Proceedings of the

Seventh International Colloquium on Differential Equations

Plovdiv, Bulgaria, 18 - 23 August, 1996

Editor: **D. Bainov**

///VSP///

UTRECHT, THE NETHERLANDS
TOKYO, JAPAN

Numerical solutions to the minimum-time problem for linear second-order conflict-controlled systems

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Abstract

The time-optimal problem for linear differential games in the plane is considered. An algorithm for constructing level sets of the value function is proposed. Numerical examples are presented.

Keywords: *Differential games; time-optimal control; value function.*

Introduction

In this paper, we consider differential games [1–3] with the linear dynamics and geometrical bounds on controls

$$\dot{x} = Ax + u + v, \quad x \in R^2, \ u \in P, \ v \in Q. \tag{1}$$

Here P and Q are convex closed polygons in the plane. The terminal set M (a convex polygon in the plane) is given. The first player who governs the control parameter u seeks to minimize the time of attaining M from some initial point m, the aim of the second player governing the control vector v is opposite. So, the payoff of the game is the time of attaining the set M. The permissible controls are feedback controls.

We are interested in finding $W(\theta, M)$, $\theta > 0$. Each of them is the set of all initial states x_0 such that the first player guarantees the transition of the state vector to M by the time θ. The set $W(\theta, M)$ is the level set (the Lebesgue set) of the value function of the minimum-time game problem.

In terms of [2, 3], the set $W(\theta, M)$ is also called t-section of the maximal u-stable bridge (the cross-section with the hyperplane $t = \theta$).

The paper is devoted to the numerical construction of $W(\theta, M)$.

In many cases, the sets $W(\theta, M)$ can be found only numerically (even for examples with very simple dynamics, for instance, $\dot{x}_1 = x_2 + v, \ \dot{x}_2 = u, \quad | u | \leq 1, \quad | v | \leq 1$).

The application of backward procedures is the typical way for solving control and differential game problems. General ideas of backward procedures were considered in papers of R.Bellman, R.Isaacs, W.Fleming, L.S.Pontryagin, and B.N.Pshenichny.

The most developed results [4–8] related to algorithmic implementations of the backward constructions to differential games were obtained for the case where a linear system should be brought to a convex terminal set M at a given time moment, and we are interested in finding the set of all states from which this transfer can be done. In this case, the application of the backward procedure gives t-sections of the maximal u-stable bridge. The algorithms use the property: the convexity of the target set implies the convexity of

t-sections of the maximal stable bridge. This makes the problem easier and enables to apply numerical methods to some important practical problems [9, 10].

The above mentioned feature is not inherent to differential games with the nonfixed time of termination: as a rule, t-sections of maximal stable bridges are not convex. Numerical methods for solving problems with the nonfixed time of termination and nonconvex problems with the fixed terminal time are studied in papers of V.N.Ushakov and his collaborators [11, 12]. Recently, numerical methods for constructing value functions and their level sets based on the notion of viscosity solutions of Hamilton-Jacobi (Bellman-Isaacs) equations were developed [13, 14].

The algorithm described below is based on the ideas of the algorithms proposed in [4] for linear differential games with the fixed time of termination.

Statement of the problem

We now define the set $W(\theta, M)$ more precisely [2, 3]. Let $\mathcal{U}$ be the set of all positional strategies U of the first player. Namely, this is the set of all functions defined on $[0, \theta] \times R^2$ and taking values in P. Let σ be an arbitrary partition of the segment $[0, \theta]$ formed by the points $0 = t_1 < t_2 < ... < t_n = \theta$, let $d(\sigma)$ be its diameter, and let $v(\cdot)$ be a measurable function of time with values in Q. Let $y(\cdot\,; \sigma, x_0, U, v(\cdot))$ denotes the Euler spline emanating from the point x_0. We denote by $W(\theta, M)$ the set of all points $x_0 \in R^2$ for each of which there exist a strategy $U \in \mathcal{U}$ and a mapping $\varepsilon \to \delta(\varepsilon)$ from R_+ to R_+ such that for any $\varepsilon > 0$, any σ with the diameter $d(\sigma) \leq \delta(\varepsilon)$, and any function $v(\cdot)$ with values in Q there exists a time $t \in [0, \theta]$ at which $y(t; \sigma, x_0, U, v(\cdot))$ belongs to the ε-neighborhood of M.

Such a definition is equivalent to other well-known definitions [3, 13, 14] of the solvability set $W(\theta, M)$ of the time-optimal game problem. We give this definition because it shows properties of the optimal guaranteeing strategy of the first player in terms of the bundle of motions generated by various controls of the second player.

The idea of the algorithm

The set $W(\theta, M)$ is formed via a step-by-step backward procedure giving a sequence of embedded sets

$$W(\Delta, M) \subset W(2\Delta, M) \subset W(3\Delta, M) \subset ... \subset W(i\Delta, M) \subset ... \subset W(\theta, M). \qquad (2)$$

Here Δ is the step of the backward procedure. Each set $W(i\Delta, M)$ consists of all initial points such that the first player brings system (1) into the set $W((i-1)\Delta, M)$ within the time duration Δ (we put $W(0, M) = M$).

Before doing the first step of the backward procedure, we find a usable part Γ_0 of the boundary of M. In accordance with [1], the usable part is a curve or several curves on the boundary of M attainable for trajectories of system (1) from points lying in the exterior of M close to the boundary of M. The usable part is defined by the following formula

$$\Gamma_0 = \mathrm{cl}\{x \in \partial M : \min_{u \in P} \max_{v \in Q} \langle \ell, Ax + u + v \rangle < 0, \ \forall \ell \in K_x\}.$$

Here K_x is the cone of outward normals to the set M at x. Since the target set is convex, each curve of the usable part is locally convex in the following sense: the normal to the curve at a point x keeps its rotation in only direction when x moves along the curve.

Let us introduce the term “front”. We put $F_0 = \Gamma_0$. The front F_i is the set of all points on the boundary of $W(i\Delta, M)$ for which the minimum guaranteeing time of the achievement of $W((i-1)\Delta, M)$ is equal to Δ. For other points of the boundary of $W(i\Delta, M)$ the

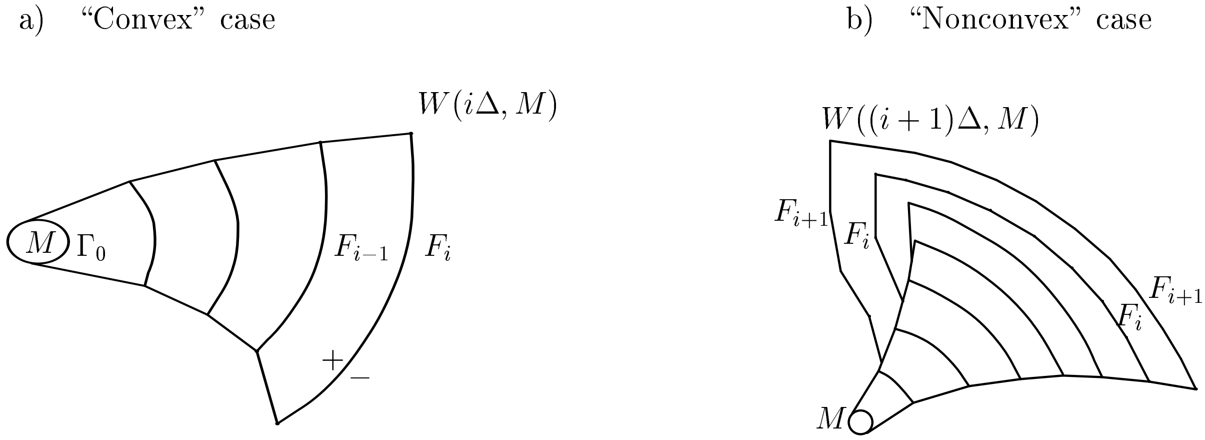

Figure 1. Construction of the sets $W(i\Delta, M)$.

optimal time is less than Δ. The line $\partial W(i\Delta, M) \setminus F_i$ possesses the properties of barriers [1]. The front F_i is designed using the previous front F_{i-1}.

Due to the linearity of system (1), the fronts F_1, $F_2, ...,$ $F_i, ...$ inherit (Figure 1a) the property of the local convexity of Γ_0 at the initial stage of constructions, and this property is kept until the current front F_i does not meet the set $W((i-1)\Delta, M)$. Straight lines connecting the endpoints of F_i with the corresponding endpoints of F_{i-1} form the continuation of the barrier lines. The boundary of $W(i\Delta, M)$ is formed by the front F_i, the above mentioned continuations of the barrier lines, and the line $\partial W((i-1)\Delta, M) \setminus F_{i-1}$. The property of the local convexity of fronts enable us to employ (with some small modifications) procedures which were developed for the construction of t-sections of maximal stable bridges in the case of linear differential games with the convex target set and the fixed time of termination [4].

If the next front F_i meets the already constructed set $W((i-1)\Delta, M)$, we say that the front collides with $W((i-1)\Delta, M)$. The situation of "collision" means that the current front meets the barrier part of the boundary of $W((i-1)\Delta, M)$ or the part $\partial M \setminus \Gamma_0$ of the boundary of M. To construct the next front F_{i+1}, we should take into account that F_i and the boundary of $W((i-1)\Delta, M)$ have the nonconvex conjunction (Figure 1b). Due to the properties of the plane, the complement of $W(i\Delta, M)$ is locally convex near the conjunction point. So, assuming that the second player seeks to bring the system to the complement, and the first player has the opposite objective, we can use the ideas of the "convex" algorithms from [4]. After combining the curve which is obtained from the locally convex part of the front with the one obtained from the nonconvex conjunction, we get a new front F_{i+1} that may be not locally convex.

So, the algorithm consists of the following operations: 1) finding the usable part on the boundary of the target set; 2) constructing the next front using the previous front; 3) testing the intersections of the current front with the barrier part of the already constructed set and the boundary of M. In the case the intersection is detected, further computations are being carried out taking into account the arising nonconvex conjunction and possible splitting the front into several parts.

Classification of singular lines

The analysis of singular lines where the value function and optimal trajectories have peculiarities is of great importance in differential games. The algorithm proposed can be provided with some simple diagnostics for finding and classification of singular lines and, in particular, for finding the most complicated among them equivocal [1, 15] lines.

In order to explain the possibility of the algorithmic classification of singular lines, we give some more detailed description of local constructions carried out in the process of finding the next front. The sets P, Q are assumed to be segments.

The front F_i is stored as an ordered collection of points. We consider each front as a polygonal line consisting of links defined by pairs of neighboring points. The normals to the links are directed to the front's outward side which is said to be negative (Figure 1). The opposite side of the front is called positive. We call an apex of the polygonal line to be the point of the local convexity if the angle between the positive sides of the neighboring links is less than π. We call this apex the point of the local concavity if the above mentioned angle is greater than π or equal to π.

Each link ab of the front F_i is assigned the segment with endpoints α, β which are defined by the formulas

$$\alpha = a - \Delta(A\,a + u^\circ + v^\circ), \quad \beta = b - \Delta(A\,b + u^\circ + v^\circ).$$

Here, u° and v° are extremal controls obtained from the following relations

$$u^\circ = arg\min_{u \in P} \langle \ell_{[ab]}, u \rangle, \quad v^\circ = arg\max_{v \in Q} \langle \ell_{[ab]}, v \rangle, \tag{3}$$

where $\ell_{[ab]}$ is the normal to the link ab. We say that the segment $\alpha\beta$ is generated by the segment ab with the use of extremal trajectories emanating from ab with the controls u°, v°. If the extremal controls of the players are the same for two neighboring links ab, bc, then the constructions done according to the above rule give four points, and two of these points coincide. So, we obtain two adjoint segments $\alpha\beta$ and $\beta\gamma$ which are generated by the links ab and bc. If the extremal controls defined by two neighboring normals do not coincide at least for one of the players, then we obtain two non-joint segments $\alpha\beta_1$ and $\beta_2\gamma$. In this case, some additional constructions should be done.

Let an apex b at which two neighboring links of the front F_i join be the point of the local convexity. The following cases are possible.

A. The extremal control of the second player is the same for the normals $\ell_{[ab]}$ and $\ell_{[bc]}$; the extremal control of the first player for $\ell_{[ab]}$ differs from that one for $\ell_{[bc]}$. In this case, two points β_1 and β_2 are associated with the point b. The segment with the endpoints β_1, β_2 is called the insertion due to P (Figure 2A).

B. The extremal control of the first player is the same for the normals $\ell_{[ab]}$ and $\ell_{[bc]}$; the extremal control of the second player for $\ell_{[ab]}$ differs from that one for $\ell_{[bc]}$. The segments $\alpha\beta_1$, $\beta_2\gamma$ generated by ab, bc intersect each other in a way shown in Figure 2B. Let ξ be the intersection point. We delete the parts $\beta_1\xi$, $\beta_2\xi$. There are two extremal trajectories that arrive at the point ξ. The first trajectory starts from the segment ab, the second one comes from bc. Assuming the motions go forward in time, we obtain that they disperse at the point ξ with respect to the segment $b\xi$.

C. The extremal controls of the first and the second players for $\ell_{[ab]}$ differ from those for $\ell_{[bc]}$. There are normals ℓ_P and ℓ_Q to P and Q that lie between $\ell_{[ab]}$, $\ell_{[bc]}$. Suppose that ℓ_Q lies between ℓ_P and $\ell_{[ab]}$. In this case, we add a segment $\beta_1\beta_3$ to the segment $\alpha\beta_1$ (Figures 2C_1, 2C_2). The endpoints β_1, β_3 are obtained from the point b using two different extremal controls of the first player and the extremal control of the second player corresponding to the normal $\ell_{[ab]}$. We call this additional segment the insertion due to P. We intersect the polygonal line $\alpha\beta_1\beta_3$ with the segment $\beta_2\gamma$. The intersection point ξ belongs either to $\beta_1\beta_3$ or $\alpha\beta_1$. We delete the parts $\beta_3\xi$, $\beta_2\xi$. If the point ξ belongs to $\alpha\beta_1$ (Figure 2C_1), then two

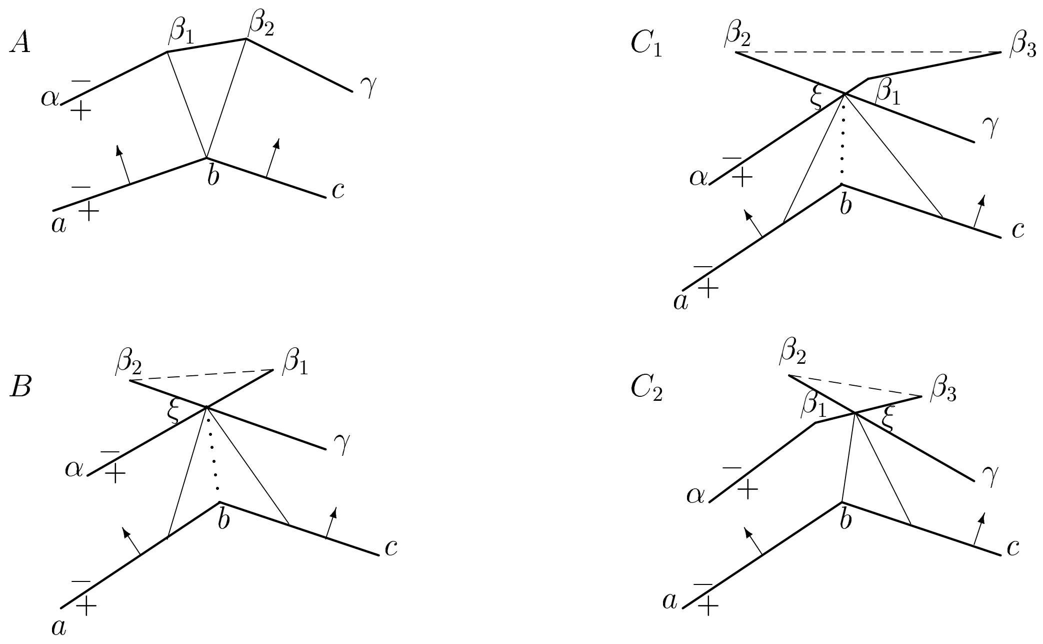

Figure 2. Constructions in the case of local convexity.

retrograde trajectories arrive at this point: one of them starts from ab, the other starts from bc. If ξ belongs to the insertion due to P (Figure $2C_2$), then one trajectory comes to ξ from the segment bc and the other trajectory comes from the point b. The latter is not extremal because it is obtained with some not extremal control of the first player ensuring the transfer to the point ξ and the extremal control of the second player corresponding to $\ell_{[ab]}$. Considering trajectories in the forward time, we obtain two trajectories emanating from ξ: one of them is extremal and the other one is not extremal.

If the normal ℓ_Q lies between $\ell_{[ab]}$ and ℓ_P, the difference is that the insertion due to P adjoins the segment $\beta_2\gamma$ generated by bc.

Practically, the algorithm we use is more complicated than the local constructions described above. Namely, we intersect polygonal lines consisting of a large number of links. Links of fronts can get smaller. Nevertheless, the algorithm does not require to decrease the step Δ in accordance with the length of links.

The classification of singular lines is done as follows.

In the case A, we have two points β_1, β_2 of the front F_{i+1} which give the insertion due to P. One of these two points (any of them) is called a switch point of the first player.

In the case B, the intersection point ξ is called a dispersal point of the second player. This name characterizes the fact that two optimal forward time trajectories obtained with different controls of the second player emanate from such a singular point. Each of these trajectories is the extremal trajectory; one of them goes to the region where the optimal control of the second player takes the first of two extremal values; the other trajectory goes to the region where the optimal control of the second player takes the second extremal value.

The intersection point ξ which appears in the case C is the dispersal point of the second player if it does not belong to the insertion due to P. Two optimal trajectories which are both extremal emanate from this point. The intersection point is called an equivocal point of the second player if it belongs to the insertion due to P. In this case, two trajectories emanate from such a point: the first one is extremal and the second one is not extremal.

It depends on the behavior of the second player which of these two trajectories is realized.

Some similar classification of singular points can be done in the case of the local concavity.

As a result, we obtain a collection of singular lines (switch lines, dispersal lines, equivocal lines) after finishing computations.

Examples

In this section, numerical examples of computing the sequence $\{W(i\Delta, M)\}$ are given. The step Δ was equal to 0.05 in all cases. The terminal set M was a small regular octagon with the center at the origin in the examples 1, 3, and with the center at the point $(-2.2, 0.4)$ in the example 2.

1. The well-known example of the time-optimal problem for oscillating systems in the theory of optimal control [16] has the form

$$\begin{aligned} \dot{x}_1 &= \quad x_2 \\ \dot{x}_2 &= \quad -x_1 + u, \quad | u | \leq 1. \end{aligned} \tag{4}$$

The sets $W(\tau, M)$, $\tau = 2\Delta k$, $k = \overline{1, 80}$, are depicted in Figure 3. Note, that $W(\tau, M)$ is convex for any τ.

2. Now, consider the following differential game:

$$\begin{aligned} \dot{x}_1 &= \quad x_2 + u_1 + v_1 \\ \dot{x}_2 &= \quad -x_1 + u_2 + v_2 \end{aligned} \tag{5}$$

$$u = (u_1, u_2)' \in P, \quad v = (v_1, v_2)' \in Q$$

whose dynamics is similar to (4). The set P is the vertical segment with the endpoints $(0, -2.5)$, $(0, 2.5)$, and Q is the segment with the apexes $(-5, 1.5)$, $(-1, -1.5)$. In Figure 4a, the sets $W(\tau, M)$, $\tau = i\Delta$, $i = \overline{1, 132}$ are depicted. The computations are carried out up to $\tau = 6.6$. At $\tau = 6.6$, the front collides with the terminal set M and is divided into two parts. Further constructions are being done independently for these two parts. The computations for the upper part are continued till $\tau = 11.6$ in Figure 4b, and we fill up the gap G. The front which corresponds to the maximal $\tau = 11.6$ lies approximately in the middle of G. In Figure 4b, only two fronts constructed from the lower part are shown. The accumulation of fronts generates black regions in Figure 4, which means very fast changing the value function (though it is continuous).

3. Figure 5 corresponds to the following system

$$\begin{aligned} \dot{x}_1 &= \quad 0.35x_1 + x_2 + v \\ \dot{x}_2 &= \quad -0.8x_1 + u \end{aligned} \tag{6}$$

$$-2 \leq u \leq 1.5, \quad -6.1 \leq v \leq -4.$$

The level sets $W(\tau, M)$ are computed for $\tau = i\Delta$, $i = \overline{1, 189}$. Up to $\tau = 5.7$, the front moves between two barriers emanating from the set M. The left barrier terminates at $\tau = 5.7$. For $\tau > 5.7$, the front begins to go around this barrier so that one of its endpoints slides along the outward side of the barrier. At $\tau = 8.15$, the front collides with some early part of the left barrier from outside. For $\tau > 8.15$, the left and right endpoints of the front move towards each other along the left barrier. The computation is finished at $\tau = 9.45$.

In this example, the set filled up with the fronts computed for $\tau \leq 9.45$ is the maximal set where the optimal guaranteeing time is less than infinity. The first player can not

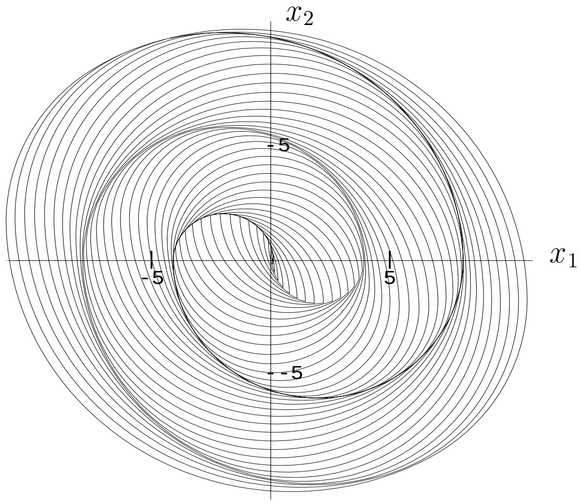

Figure 3. Canonical control problem (4).

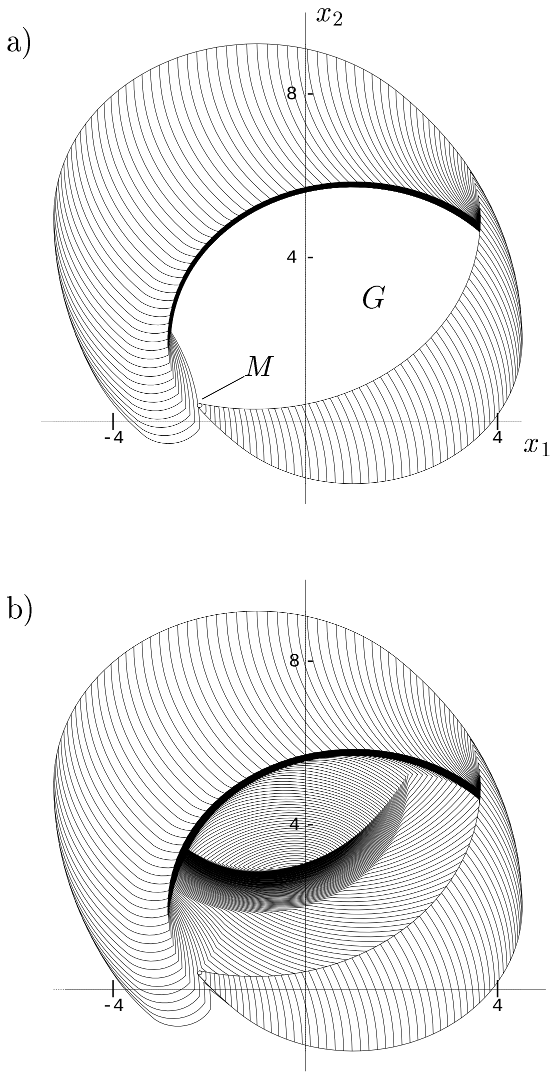

Figure 4. Level sets of value function in problem (5). Filling the gap G.

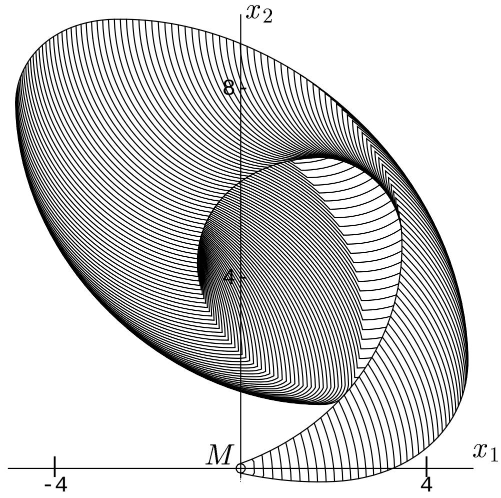

Figure 5. Level sets of value function in problem (6).

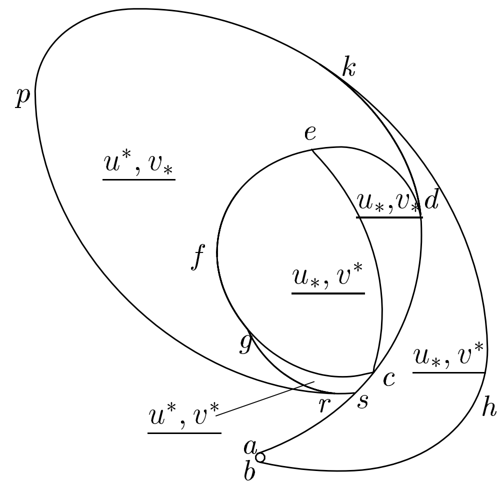

Figure 6. Structure of first and second player's optimal controls.

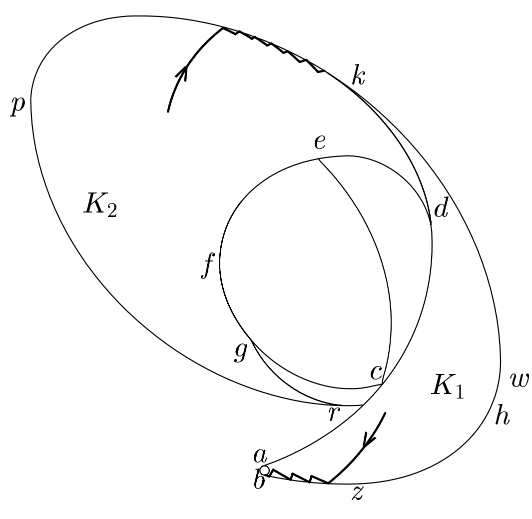

Figure 7. Sliding the motions along barrier lines.

guarantee the transferring to M from initial points lying outside this set within a finite time.

In [17], a number of numerical examples computed for the minimum-time second-order problem with different eigenvalues of the matrix A are given.

Optimal feedback control and optimal trajectories

The problem of finding optimal controls is rather independent task. Let us demonstrate the possibility of constructing optimal controls in the example (6).

The singular lines for the game (6) are depicted in Figure 6.

The barrier line $acdef$ terminates at the point f. After that, it is continued by the equivocal line fg which splits into the switch line gc of the first player and the switch line gr of the second player at the point g. The curve $bhkprs$ is the barrier, the curve dk is the equivocal line, and the curve ec is the switch line of the second player. The singular lines listed above divide the set where the problem has a solution into subsets so that the optimal controls of the players take constant values in the interiors of the subsets. These constant values are equal to the minimal and maximal admissible values of the control parameters: $u_* = -2,\ u^* = 1.5,\ v_* = -6.1,\ v^* = -4$. On the boundaries and near the boundaries of the subsets, the optimal controls are defined in a special manner.

The feedback control we specified is optimal in the interiors of the subsets (Figure 6). If the boundary of some subset contains a part of a barrier line, it can require the first and the second player to control in a special manner near this line to provide the time of termination to be close to the value of the game. The value function is discontinuous on barriers. The side of a barrier faced to the set where the value function is smaller is said to be positive. The opposite side is called negative.

Let us consider, for example, the set bounded by the curves $ascdk$ and $abhk$. Denote it by K_1 (Figure 7). In the interior of K_1, the optimal controls are $u = u_*$, $v = v^*$. The boundary of K_1 involves the arcs acd and bhk of the barriers.

The barrier acd is the trajectory of the system with $u = u_*$, $v = v_*$. The control u_* of the first player prevents trajectories from penetrating from the positive side to the negative side for all admissible controls of the second player. The control v_* of the second player prevents trajectories from penetrating from the negative side to the positive one for all admissible controls of the first player. According to the terminology of R.Isaacs [1], the controls $u = u_*$, $v = v_*$ ensure the property of semipermeability. Since the protective control of the first player coincides with the control value which is optimal in the interior of K_1, the trajectory generated by the optimal control of the first player can not come to the arc acd from the interior of K_1. So, in this case, the first player need not operate in a special manner near this arc.

The arc bhk is divided by the point w into two parts so that the protective control is $u = u^*$ for the arc bw and $u = u_*$ for wk. Near wk, the first player does not need any special way of control.

If a trajectory comes to the curve bw (or it lies very close to bw), the first player must change his control $u = u_*$ for $u = u^*$. After the changing, the trajectory can go back to the interior of K_1, then the first player switches his control to $u = u_*$, the trajectory comes to the curve bw again, and so on. As a result, a sliding mode which is schematically shown in Figure 7 occurs. Decreasing the step of the control choice of the first player, we obtain a trajectory which slides along the curve bw. Limit trajectories go along bw towards the point b. The "slowest" trajectory among them delivers the value of the game.

The arc bw is divided by the point z into two parts. For the part bz, the slowest trajectory goes with $v = v_*$, whereas the control u depends on the current point on this curve and takes values from the interval $(u_*,\ u^*)$. The motion along the arc zw is feasible for $v = v^*$ only. This value is protective for the second player. In this case, the optimal limit trajectory is obtained with $u = u^*$, $v = v^*$.

Similarly, one can consider the behavior of the first player near the barrier $kprs$ which lies on the boundary of the set K_2 contoured by the curve $kprgfedk$. Here we also have a part of the barrier which lie near the point k and for which the protective control of the first player does not coincide with the optimal control $u = u^*$ inside K_2. After arriving at this part, the first player must change his control for $u = u_*$. As a result, we obtain a sliding motion near the barrier towards the point k. The optimal limit trajectory goes exactly along the barrier. For the barrier fed, the protective control of the first player is $u = u^*$. So, the first player need not handle in a special manner near fed.

Since the second player must prevent trajectories from penetrating from the negative side of a barrier to the positive one, it can require from the second player a special way to form his control when trajectories approach the negative side of the barrier.

In Figure 8, optimal limit trajectories emanating from the initial point $x_0 \in K_3$ are shown. At the initial stage, the optimal trajectory goes with the controls u_*, v^* and it is unique. After approaching the equivocal line fg, the optimal trajectory splits into two trajectories: one goes along the equivocal line fg, the other goes to the region K_2 where $u = u^*$, $v = v_*$ are optimal. The equivocal line ends at the point f, so, there exists a trajectory going along this line up to the point f and leaving this line at the point f. Three trajectories leaving fg are shown in Figure 8. Each of them approaches the barrier pk, then goes along pk and arrives at the point k. After that, they go along the equivocal line kd, and they can bifurcate at each point of kd. After the bifurcation, one of the trajectories continues to go along the equivocal line, the other leaves this line. In Figure 8, two trajectories leaving kd are shown. Moving in the interior of K_1, where $u = u_*$, $v = v^*$ are optimal, these trajectories approach the barrier bw and go along bw.

All trajectories described are optimal. The dark region in Figure 9 is filled with the optimal trajectories starting from the point x_0. The splitting occurs on the equivocal lines fg and kd.

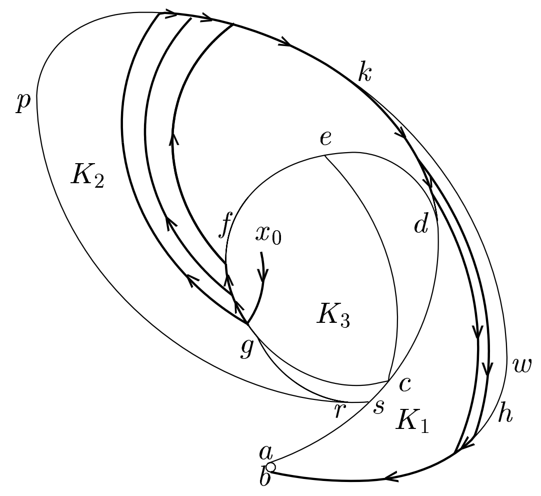

Figure 8. Splitting the optimal trajectories on equivocal lines.

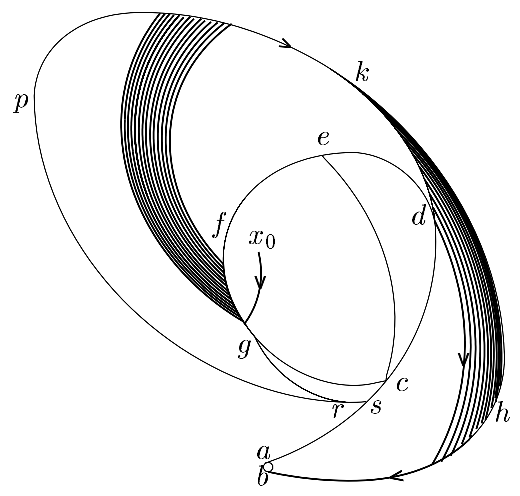

Figure 9. The set filling by optimal trajectories from $x_0 \in K_3$.

Summarizing, the principle of the construction of optimal trajectories is the following. When constructing optimal trajectories outside singular lines, we take into account that optimal controls are determined from the relations (3) with $\ell_{[ab]}$ replaced by the normal to the front at the current point. When specifying trajectories which go along singular lines, we take into account the behavior of trajectories on each of singular lines.

Conclusion

Effective algorithms can be created for solving differential games in the plane. In this paper, a general scheme of the algorithm for linear time-optimal differential games is considered. Possibility of the classification of singular lines in the computation process is demonstrated. Examples of computing level sets of the value function are given. Also, examples that serve for the explanation of the behavior of optimal trajectories are presented.

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