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Semipermeable Curves and Level Sets of the Value Function in Differential Games with the Homicidal Chauffeur Dynamics

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Abstract. A classical and a modified (acoustic) variants of the differential game "homicidal chauffeur" are considered. An interesting peculiarity of the latter variant consists, in particular, in the presence of holes located strictly inside the victory domain of the pursuit-evasion game. In the paper, an explanation to this phenomenon is given. The explanation is based on an analysis of families of semipermeable curves that are determined from only the dynamics of the system. Results of the computation of level sets of the value function are presented.

1. Introduction

The homicidal chauffeur game [4], [6] is one of the most known model differential games of pursuit-evasion. In [1], [3], an acoustic capture variant of this game proposed by P.Bernhard was considered. The evader must reduce his speed when he comes close to the pursuer in order not to be heard. Mathematically, this can be expressed in taking the restriction on the velocity of the evader that depends on the distance between the evader and pursuer.

It was shown in [1],[3] that the solvability set (victory domain) of the acoustic problem can have holes located strictly inside this set. In such a case, it is impossible to compute the boundary of the solvability set using only barrier lines emitted from the usable part [4] of the terminal set.

This paper joins the paper [8] and is devoted to the description of families of semipermeable curves arising both in the classical homicidal chauffeur problem and its acoustic modification. The families of semipermeable curves are determined from only the dynamics of the system (including constraints on the controls) and do not depend on the form of the terminal set. The knowledge of the structure of these families can be very useful when studying different properties of solutions of time-optimal games. In particular, barrier lines which bound the solvability set are composed from arcs of smooth semipermeable curves.

It was found out that for some parameters of the problem, regions where semipermeable curves are absent can arise. It is shown that certain regions of this type can cause holes in solvability sets. The boundary of the solvability set can be completely described using semipermeable curves issued from the boundary of such regions and from the boundary of the terminal set.

In the final part of the paper, results of the computation of level sets of the value function for the acoustic problem are presented.

2. Games with the Homicidal Chauffeur Dynamics

The pursuer P has a fixed speed $w^{(1)}$ but his radius of turn is bounded by a given quantity R. The evader E is inertialess. He steers by choosing his velocity vector $v = (v_1, v_2)'$ from some set. The kinematic equations are:

$$P: \quad \begin{aligned} \dot{x}_p &= w^{(1)} \sin \psi \\ \dot{y}_p &= w^{(1)} \cos \psi \\ \dot{\psi} &= w^{(1)} \varphi / R, \qquad |\varphi| \le 1 \end{aligned} \qquad\qquad E: \quad \begin{aligned} \dot{x}_e &= v_1 \\ \dot{y}_e &= v_2. \end{aligned}$$

The number of equations can be reduced to two (see [4]) if a coordinate system with the origin at P and the axis x_2 in the direction of P's velocity vector is used. The axis x_1 is orthogonal to the axis x_2.

The dynamics in the reduced coordinates is

$$\begin{aligned} \dot{x}_1 &= -w^{(1)} x_2\, \varphi / R + v_1 \\ \dot{x}_2 &= w^{(1)} x_1\, \varphi / R + v_2 - w^{(1)}, \qquad |\varphi| \le 1. \end{aligned} \tag{1}$$

The state vector $(x_1,\ x_2)'$ gives the relative position of E with respect to P.

2.1. Classical homicidal chauffeur game

The control v is chosen from a circle of radius $w^{(2)} > 0$ with the center at the origin. The objective of the control φ of the pursuer is to minimize the time of attainment of a given terminal set M by the state vector of system (1). The objective of the control v of the evader is to maximize this time. Therefore the payoff of the game is the time of attaining the terminal set.

2.2. Acoustic game

The difference is that the constraint on the control of player E depends on x. It is given by the formula

$$\mathcal{Q}(x) = k(x) Q, \quad k(x) = \min\{|x|, s\}/s, \ s > 0.$$

Here s is a parameter. We have $\mathcal{Q}(x) = Q$ if $|x| \ge s$. The objective of the control φ is to minimize the time of attaining a terminal set M. The objective of the control v is to maximize this time.

For the unification of notation, let us agree that $\mathcal{Q}(x) = Q$ for the classical homicidal chauffeur game.

3. Semipermeable Curves in Differential Games with the Homicidal Chauffeur Dynamics

The families of smooth semipermeable curves are determined from only the dynamics of the system and the bounds on the controls of the players.

We explain now what semipermeable curves mean (see also [4]). Let

$$H(\ell, x) = \min_{|\varphi|\le 1} \max_{v\in\mathcal{Q}(x)} \ell' f(x,\varphi,v) = \max_{v\in\mathcal{Q}(x)} \min_{|\varphi|\le 1} \ell' f(x,\varphi,v), \quad x \in R^2,\ \ell \in R^2. \quad (2)$$

Here $f(x,\varphi,v) = p(x)\varphi + v + g$, $p(x) = (-x_2, x_1)' \cdot w^{(1)}/R$ and $g = (0, -w^{(1)})'$. Fix $x \in R^2$ and consider ℓ such that $H(\ell,x) = 0$. Letting $\varphi^* = \operatorname{argmin}\{\ell' p(x)\varphi$: $|\varphi| \le 1\}$ and $v^* = \operatorname{argmax}\{\ell' v$: $v \in \mathcal{Q}(x)\}$, it follows that $\ell' f(x,\varphi^*,v) \le 0$ holds for any $v \in \mathcal{Q}(x)$, and $\ell' f(x,\varphi,v^*) \ge 0$ holds for any $\varphi \in [-1,1]$. This means that the direction $f(x,\varphi^*,v^*)$, which is orthogonal to ℓ, separates the vectograms $U(v^*) = \{f(x,\varphi,v^*) : \varphi \in [-1,1]\}$ and $V(\varphi^*) = \{f(x,\varphi^*,v) : v \in \mathcal{Q}(x)\}$ of players P and E. Such a direction is called semipermeable. A smooth curve is called a semipermeable curve if the tangent vector at any point of this curve is a semipermeable direction.

The number of semipermeable directions depends on the form of the function $\ell \to H(\ell,x)$ at the point x. In the case considered, the function $H(\cdot,x)$ is composed of two convex functions:

$$H(\ell,x) = \begin{cases} \max\limits_{v\in\mathcal{Q}(x)} \ell' v + \ell' p(x) + \ell' g, & \text{if } \ell' p(x) < 0 \\ \max\limits_{v\in\mathcal{Q}(x)} \ell' v - \ell' p(x) + \ell' g, & \text{if } \ell' p(x) \ge 0. \end{cases}$$

The semipermeable directions are derived from the roots of the equation $H(\ell,x)= 0$. We will distinguish the roots "$-$" to "$+$" and the roots "$+$" to "$-$". When classifying these roots, we suppose that $\ell \in \mathcal{E}$, where $\mathcal{E}$ is the boundary of a convex polygon containing the origin. We say that ℓ_* is a root $-$ to $+$ if $H(\ell_*,x) = 0$, and if $H(\ell,x) < 0$ ($H(\ell,x) > 0$) for $\ell < \ell_*$ ($\ell > \ell_*$) that are sufficiently close to ℓ_*, where the notation $\ell < \ell_*$ means that the direction of the vector ℓ can be obtained from the direction of the vector ℓ_* using a counterclockwise rotation through an angle not exceeding π. The roots $-$ to $+$ and the roots $+$ to $-$ are called roots of the first and second type, respectively.

We denote roots of the first type by $\ell^{(1),i}(x)$ and roots of the second type by $\ell^{(2),i}(x)$. The right index takes the value 1 or 2, and indicates the half-plane $\{\ell \in R^2 : \ell' p(x) < 0\}$ or $\{\ell \in R^2 : \ell' p(x) \ge 0\}$. Due to the above property of the piecewise convexity of the function $H(\cdot,x)$, the equation $H(\ell,x) = 0$ can have at most two roots of each type for any given x.

We now describe how the families of smooth semipermeable curves can be constructed.

3.1. Constraint $\mathcal{Q}$ on the control of player E does not depend on x

Assume that the constraint $\mathcal{Q}$ does not depend on x that is $\mathcal{Q}(x) = Q$. Denote

$$A_* = \{(x_1, x_2) : \ x_1 = \frac{v_2 R}{w^{(1)}} - R, \ \ x_2 = -\frac{v_1 R}{w^{(1)}}, \ (v_1, v_2)' \in Q\}, \tag{3}$$

$$B_* = \{(x_1, x_2) : \ x_1 = -\frac{v_2 R}{w^{(1)}} + R, \ \ x_2 = \frac{v_1 R}{w^{(1)}}, \ (v_1, v_2)' \in Q\}. \tag{4}$$

The set B_* is symmetric to the set A_* with respect to the origin. Let $C_* = A_* \bigcap B_*$.

3.1.1. ROOTS OF EQUATION $H(\ell, x) = 0$ Let us show for all $x \notin C_*$ that the equation $H(\ell, x) = 0$ has at least one root of the first type and one root of the second type. To prove this, it is sufficient to verify that, for any x, there exist vectors $\underline{\ell}$ and $\overline{\ell}$ such that $H(\underline{\ell}, x) < 0$ and $H(\overline{\ell}, x) > 0$.

Let $x \notin A_*$. Then there exists a vector $\widetilde{\ell}$ such that $\widetilde{\ell}' x > \widetilde{\ell}' z$ for any $z \in A_*$. That is

$$-\widetilde{\ell}' x + \max_{z \in A_*} \widetilde{\ell}' z < 0.$$

Denote by $\overline{x}$ the nearest to x point of A_*. The vector $x - \overline{x}$ can be considered as $\widetilde{\ell}$.

Assume $\underline{\ell} = \left(-\widetilde{\ell}_2 R / w^{(1)}, \ \widetilde{\ell}_1 R / w^{(1)}\right)'$. We have

$$H(\underline{\ell}, \ x) \leq \underline{\ell}' \left(\frac{w^{(1)} x_2}{R}, \ \frac{-w^{(1)} x_1}{R}\right)' + \underline{\ell}' g + \max_{v \in Q} \underline{\ell}' v =$$

$$-\widetilde{\ell}' x + \max_{v \in Q} \widetilde{\ell}' \left(\frac{v_2 R}{w^{(1)}} - R, \ \frac{-v_1 R}{w^{(1)}}\right)' = -\widetilde{\ell}' x + \max_{z \in A_*} \widetilde{\ell}' z < 0.$$

Similarly, one can show for $x \notin B_*$ that there exists a vector $\underline{\ell}$ such that $H(\underline{\ell}, x) < 0$. Hence, if $x \notin C_*$, then there exists a vector $\underline{\ell}$ such that $H(\underline{\ell}, x) < 0$.

Consider $\overline{\ell} \neq 0$ such that $\overline{\ell}' p(x) = 0$ and $\overline{\ell}' g \geq 0$. Since $0 \in \text{int} Q$, then $H(\overline{\ell}, x) = \max\{\overline{\ell}' v : \ v \in Q\} + \overline{\ell}' g > 0$. This completes the proof.

Let $x \in \text{int} C_*$. We show that $H(\ell, \ x) > 0$ for all $\ell \neq 0$. Take $\ell \neq 0$. Suppose that $\min\{\ell' p(x) \varphi : \ \varphi \in [-1, 1]\}$ occurs for $\varphi = -1$ ($\varphi = 1$). It follows from the definition of the set A_* (B_*) that for $x \in \text{int} A_*$ ($x \in \text{int} B_*$), there exists a vector $v_* \in \text{int} Q$ such that $f(x, \ -1, \ v_*) = 0$ ($f(x, \ 1, \ v_*) = 0$). Hence, $H(\ell, \ x) > 0$. Therefore, roots of the first and second type do not exist for $x \in \text{int} C_*$. Due to continuity of H, strict roots do not exist for $x \in \partial C_*$ too.

3.1.2. CASE $C_* = \emptyset$ We consider cones spanned onto the sets A_* and B_* with the apex at the origin. Denote these cones by $\text{cone} A_*$ and $\text{cone} B_*$, respectively. The part of $\text{cone} A_*$ after deleting the set

$$\{(x_1, x_2) : \ x_1 = \frac{v_2 R}{w^{(1)} \varphi} - R/\varphi, \ x_2 = -\frac{v_1 R}{w^{(1)} \varphi}, \ 1 < \varphi < \infty, \ (v_1, v_2)' \in Q\}$$

is denoted by A. Similarly, the set B as the part of $\text{cone} B_*$ is introduced.

One can find the domains of the functions $\ell^{(j),i}(\cdot)$, $j = 1, 2$, $i = 1, 2$. Figure 1 presents the sets A and B and the domains of the functions $\ell^{(j),i}(\cdot)$, $j = 1, 2$, $i =$

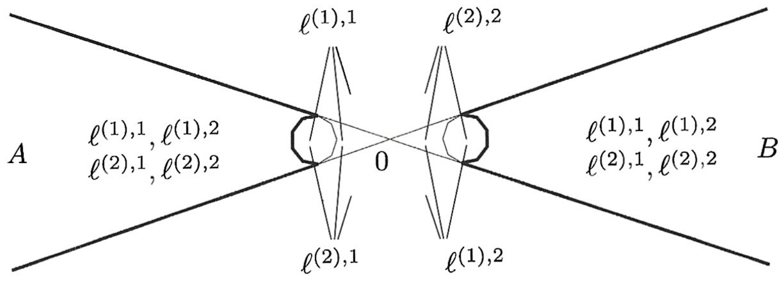

FIGURE 1. Domains of $\ell^{(j),i}$. Set Q does not depend on x; $C_* = \emptyset$.

$1, 2$, for the case where the set Q is a polygonal approximation of a circle of some radius $w^{(2)}$. The boundaries of A and B are drawn with the thick lines. There exist two roots of the first type and two roots of the second type at each internal point of the sets A and B. For any point in the exterior of A and B, there exist one root of the first type and one root of the second type.

The function $\ell^{(j),i}(\cdot)$ is Lipschitz continuous on any closed bounded subset of the interior of its domain. Consider the two-dimensional differential equation

$$dx/dt = \Pi \ell^{(j),i}(x), \tag{5}$$

where Π is the matrix of rotation through the angle $\pi/2$, the rotation being clockwise or counterclockwise if $j = 1$ or $j = 2$, respectively. Since the tangent vector at each point of the trajectory defined by this equation is a semipermeable direction, the trajectories are semipermeable curves. Therefore player P can keep the state vector x on one side of the curve (positive side), and player E can keep x on the other (negative) side. Equation (5) specifies a family $\Lambda^{(j),i}$ of smooth semipermeable curves. Pictures of the families $\Lambda^{(j),i}$ for the case $C_* = \emptyset$ are given in [7].

3.1.3. CASE $C_* \neq \emptyset$ There are no roots in the set C_*, there are four roots in the set $R^2 \setminus (A_* \bigcup B_*)$, and there are two roots (one root of the first type and one root of the second type) in the rest part of the plane. Figure 2 shows the domains of the functions $\ell^{(j),i}(\cdot)$ for this case. The set Q is a circle of some radius $w^{(2)} > w^{(1)}$. The digits 4, 2 and 0 state the number of roots. Using (5), one can produce the families $\Lambda^{(j),i}$ for the case where $C_* \neq \emptyset$.

The following important property holds true for any point $x \in C_* = A_* \bigcap B_*$: for any $\varphi \in [-1, 1]$ there exists $v \in Q$ such that $f(x, \varphi, v) = 0$. Therefore, in the region C_*, player E can counter any control of player P, so the state remains immovable all the time. Further, if a point x with the above property does not belong to the terminal set M, then M cannot be reached from x. We call regions of such points the superiority sets of player E.

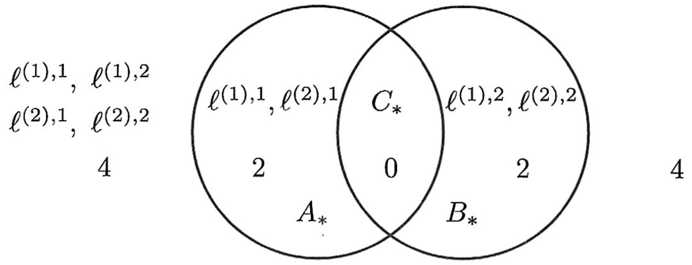

FIGURE 2. Domains of $\ell^{(j),i}(\cdot)$. Set $\mathcal{Q}$ does not depend on x; $C_* \neq \emptyset$.

3.2. Constraint $\mathcal{Q}$ on the control of player E depends on x

Using the form of the domains of $\ell^{(j),i}(\cdot)$ from section 3.1, one can construct the domains for the case $\mathcal{Q}(x) = k(x)Q$. Let us describe briefly how it can be done.

First note that $k(x)$ = const for the points x of any circumference of some fixed radius with the center at $(0,0)$. It holds $k(x) = 1$ outside the circle of radius s. Take a circumference $\Omega(r)$ of radius r with the center at $(0,0)$. Set $k(r) = \min\{r,\ s\}/s$ and $Q(r) = k(r)Q$. We have $\mathcal{Q}(x) = Q(|x|)$.

Form the sets $A_*(r)$ and $B_*(r)$ substituting the set $Q(r)$ instead of Q in formulae (3) and (4) for A_* and B_*. Let $C_*(r) = A_*(r) \bigcap B_*(r)$. Using $A_*(r)$ and $B_*(r)$, construct domains of $\ell^{(j),i}(\cdot)$, the cases $C_*(r) = \emptyset$ and $C_*(r) \neq \emptyset$ being distinguished. Put the circumference $\Omega(r)$ onto the constructed domains. As a result, a division of the circumference onto arcs is obtained. The number and the type of roots are the same for all points of each arc. This technique is applied for every r in $[0, s]$, and identically named division points are connected. Thus the circle of radius s is divided into parts according to the kinds of roots. Outside this circle, the dividing lines coincide with the lines constructed for the case when $\mathcal{Q}$ does not depend on x.

Since Q is a circle of radius $w^{(2)}$, then $Q(r)$ is a circle of radius $w^{(2)}(r) = \min\{r,\ s\}w^{(2)}/s$. The condition $C_*(r) = \emptyset$ means $w^{(2)}(r) < w^{(1)}$, and the condition $C_*(r) \neq \emptyset$ is equivalent to the relation $w^{(2)}(r) \geq w^{(1)}$. If $w^{(2)}(r) \leq w^{(1)}$, we put the points $x \in \Omega(r)$ onto the domains of Figure 1 constructed for $w^{(2)} = w^{(2)}(r)$. Otherwise, if $w^{(2)}(r) > w^{(1)}$, we put these points onto the domains of Figure 2.

Figures 3 and 4 were constructed in this way for the parameters $w^{(1)} = 1$, $R = 0.8$, $s = 0.75$ and $w^{(2)} = 1.8$ and 2. In Figure 3, two symmetric superiority sets of player E arise, the upper set being denoted by C_U and the lower set by C_L. If we increase $w^{(2)}$, the sets C_U and C_L expand and form a doubly connected region that is denoted by C_* in Figure 4. The number of roots of the equation $H(\ell, x) = 0$ is also given in Figures 3 and 4. A picture of the family $\Lambda^{(1),1}$ corresponding to the parameters of Figure 3 is given in [8].

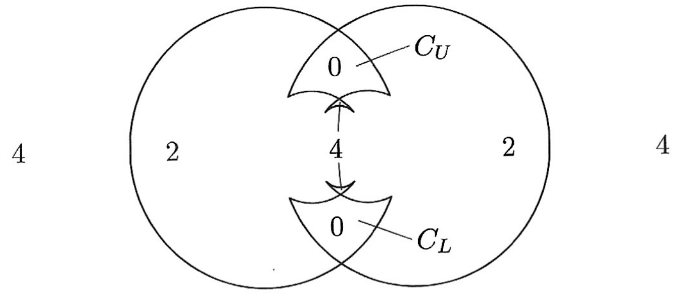

FIGURE 3. Superiority sets C_U and C_L of player E. Set $\mathcal{Q}$ depends on x; $w^{(2)} = 1.8$.

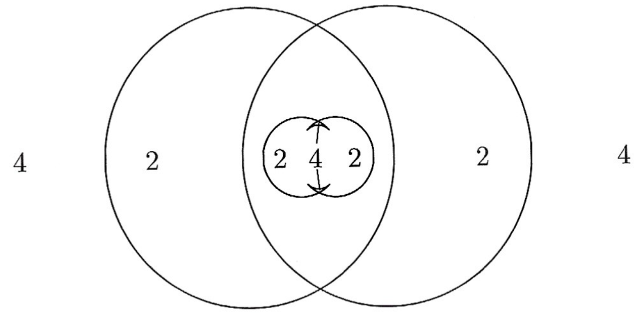

FIGURE 4. Superiority set C_* of player E. Set $\mathcal{Q}$ depends on x; $w^{(2)} = 2$.

4. Formation of Holes in Solvability Sets Due to Superiority Sets

The role of superiority sets in the appearance of holes within the solvability sets will be explained in this section. As noted above, in the case of problem 2.2, there can be one doubly connected superiority set C_* of player E, or two simply connected sets C_U and C_L, or the superiority set can be empty. In the case of problem 2.1, the superiority set of player E can be simply connected or empty.

4.1. Stable set $\hat{D}$

Let D be a closed set. Assume that the objective of player E is to bring the state of the system to the set D. Denote by $\hat{D}$ the solvability set (victory domain of player E) for this problem. It follows from the definition of $\hat{D}$ that E can bring the state of the system to D from any point $x \in \hat{D}$, but player P can prevent the state of the system from approaching the set D for any point $x \notin \hat{D}$. The boundary of $\hat{D}$ is composed of smooth semipermeable curves of the families $\Lambda^{(j),i}$. The sewing points possess the semipermeability property (see [2]). In some cases, a part of the boundary of $\hat{D}$ can coincide with a part of the boundary of D.

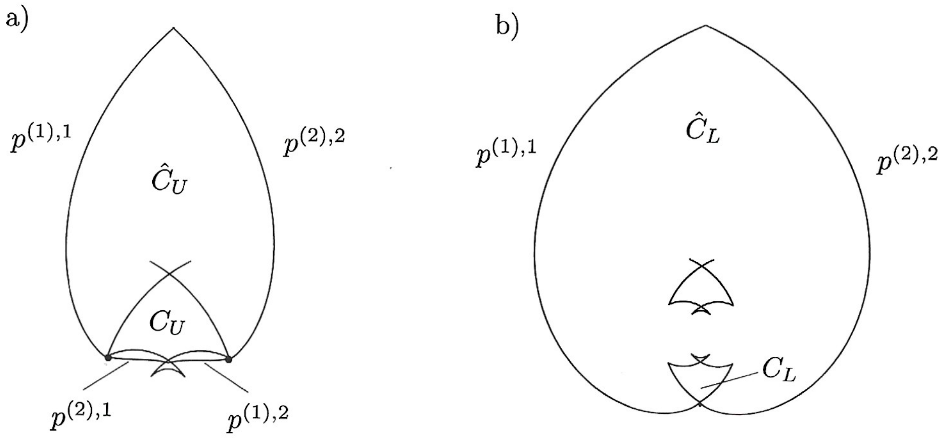

FIGURE 5. Construction of the sets $\hat{C}_U$ and $\hat{C}_L$ on the base of the superiority sets C_U and C_L.

Below, the set C_* or one of the sets C_U and C_L is used as the set D. Since in this case, D is a superiority set of E, it possesses the property of v-stability (see [5] for the definition) or, in other terms, the property of viability for E (see [1]), and the set $\hat{D}$ is v-stable too. This means [5] that player E can hold the trajectories of the system in $\hat{D}$ for infinite time. Hence, if $\hat{D} \bigcap M = \emptyset$, then the time for achieving the terminal set M in the main problem is infinite for any point x in $\hat{D}$. For this reason, level lines of the value function cannot "penetrate" into the set $\hat{D}$.

Due to the simple geometry of the sets D of the problems considered, the sets $\hat{D}$ can be obtained easily using the families of semipermeable curves. For example, Figure 5a presents the configuration of $\hat{C}_U$. The values of parameters correspond to Figure 3. The sewing point of the semipermeable curves $p^{(2),2}$ and $p^{(1),2}$ from the families $\Lambda^{(2),2}$ and $\Lambda^{(1),2}$, and symmetric to it sewing point of the curves $p^{(1),1}$ and $p^{(2),1}$ from the families $\Lambda^{(1),1}$ and $\Lambda^{(2),1}$, lie on the boundary of C_U. In Figure 5b, an example of the set $\hat{C}_L$ for the same values of parameters is given.

Since level lines of the value function cannot penetrate into the set $\hat{D}$ in the case $\hat{D} \bigcap M = \emptyset$, one can try to generate examples with holes in solvability sets using the knowledge of the geometry of the sets $\hat{D}$. We will show that the sets $\hat{C}_U$, but not the sets $\hat{C}_L$ or $\hat{C}_*$ can appear as holes.

4.2. Set $\hat{C}_L$ cannot be a hole

For the set C_L, a collection of expanding v-stable sets can be easily obtained. Figure 6a shows such a collection computed for the set C_L from Figures 3 and 5b. The first set of the collection is $\hat{C}_L$. The boundaries of the sets are formed by semipermeable curves $p^{(1),1}$ and $p^{(2),2}$.

Figure 6b shows the semipermeable curves that form the boundary of some set S from the above collection. The curve $p^{(1),1}$ corresponds to the control $\varphi = 1$,

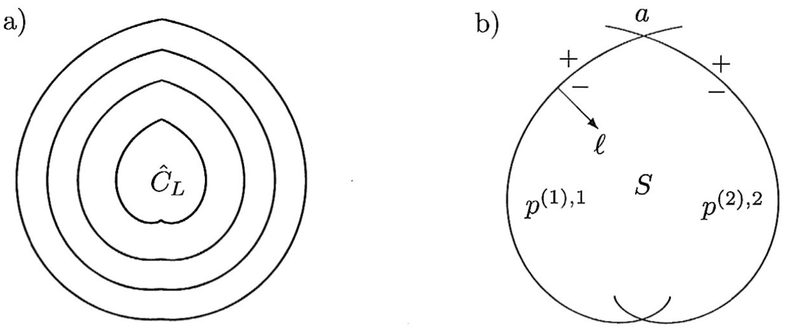

FIGURE 6. a: Collection of expanding v-stable sets for the set C_L.
b: Explanation of v-stability.

but the curve $p^{(2),2}$ corresponds to the control $\varphi = -1$. The sign "+" ("−") marks those sides of curves that player P (E) keeps. The curves $p^{(1),1}$ and $p^{(2),2}$ are faced with negative sides at the intersection point a. The property of v-stability means the following: for any $x \in \partial S$ and any $\varphi \in [-1,\ 1]$ there exists $v \in \mathcal{Q}(x)$ such that the vector $f(x,\ \varphi,\ v)$ is directed inside the set S or it is tangent to the boundary of S at x. For any point $x \in \partial S$ excluding the point a, a vector $v \in \mathcal{Q}(x)$ that gives the maximum in (2) would be appropriate. A normal vector to the curve in the negative side direction is considered as ℓ when computing the maximum in (2). For the point a, the choice of an appropriate v depends on φ.

Let us assume that there exists a hole $\hat{C}_L$ which is located strictly inside the solvability set. It follows from this assumption that: 1) $\hat{C}_L \bigcap M = \emptyset$, 2) for any boundary point x of $\hat{C}_L$, there exist points of the fronts that are arbitrarily close to x. Consider a v-stable set $\bar{S}$ from the expanding collection generated by the set C_L and such that $\bar{S}$ and M have common points on the boundaries of $\bar{S}$ and M only. Take a point x on a front strictly inside the set $\bar{S}$. Such a point exists because the set $\hat{C}_L$ belongs to the interior of the set $\bar{S}$ and the fronts come arbitrarily close to the set $\hat{C}_L$. Then, player E can keep the trajectories of the system within a set $\widetilde{S}$, which is a subset of $\bar{S}$ and contains the point x on its boundary, for infinite time. This contradicts to the fact that x lies on the front and, therefore, player P brings the system to M for a finite time.

Similar arguments are true for the sets $\hat{C}_*$ in the acoustic or classical game.

4.3. Set $\hat{C}_U$ can be a hole

We show that the set C_U cannot generate an expanding collection of v-stable sets.

Denote by $r^b = w^{(1)}s/w^{(2)}$ the minimal r for which $C_*(r) \neq \emptyset$. Consider the circle $F(\widetilde{r})$ of radius $\widetilde{r} = r^b/2$ with the center at the origin. We have

$$w^{(1)} - w^{(2)}(|x|) \geq w^{(1)} - w^{(2)}(\widetilde{r}) = w^{(1)}/2, \quad x \in F(\widetilde{r}). \tag{6}$$

Let $\xi(r) = -R + w^{(2)}(r)R/w^{(1)}, \quad r \geq 0.$

Since the set C_U is strictly above the axis x_1, then, for any $r \geq 0$, the set $C_*(r)$ does not contain the points of intersection of the circumference $\Omega(r)$ of radius r and the center at the origin with the axis x_1. Hence $r > \xi(r)$.

Let $x^{\#}(r,\ \alpha)$ and $x^{\diamondsuit}(r,\ \alpha)$ are right and left intersection points of the straight line $x_2 = \alpha$ with the circumference $\Omega(r)$, $0 \leq \alpha \leq \widetilde{r}$, $r \geq \widetilde{r}$. Using inequality $r > \xi(r)$, choose positive $\widetilde{\beta}$ and $\widetilde{\alpha}$, $\widetilde{\alpha} \leq \widetilde{r}$, so that $x_1^{\#}(r,\ \alpha) \geq \widetilde{\beta} + \xi(r)$, $r \geq \widetilde{r}$, $0 \leq \alpha \leq \widetilde{\alpha}$. We also obtain $x_1^{\diamondsuit}(r,\ \alpha) \leq -\widetilde{\beta} - \xi(r)$, $r \geq \widetilde{r}$, $0 \leq \alpha \leq \widetilde{\alpha}$.

Denote by $X(\alpha) = \{x :\ 0 < x_2 \leq \alpha\}$, $\alpha \leq \widetilde{\alpha}$, a horizontal strip of the width α over the axis x_1.

Using the inequality for $x_1^{\#}(r, \alpha)$, we obtain $x_1 \geq \widetilde{\beta} + \xi(|x|)$ for the points $x \in X(\widetilde{\alpha})$ on the right of the circle $F(\widetilde{r})$. Hence, it holds

$$\dot{x}_2|_{\varphi=-1} = -x_1 w^{(1)}/R + v_2 - w^{(1)} \leq -\widetilde{\beta} w^{(1)}/R + w^{(1)} -$$
$$w^{(2)}(|x|) + v_2 - w^{(1)} \leq -\widetilde{\beta} w^{(1)}/R \qquad (7)$$

for any $v \in \mathcal{Q}(x)$ and $\varphi = -1$. Similarly, using the inequality for $x_1^{\diamondsuit}(r, \alpha)$, we get

$$\dot{x}_2|_{\varphi=1} = x_1 w^{(1)}/R + v_2 - w^{(1)} \leq -\widetilde{\beta} w^{(1)}/R \qquad (8)$$

for $x \in X(\widetilde{\alpha})$ on the left of the circle $F(\widetilde{r})$, any $v \in \mathcal{Q}(x)$ and $\varphi = 1$.

If a point $x \in X(\widetilde{\alpha})$ belongs to the circle $F(\widetilde{r})$ and satisfies the inequality $x_1 \geq \xi(\widetilde{r})/2 = -R/4$, then we obtain

$$\dot{x}_2|_{\varphi=-1} = -x_1 w^{(1)}/R + v_2 - w^{(1)} \leq w^{(1)}/4 + v_2 - w^{(1)} \leq -w^{(1)}/4 \qquad (9)$$

for any $v \in \mathcal{Q}(x)$ and $\varphi = -1$. It was taken into account here that, using (6), the relation $|v_2| \leq w^{(2)}(|x|) \leq w^{(2)}(\widetilde{r}) = w^{(1)}/2$ holds for $x \in F(\widetilde{r})$. Similarly, if a point $x \in X(\widetilde{\alpha})$ belongs to the circle $F(\widetilde{r})$ and satisfies the inequality $x_1 \leq R/4$, then for any $v \in \mathcal{Q}(x)$ and $\varphi = 1$, we get

$$\dot{x}_2|_{\varphi=1} = x_1 w^{(1)}/R + v_2 - w^{(1)} \leq -w^{(1)}/4. \qquad (10)$$

Let $\widetilde{\gamma} = \min\{\widetilde{\beta} w^{(1)}/R,\ w^{(1)}/4\}$. Take positive $\overline{\alpha} \leq \min\{\widetilde{\alpha},\ \widetilde{r}/2\}$ such that

$$\overline{\alpha} w^{(2)}/\widetilde{\gamma} \leq \min\{R/4,\ \widetilde{r}/2\}. \qquad (11)$$

Put $\varphi \equiv -1$ for the states $x_0 \in X(\overline{\alpha})$ with $x_{01} \geq 0$. Taking into account (11) and the estimate $\dot{x}_1 = w^{(1)} x_2/R + v_1 \geq v_1 \geq -w^{(2)}$ for $x_2 \geq 0$, we obtain that any trajectory emanated from the point x_0 remains on the right side from the vertical straight line $x_1 = \max\{-R/4,\ -\widetilde{r}/2\}$ within the time $\overline{\alpha}/\widetilde{\gamma}$. Using (7) and (9), we get from here that the trajectory arrives at the axis x_1 within this time. Similarly, setting $\varphi = 1$ and using (8), (10) and (11), one obtains that any trajectory emanated from the point $x_0 \in X(\overline{\alpha}), x_{01} \leq 0$, arrives at the axis x_1 within the time $\overline{\alpha}/\widetilde{\gamma}$ remaining on the left side from the straight line $x_1 = \min\{R/4,\ \widetilde{r}/2\}$.

Thus player P can bring trajectories to the axis x_1 from any initial point x that belongs to the strip $X(\overline{\alpha})$. It follows from this property that $\hat{C}_U \bigcap X(\overline{\alpha}) = \emptyset$. Using the latter, one obtains that there is not any collection of v-stable sets that monotonically expands from the set $\hat{C}_U$ and fills out the whole plane.

5. Holes and the Boundary of Solvability Set

The results of the previous section show that holes located strictly inside solvability sets (victory domains) cannot be formed due to the sets $\hat{C}_*$ and $\hat{C}_L$. On the contrary, the set $\hat{C}_U$ being generated in a way similar to that for $\hat{C}_*$ and $\hat{C}_L$ can be a hole in the victory domain.

In the papers [1], [3], the acoustic game is considered for a terminal set M in the form of a rectangle $\{(x_1, x_2) : -3.5 \leq x_1 \leq 3.5,\ -0.2 \leq x_2 \leq 0\}$. Figure 7 presents level sets of the value function of the acoustic game with the above terminal set and the following values of parameters: $w^{(1)} = 1$, $w^{(2)} = 1.5$ and $s = 0.8$. The computation was done using an algorithm [7], [8] developed by the authors. One can see that in fact the hole coincides with the set $\hat{C}_U$.

It is emphasized in [1], [3] that the victory domain in similar examples with holes cannot be obtained using semipermeable curves (barriers) emitted from the boundary of the terminal set only. Now this conclusion can be formulated more precisely: the boundary of the victory domain is composed not only of semipermeable curves issued from the boundary of the terminal set but also of semipermeable curves emitted from the boundary of the set C_U.

If one increases $w^{(2)}$, the hole is being inflated and becomes "open" (see, for example, Figure 22 in [8]). The boundary of the victory domain transforms into

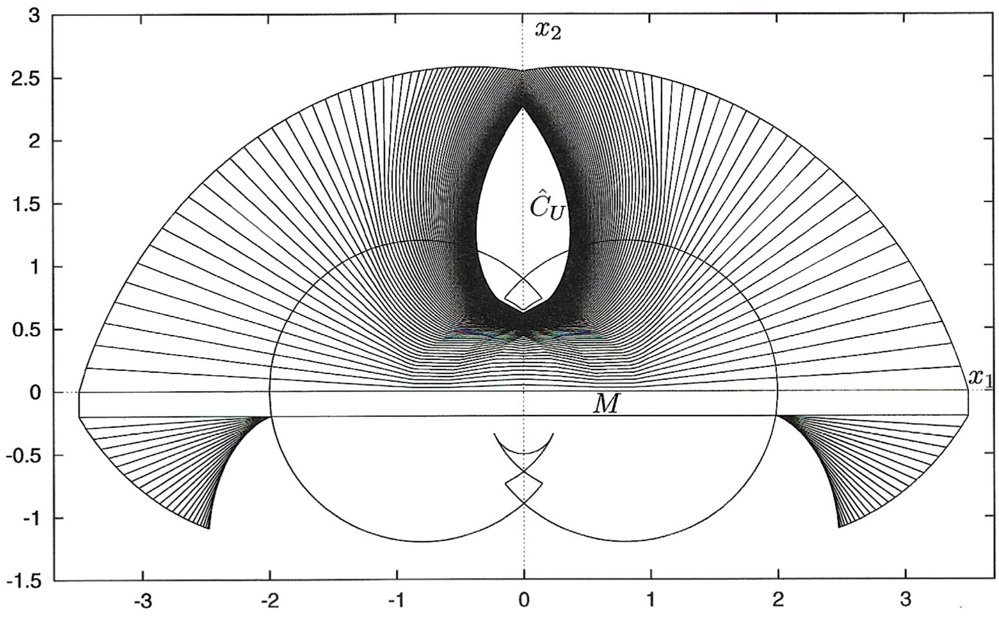

FIGURE 7. Level sets for $w^{(2)} = 1.5$; 746 upper fronts, 340 lower fronts, every 10th front is plotted.

a connected curve but even in this case, it is composed of semipermeable curves emitted both from the boundary of the terminal set and boundary of the set C_U.

The following question can be formulated. Does an example with the homicidal chauffeur dynamics exist where a hole, which is strictly inside the victory domain, does not coincide with the set $\hat{C}_U$? (In this paper, it is shown that such holes cannot coincide with the sets $\hat{C}_*$ and $\hat{C}_L$.)

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