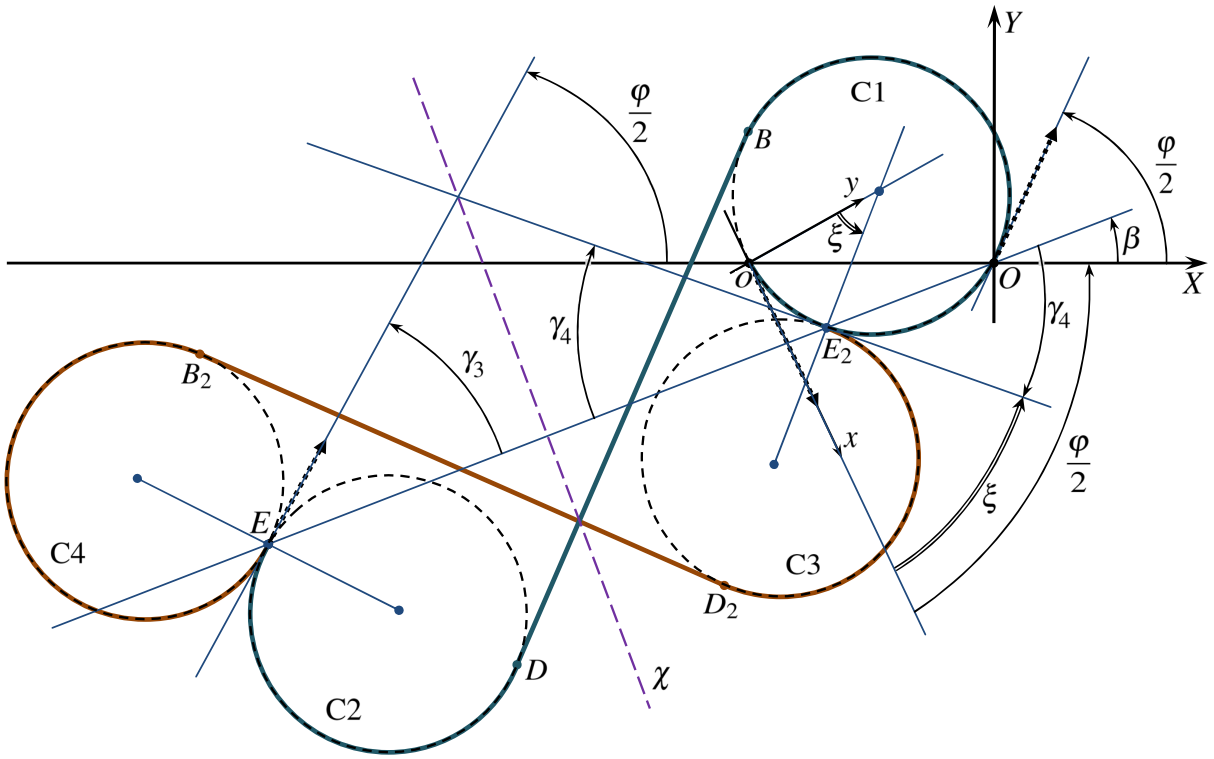

FIGURE 8. Additional motion for the condition $\beta < \varphi/2$

point E lies below the axis X, then $\beta \in (0, \pi)$. Generally speaking, in addition to the point O, the straght line EO has a second intersection with the first circumference. Let us denote such a point by the symbol E_2. For $\beta = \varphi/2$, the straght line EO touches the first circumference. In this situation, we suppose $E_2 = O$.

Denote by the symbol ξ the central angle of the first circumference based on the arc oE_2 corresponding to the rotation from the point o to the point E_2. Let us show that $\xi = 2\beta$. If $\beta = \varphi/2$, then the equality $\xi = 2\beta$ is fulfilled by the construction. Now we consider two variants: $\beta < \varphi/2$ (Fig. 8) and $\beta > \varphi/2$ (Fig. 9). In the first of them, the equality $\xi = 2\beta$ follows from the fact that the angle inscribed in the first circumference with a vertex at the point O and based on the arc oE_2 is equal to β. In the second variant, consider the central angle of the first circumference corresponding to the rotation from the point O to the point E_2. It is equal to $\xi - \varphi$. The corresponding inscribed angle with the vertex O is equal to $\beta - \varphi/2$. Therefore, $\xi - \varphi = 2(\beta - \varphi/2)$. Thus, $\xi = 2\beta$.

3) Let us show that $2\beta < t_1$. We base on Fig. 10.

We rotate the tangent to the first circumference clockwise starting from touching at the point B until passing it through the point E. The corresponding point of tangency with the first circumference is denoted by K.

The angle α is defined as the angle counted from the negative direction of the axis X counterclockwise to the direction of the vector from the point B to the point D (the direction of the rectilinear part of the motion). The inequality $\varphi \leq t_1$ is fulfilled, since the original motion is

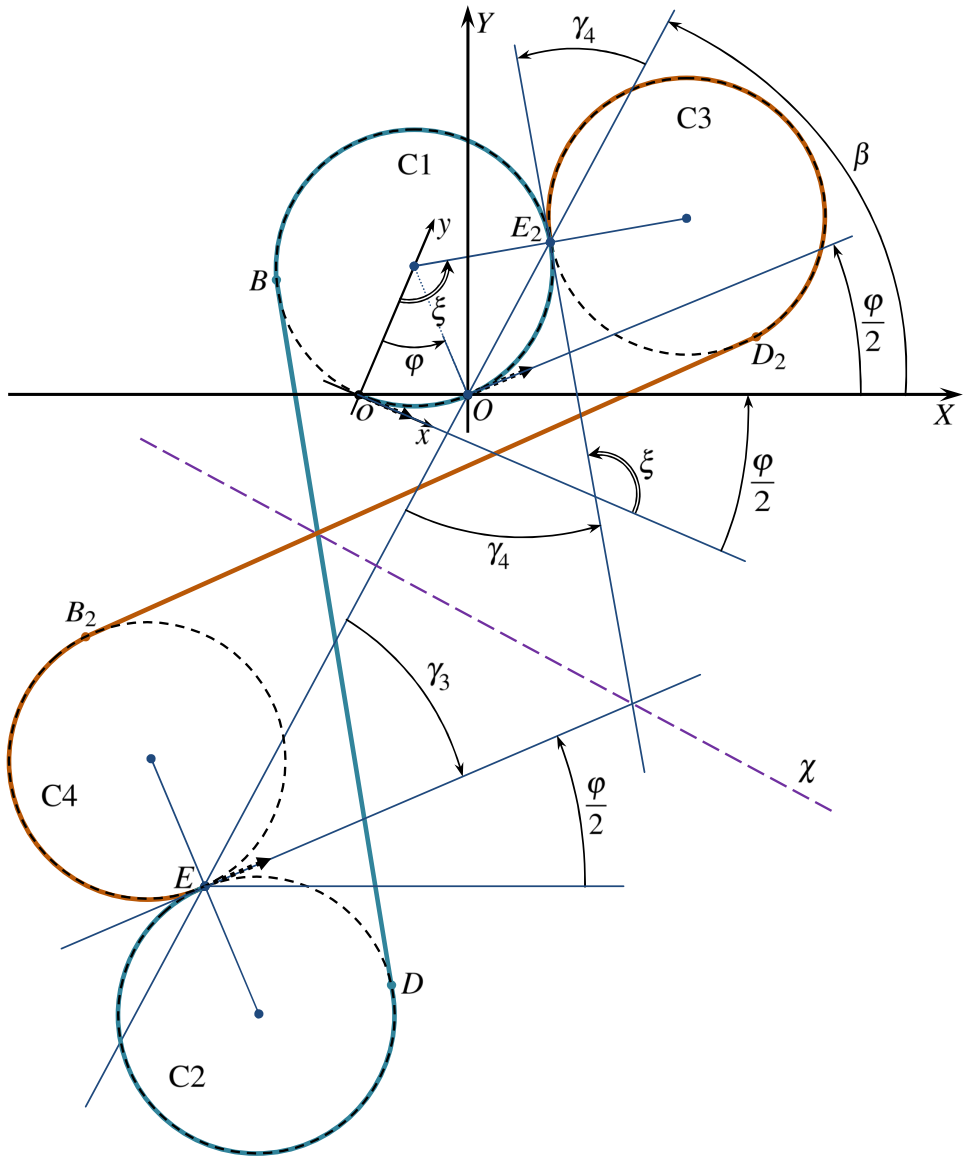

FIGURE 9. Additional motion for the condition $\beta > \varphi/2$

obtained by virtue of the control of the type U3. The inequality $t_1 < 2\pi$ is satisfied due to the assumption that there are no cycles on the original motion. Considering additionally that the point E lies below the axis X, we obtain the range of admissible values of α in the form $(0, 2\pi - \varphi/2)$.

Let us show that $\alpha > \beta$. Indeed, the point E does not coincide with the point D, since on the original motion there are no degenerate time intervals of constant control and there are no cycles. Hence, $E \neq K$ and $E \neq O$. Therefore, the angles γ_1 and γ_2 are uniquely defined, and the inequalities $\gamma_1 \geq 0$ and $\gamma_2 > 0$ are satisfied. In addition, one can see from Fig. 10 that $\alpha = \beta + \gamma_1 + \gamma_2$. So, it follows that $\alpha > \beta$.

The angle α can be calculated by the formula $\alpha = t_1 - \pi - \varphi/2$ (see Fig. 10). From the inequality $\alpha > \beta$, we get $t_1 - \pi - \varphi/2 > \beta$. Since $\varphi \geq 0$ and $0 < \beta < \pi$, we have

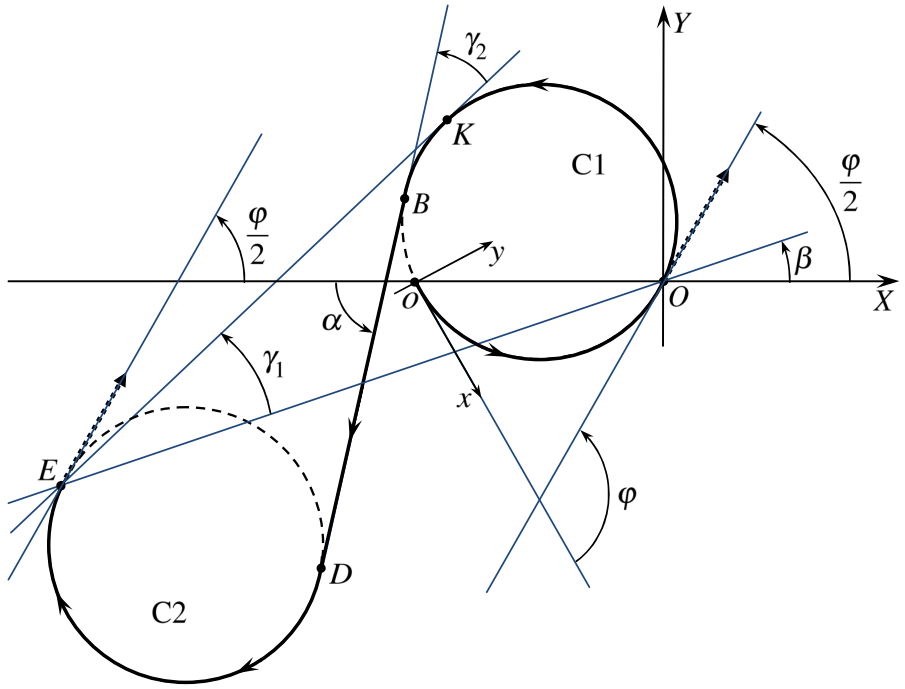

FIGURE 10. Illustration to the proof of the inequality $\alpha > \beta$

$t_1 \geq t_1 - \varphi/2 > \beta + \pi > 2\beta$. Thus, the inequality $0 < 2\beta < t_1$ is satisfied. Therefore, the point E_2 lies on the first part of the original motion.

4) In addition to the circumferences C1 and C2, consider the auxiliary circumferences C3 and C4 (Figs. 8 and 9). The circumference C3 (and, correspondingly, C4) touches the circumference C1 (C2) at the point E_2 (E). The circumferences C2 and C3 do not coincide, since the rectilinear part of the original motion is non-degenerate. We draw a perpendicular through the middle of the segment $[E, E_2]$ and denote it by the letter χ. Let us introduce the angle γ_3 between the motion velocity vector at the point E and the straight line EO. The angle γ_3 is counted from the straight line EO with the positive counterclockwise direction. Similarly, consider the angle γ_4 between the motion velocity vector at the point E_2 and the straight line EO. The angle γ_4 is also counted from the straight line EO, but with the positive direction of counting clockwise. In Fig. 8, the values γ_3 and γ_4 are positive, and in Fig. 9, they are negative.

Let us show that $\gamma_3 = \gamma_4$. By the construction, we have $\gamma_3 = \varphi/2 - \beta$ and $\gamma_4 = \varphi/2 - \xi + \beta$. Taking into account that $\xi = 2\beta$, we get $\gamma_4 = \varphi/2 - \beta = \gamma_3$.

The equality $\gamma_3 = \gamma_4$ implies the mirror symmetry of arrangement of the circumferences C1 and C4 as well as the circumferences C3 and C2 relatively to the straight line ξ. With this symmetry, the internal tangent BD to the circumferences C1, C2 goes to the tangent B_2D_2 to the circumferences C4, C3 (Figs. 8 and 9).

5) Let us introduce an additional motion. It starts from the same point o and with the same direction $\varphi(t_0) = 0$ as the original motion. On the initial time interval of the length 2β, the additional motion (like the original one) goes with the control $u = 1$ up to the point E_2. We denote the instant of arriving to the point E_2 by t_1^*. We have $t_1^* = 2\beta$. After this instant, the trajectory of the additional motion is mirror-symmetrical to the original trajectory (Figs. 8 and 9)

relatively to the straight line χ. Namely, in the time interval from t_1^* to $t_2^* = 2\beta + t_f - t_2$, the additional motion goes with the control $u = -1$ along the circumference C3 clockwise from the point E_2 to the point D_2. In the interval from t_2^* to $t_3^* = 2\beta + t_f - t_1$, the motion proceeds with zero control along the straight line segment of the length $t_3^* - t_2^* = t_2 - t_1$ from the point D_2 to the point B_2. The final stage of the motion from the instant t_3^* to the instant t_f occurs with the control $u = 1$ along the circumference C4 counterclockwise from the point B_2 to the point E. The constructed curve leads to the same point E and with the same direction φ as the original one. Its length coincides with the length of the original curve.

According to the assumption, there are three non-degenerate intervals of the control constancy on the original motion, and there are no cycles on the first and third time intervals. Considering the inequality $t_1 > 2\beta > 0$, we get that there are three switch instants on the additional motion that satisfy the inequalities $0 < t_1^* < t_2^* < t_3^* < t_f$. At the same time, there are no cycles in the second interval of the control constancy (motion from the point E_2 to the point D_2). From this, it follows that the points of the geometric position of the additional motion at the switch instants do not lie on the same straight line. The latter contradicts to the PMP. Therefore, $z(t_f) \in \mathrm{int} G(t_f)$. □

Remark 4.4. Figures 8-10 (explaining the proof of Lemma 4.3) are made for $0 < \varphi < \pi$. If $\varphi = 0$, then we consider only the variant $\beta > \varphi/2 = 0$. All elements of the proof based on Fig. 9 are preserved. If $\pi \le \varphi < 2\pi$, then we are dealing only with the variant $\beta < \varphi/2$. This follows from the fact that the inequality $\beta < \pi/2$ is satisfied. Indeed, for $\beta \ge \pi/2$, duration of the first time interval of control constancy exceeds 2π. It contradicts the assumption that there are no cycles on the original motion. When analyzing the variant $\beta < \varphi/2$, all elements of the proof that were made with reference to Fig. 8 are preserved.

5. Statements about the structure of φ-sections

Proofs of Lemmas 5.1 and 5.2 formulated below are based on the Jordan curve theorem [48, 49], which characterizes partition of the plane by a continuous closed curve S without self-intersections. According to the Jordan theorem, the set $\mathbb{R}^2 \setminus S$ consists of two open, connected and disjoint components S^+ (external unlimited) and S^- (internal limited). The curve S is the boundary of these components.

Lemmas 5.1 and 5.2 are used later to construct the boundary of the φ-sections. Lemma 5.3 will also be used in the sequel.

Lemma 5.1. *Consider some values $t_f > 0$ and $\varphi \in [0, t_f)$. Let S be a continuous closed curve without self-intersections in the plane x, y, and to any point of S at least one of the controls of the types* U1, U2, U3, *and* U6 *leads at the instant t_f (with the given value φ). Suppose that the set S^+ contains no points formed by controls of the types* U1, U2, U3, *and* U6 *for the mentioned values t_f and φ. Then $S \subset \partial G_\varphi(t_f)$ and*

$$G_\varphi(t_f) \cap S^+ = \varnothing. \tag{5.1}$$

Proof. Let us show that relation (5.1) holds. On the contrary, let the set S^+ contain some point $P \in G_\varphi(t_f)$. The reachable set $G(t_f)$ of system (2.1), and, therefore, its section $G_\varphi(t_f)$ are bounded. Let us choose in the unlimited set S^+ some point $Q \notin G_\varphi(t_f)$. We connect the points

P and Q by a continuous curve γ that lies entirely in S^+. It is possible due to the connectedness of the set S^+. By virtue of continuity, the curve γ contains a point $E \in \partial G_\varphi(t_f)$. According to Lemma 3.1, at least one of the controls of the type U1, U2, U3, U6 leads to the point E. But it is impossible by the hypothesis of the lemma being proved. Therefore, $E \notin \partial G_\varphi(t_f)$. We came to a contradiction, i.e., the initial assumption that $P \in G_\varphi(t_f)$ is wrong. Thus, relation (5.1) is established.

We have $S = \partial S^+$ and (by the condition of this lemma) for any point of the curve S, some admissible control exists leading to this point. Taking into account (5.1), we get that $S \subset \partial G_\varphi(t_f)$. □

Lemma 5.2. *Consider some values $t_f > 0$ and $\varphi \in [0, t_f)$. Let S be a continuous closed curve without self-intersections in the plane x, y, and to any point of S at least one of the controls of the types* U1, U2, U3, *and* U6 *leads at the instant t_f (with the given value φ). Suppose that the set S^- contains no points formed by controls of the types* U1, U2, U3, *and* U6 *for the mentioned values t_f and φ. Also assume that there is at least one point $F \in S^-$ that does not belong to the set $G_\varphi(t_f)$. Then $S \subset \partial G_\varphi(t_f)$ and $G_\varphi(t_f) \cap S^- = \varnothing$.*

The proof is similar to the proof of the previous lemma with the replacement of the set S^+ by the set S^- and the point Q by the point F.

Lemma 5.3. *Consider some values $t_f > 0$ and $\varphi \in [0, t_f)$. Let a set M in the plane of geometric coordinates x, y be open and connected. Assume that one of the following conditions is valid*: 1) *there are no points of curves* A1, A2, A3, *and* A6 *in the set M, but there is at least one point $P \in \operatorname{int} G_\varphi(t_f)$*; 2) *the set M contains some points of the curves* A1, A2, A3, A6, *and for all such points, except, perhaps, only one, it is known that they belong to the set* $\operatorname{int} G_\varphi(t_f)$. *Then $M \subset \operatorname{int} G_\varphi(t_f)$.*

Proof. Let condition 1) be satisfied. Suppose that the set M contains at least one point $K \in \partial G_\varphi(t_f)$. Taking into account Lemma 3.1, at least one of the controls of the type U1, U2, U3, and U6 leads to this point. But then the point K lies on one of the curves A1, A2, A3, and A6 that contradicts condition 1). Therefore, $M \cap \partial G_\varphi(t_f) = \varnothing$.

Suppose that in the set M there is some point $L \notin G_\varphi(t_f)$. Let us connect the points L and P by a continuous curve belonging to M. Then this curve will have at least one point from $\partial G_\varphi(t_f)$ that is impossible.

Thus, $M \subset \operatorname{int} G_\varphi(t_f)$.

Let condition 2) be satisfied. Denote by the symbol E the special point referred to in this condition. Consider the set $V = M \setminus \{E\}$. If the point E is missing, we consider $V = M$. The set V is open and connected. Repeating the reasoning scheme used in the first paragraph of the proof and taking into account the specifics of condition 2), we establish that $V \cap \partial G_\varphi(t_f) = \varnothing$.

From condition 2), taking into account the continuity of the curves A1, A2, A3, and A6, it follows that in the set V some point $F \in \operatorname{int} G_\varphi(t_f)$ exists. Suppose that there is a point $L \notin G_\varphi(t_f)$ in the set V. We connect the points L and F by a continuous curve lying in V. Then this curve contains a point from $\partial G_\varphi(t_f)$ that is impossible. Therefore, $V \subset \operatorname{int} G_\varphi(t_f)$.

Since the open, connected sets V and M differ, perhaps, by only one point E, then $M \subset \operatorname{int} G_\varphi(t_f)$. □

6. Classification of φ-sections for $\varphi \geq 0$

The shape of the φ-sections is determined by the values of φ and t_f. Assume that $t_f > 0$ and $\varphi \in [0, t_f]$. We shall distinguish five sets in the space of values φ, t_f. Respectively, we consider five cases

$$\text{Case 1:} \quad 0 \leq \varphi < t_f, \quad t_f < 4\pi - \varphi, \quad t_f < 3\pi + 2\cos(\varphi/2). \tag{6.1}$$

$$\text{Case 2:} \quad 0 \leq \varphi < \pi, \quad t_f < 4\pi - \varphi, \quad t_f \geq 3\pi + 2\cos(\varphi/2). \tag{6.2}$$

$$\text{Case 3:} \quad 0 \leq \varphi < 2\pi, \quad t_f \geq 4\pi - \varphi. \tag{6.3}$$

$$\text{Case 4:} \quad 2\pi \leq \varphi < t_f. \tag{6.4}$$

$$\text{Case 5:} \quad \varphi = t_f. \tag{6.5}$$

Let us establish that any point (φ, t_f) satisfying the conditions $t_f > 0$ and $0 \leq \varphi \leq t_f$ belongs to one and only one of the specified sets.

If $\varphi = t_f$, then the point (φ, t_f) belongs to Set 5 (Case 5). With that, it is obvious that the point (φ, t_f) does not come into Set 1 and Set 4. From the first and third conditions (6.2) in Case 2, it follows that $t_f > 3\pi$; at the same time, $\varphi < \pi$. In Case 3, we have $t_f > 2\pi$. Thus, the point under consideration does not belong to Sets $1-4$.

From the first two inequalities in (6.1), it follows that in Case 1 the inequality $\varphi < 2\pi$ holds. Therefore, under condition (6.4), the point (φ, t_f) comes only into Set 4.

If $0 \leq \varphi < \pi$, then the inequality

$$4\pi - \varphi > 3\pi + 2\cos(\varphi/2) \tag{6.6}$$

holds. Indeed, the condition $0 \leq \varphi < \pi$ can be rewritten as $0 < (\pi - \varphi)/2 \leq \pi/2$. Therefore, the inequality $(\pi - \varphi)/2 > \sin((\pi - \varphi)/2)$ holds that is equivalent to (6.6). Inequality (6.6) means that for $0 \leq \varphi < \pi$ in (6.1), the second condition follows from the third condition. Similarly, for $\varphi \in [\pi, 2\pi)$, we have

$$4\pi - \varphi \leq 3\pi + 2\cos(\varphi/2), \tag{6.7}$$

i.e., in (6.1), the third condition follows from the second one.

Let $0 \leq \varphi < \pi$ and $\varphi < t_f$. Based on inequality (6.6), we establish that the point (φ, t_f) comes at $t_f \in (\varphi, 3\pi + 2\cos(\varphi/2))$ only to Set 1; correspondingly, for $t_f \in [3\pi + 2\cos(\varphi/2), 4\pi - \varphi)$, it comes only to Set 2, and only to Set 3 for $t_f \in [4\pi - \varphi, \infty)$. Let $\varphi \in [\pi, 2\pi)$ and $\varphi < t_f$. Using inequality (6.7), we get that the point (φ, t_f) belongs only to Set 1 for $t_f < 4\pi - \varphi$ and only to Set 3 for $t_f \geq 4\pi - \varphi$.

Thus, the property of partition of the area $\{(\varphi, t_f) : 0 \leq \varphi \leq t_f, t_f > 0\}$ into Sets $1-5$ is established.

Sets $1-5$ are shown in Fig. 11. For $0 \leq \varphi < \pi$, Set 1 is separated from Set 2 by the line $t_f = 3\pi + 2\cos(\varphi/2)$ (the line is included into Set 2), and Set 2 is separated from Set 3 by a segment of the straight line $t_f = 4\pi - \varphi$ (that is included into Set 3). For $\pi \leq \varphi < 2\pi$, Set 1 is separated from Set 3 by a segment of the straight line $t_f = 4\pi - \varphi$. This segment is included into Set 3. Set 3 is separated from Set 4 by an unlimited ray $\varphi = 2\pi$, $t_f > \varphi$. This ray is included into Set 4. Set 5 is an unlimited ray $0 < \varphi = t_f$.

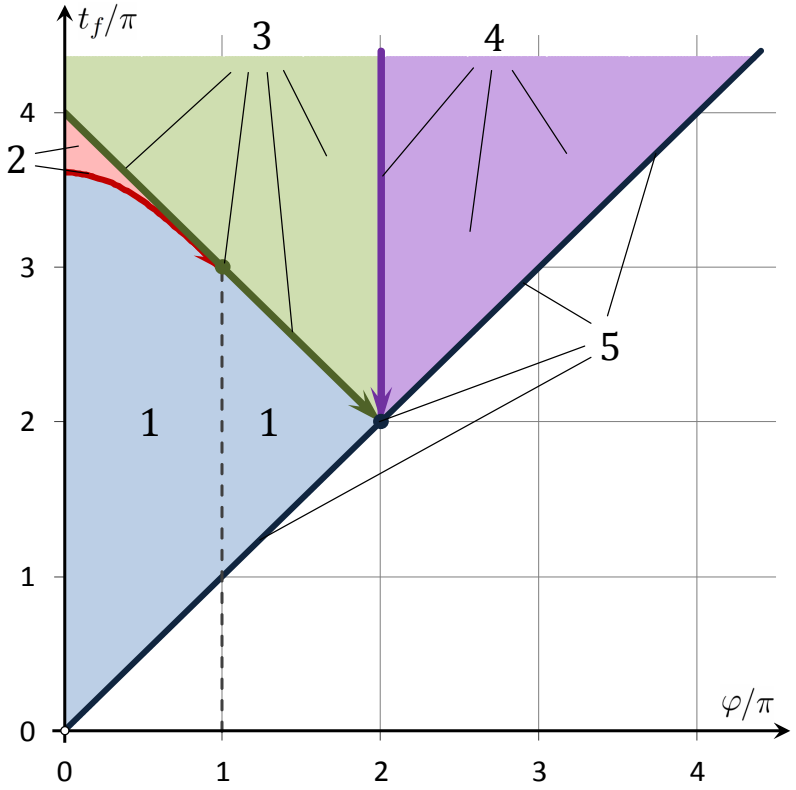

FIGURE 11. Classification of the φ-sections of the reachable set $G(t_f)$ for values $t_f > 0$ and $0 \le \varphi \le t_f$

In Case 5, as noted in Section 3.1, any φ-section is a one-point set. Geometric coordinates in the original system are represented by the formulas $x(t_f) = \sin t_f$ and $y(t_f) = 1 - \cos t_f$. Position of such a point in the auxiliary coordinate system has the form $X(t_f) = 0$, $Y(t_f) = 0$.

In the following Sections, we describe the φ-sections for Cases 1 – 4.

7. BOUNDARY OF φ-SECTIONS FOR CASE 1

We assume that (φ, t_f) belongs to Set 1. The curves A1, A2, A3, and A6 are used as the basic elements of the boundary of the φ-sections. Each of them is fully included into the description of the boundary. This case is partially considered in [44] where it was assumed that $t_f \le 2\pi$.

From the first two relations in (6.1), it follows that $0 \le \varphi < 2\pi$ and $0 < \theta < 2\pi$ where $\theta = (t_f - \varphi)/2$. Considering formulas (3.10), (3.13) for the curves A1 and A6 (they are the arcs of circumferences), as well as the corresponding ranges of parameters s_1 and s_3 in (3.4), we conclude that in Case 1, the span of each of the arcs A1 and A6 is less than 2π.

If $\varphi = 0$, then the arc A1 degenerates into a point coinciding with the starting point of the curve A3 and the end point of the curve A2. If $\varphi > 0$, the arc A1 is not degenerate. Its starting point is below the axis X, and its end point is above the axis X.

1) Let us show that the curves A2 and A3 in Case 1 have a common point only for $\varphi = 0$. Herewith, there is only one common point, which is their mutual joining point. It coincides with the degeneration point of the arc A1.

Investigate the curve A3. For the extreme values $s_3 = 0$ and $s_3 = \theta$, we have

$$Y_{U3}(0) = (t_f - \varphi)\sin\left(\frac{\varphi}{2}\right) \geq 0, \quad Y_{U3}(\theta) = 4\sin\left(\frac{t_f - \varphi}{4}\right)\sin\left(\frac{t_f + \varphi}{4}\right) > 0. \tag{7.1}$$

The first inequality follows from the fact that $\varphi \in [0, 2\pi)$ and $\varphi < t_f$. Using the condition $t_f < 4\pi - \varphi$, we obtain the second inequality. The first inequality turns into the equality only for $\varphi = 0$.

Let us establish that $Y_{U3}(s_3) > 0$ for any $s_3 \in (0, \theta)$. We calculate the derivative of the function $Y_{U3}(s_3)$ with respect to s_3:

$$\left(Y_{U3}(s_3)\right)'_{s_3} = 2(\theta - s_3)\cos\left(s_3 + \frac{\varphi}{2}\right). \tag{7.2}$$

For $s_3 \in (0, \theta)$, the sign of the derivative is determined by the last factor, and the value $s_3 + \varphi/2$ varies in the range $\left(\varphi/2, t_f/2\right)$. From the first two relations in (6.1), it follows that

$$0 \leq \varphi < t_f < 4\pi.$$

Therefore, $(s_3 + \varphi/2) \in (0, 2\pi)$. Within this extended range, the expression $\cos(s_3 + \varphi/2)$ equals zero for only two values of the parameter s_3: $s_3^* = (\pi - \varphi)/2$ and $s_{3*} = (3\pi - \varphi)/2$. These points do not always belong to the range $(0, \theta)$ of admissible values s_3.

In Fig. 12, the curves A1, A2, A3, and A6 are shown for the same value $\varphi = 0.5\pi$ and four values of t_f: 0.99π, 2.3π, 2.8π, 3.3π. If $t_f = 0.99\pi$, then there are no values s_3^* and s_{3*} (defining horizontal tangents to the curve A3) in the interval $(0, \theta)$. If $t_f = 2.3\pi$ or $t_f = 2.8\pi$, then $s_3^* \in (0, \theta)$, $s_{3*} \notin (0, \theta)$. For $t_f = 3.3\pi$, we get $s_3^* \in (0, \theta)$ and $s_{3*} \in (0, \theta)$ (respectively, the curve A3 has two horizontal tangents).

Consider the value s_3^*. Due to (3.12), the value $Y_{U3}(s_3^*)$ is calculated by the formula

$$Y_{U3}(s_3^*) = 2(\theta - s_3^*)\sin\left(s_3^* + \frac{\pi}{2}\right) + 4\sin\frac{s_3^*}{2}\sin\left(\frac{s_3^*}{2} + \frac{\pi}{2}\right). \tag{7.3}$$

Condition $s_3^* \in (0, \theta)$ can be written as $0 < (\pi - \varphi)/2 < (t_f - \varphi)/2$. It implies the fulfillment of the inequalities $\varphi < \pi < t_f$. Due to these inequalities, substituting the values of s_3^* and θ into (7.3), we obtain

$$Y_{U3}(s_3^*) = (t_f - \pi) + 4\sin\left(\frac{\pi}{4} - \frac{\varphi}{4}\right)\sin\left(\frac{\pi}{4} + \frac{\varphi}{4}\right) > 0. \tag{7.4}$$

Consider the value s_{3*}. The condition $s_{3*} \in (0, \theta)$ implies the inequality $t_f > 3\pi$. Taking into account the second relation in (6.1), we obtain the inequality $\varphi < \pi$. We calculate $Y_{U3}(s_{3*})$:

$$Y_{U3}(s_{3*}) = 3\pi - t_f + 4\sin\left(\frac{3\pi}{4} - \frac{\varphi}{4}\right)\sin\left(\frac{3\pi}{4} + \frac{\varphi}{4}\right) = 3\pi - t_f + 2\cos\left(\frac{\varphi}{2}\right). \tag{7.5}$$

Using the third relation in (6.1), we have

$$Y_{U3}(s_{3*}) > 0. \tag{7.6}$$

We have inequalities (7.1) for the values of $Y_{U3}(s_3)$ at the extreme points of the curve A3 as well as inequalities (7.4), (7.6) for the interior points where the derivative $\left(Y_{U3}(s_3)\right)'_{s_3}$ equals to zero. From it, we conclude that the curve A3 lies above the axis X (see two examples in Fig. 13) with the exception of the starting point at $\varphi = 0$.

Due to the symmetry of the curves A2 and A3 w.r.t. the axis X, the curve A2 lies below the axis X with the exception of the end point for $\varphi = 0$. Thus, the curves A2 and A3 for $\varphi > 0$

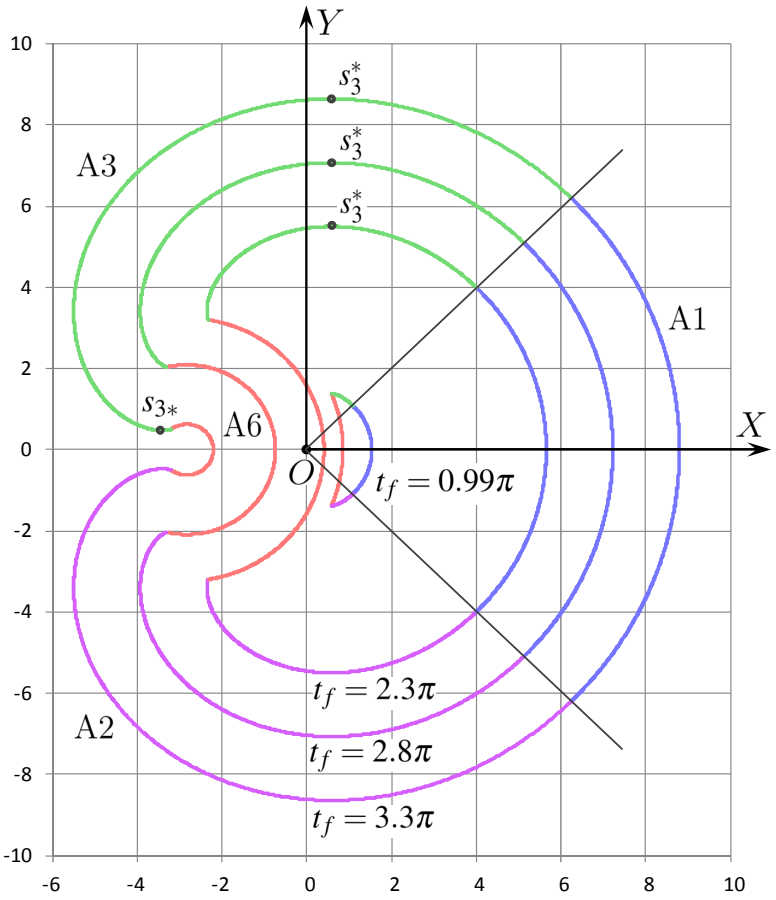

FIGURE 12. Glued curves $\mathrm{A}_\varphi(t_f)$ for the angle $\varphi = 0.5\pi$ and four values $t_f = 0.99\pi, 2.3\pi, 2.8\pi, 3.3\pi$. Location of the points s_3^*, s_{3*} on the curves A3

have no common points, and for $\varphi = 0$ the starting point of the curve A3 coincides with the end point of the curve A2.

2) In general, taking into account the results of Section 3, we see that the "glued" curve $\mathrm{A}_\varphi(t_f) = \mathrm{A1} \cup \mathrm{A3} \cup \mathrm{A6} \cup \mathrm{A2}$ is a piecewise smooth closed one without self-intersections. The smoothness is broken only at the points of joining the arcs A2 and A3 with the arc A6 (Fig. 12). The glued curve contains *all* points of motion by the controls U1, U2, U3, and U6 for the given t_f and φ.

Let the symbol $\mathbf{A}_\varphi(t_f)$ denote the closed set bounded by the curve $\mathrm{A}_\varphi(t_f)$. By Lemma 5.1, this curve belongs to $\partial G_\varphi(t_f)$ and the inclusion $G_\varphi(t_f) \subset \mathbf{A}_\varphi(t_f)$ is true.

For $\varphi \in (0, t_f)$, consider the motion generated on the interval $[0, t_f]$ by the constant control $u(t) \equiv \varphi/t_f$. Such the motion does not satisfy the PMP. Therefore, by virtue of [42], it leads to the $\mathrm{int}G(t_f)$, and, hence, to the $\mathrm{int}G_\varphi(t_f)$. For $\varphi = 0$, as the similar motion (but with one switch), we take the motion with the control $u(t) = -0.5$ for $t \in [0, t_f/2)$ and $u(t) = +0.5$ for $t \in [t_f/2, t_f]$. It also leads into $\mathrm{int}G_\varphi(t_f)$. Turning to Lemma 5.3, we see that for the set $M = \mathrm{int}\mathbf{A}_\varphi(t_f)$ the condition 1) is satisfied. Therefore, $\mathrm{int}\mathbf{A}_\varphi(t_f) \subset G_\varphi(t_f)$.

As a result, we get $\mathbf{A}_\varphi(t_f) = G_\varphi(t_f)$.

Two examples of the sets $G_\varphi(t_f)$ for Case 1 are shown in Fig. 13. Scale of the image is determined by the radius of the circle C_{A1}.

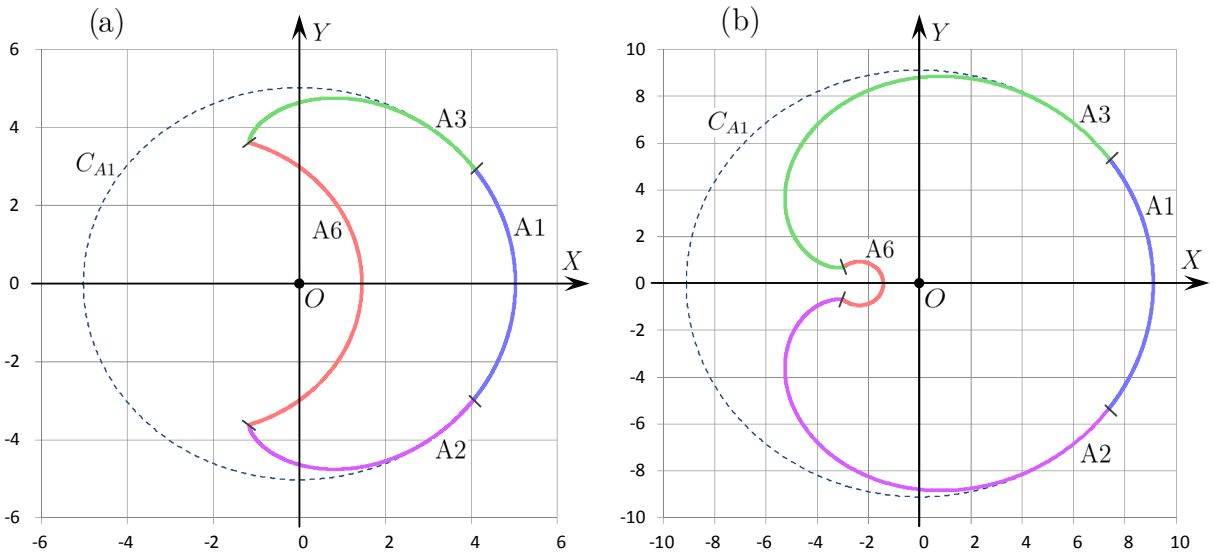

FIGURE 13. Variants of the sets $G_\varphi(t_f)$ for Case 1 with $\varphi = 0.4\pi$ and two values $t_f = 2\pi$ (a) and $t_f = 3.3\pi$ (b)

8. BOUNDARY OF φ-SECTIONS FOR CASE 2

Let fix φ and t_f satisfying the conditions of Case 2. As in Case 1, the boundary of the φ-section will be constructed on the basis of the curves A1, A2, A3, and A6. The main feature is that any φ-section related to Case 2 is not simply connected.

8.1. **Points of intersection of the curves A2 and A3.** Each of the curves A2 and A3 is an involute arc. The curve A1 joins with the curves A2 and A3 with keeping of curvature continuity (Section 3.7, Property 7).

Based on (6.2), we prove that there are one or two points of intersection of the curves A2 and A3.

Consider the curve A3. For the extreme points (by analogy with Case 1 using the relations (7.1)), we obtain $Y_{U3}(0) \geq 0$, $Y_{U3}(\theta) > 0$. The first inequality turns into the equality only for $\varphi = 0$.

For $s_3 \in (0,\, \theta)$, the derivative $(Y_{U3}(s_3))'_{s_3}$ calculated by formula (7.2) becomes zero for values $s_3^* = (\pi - \varphi)/2$ and $s_{3*} = (3\pi - \varphi)/2$. In Case 2 (unlike Case 1), these values always belong to the interval $(0,\, \theta)$ due to (6.2). Therefore, the corresponding points on the curve A3 always exist.

As noted in Section 7, formula (7.4), we have $Y_{U3}(s_3^*) > 0$ for the first point. For the second point, we get $Y_{U3}(s_{3*}) = 3\pi - t_f + 2\cos(\varphi/2) \leq 0$ by virtue of (7.5) and the third inequality in (6.2).

Let

$$t_f > 3\pi + 2\cos(\varphi/2). \tag{8.1}$$

We show that under this condition, the curves A2 and A3 intersect at two points located on the negative part of the axis X (Fig. 14b). To do this, it is sufficient to consider the intersection points of the curve A3 with the axis X due to the symmetry of the curves A2 and A3 relatively to this axis.

Let us select three parts on the curve A3: $s_3 \in (0, s_3^*)$, $s_3 \in (s_3^*, s_{3*})$, and $s_3 \in (s_{3*}, \theta)$. The function $Y_{U3}(s_3)$ is strictly monotone on each of these parts. In the first part, there is no intersection with the axis X. On the second, there is one intersection point, let us designate it P_1. The third part also has one intersection point, we denote it by P_2. In general, we get two points of intersection of the curve A3 (with extreme points excluded) and the axis X. This property is also true for the curve A2. For $\varphi = 0$, in addition to the indicated points, the curves A2 and A3 are joined at the point of the curve A1 degeneration.

Let the equality

$$t_f = 3\pi + 2\cos(\varphi/2) \tag{8.2}$$

be fulfilled now. This equality corresponds to the situation when the curve A3 touches the axis X (and hence the curve A2) at the point $(X_{U3}(s_{3*}), Y_{U3}(s_{3*}))^\mathsf{T}$. Herein, with the exception of the point of tangency, the interior points of the curve A3 lie above the axis X, and the interior points of the curve A2 are below the axis X (Fig. 14a). Thus, under condition (8.2) for $\varphi > 0$, the curves A2 and A3 have only one common point $P_1 = P_2$ (this is the point of tangency defined on the curve A3 by the parameter $s_3 = s_{3*}$). If $\varphi = 0$, then (as well as under condition (8.1)) the curves A2 and A3 are additionally joined at the point of degeneration of the curve A1.

8.2. **Analysis of arcs of the curves A2 and A3 between the points of their intersection.** Consider under condition (8.1) an open arc of the curve A3 between the points P_1 and P_2 in the plane X,Y. This arc is located below the axis X. Let us show that it belongs to $\mathrm{int}G_\varphi(t_f)$. We shall use Lemma 4.3. To ensure that the conditions of this Lemma are fulfilled, we establish that for any point P of this arc the corresponding motion (with a control of the type U3) has no cycles and each of three intervals of the control constancy is not degenerate.

For controls of the type U3, the switch instants t_1 and t_2 satisfy the relation $\varphi = t_1 - (t_f - t_2)$. The sum of the lengths of the first and third intervals does not exceed t_f. Therefore, $2t_1 - \varphi = t_1 + (t_f - t_2) \le t_f$. For Case 2 (see (6.2)), the inequality $t_f < 4\pi - \varphi$ holds. So, $2t_1 - \varphi < 4\pi - \varphi$, i.e. $t_1 < 2\pi$. It means that there are no cycles on the first interval. For $\varphi \ge 0$, the length of the third interval does not exceed the length of the first one. So, there are no cycles on the third interval also.

As noted in Section 3.7, Property 4, any control of the type U3 leading to an interior point P of the arc A3 has three non-degenerate intervals of constancy. Applying Lemma 4.3, we obtain $(P,\varphi)^\mathsf{T} \in \mathrm{int}G(t_f)$. Hence, $P \in \mathrm{int}G_\varphi(t_f)$.

8.3. **Description of the boundary of φ-sections for Case 2 in the form of outer and inner contours.** The curve A6 is a circular arc. Its center (in the auxiliary coordinate system) is located at the point H by formula (3.15), and the radius R_{A6} is described by formula (3.16). From the second inequality in the definition of Case 2, it follows that the arc A6 is non-degenerate (the radius is not equal to zero).

The arc A6 (with parts of the curves A2 and A3 attached to it up to the second point P_2 of their intersection) forms a closed curve. Let us denote it by $\mathrm{B}_\varphi(t_f)$. The symbol $\mathrm{A}_\varphi(t_f)$ denotes the closed curve composed of the arc A1 with parts of the curves A2 and A3 attached to it at the

first point P_1 of their intersection (see Fig. 14). When $\varphi = 0$, the arc A1 degenerates into a point that becomes the junction point of the curves A2 and A3. Therefore, for $\varphi = 0$, we can assume that the closed curve $\mathrm{A}_\varphi(t_f)$ is formed by two curves A2 and A3 (up to the point P_1). Each of the curves $\mathrm{A}_\varphi(t_f)$ and $\mathrm{B}_\varphi(t_f)$ has no self-intersections. Let $\mathbf{A}_\varphi(t_f)$ (respectively, $\mathbf{B}_\varphi(t_f)$) be a closed set bounded by the curve $\mathrm{A}_\varphi(t_f)$ (respectively, curve $\mathrm{B}_\varphi(t_f)$).

The examples of the sets $\mathbf{A}_\varphi(t_f)$ and $\mathbf{B}_\varphi(t_f)$ are shown in Fig. 14.

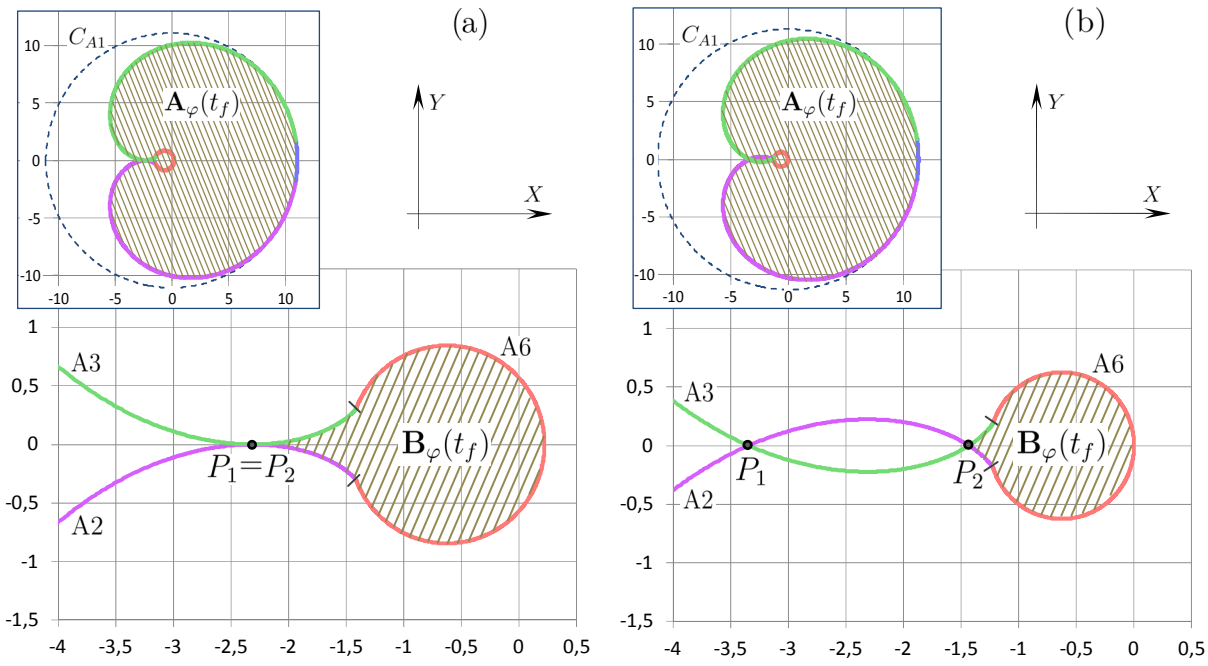

FIGURE 14. The sets $\mathbf{A}_\varphi(t_f)$ and $\mathbf{B}_\varphi(t_f)$ for Case 2 with $\varphi = 0.1\pi$ and two values $t_f = 3\pi + 2\cos(0.05\pi)$ (a) and $t_f = 3.7\pi$ (b)

When fulfilling inequality (8.1), the arcs of the curves A2 and A3 between the points P_1 and P_2 of their intersection lie in the int$G_\varphi(t_f)$.

1) Consider two circles centered at the point H: one passes through the point P_1, the other goes through the point P_2. The corresponding circles will be denoted by C_{P1} and C_{P2}. In Section 3.7 (Properties 5 and 6), it was noted that, for the parametrically defined curves A2 and A3, there is the monotonic change of the distance from the point H to the points of the curves A2 and A3. From this, it follows that the circle C_{A6} (with the center H), on the boundary of which the curve A6 lies, belongs to the set $\mathbf{B}_\varphi(t_f)$. Besides, for (8.1), the circle C_{P1} covers the circle C_{P2}, and for (8.2), these circles coincide.

Additionally, taking into account Property 10 of Section 3.7, we get that the curve $\mathrm{A}_\varphi(t_f)$ (excepting the point P_1) lies outside the circle C_{P1}. The remaining parts of the curves A2, A3, and, also, the curve A6 are located in the intC_{P1}. Similarly, the curve $\mathrm{B}_\varphi(t_f)$ (excepting the point P_2) lies in the intC_{P2}; the remaining parts of the curves A2, A3, and the whole curve A1 are outside the circle C_{P2}.

Outside the set $\mathbf{A}_\varphi(t_f)$, there are no points generated by controls of the types U1, U2, U3, and U6. Therefore, by virtue of Lemma 5.1, the curve $\mathrm{A}_\varphi(t_f)$ forms the "outer" boundary of the set $G_\varphi(t_f)$ for Case 2.

2) Now, let us establish that the curve $\mathrm{B}_\varphi(t_f)$ is the "inner" boundary of the set $G_\varphi(t_f)$.

2.1) Consider in the set $\mathrm{int}\mathbf{B}_\varphi(t_f)$ a point J located on the axis X and spaced from the point H to the left by $\frac{1}{2}R_{A6}(t_f)$. We have

$$J = \left(-4\sin\left(\frac{\varphi}{2}\right) - 2\sin\left(\frac{t_f+\varphi}{4}\right), 0\right)^{\mathsf{T}}.$$

We shall show that $J \notin G_\varphi(t_f)$. Suppose the opposite, i.e., $J \in G_\varphi(t_f)$. Then by Filippov's theorem [50], there is an *time-optimal* control leading to the phase point $(J,\varphi)^{\mathsf{T}}$. Let t_f^* be the corresponding time-optimal instant. We have $0 \le t_f^* \le t_f$.

By virtue of Theorem 1 of [1, p. 515], the time-optimal control can be taken in the form of one of six variants U1 – U6. Let us consider them. At the same time, for the U5 and U6 control types, we do not suppose obligatory to fulfill condition (3.2).

Any control of the type U4 gives the value $\varphi \le 0$ at the instant t_f^*. Since we assume $0 \le \varphi < \pi$ (see. (6.2)), then for U4 we have to consider only the value $\varphi = 0$ with the control $u(\cdot)\equiv 0$. Here, we get the point $\left(x(t_f^*), y(t_f^*)\right)^{\mathsf{T}} = \left(t_f^*, 0\right)^{\mathsf{T}}$. It does not coincide with the point J for any $t_f^* \in [0, t_f]$.

To provide the inequality $\varphi > 0$ at the instant t_f^* with a control of the type U5, it is required that the length of the middle interval of the control constancy to be less than the sum of the lengths of the first and third intervals. In paper [40] (the proof of Lemma 2) for such a condition, it was established that there is another motion that comes exactly to the same phase point at the same instant t_f^*, but it does not satisfy the PMP. So, the instant t_f^* is not time-optimal. Thus, any controls of the type U5 cannot be time-optimal to arrive to the point $(J,\varphi)^{\mathsf{T}}$ for $\varphi > 0$.

If $\varphi = 0$, then controls of the type U5 generate the same set of points (x,y) at the instant t_f^* as controls of the type U6. Therefore, controls of the type U5 need not be considered.

Controls of four types remain: U1, U2, U3, and U6. Let us introduce the notations $\mathrm{A1}^*$, $\mathrm{A2}^*$, $\mathrm{A3}^*$, and $\mathrm{A6}^*$ for the curves generated by the indicated controls at the time-optimal instant $t_f^* \in [\varphi, t_f]$ with the fixed value φ. These curves are calculated using formulas (3.10)–(3.13) and (3.4) with the t_f symbol replaced by t_f^* one. Let us show that the curves $\mathrm{A1}^*$, $\mathrm{A2}^*$, $\mathrm{A3}^*$, and $\mathrm{A6}^*$ do not contain the point J.

Consider two variants: $t_f^* \in [\varphi, \pi)$, $t_f^* \in [\pi, t_f]$.

In the first variant, you can make sure that the curves $\mathrm{A1}^*$, $\mathrm{A2}^*$, $\mathrm{A3}^*$, and $\mathrm{A6}^*$ are located in the half-plane $X \ge 0$. At the same time, the point J is in the half-plane $X < 0$. Therefore, any controls of the types U1, U2, U3, and U6 cannot be the solution of the time-optimal problem for t_f^*.

In the second variant, the inequality $\pi/4 \le (t_f^* + \varphi)/4$ is true. We show that $J \in \mathrm{int}C_{A6^*}$. To do this, it is enough to establish that the distance $2\sin\left((t_f+\varphi)/4\right)$ between the points J and H is less than the radius $R_{A6}(t_f^*) = 4\sin\left((t_f^*+\varphi)/4\right)$ of the circumference corresponding to the arc $\mathrm{A6}^*$ (see (3.16)). The modulus sign in the formula for $R_{A6}(t_f^*)$ is omitted, since (by virtue of $0 \le t_f^* + \varphi \le t_f + \varphi < 4\pi$) the inequality $\sin\left((t_f^*+\varphi)/4\right) \ge 0$ is fulfilled.

So, let us check the inequality

$$2\sin\big((t_f+\varphi)/4\big) < 4\sin\big((t_f^*+\varphi)/4\big). \tag{8.3}$$

If $(t_f^*+\varphi)/4 \geq \pi/2$, then inequality (8.3) is satisfied due to the monotonicity of the sine for the values $\pi/2 \leq (t_f^*+\varphi)/4 \leq (t_f+\varphi)/4 < \pi$. The last inequality follows from the definition of Case 2. If $(t_f^*+\varphi)/4 < \pi/2$, then $4\sin\Big((t_f^*+\varphi)/4\Big) \geq 4\sin(\pi/4) = 2\sqrt{2}$. From here, taking into account $2\sin\big((t_f+\varphi)/4\big) \leq 2$, inequality (8.3) follows.

Since the arc A6* lies on the boundary of the circle C_{A6^*}, any control of the type U6 cannot lead to the phase point $(J,\varphi)^\mathsf{T}$ at the instant t_f^*.

From Section 3.7, Property 9, it follows that the curves A1*, A2*, and A3* are located outside the intC_{A6^*}. Therefore, controls of the types U1, U2, and U3, also, cannot lead to the phase point $(J,\varphi)^\mathsf{T}$ at the instant t_f^*.

Thus, none of controls of the types U1–U6 can give solutions to the time-optimal problem to the point $(J,\varphi)^\mathsf{T}$ at the instant t_f^*. Therefore, the instant $t_f^* \in [0,t_f]$ cannot be the time-optimal one. We came to a contradiction. Therefore, $J \notin G_\varphi(t_f)$.

2.2) As noted above, the curve $\mathrm{B}_\varphi(t_f)$ (excepting the point P_2) is located in the intC_{P2}. Besides the curve $\mathrm{B}_\varphi(t_f)$, there are no other points in this circle formed by controls of the types U1, U2, U3, and U6. So, in the set int$\mathbf{B}_\varphi(t_f)$, there are no points generated by these controls for the considered values t_f and φ.

Using Lemma 5.2, we see that all its conditions are fulfilled. As a result, we get that the curve $\mathrm{B}_\varphi(t_f)$ forms the "inner" boundary of the φ-section of the set $G(t_f)$ for Case 2.

Consider the open set $\mathrm{M}_\varphi(t_f) = \big(\mathrm{A}_\varphi(t_f)\big)^- \cap \big(\mathrm{B}_\varphi(t_f)\big)^+$. The curves $\mathrm{A}_\varphi(t_f)$ and $\mathrm{B}_\varphi(t_f)$ have at most one common point. Such a point is one of tangency of the curves A2 and A3 under condition (8.2). If this condition is not true, then the curves $\mathrm{A}_\varphi(t_f)$ and $\mathrm{B}_\varphi(t_f)$ have not the common points. Based on the Jordan and Schoenflies theorems [48, 49], it can be shown that the set $\mathrm{M}_\varphi(t_f)$ is connected.

Thus, the conditions of Lemma 5.3 are satisfied for the set $M = \mathrm{M}_\varphi(t_f)$. We conclude that the set $G_\varphi(t_f)$ in Case 2 has the external boundary in the form of the curve $\mathrm{A}_\varphi(t_f)$ and the inner boundary in the form of the curve $\mathrm{B}_\varphi(t_f)$. We have $G_\varphi(t_f) = \mathbf{A}_\varphi(t_f)\backslash\mathrm{int}\mathbf{B}_\varphi(t_f)$ (Fig. 14).

9. Boundary of φ-sections for Case 3

The peculiarity of Case 3 is that the curve A6 will not be used in the description of the boundary of the φ-sections. To construct the boundary, some partition of the curves A2 and A3 will be required.

9.1. **Partition of the curves A2 and A3.** The required partition of the curve A3 is described below in three subpoints. Due to the symmetry about the axis X, the similar partition takes place for the curve A2.

1) The curve A3 is defined by formula (3.12) and given with the help of the parameter $s_3 \in [0,\theta]$. Let us put $s_3^c = 2\pi - \varphi$. Now, we show that $s_3^c \in (0,\theta]$. Indeed, it follows from the inequality $\varphi < 2\pi$ in the definition of Case 3 that $s_3^c > 0$. The second relation in the definition of Case 3 is written as $t_f - \varphi \geq 4\pi - 2\varphi$. Therefore, $s_3^c \leq (t_f-\varphi)/2 = \theta$. Thus, the value $s_3 = s_3^c$ defines some point on the curve A3, which we denote as $\mathcal{P}_3^c$. The point $\mathcal{P}_3^c$ does not coincide with the starting point of the curve A3, but it may coincide with the curve last point.

Meaningfully, the parameter s_3^c corresponds to the value $t_1 = 2\pi$ (by virtue of (3.3)). Therefore, there is a cycle on the first constancy interval of the arising control of the type U3. For $s_3 < s_3^c$, the corresponding motion does not contain cycles. For $s_3 > s_3^c$, the duration t_1 of the first interval satisfies the relation $t_1 > 2\pi$. Thus, there is at least one cycle on the first interval.

Figure 15 shows the example of the curve A3 corresponding to $t_f = 3.5\pi$, $\varphi = 1.2\pi$. Here, $\theta = 1.15\pi$. Dashed trajectories are constructed for five values of the parameter s_3. The trajectory leading to the point $\mathcal{P}_3^c$ corresponds to the value $s_3^c = 0.8\pi$. For $s_3 \in [0, 0.8\pi)$, the first interval of the control constancy has the duration less than 2π. Therefore, there are no cycles on the corresponding trajectories. For $s_3 \in [0.8\pi, 1.15\pi]$, there is a cycle on the first interval and there is no cycle on the third interval.

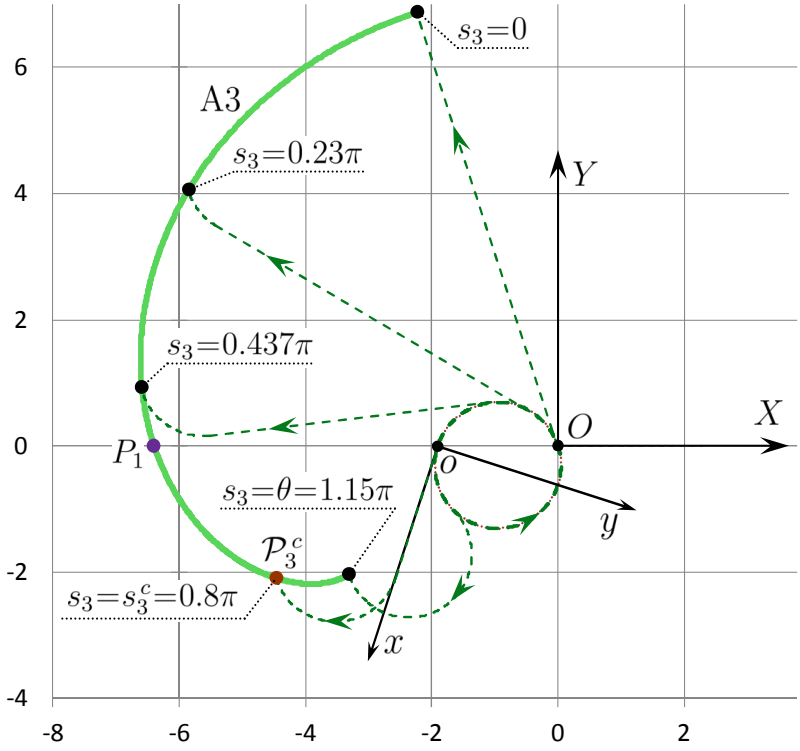

FIGURE 15. Example of the point $\mathcal{P}_3^c$ location on the curve A3 ($t_f = 3.5\pi$ and $\varphi = 1.2\pi$)

There may be cycles on the third interval of the control constancy, but then there should be also a cycle on the first interval, since $\varphi = t_1 - (t_f - t_2) \geq 0$.

The corresponding example of the curve A3 for $t_f = 7.2\pi$, $\varphi = 1.2\pi$ is shown in Fig. 16. Dashed trajectories are constructed for seven values of the parameter s_3. Among them, there is a trajectory leading to the point $\mathcal{P}_3^c$. Since the taken value of φ is the same as for Fig. 15, then the values $s_3^c = 2\pi - \varphi$ are also the same. If $s_3 \in [0, 0.8\pi)$, then there are no cycles on the corresponding motion. For $s_3 \in [0.8\pi, 2\pi)$, there is only one cycle and it is located in the first interval. If $s_3 = 2\pi$, then the additional cycle arises on the third interval. For $s_3 > 2\pi$, there are cycles in both the first and third intervals.

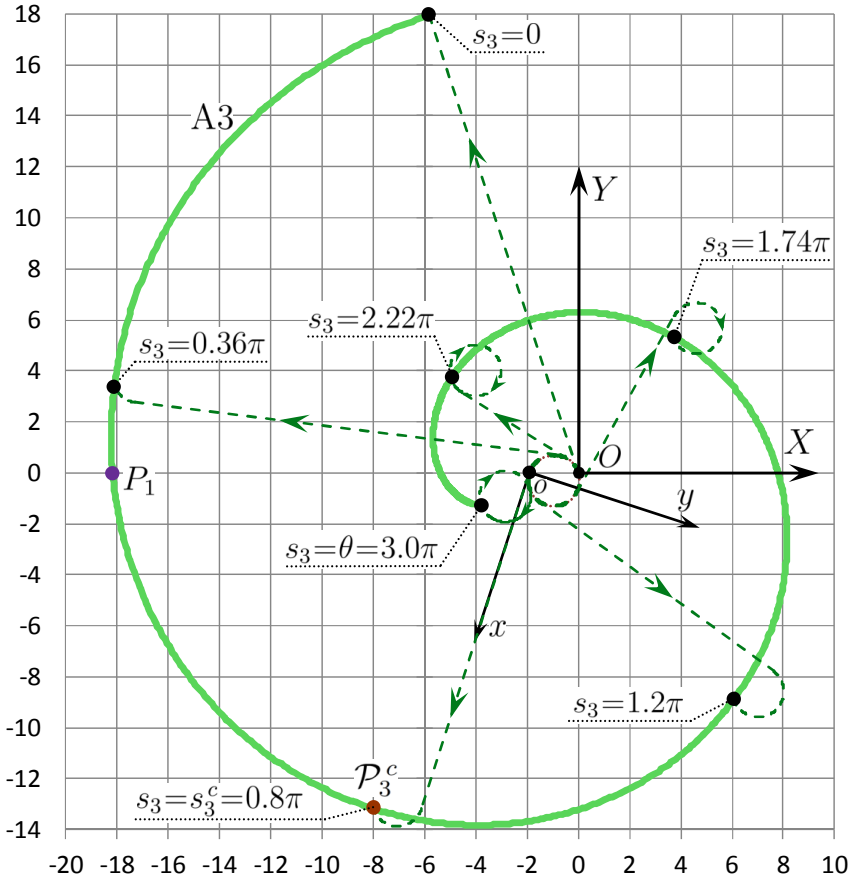

FIGURE 16. Example of the curve A3 with cycles on the first and third intervals of its generating trajectories ($t_f = 7.2\pi$ and $\varphi = 1.2\pi$)

The point $\mathcal{P}_3^c$ cannot lie above the axis X, i.e., $Y_{U3}(s_3^c) \le 0$. Indeed, let us substitute $s_3^c = 2\pi - \varphi$ in (3.12). For the coordinate Y of the resulting point on curve A3, we have

$$Y_{U3}(s_3^c) = 2\left((\theta - 2\pi + \varphi)\sin\left(2\pi - \varphi + \frac{\varphi}{2}\right) + 2\sin\left(\pi - \frac{\varphi}{2}\right)\sin\left(\pi - \frac{\varphi}{2} + \frac{\varphi}{2}\right)\right).$$

Revealing the value θ after simplifications, we get

$$Y_{U3}(s_3^c) = 2\left(\frac{\varphi - t_f}{2} + 2\pi - \varphi\right)\sin\left(\frac{\varphi}{2}\right) = (4\pi - \varphi - t_f)\sin\left(\frac{\varphi}{2}\right).$$

It follows from the second relation in the definition of Case 3 that the first factor of the last expression is not positive. The second factor is not negative because $0 \le \varphi < 2\pi$. Therefore, $Y_{U3}(s_3^c) \le 0$. The equality $Y_{U3}(s_3^c) = 0$ is equivalent to fulfillment of at least one of the conditions $\varphi = 0$, $t_f = 4\pi - \varphi$.

2) From the definition of Case 3, it follows that $t_f \ge 2\pi$. Thus, in Case 3, the relations $0 \le \varphi < 2\pi$, $\varphi < t_f$ hold. Therefore, the first inequality from (7.1) (established for Case 1) is valid. This means that the starting point $\mathcal{P}_{1,3}$ of the curve A3 is above the axis X (for $\varphi > 0$), or on this axis (for $\varphi = 0$).

Consider a part of the curve A3 from the starting point $\mathcal{P}_{1,3}$ to the point $\mathcal{P}_3^c$. We shall show that on this part there is a point P_1 lying on the axis X such that the open arc $(\mathcal{P}_{1,3}, P_1)$ is located

above the axis X, and then, the open arc $(P_1, \mathcal{P}_3^c)$ is below the axis X. The point P_1 can coincide with the point $\mathcal{P}_3^c$.

To establish existence of the point P_1 (by analogy with Case 2), we investigate the derivative $(Y_{U3}(s_3))'_{s_3}$, which is calculated by formula (7.2). Let us consider the function $s_3 \to (Y_{U3}(s_3))'_{s_3}$ on the interval $(0, s_3^c)$, which corresponds to the arc $(\mathcal{P}_{1,3}, \mathcal{P}_3^c)$.

If $\varphi \geq \pi$, we get $(Y_{U3}(s_3))'_{s_3} < 0$ for $s_3 \in (0, s_3^c)$. We have $Y_{U3}(0) > 0$ and $Y_{U3}(s_3^c) \leq 0$. Therefore, it is possible to uniquely determine the point P_1 of intersection of the arc $(\mathcal{P}_{1,3}, \mathcal{P}_3^c]$ with the axis X.

For $\varphi < \pi$, the function $s_3 \to (Y_{U3}(s_3))'_{s_3}$ (as in Case 2) on the interval $(0, s_3^c)$ vanishes only at two points $s_3^* = (\pi - \varphi)/2$ and $s_{3*} = (3\pi - \varphi)/2$. It is not difficult to establish that $0 < s_3^* < s_{3*} < s_3^c$. We have $(Y_{U3}(s_3))'_{s_3} > 0$ on the interval $(0, s_3^*)$. Therefore, $Y_{U3}(s_3^*) > 0$. Similarly, we have $(Y_{U3}(s_3))'_{s_3} > 0$ on the interval (s_{3*}, s_3^c) that means $Y_{U3}(s_{3*}) < 0$. In turn, on the interval (s_3^*, s_{3*}), the derivative satisfies the relation $(Y_{U3}(s_3))'_{s_3} < 0$. Therefore, on this part of the curve A3 (as well as for $\varphi \geq \pi$), it is possible to uniquely define the point P_1 of intersection with the axis X.

Thus, on the arc $(\mathcal{P}_{1,3}, \mathcal{P}_3^c]$ of the curve A3, you can specify the point P_1 on the X axis so that the arc $(\mathcal{P}_{1,3}, P_1)$ of the curve A3 lies above the axis X. With that, the remaining part $(P_1, \mathcal{P}_3^c)$ of the curve A3 (if $P_1 \neq \mathcal{P}_3^c$) is located below the axis X. It can be done both for $\varphi \geq \pi$ and for $\varphi < \pi$.

3) If the point $\mathcal{P}_3^c$ does not coincide with the point P_1, then consider an open arc $(P_1, \mathcal{P}_3^c)$ of the curve A3 (see Figs. 15 and 16). For any point P of this arc, the control of the type U3 leading to it has three non-degenerate intervals of control constancy (see Section 3.7, Property 4). The corresponding trajectory has no cycles by definition of the point $\mathcal{P}_3^c$. The arc $(P_1, \mathcal{P}_3^c)$ lies below the axis X. By Lemma 4.3, we see that the point $(P, \varphi)^\mathsf{T}$ belongs to the int$G(t_f)$.

9.2. **Non-degenerate and degenerate subcases.** We shall distinguish two subcases of Case 3:

$$t_f > 4\pi - \varphi, \tag{9.1}$$

$$t_f = 4\pi - \varphi. \tag{9.2}$$

The first one will be called non-degenerate, the second one is degenerate.

1) Non-degenerate subcase. Rewrite condition (9.1) as $t_f - \varphi > 4\pi - 2\varphi$. Then, it follows that the point $\mathcal{P}_3^c$ defined by the parameter $s_3^c = 2\pi - \varphi$ does not coinside with the point $\mathcal{P}_{3,6}$ corresponding to the parameter $s_3 = \theta$. Consider the half-open arc $[\mathcal{P}_3^c, \mathcal{P}_{3,6})$ of the curve A3. For any point P of this arc, the respective control of the type U3 leading to this point has three non-degenerate intervals of control constancy. Moreover, the first interval contains a cycle due to the choice of the point $\mathcal{P}_3^c$. By Lemma 4.1, we obtain that $(P, \varphi)^\mathsf{T} \in \text{int}G(t_f)$.

Thus, the arc $(P_1, \mathcal{P}_3^c) \cup [\mathcal{P}_3^c, \mathcal{P}_{3,6})$ is located in the int$G_\varphi(t_f)$.

The last point $\mathcal{P}_{3,6}$ of the curve A3 is simultaneously the extreme point of the curve A6.

Consider the curve A6. Take an arbitrary point P on it. For the switch instants t_1 and t_2 of the control of the type U3 (which leads to this point), the following relations hold:

$$t_f = (t_1 - t_0) + (t_2 - t_1) + (t_f - t_2), \qquad \varphi = -(t_1 - t_0) + (t_2 - t_1) - (t_f - t_2).$$

Substituting these expressions into the inequality $t_f > 4\pi - \varphi$, we get $(t_2 - t_1) > 2\pi$. Therefore, the length of the middle interval of the control U6 constancy is greater than 2π. From the

condition $\varphi < t_f$ satisfied to Case 3 (see (6.3)), it follows that the first and third intervals of control constancy cannot simultaneously degenerate.

Using Lemma 4.2, we obtain $(P,\varphi)^{\mathsf{T}} \in \operatorname{int} G(t_f)$. Therefore, $P \in \operatorname{int} G_\varphi(t_f)$. Thus, the whole curve A6 belongs to the $\operatorname{int} G_\varphi(t_f)$.

For subcase (9.1), we get the following result. The curve A1 together with the part of the curve A3 (from the point $\mathcal{P}_{1,3}$ to the point P_1) and the part of the curve A2 (from the point P_1 to the point $\mathcal{P}_{1,2}$) form a closed curve without self-intersections. Denote it as $\mathrm{A}_\varphi(t_f)$. Let $\mathbf{A}_\varphi(t_f)$ be the closed set bounded by this curve. The curve A6 and the remaining parts of the curves A2 and A3 belong to the set $\operatorname{int}\mathbf{A}_\varphi(t_f)$, and they are also in the $\operatorname{int} G_\varphi(t_f)$.

Figure 4b in Section 3 refers to Case 3, the non-degenerate subcase. It can be seen that the curve A3 after the first crossing the axis X (point P_1) makes several revolutions. The part of this curve after the point P_1 lies in the $\operatorname{int} G_\varphi(t_f)$. Similarly, the same fact takes place for the part of the curve A2 symmetric to A3.

2) Degenerate subcase. From condition (9.2) and by analogy with the non-degenerate subcase, it follows that $\mathcal{P}_3^c = \mathcal{P}_{3,6}$. Here, the curve A6 degenerates (the radius of the circle becomes equal to zero) into the point $H = \mathcal{P}_{2.6} = \mathcal{P}_{3.6}$. If $\varphi = 0$, then the curve A1 degenerates into the point coinciding with the points $\mathcal{P}_{1,2}$ and $\mathcal{P}_{1,3}$.

For the degenerate subcase in Fig. 17, the curve $\mathrm{A}_\varphi(t_f)$ composed of the curves A1, A3, A6 (degenerate one), and A2 is shown. The point $\mathcal{P}_3^c$ coincides with the point H. Three variants are presented here: $\varphi = 0.3\pi$, $t_f = 3.7\pi$ (Fig. 17a); $\varphi = 1.0\pi$, $t_f = 3.0\pi$ (Fig. 17b); $\varphi = 1.4\pi$, $t_f = 2.6\pi$ (Fig. 17c).

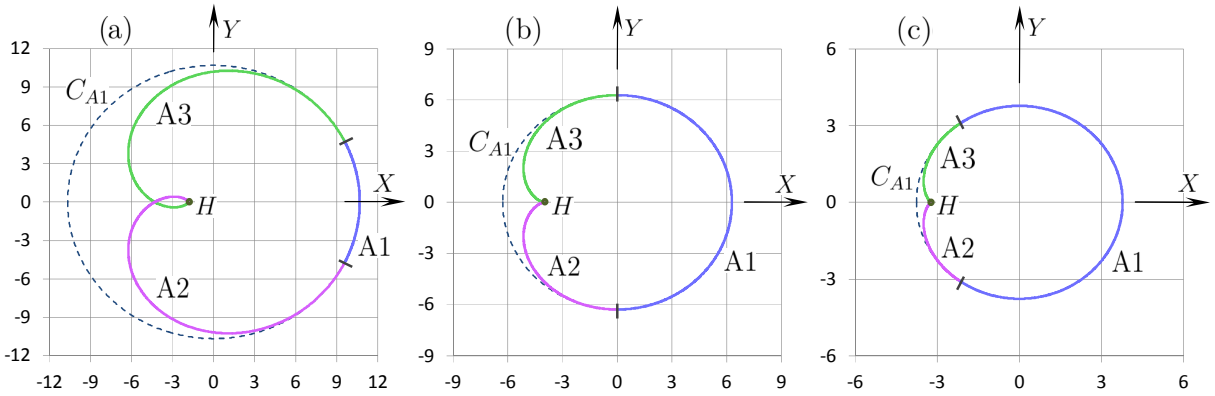

FIGURE 17. Case 3. Variants of the curve $\mathrm{A}_\varphi(t_f)$ for the degenerate subcase

2.1) Suppose that $\varphi < \pi$ (Fig. 17a). Then $P_1 \neq \mathcal{P}_{3,6}$. As in the non-degenerate subcase, the curve A1 together with a part of the curve A3 (from the point $\mathcal{P}_{1,3}$ to the point P_1) and a part of the curve A2 (from the point P_1 to the point $\mathcal{P}_{1,2}$) form the closed curve without self-intersections. Let $\mathbf{A}_\varphi(t_f)$ be the closed set bounded by this curve. Under this, the arc $(\mathcal{P}_{2,6}, P_1)$ of the curve A2 and the arc $(P_1, \mathcal{P}_{3,6})$ of the curve A3 are in the $\operatorname{int} G_\varphi(t_f)$. The similar property for the point $\mathcal{P}_{2.6} = \mathcal{P}_{3.6}$ will be established in the next subsection.

2.2) Suppose $\varphi \geq \pi$. Then, $P_1 = \mathcal{P}_{3,6}$. We get that the curves A1, A2, and A3 form the closed curve without self-intersections. Let $\mathbf{A}_\varphi(t_f)$ be the closed set bounded by this curve.

Geometrically, the situations $\varphi = \pi$ (Fig. 17b) and $\varphi > \pi$ (Fig. 17c) differ: in the first of them, the curves A2 and A3 touch at the point $H = P_1$, and in the second situation, they join at this point at some angle.

Outside the set $\mathbf{A}_\varphi(t_f)$, as well as in its interior, there are no points generated by controls of the types U1, U2, U3, and U6.

9.3. **Constructing the boundary of φ-section.** Applying Lemma 5.1 to the set $\mathbf{A}_\varphi(t_f)$ introduced in the previous Section 9.2, we obtain that

$$G_\varphi(t_f) \subset \mathbf{A}_\varphi(t_f), \qquad \partial\mathbf{A}_\varphi(t_f) \subset \partial G_\varphi(t_f). \tag{9.3}$$

Let us show that for the set $\mathrm{int}\mathbf{A}_\varphi(t_f)$ the conditions of Lemma 5.3 are satisfied. We put $\mathrm{M}_\varphi(t_f) = \mathrm{int}\mathbf{A}_\varphi(t_f)$.

In the non-degenerate subcase, all arcs of the curves A2, A3, and A6 that do not participate in formation of boundary of the set $\mathbf{A}_\varphi(t_f)$ belong to the set $\mathrm{M}_\varphi(t_f)$. Moreover, these arcs belong to the $\mathrm{int}G_\varphi(t_f)$. Therefore, condition 2) of Lemma 5.3 is satisfied for the set $M = \mathrm{M}_\varphi(t_f)$.

In the degenerate subcase for $\varphi < \pi$, condition 2) of Lemma 5.3 is satisfied. Indeed, for the parts of the curves A2, A3, and A6 that belong to the set $\mathrm{M}_\varphi(t_f)$, it has been established that any point of them (excepting, perhaps, only the point $\mathcal{P}_{3,6}$) belongs to the $\mathrm{int}G_\varphi(t_f)$. For $\varphi \geq \pi$ there are no the curves A1, A2, A3 and A6 outside the set $\partial\mathbf{A}_\varphi(t_f)$, and, therefore, these curves are absent in the set $\mathrm{M}_\varphi(t_f)$. To apply Lemma 5.3 (under condition 1)), it is necessary to specify a point belonging to the $\mathrm{int}G_\varphi(t_f)$. As such a point (similarly to Section 7), take the point generated by the constant control $u(t) \equiv \varphi/t_f$. Thus, condition 1) of Lemma 5.3 is satisfied for $M = \mathrm{M}_\varphi(t_f)$.

As a result, for all the variants (considered in Section 9.2) of Case 3, we obtain the fulfillment of Lemma 5.3 conditions. So, $\mathrm{M}_\varphi(t_f) = \mathrm{int}\mathbf{A}_\varphi(t_f) \subset G_\varphi(t_f)$. Hence, taking into account (9.3), we get $G_\varphi(t_f) = \mathbf{A}_\varphi(t_f)$ in Case 3. As a consequence, for the degenerate subcase under $\varphi < \pi$, we have $\mathcal{P}_{2,6} = \mathcal{P}_{3,6} \in \mathrm{int}G_\varphi(t_f)$.

The boundary of the set $G_\varphi(t_f)$ is made up of the curve A1 and the part of the curve A3 to the first point of its intersection with the axis X as well as the symmetric part of the curve A2 (see Figs. 17 and 18). Fig. 18a (Fig. 18b) corresponds to $t_f = 4.5\pi$, $\varphi = \pi$ ($t_f = 8.5\pi$, $\varphi = \pi$).

Remark 9.1. Consider the degenerate subcase (9.2) of Case 3.

Under the condition $\varphi \geq \pi$ (Section 9.2, subpoint 2.2, Fig. 17b and 17c), the point H lies on the boundary of the set $G_\varphi(t_f)$. Therefore, the point $(H,\varphi)^\mathsf{T}$ belongs to the boundary of the three-dimensional set $G(t_f)$.

Under the condition $0 \leq \varphi < \pi$ (Section 9.2, subpoint 2.1, Fig. 17a), we have $H \in \mathrm{int}G_\varphi(t_f)$. If $0 < \varphi < \pi$, then with a fixed value of t_f, the point H is the limit for the arc $\mathrm{A6}(\widetilde{\varphi})$, where $0 < \widetilde{\varphi} < \varphi$ and $\widetilde{\varphi} \to \varphi$. Since for any point $P \in \mathrm{A6}(\widetilde{\varphi})$, the inclusion $P \in \partial G_{\widetilde{\varphi}}(t_f)$ is performed, then $(P,\widetilde{\varphi})^\mathsf{T} \in \partial G(t_f)$. Hence, taking into account the closedness of the set $G(t_f)$, we obtain $(H,\varphi)^\mathsf{T} \in \partial G(t_f)$.

Thus, in the whole, for the degenerate subcase (9.2) of Case 3, the point $(H,\varphi)^\mathsf{T}$ belongs to the boundary of the three-dimensional reachable set $G(t_f)$ for $\varphi > 0$. If $\varphi = 0$, then the point $(H,\varphi)^\mathsf{T}$ belongs to the interior of the three-dimensional reachable set $G(t_f)$.

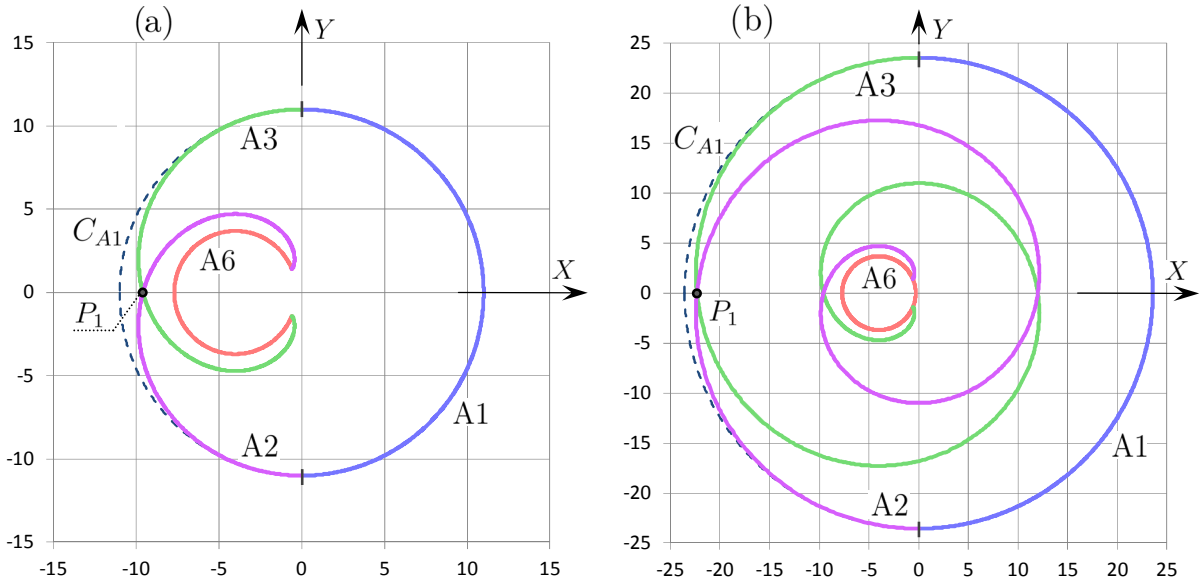

FIGURE 18. Case 3. The set $G_\varphi(t_f)$ in the non-degenerate subcase

10. BOUNDARY OF φ-SECTIONS FOR CASE 4

Let us show that in Case 4 any φ-section is the circle C_{A1} given by the curve A1. The position of the center of the circle depends on φ, and its radius depends on φ and t_f.

The curve A1 is defined by formula (3.10) and is an arc of the circle with the radius $R_{A1} = (t_f - \varphi)$ and with the center at the origin of the auxiliary coordinate system X, Y. By the definition of Case 4 (see (6.4)), the inequality $\varphi \geq 2\pi$ holds. The arc A1, the span of which is equal to φ, forms a circumference with an "overlap". The curves A2, A3, and A6 belong to the circle C_{A1}. Moreover, the points $\mathcal{P}_{1,2}$ and $\mathcal{P}_{1,3}$ (where the curves A2 and A3 are joining with the curve A1) lie on its boundary. All other points of the curves A2 and A3, as well as all the curve A6, are located in the intC_{A1} (Section 3.7, Property 8).

Let us establish that all points of the curves A2, A3, and A6 lying in the intC_{A1} belong to the int$G_\varphi(t_f)$. The interior points of the curve A3 are generated by controls of the type U3 with three nondegenerate constancy intervals (Section 3.7, Property 4). Here, the switch instans t_1 and t_2 are defined by the relation $\varphi = t_1 - (t_f - t_2)$. Since $\varphi \geq 2\pi$, so, $t_1 > t_1 - (t_f - t_2) \geq 2\pi$. Therefore, on all motions leading to the inner points of the curve A3, there is a cycle on the first interval of control constancy. Thus, in accordance with Lemma 4.1, we obtain that the points of the curve A3 under consideration belong to int$G_\varphi(t_f)$. The similar property is valid for the curve A2.

Consider the curve A6. It is generated by controls of the type U6. The corresponding switch instans t_1 and t_2 are defined by the relations

$$t_f = (t_1 - t_0) + (t_2 - t_1) + (t_f - t_2), \qquad \varphi = -(t_1 - t_0) + (t_2 - t_1) - (t_f - t_2).$$

From here, $(t_2 - t_1) = (t_f + \varphi)/2$. From condition (6.4), we get $2\pi < (t_2 - t_1) < t_f$. Thus, for any point of the curve A6 on the corresponding control, duration of the second interval is greater than 2π, and at least one of the two adjacent extreme intervals is not degenerate. By Lemma 4.2, we conclude that the curve A6 entirely belongs to the int$G_\varphi(t_f)$.

Using Lemma 5.1 and Lemma 5.3, we obtain $G_\varphi(t_f) = C_{A1}$.

Figure 19a (19b) shows the set $G_\varphi(t_f)$ for the values $t_f = 5\pi$ and $\varphi = 2.5\pi$ (correspondingly, $t_f = 8\pi$ and $\varphi = 2.5\pi$). It coincides with the circle C_{A1}. The curve A1 is the circumference with an "overlap". The curves A2, A3, and A6 are also shown.

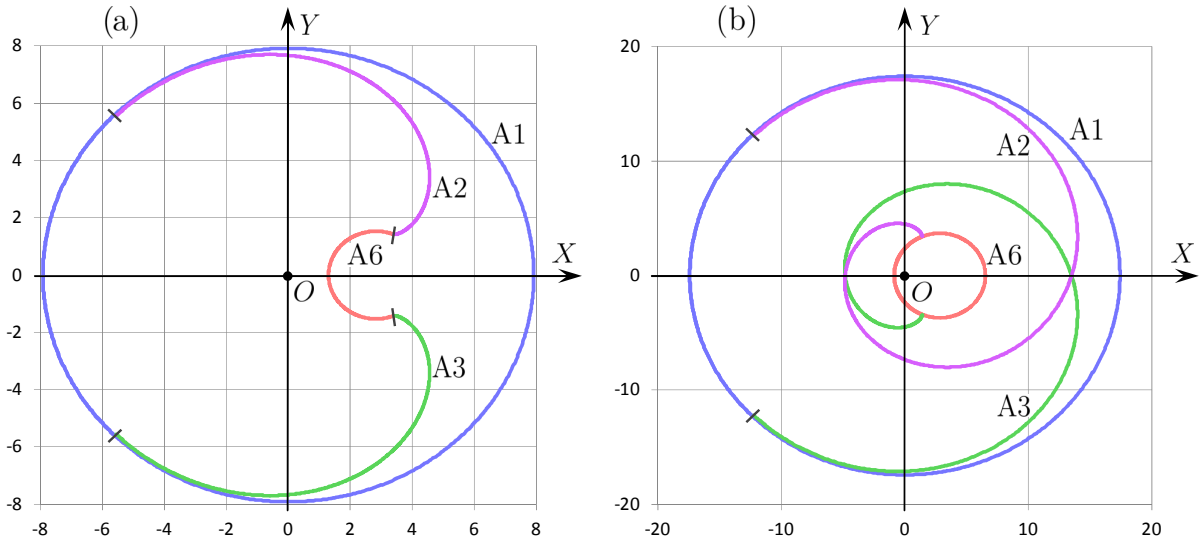

FIGURE 19. Case 4. The set $G_\varphi(t_f)$ for two variants t_f and φ

11. GENERAL DESCRIPTION OF THE STRUCTURE OF φ-SECTIONS FOR $\varphi \geq 0$

We summarize the results of Sections 6 – 10 on the structure of the φ-sections of the reachable set for $\varphi \geq 0$ in the following statement.

Theorem 11.1. *Let $\varphi \geq 0$. Then*

1) *the φ-sections $G_\varphi(t_f)$ of the reachable set $G(t_f)$ are simply connected in Cases* 1, 3, 4, 5. *In Case* 2, *the φ-section $G_\varphi(t_f)$ is not simply connected*;

2) *in Case* 1, *the boundary of the φ-section $G_\varphi(t_f)$ is a simple closed curve $A_\varphi(t_f)$ composed of successively connected curves* A1, A3, A6, *and* A2;

3) *in Case* 2, *the boundary of the φ-section $G_\varphi(t_f)$ splits into two parts: the outer contour (the curve $A_\varphi(t_f)$) and the inner contour (the curve $B_\varphi(t_f)$). The outer contour is made up of the curve* A1 *and adjacent parts of the curves* A2 *and* A3 *up to the point P_1 of their first intersection. The inner contour is made up of the curve* A6 *and adjacent parts of the curves* A2 *and* A3 *up to the point P_2 of their last intersection*;

4) *in Case* 3, *the boundary of the φ-section $G_\varphi(t_f)$ is a simple closed curve $A_\varphi(t_f)$. It consists of the curve* A1 *and adjacent parts of the curves* A2 *and* A3 *up to the point P_1 of their first intersection*;

5) *in Case* 4, *the boundary of the φ-section $G_\varphi(t_f)$ is a circumference defined by the curve* A1;

6) *in Case* 5, *the φ-section $G_\varphi(t_f)$ degenerates to a point.*

12. Case $\varphi < 0$

Study of the case $\varphi < 0$ is based on the symmetry property of system (2.1), which is the following. Consider the motion $(x^*(t), y^*(t), \varphi^*(t))^{\mathsf{T}}$ of the system on the interval $[0, t_f]$ from the initial zero phase point generated by some admissible control $u^*(t)$. At the instant t_f, we get the point $\big(x^*(t_f), y^*(t_f), \varphi^*(t_f)\big)^{\mathsf{T}}$. The control $u_*(t) = -u^*(t)$ (i.e., it differs from the initial one only in sign) with the same zero initial phase point gives the phase state $\big(x_*(t_f), y_*(t_f), \varphi_*(t_f)\big)^{\mathsf{T}}$ at the instant t_f where $x_*(t_f) = x^*(t_f)$, $y_*(t_f) = -y^*(t_f)$, $\varphi_*(t_f) = -\varphi^*(t_f)$.

This leads to the fact that the φ^*-section $G_{\varphi^*}(t_f)$ of the reachable set $G(t_f)$ for any φ^* is associated with the corresponding section for $\varphi_* = -\varphi^*$ by mirroring about the axis x of the original coordinate system. Within the framework of the auxiliary coordinate system (if it is also introduced for $\varphi < 0$), the new section coincides with the old one. Such facts make it possible not to consider separately the case $\varphi < 0$. The result for it is determined by the case $\varphi > 0$.

For $\varphi = 0$, the original coordinate system x, y coincides with the auxiliary system X, Y. Therefore, the corresponding φ-section $G_\varphi(t_f)$ is symmetric w.r.t. the axis x of the original coordinate system.

Remark 12.1. The material of Sections 6–10 can be interpreted as a description of additional conditions imposed onto controls of the type U1 – U6 from (3.1). Such conditions (along with relation (3.2)) select controls leading to the boundary of the φ-sections of the reachable set $G(t_f)$. These conditions depend on t_f and φ (see formulas (6.1)–(6.5) and Fig. 11). Taking them into account completely characterizes not only the boundary of the φ-sections, but, also, the boundary of the entire three-dimensional reachable set $G(t_f)$.

Conclusion

For the Dubins car with symmetric control constraint, the analytical description is obtained for the sections along the angular coordinate φ of the three-dimensional reachable set $G(t_f)$. Possible types of the two-dimensional φ-sections of the set $G(t_f)$ with different collections of boundary arcs are classified. The case is highlighted when the φ-section is not simply connected.

A certain class of piecewise constant controls with at most two switches was described in [40]. It was shown that we can restrict ourselves to such controls when constructing the boundary of the three-dimensional reachable set. However, in this class, there were also controls leading to the interior of the reachable set. The relations obtained in this paper filter out such controls. Thus, a subclass of controls is singled out, each of which leads to the boundary of the reachable set. On the contrary, for each point of the boundary, there is at least one control from the given subclass leading to this point.

The paper results open up the possibility (based on the effective numerical construction of the set $G(t_f)$ boundary with the growth of t_f) to solve many problems of optimal control with the dynamics of the Dubins car including the time-optimal problems. It is also possible to use the results obtained for solving some game problems.

The paper results are planned to carry over onto the case of an asymmetric constraint on the control $u \in [u_1, u_2]$ for $u_1 < 0$, $u_2 > 0$.

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