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Chapter 14
Study of Linear Game with Two Pursuers and One Evader: Different Strength of Pursuers

Sergey A. Ganebny, Sergey S. Kumkov, Stéphane Le Ménec, and Valerii S. Patsko

Abstract The paper deals with a problem of pursuit-evasion with two pursuers and one evader having linear dynamics. The pursuers try to minimize the final miss (an ideal situation is to get exact capture), the evader counteracts them. Results of numerical construction of level sets (Lebesgue sets) of the value function are given. A feedback method for producing optimal control is suggested. The paper includes also numerical simulations of optimal motions of the objects in various situations.

Keywords Game theory • Differential games • Group pursuit-evasion games • Maximal stable bridges • Numerical schemes for differential games

14.1 Introduction

Group pursuit-evasion games (several pursuers and/or several evaders) are studied intensively in the theory of differential games [2, 4, 6, 7, 11, 16, 17, 20].

From a general point of view, a group pursuit-evasion game (without any hierarchy among players) can be often treated as an antagonistic differential game where all pursuers are joined into one player, whose objective is to minimize some functional, and, similarly, all evaders are joined into another player, who is the opponent to the first one. The theory of differential games gives an existence theorem for the value function of such a game. But, usually, any more concrete results

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(for example, concerning effective construction of the value function) cannot be obtained. This is due to high dimension of the state vector of the corresponding game and absence of convexity of time sections of level sets (Lebesgue sets) of the value function. Just these reasons can explain why group pursuit-evasion games are very difficult and are usually investigated by means of specific methods and under very strict assumptions.

In this paper, we consider a pursuit-evasion game with two pursuers and one evader. Such a model formulation arises from analysis of an applied problem where two aircrafts (or missiles) intercept another one in the horizontal plane. The peculiarity of the game explored in the paper is that solvability sets (the sets wherefrom the interception can be guaranteed with a miss which is not greater than some given value) and optimal feedback controls of the objects can be built numerically in the framework of a one-to-one antagonistic game. Such an investigation is the aim of this paper.

The paper is based on the problem formulation suggested in [12, 13]. In these works, a case is studied when each pursuer is "stronger" than the evader. In our paper, we research the game without this assumption.

14.2 Formulation of Problem

In Fig. 14.1, one can see a possible initial location of the pursuers and evader when they move towards each other. Also, the evader can move from both pursuers, or from one of them, but towards another pursuer.

Let us assume that the initial velocities are parallel and quite large, and control accelerations affect only lateral components of object velocities. Thus, one can suppose that instants of passages of the evader by each of the pursuers are fixed. Below, we call them termination instants and denote by T_{f1} and T_{f2}, respectively. We consider both cases of equal and different termination instants. The players' controls define the lateral deviations of the evader from the first and second pursuers at the termination instants. Minimum of absolute values of these deviations is called *the resulting miss*. The objective of the pursuers is minimization of the resulting miss, the evader maximizes it. The pursuers generate their controls by a coordinated effort (from one control center).

Fig. 14.1 Schematic initial positions of the pursuers and evader

In the relative linearized system, the dynamics is the following (see [12, 13]):

$$\begin{aligned} \ddot{y}_1 &= -a_{P1} + a_E, & \ddot{y}_2 &= -a_{P2} + a_E, \\ \dot{a}_{P1} &= (A_{P1}u_1 - a_{P1})/l_{P1}, & \dot{a}_{P2} &= (A_{P2}u_2 - a_{P2})/l_{P2}, \\ \dot{a}_E &= (A_E v - a_E)/l_E. \end{aligned} \tag{14.1}$$

Here, y_1 and y_2 are the current lateral deviations of the evader from the first and second pursuers; a_{P1}, a_{P2}, a_E are the lateral accelerations of the pursuers and evader; u_1, u_2, v are the players' command controls; A_{P1}, A_{P2}, A_E are the maximal values of the accelerations; l_{P1}, l_{P2}, l_E are the time constants describing the inertiality of servomechanisms.

The controls have bounded absolute values:

$$|u_1| \le 1, \quad |u_2| \le 1, \quad |v| \le 1.$$

The linearized dynamics of the objects in the problem under consideration is typical (see, for example, [19]).

Consider new coordinates x_1 and x_2 which are the values of y_1 and y_2 forecasted to the corresponding termination instants T_{f1} and T_{f2} under zero players' controls. One has

$$x_i = y_i + \dot{y}_i \tau_i - a_{Pi} l_{Pi}^2 h(\tau_i / l_{Pi}) + a_E l_E^2 h(\tau_i / l_E), \quad i = 1, 2.$$

Here, x_i, y_i, a_{Pi}, and a_E depend on t, and

$$\tau_i = T_{fi} - t, \quad h(\alpha) = \mathrm{e}^{-\alpha} + \alpha - 1.$$

We have $x_i(T_{fi}) = y_i(T_{fi})$.

Passing to a new dynamics in "equivalent" coordinates x_1 and x_2 (see [12, 13]), we obtain

$$\begin{aligned} \dot{x}_1 &= -A_{P1} l_{P1} h(\tau_1 / l_{P1}) u_1 + A_E l_E h(\tau_1 / l_E) v, \\ \dot{x}_2 &= -A_{P2} l_{P2} h(\tau_2 / l_{P2}) u_2 + A_E l_E h(\tau_2 / l_E) v. \end{aligned} \tag{14.2}$$

Join both pursuers $P1$ and $P2$ into one player which will be called the *first player*. The evader E is the *second player*. The first player governs the controls u_1 and u_2; the second one governs the control v. We introduce the following payoff functional:

$$\varphi\big(x_1(T_{f1}), x_2(T_{f2})\big) = \min\big(\left|x_1(T_{f1})\right|, \left|x_2(T_{f2})\right|\big). \tag{14.3}$$

It is minimized by the first player and maximized by the second one. Thus, we get a standard antagonistic game with dynamics (14.2) and payoff functional (14.3). This game has [1, 8–10] the value function $V(t,x)$, where $x = (x_1, x_2)$. For each initial position (t_0, x_0), the value $V(t_0, x_0)$ of the value function V equals the payoff guaranteed for the first (second) player by its optimal feedback control. Each level set

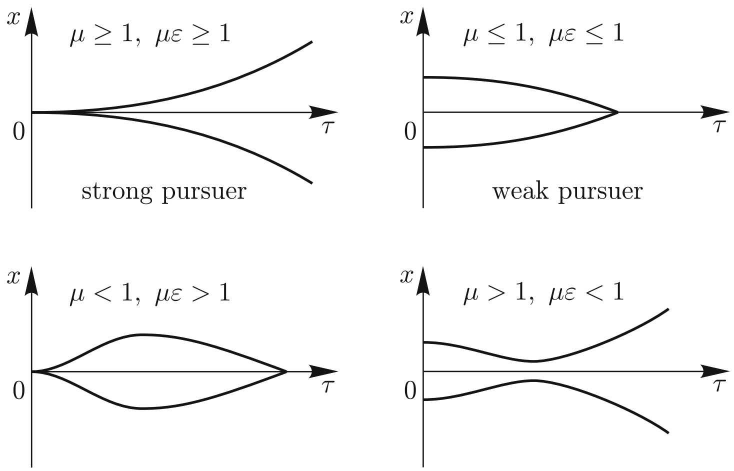

Fig. 14.2 Various variants of the stable bridge evolution in an individual game

$$W_c = \{(t,x) : V(t,x) \leq c\}$$

of the value function coincides with the maximal stable bridge (see [9, 10]) built from the target set

$$M_c = \{(t,x) : t = T_{f1},\ |x_1| \leq c\} \cup \{(t,x) :\ t = T_{f2},\ |x_2| \leq c\}.$$

The set W_c can be treated as the solvability set for the pursuit-evasion game with the result c.

When $c = 0$, we have the situation of the exact capture. The exact capture implies equality to zero of at least one of y_i at the instant T_{fi}, $i = 1, 2$.

The works [12, 13] consider only cases with the exact capture, and pursuers "stronger" than the evader. The latter means that the parameters A_{Pi}, A_E, and l_{Pi}, l_E ($i = 1, 2$) are such that the maximal stable bridges in the individual games ($P1$ vs. E and $P2$ vs. E) grow monotonically in the backward time.

Considering individual games of each pursuer vs. the evader, one can introduce parameters [18] $\mu_i = A_{Pi}/A_E$ and $\varepsilon_i = l_E/l_{Pi}$. They and only they define the structure of the maximal stable bridges in the individual games. Namely, depending on values of μ_i and $\mu_i\varepsilon_i$, there are four cases of the bridge evolution (see Fig. 14.2):

- Expansion in the backward time (a strong pursuer)
- Contraction in the backward time (a weak pursuer)
- Expansion of the bridge until some backward time instant and further contraction
- Contraction of the bridge until some backward time instant and further expansion (if the bridge still has not broken).

Respectively, given combinations of pursuers' capabilities and individual games durations (equal/different), there are significant number of variants for the problem with two pursuers and one evader. Some of them are considered below.

The main objective of this paper is to construct the sets W_c for typical cases of the game under consideration. The difficulty of the problem is that the *time sections* $W_c(t)$ of these sets are non-convex. Constructions are made by means of an algorithm for constructing maximal stable bridges worked out by the authors for problems with two-dimensional state variable. The algorithm is similar to one used in [15]. Another objective is to build optimal feedback controls of the first player (that is, of the pursuers $P1$ and $P2$) and the second one (the evader E).

14.3 Idea of Numerical Method

As it was mentioned above, a level set W_c of the value function is the maximal stable bridge for dynamics (14.2) built in the space t, x from the target set M_c. A time section $W_c(t)$ of the bridge W_c at the instant t is a set in the plane of two-dimensional variable x.

To be definite, let $T_{f1} \geq T_{f2}$. Then for any $t \in (T_{f2}, T_{f1}]$, the set $W_c(t)$ is a vertical strip around the axis x_2. Its width along the axis x_1 equals the width of the bridge in the individual game $P1$ vs. E at the instant $\tau = T_{f1} - t$ of the backward time. At the instant $t = T_{f1}$, the half-width of $W_c(T_{f1})$ is equal to c.

Denote by $W_c(T_{f2}+0)$ the right limit of the set $W_c(t)$ as $t \to T_{f2}+0$. Then the set $W_c(T_{f2})$ is cross-like obtained by union of the vertical strip $W_c(T_{f2}+0)$ and a horizontal one around the axis x_1 with the width equal $2c$ along the axis x_2.

When $t \leq T_{f2}$, the backward construction of the sets $W_c(t)$ is made starting from the set $W_c(T_{f2})$.

The algorithm which is suggested by the authors for constructing the approximating sets $\widetilde{W}_c(t)$, uses a time grid in the interval $[0, T_{f1}]$: $t_N = T_{f1}$, t_{N-1}, $\ldots$, $t_S = T_{f2}$, t_{S-1}, t_{S-2}, $\ldots$. For any instant t_k from the taken grid, the set $\widetilde{W}_c(t_k)$ is built on the basis of the previous set $\widetilde{W}_c(t_{k+1})$ and a dynamics obtained from (14.2) by fixing its value at the instant t_{k+1}. So, dynamics (14.2) which varies in the interval $(t_k, t_{k+1}]$ is changed by a dynamics with simple motions [8]. The set $\widetilde{W}_c(t_k)$ is regarded as a collection of all positions at the instant t_k, wherefrom the first player guarantees guiding the system to the set $\widetilde{W}_c(t_{k+1})$ under "frozen" dynamics (14.2) and discrimination of the second player, that is, when the second player announces its constant control v, $|v| \leq 1$, in the interval $[t_k, t_{k+1}]$.

Due to symmetry of dynamics (14.2) and the sets $W_c(T_{f1})$, $W_c(T_{f2})$ with respect to the origin, one gets that for any $t \leq T_{f1}$ the time section $W_c(t)$ is symmetric also.

Up to now, different workgroups suggested many algorithms for constructing the value function in differential games of quite generic type (see, for example, [3, 5, 14, 21]). The problem under consideration has linear dynamics and the second order on the phase variable. Due to this, we use a specific method. This allows us to make very fast computations of many variants of the game.

14.4 Strong Pursuers, Equal Termination Instants

Add dynamics (14.2) by a "cross-like" target set

$$M_c = \{|x_1| \le c\} \cup \{|x_2| \le c\}, \quad c \ge 0,$$

at the instant $T_f = T_{f1} = T_{f2}$. Then we get a standard linear differential game with fixed termination instant and non-convex target set. The collection $\{W_c\}$ of maximal stable bridges describes the value function of the game (14.2) with payoff functional (14.3).

For the considered case of two stronger pursuers, choose the following parameters:

$$\begin{aligned} &A_{P1} = 2, && A_{P2} = 3, && A_E = 1, \\ &l_{P1} = 1/2, && l_{P2} = 1/0.857, && l_E = 1, \\ &T_{f1} = T_{f2} = 6. \end{aligned}$$

1. *Structure of maximal stable bridges.* Figure 14.3 shows results of constructing the set $W = W_0$ (that is, with $c = 0$). In the figure, one can see several time sections $W(t)$ of this set. The bridge has a quite simple structure. At the initial instant $\tau = 0$ of the backward time (when $t = 6$), its section coincides with the

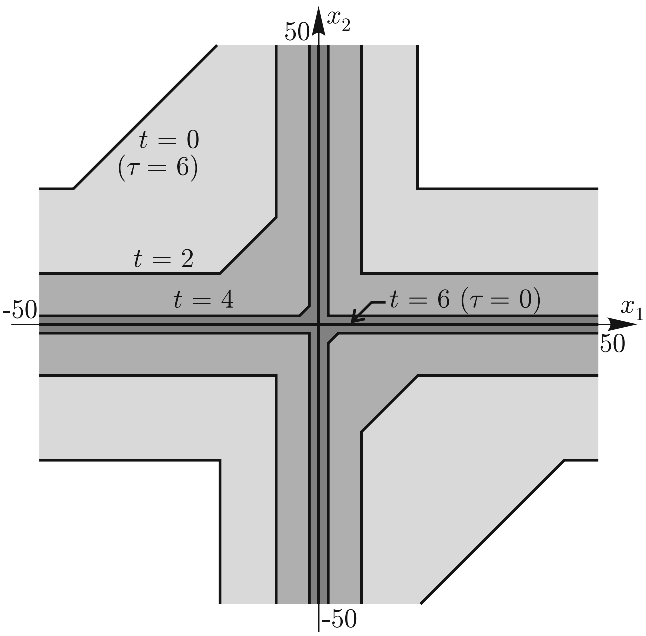

Fig. 14.3 Two strong pursuers, equal termination instants: time sections of the bridge W

target set M_0 which is the union of two coordinate axes. Further, at the instants $t = 4, 2, 0$, the cross thickens, and two triangles are added to it. The widths of the vertical and horizontal parts of the cross correspond to sizes of the maximal stable bridges in the individual games with the first and second pursuers. These triangles are located in the II and IV quadrants (where the signs of x_1 and x_2 are different, in other words, when the evader is between the pursuers) and give the zone where the capture is possible only under collective actions of both pursuers (trying to avoid one of the pursuer, the evader is captured by another one).

These additional triangles have a simple explanation from the point of view of problem (14.1). Their hypotenuses have slope equal to 45°, that is, are described by the equation $|x_1| + |x_2| = \text{const}$. Consider the instant τ when the hypotenuse reaches a point (x_1, x_2). It corresponds to the instant when the pursuers cover together the distance $|x_1(0)| + |x_2(0)|$ which is between them at the initial instant $t = 0$. Therefore, at this instant, both pursuers come to the same point. Since the evader was initially between the pursuers, it is captured at this instant.

The set W (maximal stable bridge) built in the coordinates of system (14.2) coincides with the description of the solvability set obtained analytically in [12, 13]. The solvability set for system (14.1) is defined as follows: if in the current position of system (14.1) at the instant t, the forecasted coordinates x_1, x_2 are inside the time section $W(t)$, then under the controls u_1, u_2 the motion is guided to the target set M_0; on the contrary, if the forecasted coordinates are outside the set $W(t)$, then there is an evader's control v which deviates system (14.2) from the target set. Therefore, there is no exact capture in the original system (14.1).

Time sections $W_c(t)$ of other bridges W_c, $c > 0$, have the shape similar to $W(t)$. In Fig. 14.4, one can see the sections $W_c(t)$ at $t = 2$ ($\tau = 4$) for a collection $\{W_c\}$ corresponding to some series of values of the parameter c. For other instants t, the structure of the sections $W_c(t)$ is similar. The sets $W_c(t)$ describe the value function $x \to V(t,x)$.

2. *Feedback control of the first player.* Rewrite system (14.2) as

$$\dot{x} = \mathcal{D}_1(t)u_1 + \mathcal{D}_2(t)u_2 + \mathcal{E}(t)v,$$
$$|u_1| \le 1, \ |u_2| \le 1, \ |v| \le 1.$$

Here, $x = (x_1, x_2)$; vectors $\mathcal{D}_1(t)$, $\mathcal{D}_2(t)$, and $\mathcal{E}(t)$ look like

$$\mathcal{D}_1(t) = \big(-A_{P1}l_{P1}h((T_{f1} - t)/l_{P1}), 0\big), \quad \mathcal{D}_2(t) = \big(0, -A_{P2}l_{P2}h((T_{f2} - t)/l_{P2})\big),$$
$$\mathcal{E}(t) = \big(A_E l_E h((T_{f1} - t)/l_E), A_E l_E h((T_{f2} - t)/l_E)\big).$$

We see that the vector $\mathcal{D}_1(t)$ ($\mathcal{D}_2(t)$) is directed along the horizontal (vertical) axis; when $T_{f1} = T_{f2}$, the angle between the axis x_1 and the vector $\mathcal{E}(t)$ equals 45°; when $T_{f1} \ne T_{f2}$, the angle changes in time.

Analyzing the change of the value function V along a horizontal line in the plane x_1, x_2 for a fixed instant t, one can conclude that the minimum of the function is reached in the segment of intersection of this line and the set $W(t)$.

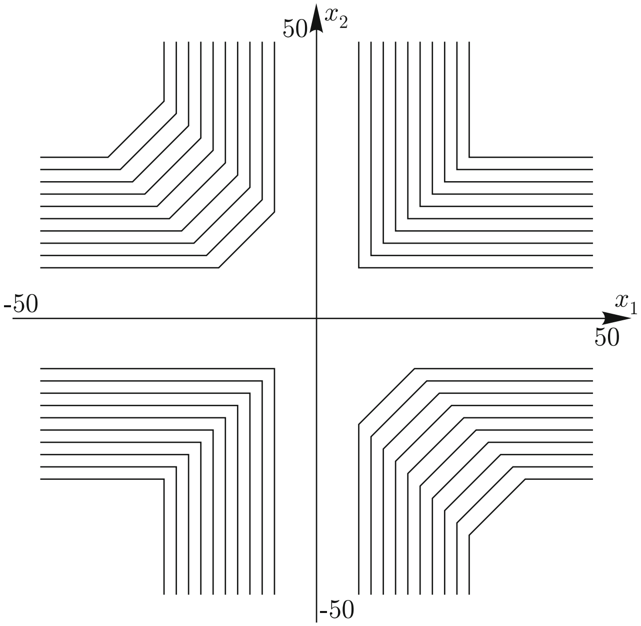

Fig. 14.4 Two strong pursuers, equal termination instants: level sets of the value function, $t = 2$

With that, the function is monotonic at both sides of the segment. For points at the right (at the left) from the segment, the control $u_1 = 1$ ($u_1 = -1$) directs the vector $\mathcal{D}_1(t)u_1$ to the minimum.

Splitting the plane into horizontal lines and extracting for each line the segment of minimum of the value function, one can gather these segments into a set in the plane and draw a *switching line* through this set which separates the plane into two parts at the instant t. At the right from this switching line, we choose the control $u_1 = 1$, and at the left the control is $u_1 = -1$. On the switching line, the control u_1 can be arbitrary obeying the constraint $|u_1| \leq 1$. The easiest way is to take the vertical axis x_2 as the switching line.

In the same way, using the vector $\mathcal{D}_2(t)$, we can conclude that the horizontal axis x_1 can be taken as the switching line for the control u_2.

Thus,

$$u_i^*(t,x) = \begin{cases} 1, & \text{if } x_i > 0, \\ -1, & \text{if } x_i < 0, \\ \text{any } u_i \in [-1,1], & \text{if } x_i = 0. \end{cases} \tag{14.4}$$

The switching lines (the coordinate axes) at any t divide the plane x_1, x_2 into 4 cells. In each of these cells, the optimal control (u_1^*, u_2^*) of the first player is constant.

The vector control $\big(u_1^*(t,x), u_2^*(t,x)\big)$ is applied in a discrete scheme (see [9, 10]) with some time step Δ: a chosen control is kept constant during a time step Δ. Then, on the basis of the new position, a new control is chosen, etc. When $\Delta \to 0$, this control guarantees to the first player a result not greater than $V(t_0, x_0)$ for any initial position (t_0, x_0).

3. *Feedback control of the second player.* Now let us describe the optimal control of the second player. When $T_{f1} = T_{f2}$, the vectogram $\big\{\mathcal{E}(t)v : v \in [-1,1]\big\}$ of the second player in system (14.2) is a segment parallel to the diagonal of I and III quadrants. Thus, the second player can shift the system along this line only.

   Using the sets $W_c(t)$ at some instant t, let us analyze the change of the function $x \to V(t,x)$ along the lines parallel to this diagonal. Consider an arbitrary line from this collection such that it passes through the II quadrant. One can see that local minima are attained at points where the line crosses the axes Ox_1 and Ox_2, and a local maximum is in the segment where the line passes through the rectilinear diagonal part of the boundary of some level set of the value function. The situation is similar for lines passing through the IV quadrant.

   Thus, the switching lines for the second player's control v can be constructed from three parts: the axes Ox_1 and Ox_2, and some slope line $\Pi(t)$. The latter has two half-lines passing through the middles of the diagonal parts on the level set boundaries in the II and IV quadrants. In our case, when the position of the system is on the switching line, the control v can take arbitrary values $|v| \leq 1$. Inside each of 6 cells, to which the plane is divided by the switching lines, the control is taken either $v = +1$, or $v = -1$. Such a control pulls the system towards the points of maximum. Applying this control in a discrete scheme with time step Δ, the second player guarantees that the result will be not less than $V(t_0, x_0)$ for any initial position (t_0, x_0) as $\Delta \to 0$.

   Note. Since $W(t) \neq \varnothing$, then the global minimum of the function $x \to V(t,x)$ is attained at any $x \in W(t)$ and equal 0. Thus, when the position (t,x) of the system is such that $x \in W(t)$, the players can choose, generally speaking, any controls under their constraints. If $x \notin W(t)$, the choices should be made according to the rules described above and based on the switching lines.

4. *Optimal motions.* In Fig. 14.5, one can see the results of optimal motion simulations. This figure contains time sections $W(t)$ (thin solid lines; the same sections as in Fig. 14.3), switching lines $\Pi(0)$ at the initial instant and $\Pi(6)$ at the termination instant of the direct time (dotted lines), and two trajectories for two different initial positions: $\xi_{\mathrm{I}}(t)$ (thick solid line) and $\xi_{\mathrm{II}}(t)$ (dashed line). The motion $\xi_{\mathrm{I}}(t)$ starts from the point $x_1^0 = 40$, $x_2^0 = -25$ (marked by a black circle) which is inside the initial section $W(0)$ of the set W. So, the evader is captured: the endpoint of the motion (also marked by a black circle) is at the origin. The initial point of the motion $\xi_{\mathrm{II}}(t)$ has coordinates $x_1^0 = 25$, $x_2^0 = -50$ (marked by a star). This position is outside the section $W(0)$, and the evader escapes from the exact capture: the endpoint of the motion (also marked by a star) has non-zero coordinates.

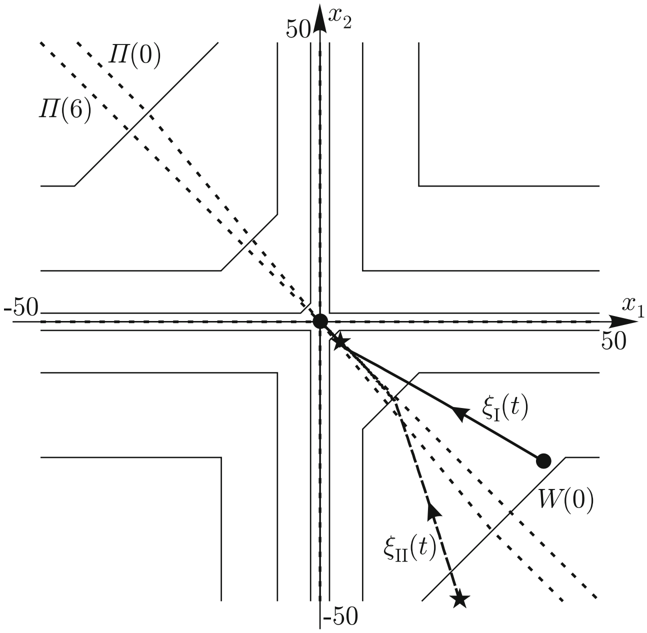

Fig. 14.5 Two strong pursuers, equal termination instants: result of optimal motion simulation

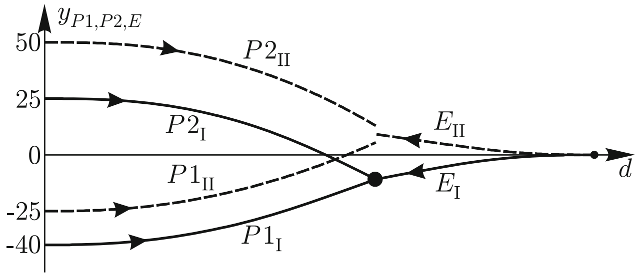

Fig. 14.6 Two strong pursuers, equal termination instants: trajectories in the original space

Figure 14.6 gives the trajectories of the objects in the original space. Values of longitudinal components of the velocities are taken such that the evader moves towards the pursuers. For all simulations here and below, we take

$$y_1^0 = -x_1^0, \quad y_2^0 = -x_2^0, \quad \dot{y}_1^0 = \dot{y}_2^0 = 0, \quad a_{P1}^0 = a_{P2}^0 = a_E^0 = 0.$$

Solid lines correspond to the first case when the evader is successfully captured (at the termination instant, the positions of both pursuers are the same as the position of the evader). Dashed lines show the case when the evader escapes: at the termination instant no one of the pursuers superposes with the evader. In this case, one can see that the evader aims itself to the middle between the terminal positions of the pursuers (this guarantees the maximum of the payoff functional φ).

14.5 Strong Pursuers, Different Termination Instants

Take the parameters as in the previous section, except the termination instants. Now they are $T_{f1} = 7$ and $T_{f2} = 5$. Investigation results are shown in Figs. 14.7–14.9.

The maximal stable bridge $W = W_0$ for system (14.2) with the taken target set

$$M_0 = \{t = T_{f1},\ x_1 = 0\} \cup \{t = T_{f2},\ x_2 = 0\}$$

is built in the following way. At the instant $\tau_1 = 0$ (that is, $t = T_{f1}$), the section of the bridge coincides with the vertical axis $x_1 = 0$. At the instant $\tau_1 = 2$ (that is, $t = T_{f2}$), we add the horizontal axis $x_2 = 0$ to the bridge expanded during passed time period. Further, the time sections of the bridge are constructed using standard procedure under relation $\tau_2 = \tau_1 - 2$.

In the same way, bridges W_c, $c > 0$, corresponding to the target sets

$$M_c = \{t = T_{f1},\ |x_1| \le c\} \cup \{t = T_{f2},\ |x_2| \le c\}$$

can be built.

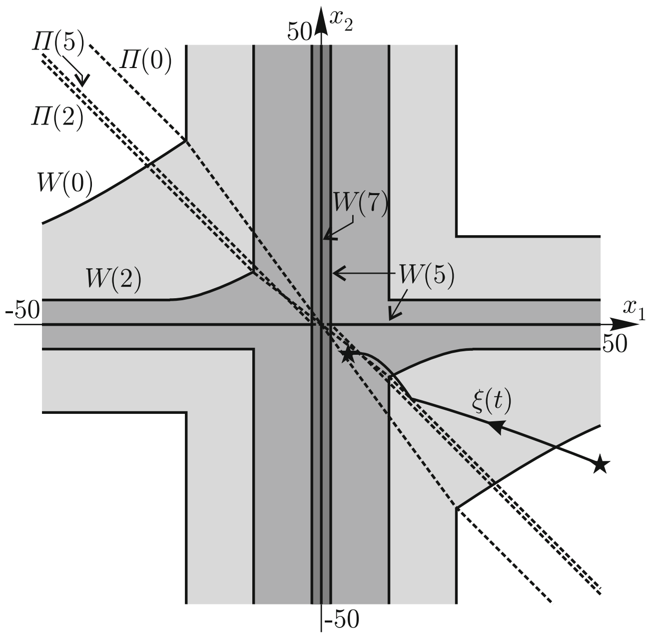

Fig. 14.7 Two strong pursuers, different termination instants: the bridge W and optimal motions

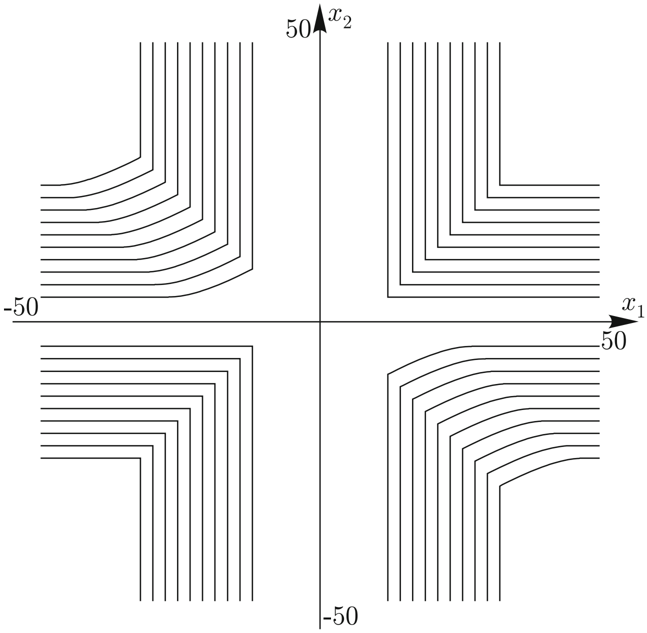

Fig. 14.8 Two strong pursuers, different termination instants: level sets of the value function, $t = 2$

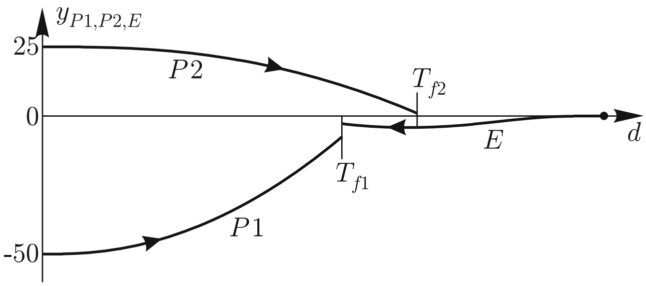

Fig. 14.9 Two strong pursuers, different termination instants: trajectories in the original space

Results of construction of the set W are given in Fig. 14.7. When $\tau_1 > 2$, time sections $W(t)$ grow both horizontally and vertically; two additional triangles appear, but now they are curvilinear. Analytical description of these curvilinear parts of the boundary is difficult. Due to this, in [12, 13], there is only an upper estimation for the solvability set for this variant of the game.

Total structure of the sections $W_c(t)$ at $t = 2$ ($\tau_1 = 5$, $\tau_2 = 3$) is shown in Fig. 14.8. Optimal feedback controls of the pursuers and evader are constructed in the same way as in the previous example, except that the switch line $\Pi(t)$ for the evader is formed by the corner points of the additional curvilinear triangles of the sets $W_c(t)$, $c \geq 0$.

In Fig. 14.7, the trajectory for the initial point $x_1^0 = 50$, $x_2^0 = -25$ is shown as a solid line between two points marked by starts. The trajectories in the original space are shown in Fig. 14.9. One can see that at the beginning the evader escapes from the second pursuer and goes down, after that the evader's control is changed to escape from the first pursuer and the evader goes up.

14.6 Two Weak Pursuers, Different Termination Instants

Now we consider a variant of the game when both pursuers are weaker than the evader. Let us take the parameters

$$A_{P1} = 0.9, \quad A_{P2} = 0.8, \quad A_E = 1, \quad l_{P1} = l_{P2} = 1/0.7, \quad l_E = 1,$$

and different termination instants $T_{f1} = 7$, $T_{f2} = 5$.

Since in this variant, the evader is more maneuverable than the pursuers, they cannot guarantee the exact capture.

Fix some level of the miss, namely, $\left|x_1(T_{f1})\right| \le 2.0$, $\left|x_2(T_{f2})\right| \le 2.0$. Time sections $W_{2.0}(t)$ of the corresponding maximal stable bridge are shown in Fig. 14.10. The upper-left subfigure corresponds to the instant $t = 7$ when the first pursuer stops to act. The upper-right subfigure shows the picture for the instant $t = 5$ when the second pursuer finishes its pursuit. At this instant, the horizontal strip is added which is a little wider than the vertical one contracted during the passed period of the backward time. Then, the bridges contracts both in horizontal and vertical directions, and two additional curvilinear triangles appear (see middle-left subfigure). The middle-right subfigure gives the view of the section when the vertical strip collapses, and the lower-left subfigure shows the configuration just after the collapse of the horizontal strip. At this instant, the section loses connectivity and disjoins into two parts symmetrical with respect to the origin. Further, these parts continue to contract (as it can be seen in the lower-right subfigure) and finally disappear.

Time sections $\{W_c(t)\}$ and corresponding switching lines of the first player are given in Fig. 14.11 at the instant $t = 0$ ($\tau_1 = 7$, $\tau_2 = 5$). The dashed line is the switching line for the component u_1; the dotted one is for the component u_2. The switching lines are obtained as a result of the analysis of the function $x \to V(t,x)$ in horizontal (for u_1) and vertical (for u_2) lines. In some region around the origin, the switching line for u_1 (respectively, for u_2) differs from the vertical (horizontal) axis. If in the considered horizontal (vertical) line the minimum of the value function is attained in a segment, then the middle of such a segment is taken as a point for the switching line. Arrows show directions of components of the control in four cells. Similarly, in Fig. 14.12, switching lines and optimal controls are displayed for the second player. Here, the switching lines are drawn with thick solid lines. We have four cells where the second player's control is constant.

For simulations, let us take the initial position $x_1^0 = 12, x_2^0 = -12$ for system (14.2). In Fig. 14.13, trajectories of the objects are shown in the original space. At the

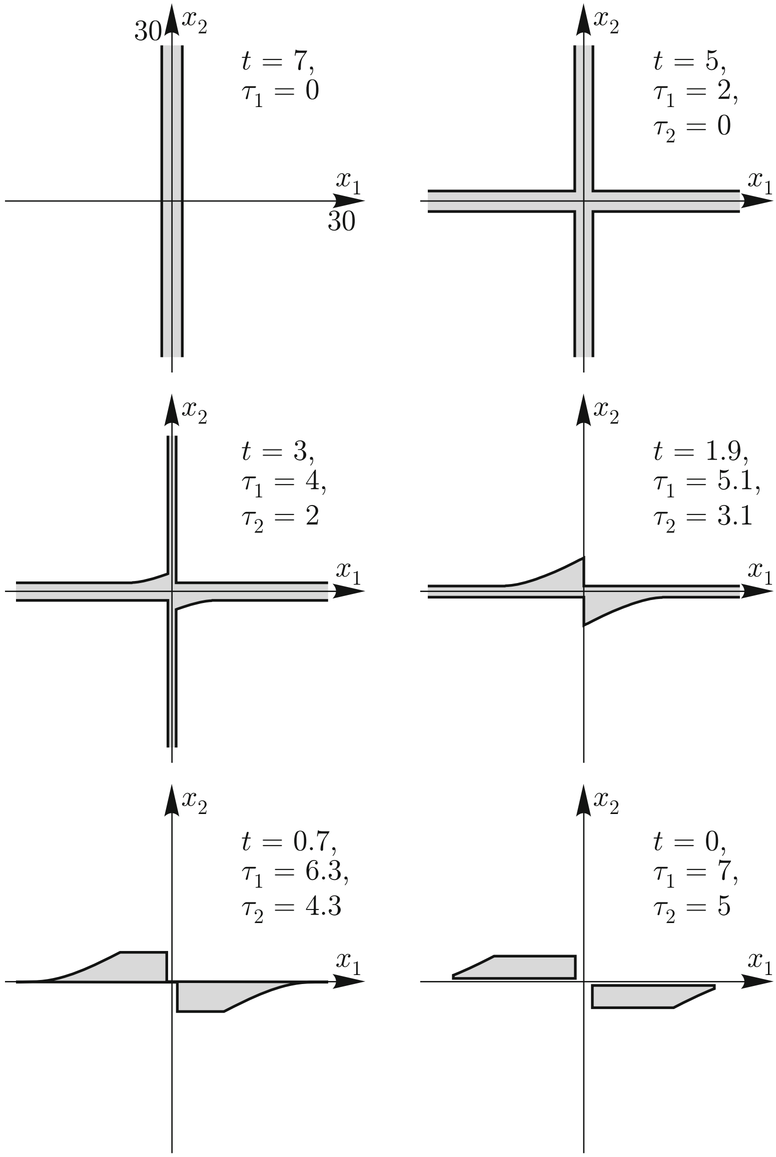

Fig. 14.10 Two weak pursuers, different termination instants: time sections of the maximal stable bridge $W_{2.0}$

beginning of the pursuit, the evader closes to the first (lower) pursuer. It is done to increase the miss from the second (upper) pursuer at the instant T_{f2}. Further closing is not reasonable, and the evader switches its control to increase the miss from the first pursuer at the instant T_{f1}.

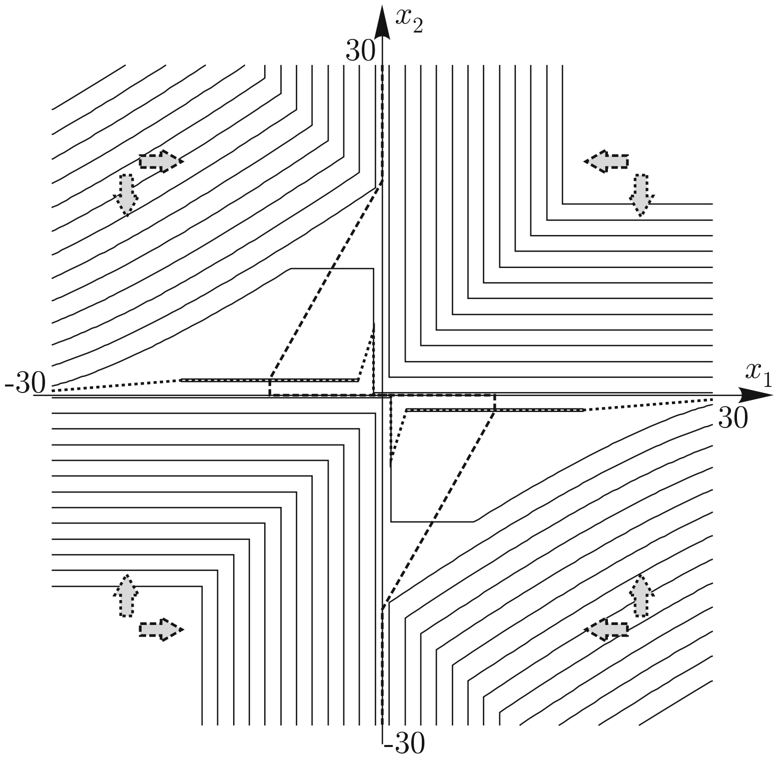

Fig. 14.11 Two weak pursuers, different termination instants: switching lines and optimal controls for the first player (the pursuers), $t = 0$

14.7 One Strong and One Weak Pursuers, Different Termination Instants

Let us change the parameters of the second pursuer in the previous example and take the following parameters of the game:

$$A_{P1} = 2, \quad A_{P2} = 1, \quad A_E = 1, \quad l_{P1} = 1/2, \quad l_{P2} = 1/0.3, \quad l_E = 1.$$

Now the evader is more maneuverable than the second pursuer, and an exact capture by this pursuer is unavailable. Assume $T_{f1} = 5$, $T_{f2} = 7$.

In Fig. 14.14, there are sections of the maximal stable bridge $W_{5.0}$ (that is, for $c = 5.0$) for six instants: $t = 7.0$, 5.0, 2.5, 1.4, 1.0, 0.0. The horizontal part of its time section $W_{5.0}(\tau)$ decreases with growth of τ, and breaks further. The vertical part grows. Even after breaking the individual stable bridge of the second pursuer (and respective collapse of the horizontal part of the cross), additional capture zones still exist and are kept in time.

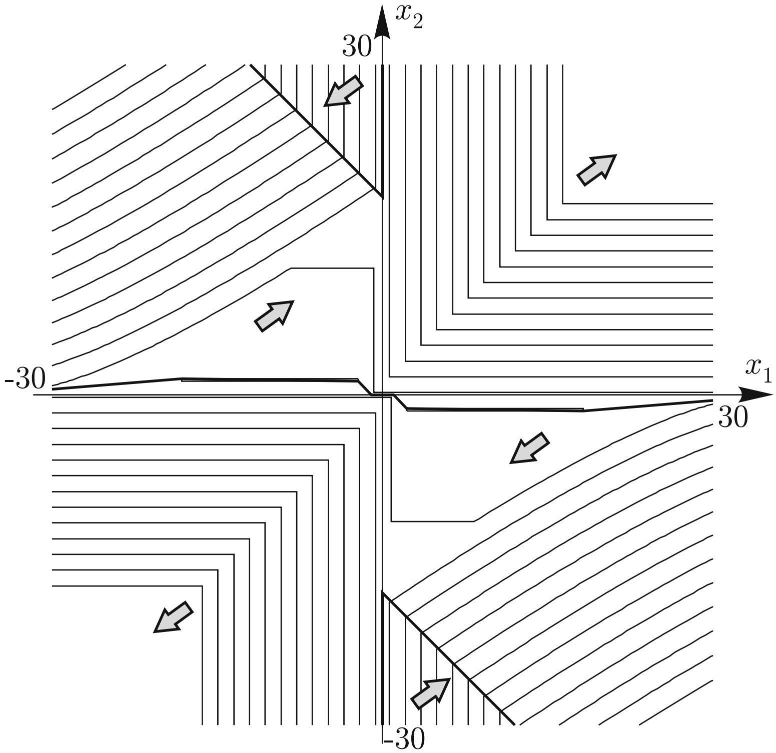

Fig. 14.12 Two weak pursuers, different termination instants: switching lines and optimal controls for the second player (the evader), $t = 0$

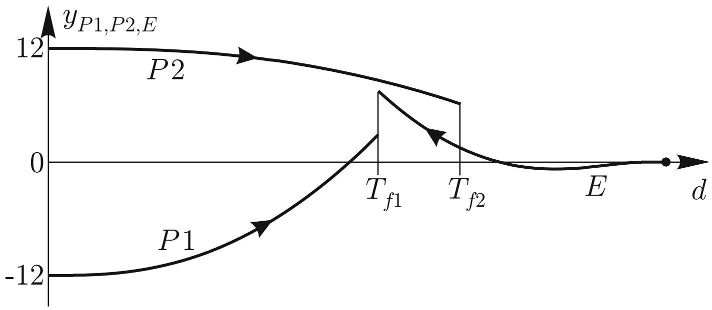

Fig. 14.13 Two weak pursuers, different termination instants: trajectories of the objects in the original space

Switching lines of the first and second players for the instant $t = 1$ are given in Figs. 14.15 and 14.16. These lines are obtained by processing collection $\{W_c(t = 1)\}$ computed for different values of c. In comparison with the previous case of two weak pursuers, the switching lines for the first player have simpler structure.

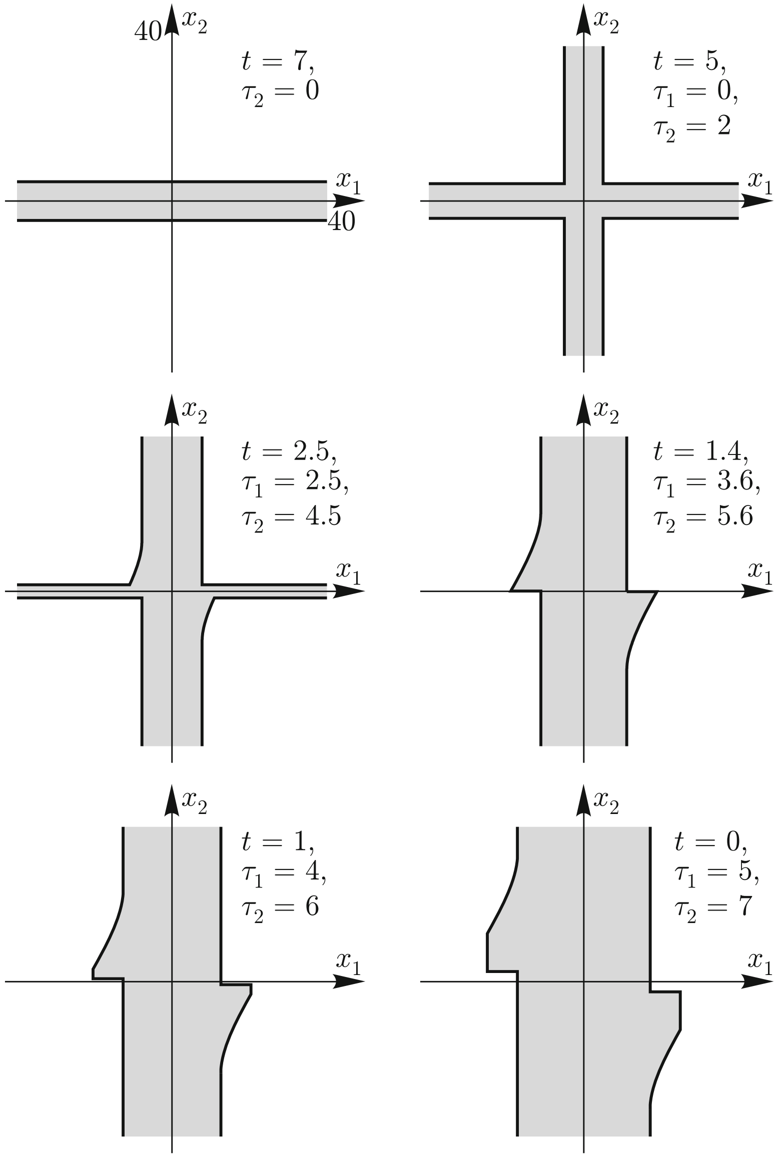

Fig. 14.14 One strong and one weak pursuers, different termination instants: time sections of the maximal stable bridge $W_{5.0}$

Here, as in the previous section, the trajectories of the objects are drawn in the original space only (see Fig. 14.17). For simulations, the initial lateral deviations are taken as $x_1^0 = 20$, $x_2^0 = -20$. Longitudinal components of the velocities are such that the evader moves towards one pursuer, but from another.

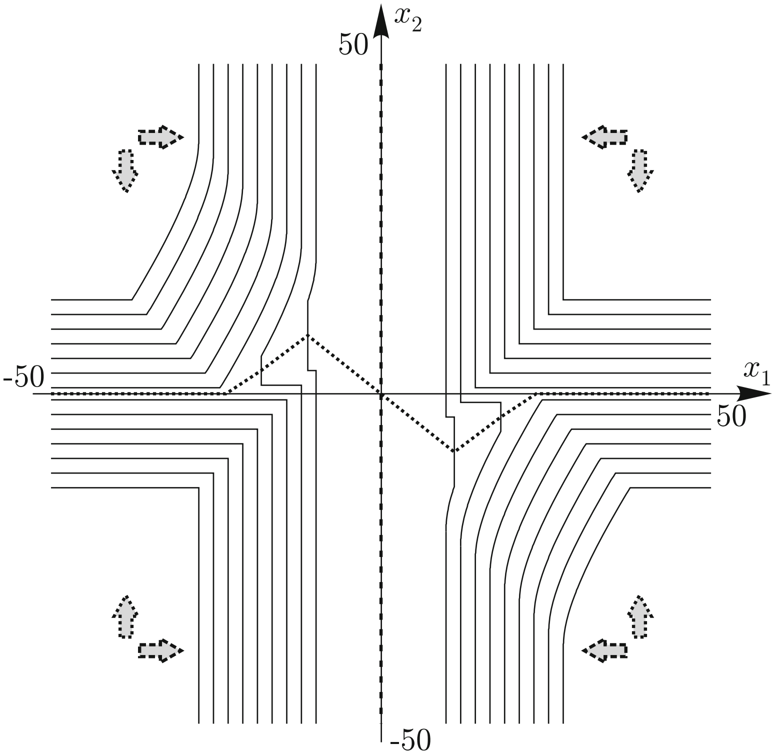

Fig. 14.15 One strong and one weak pursuers, different termination instants: switching lines and optimal controls for the first player (the pursuers), $t = 1$

14.8 Varying Advantage of Pursuers, Equal Termination Instants

Another interesting case is when the pursuers have equal capabilities such that, at the beginning of the backward time, the bridges in the individual games contract and further expand. That is, at the beginning of the direct time, the pursuers have advantage over the evader, but at the final stage the evader is stronger.

Parameters of the game are taken as follows:

$$A_{P1} = A_{P2} = 1.5, \quad A_E = 1, \quad l_{P1} = l_{P2} = 1/0.3, \quad l_E = 1.$$

Termination instants are equal: $T_{f1} = T_{f2} = 10$.

In Fig. 14.18, time sections of the maximal stable bridge $W_{1.5}$ built for $c = 1.5$ are shown for six instants: $t = 10.0, 7.0, 5.7, 4.5, 1.3, 0.0$. At the termination instant, the terminal set is taken as a cross (the upper-left subfigure).

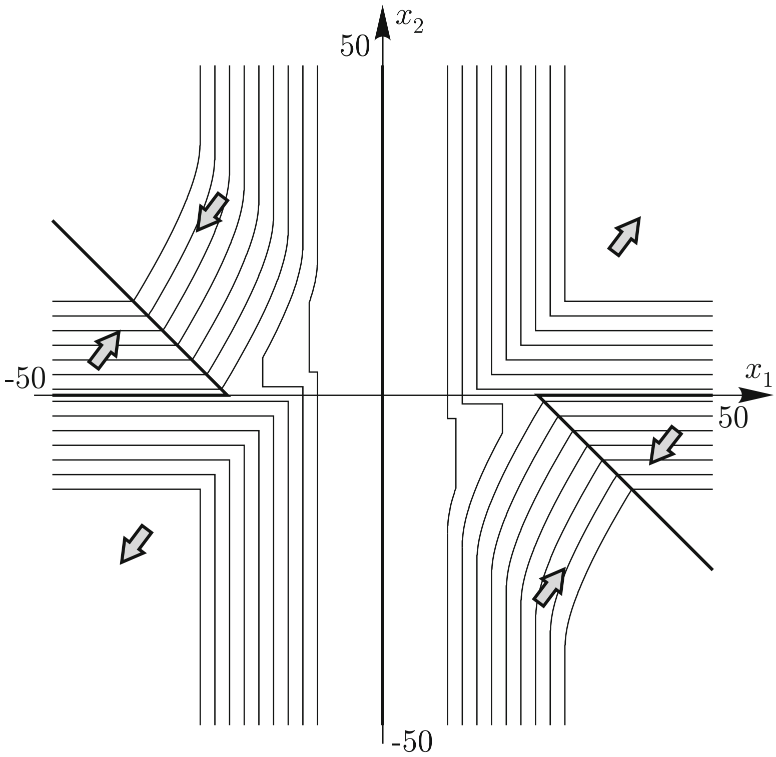

Fig. 14.16 One strong and one weak pursuers, different termination instants: switching lines and optimal controls for the second player (the evader), $t = 1$

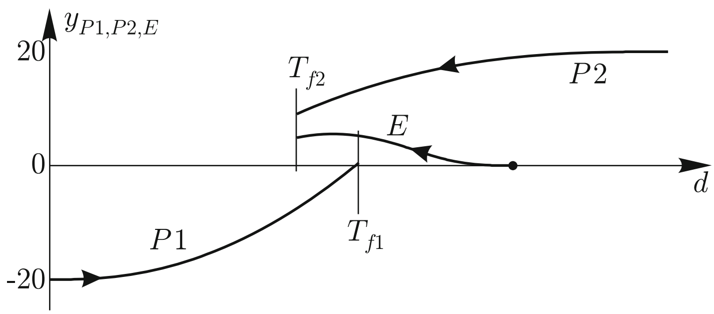

Fig. 14.17 One strong and one weak pursuers, different termination instants: trajectories of the objects in the original space

At the beginning of backward time, the structure of the bridges is similar to the case of two weak pursuers: widths of both vertical and horizontal strips of the "cross" decreases, and two straight-linear additional triangles of joint capture zone appear (the upper-right subfigure). Then at some instant, both strips collapse, and

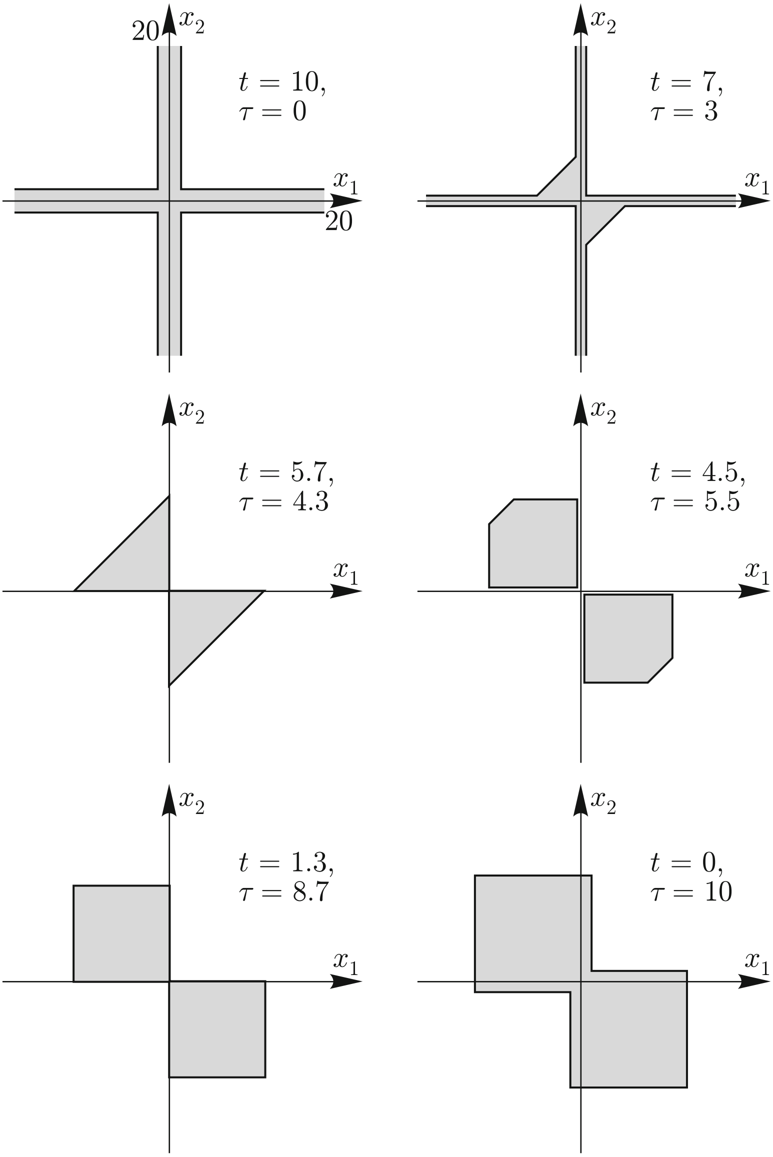

Fig. 14.18 Varying advantage of the pursuers, equal termination instants: time sections of the maximal stable bridge $W_{1.5}$

only the triangles constitute the time section of the bridge (the central left subfigure). Further, the triangles continue to contract, so they become two pentagons separated by an empty space near the origin (the central right subfigure in Fig. 14.18). Transformation to pentagons can be explained in the following way: the first player using

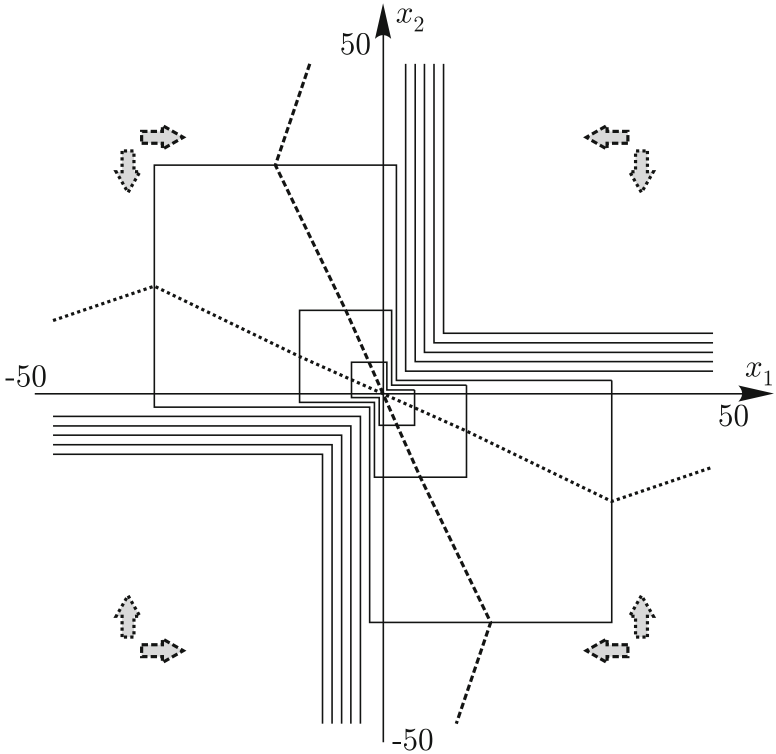

Fig. 14.19 Varying advantage of the pursuers, equal termination instants: switching lines and optimal controls for the first player (the pursuers), $t = 0$

its controls expands the triangles vertically and horizontally, and the second player contracts them in diagonal direction. So, vertical and horizontal edges appear, but the diagonal part becomes shorter. Also, in general, size of each figure decreases slowly.

Due to action of the second player, at some instant, the diagonal disappears, and the pentagons convert to squares (this is not shown in Fig. 14.18). After that, the pursuers take advantage, and total contraction is changed by growth: the squares start to enlarge. When some time passes, due to the growth, the squares touch each other at the origin (the lower-left subfigure in Fig. 14.18). Since the enlargement continues, their sizes grow, and the squares start to overlap forming one "eight-like" shape (the lower-right subfigure in Fig. 14.18).

Figures 14.19 and 14.20 show time sections of a collection of maximal stable bridges and switching lines for the first and second players, respectively, for the instant $t = 0$.

As above, the simulated trajectories are shown in the original space only. For simulation, the following initial conditions are taken: $x_1^0 = 5$, $x_2^0 = -20$. Longitudinal components of the velocities are such that the evader moves from both pursuers.

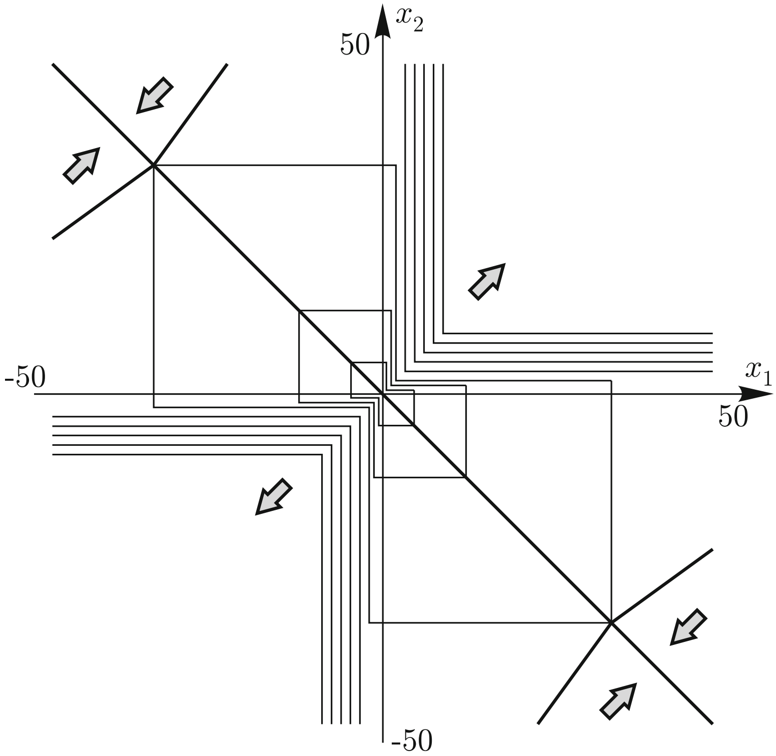

Fig. 14.20 Varying advantage of the pursuers, equal termination instants: switching lines and optimal controls for the second player (the evader), $t = 0$

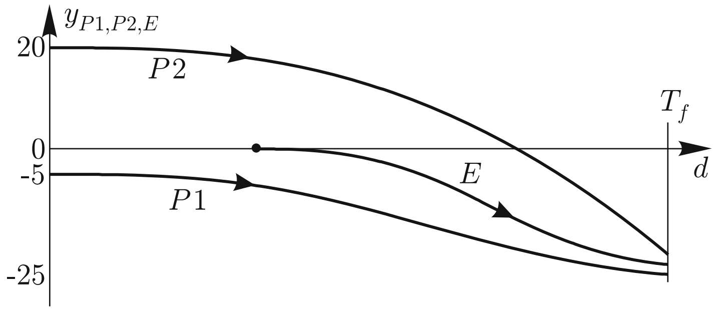

Fig. 14.21 Varying advantage of the pursuers, equal termination instants: trajectories of the objects in the original space

The computed trajectories are given in Fig. 14.21. As it was said earlier, since at the final stage of interception the pursuers are weaker than the evader, they cannot guarantee the exact capture but only some non-zero level of the miss.

14.9 Conclusion

Presence of two pursuers acting together and minimizing the miss from the evader leads to non-convexity of time sections of the value function when the situation is considered as a standard antagonistic differential game where both pursuers are joined into one player. In the paper, results of numerical study of this problem are given for several variants of the parameters. The structure of the solution depends on the presence or absence of dynamic advantage of one or both pursuers over the evader. Optimal feedback control methods of the pursuers and evader are built by preliminary construction and processing the level (Lebesgue) sets of the value function (maximal stable bridges) for some quite fine grid of values of the payoff. Switching lines obtained for each scalar component of controls depend on time, and only they, not the level sets, are used for generating controls. Optimal controls are produced at any current instant depending on the location of the state point respectively to the switching lines at this instant. Accurate proof of the suggested optimal control method needs for some additional study.

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