Computation of solvability set for differential games in the plane with simple motion and non-convex terminal set

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The paper suggests an algorithm for an exact construction of solvability set in a differential game with simple motion in the plane, with a fixed terminal time and a polygonal (in the general case, non-convex) terminal set. Some examples of solvability sets are computed.

Introduction

A system with simple motion introduced by R. Isaacs in differential games is a simplest controlled system with geometric constraints of players' controls, which do not depend on time. The phase velocity for the system at a current instant of time is determined only by controls of the players selected at that instant. Researches of many meaningful game problems in the literature on differential games are based on the assumption that dynamics of simple motion is used. In addition, the dynamics of simple motion is often used in computational methods as an approximation. The standard problem in the theory of differential games with zero sum is a problem with a fixed terminal time and a given terminal set, to which the first player tries to bring the system at a terminal instant, and the second one prevents it. In the framework of this problem, it is important to develop an algorithm for constructing the maximal set (solvability set) of initial phase points, from which such a transfer is guaranteed for the first player. In the case of convex terminal set, the solution of this problem is known. In this paper, for problems with simple motion in the plane, an algorithm for an exact construction of solvability set for a polygonal terminal set (convex or non-convex) is described.

I. Problem statement

We consider the following differential game with fixed terminal time. A dynamical system in $\mathbb{R}^2$ is described by differential equation with simple motion [1]

$$\dot{x} = u + v, \quad u \in P, \quad v \in Q, \quad t \in [0, \vartheta], \quad \vartheta > 0, \tag{1}$$

where u and v are controls of the first and second players, each of the sets P and Q is either a convex simple polygon or a linear segment. Let $M \subset \mathbb{R}^2$ be either a simple polygon, or the closure of complement of a simple polygon. The last means the set

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$M' = \overline{\mathbb{R}^2 \setminus M}$ is a simple polygon. Here, the overline denotes the closure of a set. The set M is considered as a terminal set by the first player at the instant ϑ. The second player aims to prevent the system from reaching M at the instant ϑ.

Such dynamics and terminal set are usually the result of simplifying dynamics in a more complex control problem with uncertainty and conflict, applied to a small period of time.

The problem is to construct *the solvability set* $G_\vartheta(M) \subset \mathbb{R}^2$ of the first player, i.e. the set of all initial points $x(0) = x_0 \in \mathbb{R}^2$ for the system (1), from which the first player guarantees attainability of the terminal set M at the instant ϑ. Each player chooses his control from the corresponding set of constraints (P or Q) basing on a current position $(t, x(t))$ of the system. Definitions of the players' strategies and trajectories of the dynamic system under the strategies can be introduced in various ways [2–4], but lead to equivalent notions of solvability set.

II. Basic definitions

Let us introduce some definitions related to polygons and its representations.

A. Polygonal chain

Let $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ denote a finite cyclic sequence of points in the plane, that is each element $a_i \in \mathbf{a}$ has the next element a_{i+1} and the previous element a_{i-1}, where $a_0 = a_n$, $a_{n+1} = a_1$, $n \in \mathbb{N}$.

A *closed polygonal chain* is a non-empty finite sequence of line segments (*edges*) $\overrightarrow{a_1a_2}$, $\overrightarrow{a_2a_3}$, ..., $\overrightarrow{a_na_{n+1}}$, which join adjacent points in a finite cyclic sequence $[a_1, a_2, \ldots, a_n]$ of points (*vertices*) in the plane, and consecutive vertices are distinct, i.e. $a_i \neq a_{i+1}$, $i = 1, \ldots, n$. Each edge $\overrightarrow{a_ia_{i+1}}$ of a polygonal chain is directed from a_i to a_{i+1}. We assume also that there are no collinear consecutive edges. Let us denote by $\mathcal{C}$ the set of all closed polygonal chains.

We will use the notation $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ also for a closed polygonal chain that means the inclusion $\mathbf{a} \in \mathcal{C}$ makes sense.

We define a chain $\widehat{\mathbf{a}}$ based on a finite cyclic sequence $\mathbf{a}$ of points as follows. First we remove from $\mathbf{a}$ all vertices that repeat the previous one. Then we delete all vertices adjacent to collinear edges. The resulting sequence $\widehat{\mathbf{a}}$ is either a singleton (that is, it determines a point in the plane) or $\widehat{\mathbf{a}} \in \mathcal{C}$.

B. Simple polygon

A *simple polygon* means a region enclosed by a single closed polygonal chain that does not intersect itself. For brevity, we call a closed polygonal chain without self-intersections a *simple polygonal chain*. Let $\mathcal{C}_0$ denote the set of all simple (closed) polygonal chains.

Any simple polygon $A \subset \mathbb{R}^2$ is defined by a cyclic counterclockwise sequence of its vertices a_1, a_2, ..., a_n, that is the interior of A lies to the left of $\overrightarrow{a_ia_{i+1}}$ as we move from a_i to a_{i+1}. The polygonal chain formed by the vertices of A in reverse (clockwise) order describes the set $A' = \overline{\mathbb{R}^2 \setminus A}$. We will use the notations $A \sim [a_1, a_2, \ldots, a_n]$ and $A' \sim [a_n, a_{n-1}, \ldots, a_1]$.

C. Base list of half-planes

We associate with a polygonal chain $\mathbf{a} = [a_1, a_2, \ldots, a_n] \in \mathcal{C}$ a cyclic list $[\mathbf{a}]^*$ of half-planes as follows. The edge $\overrightarrow{a_i a_{i+1}}$ describes a half-plane Π_i that lies to the left of the straight line passing through the points a_i and a_{i+1} $(i = 1, \ldots, n)$. This view gives us a cyclic list

$$[\mathbf{a}]^* := [\Pi_1, \Pi_2, \ldots, \Pi_n].$$

The list $[\mathbf{a}]^*$ is called *base list* of half-planes for $\mathbf{a} \in \mathcal{C}$.

Any pair of adjacent half-planes in the list $[\mathbf{a}]^*$ has a unique point of intersection of their boundaries. Therefore the polygonal chain $\mathbf{a} \in \mathcal{C}$ is uniquely restored from the list $[\mathbf{a}]^*$, namely $a_i = \partial\Pi_{i-1} \cap \partial\Pi_i$, $i = 1, \ldots, n$, $\Pi_0 = \Pi_n$. Here, the symbol ∂ denotes the boundary of a set.

D. Regular list and consistent list of half-planes

Suppose now that a cyclic list $\mathbf{L} = [\Pi_1, \Pi_2, \ldots, \Pi_n]$, $n \geq 2$, of half-planes is initially given, where the boundaries of adjacent half-planes have a unique intersection point. Such a list will be called *regular*. Further we will consider only regular lists of half-planes.

A regular list $\mathbf{L}$ corresponds uniquely to a cyclic sequence $[\mathbf{L}]^* = [a_1, a_2, \ldots, a_n]$ of points $a_i = \partial\Pi_{i-1} \cap \partial\Pi_i$, $i = 1, \ldots, n$, $\Pi_0 = \Pi_n$. We note that the cyclic sequence $\mathbf{a} = [\mathbf{L}]^*$ is a closed polygonal chain (that is, $\mathbf{a} \in \mathcal{C}$) under the condition $a_i \neq a_{i+1}$, $i = 1, \ldots, n$, $a_{n+1} = a_1$, since the absence of collinear adjacent edges follows from the regularity of the list $\mathbf{L}$.

Let us point out the conditions for the list $\mathbf{L}$ that ensure $[\mathbf{L}]^* \in \mathcal{C}$ and $[[\mathbf{L}]^*]^* = \mathbf{L}$. To do this, we classify the types of pairs and triples of half-planes that can appear in the list $\mathbf{L}$.

An ordered pair of different half-planes (Π_1, Π_2) is called *convex* (*concave*) if $\Pi_1 \cap \Pi_2$ is an angle in the plane and this angle lies on the left-hand side (right-hand side) when we first move along the straight line $\partial\Pi_1$ up to the intersection with $\partial\Pi_2$ and then along the straight line $\partial\Pi_2$. In all other cases, the straight lines $\partial\Pi_1$ and $\partial\Pi_2$ are *parallel*, which we denote as $\Pi_1 \parallel \Pi_2$. We will also distinguish a *collinear* pair of half-planes.

Let us note that adjacent edges $\overrightarrow{a_1 a_2}$ and $\overrightarrow{a_2 a_3}$ of a polygonal chain form a convex (concave) pair of half-planes if and only if the direction $\overrightarrow{a_2 a_3}$ is a turn to the left (to the right) with respect to the direction $\overrightarrow{a_1 a_2}$.

An ordered triple (Π_a, Π, Π_b) of different half-planes is called *convex* (*concave*) if the pairs (Π_a, Π) and (Π, Π_b) are convex (concave). An ordered triple (Π_a, Π, Π_b) of different half-planes is called *a zigzag triple* if the pair (Π_a, Π) is convex and (Π, Π_b) is concave or vise versa.

We start with definition of consistent convex, concave and zigzag triples.

A convex triple $\sigma = (\Pi_a, \Pi, \Pi_b)$ is said to be *consistent*, if $\Pi_a \nparallel \Pi$, $\Pi \nparallel \Pi_b$, and the set $\Pi_a \cap \Pi_b \cap \partial\Pi$ is a non-degenerate linear segment (Fig. 1a).

Observe that consistent convex triple $\sigma = (\Pi_a, \Pi, \Pi_b)$ defines points $a = \partial\Pi_a \cap \partial\Pi$ and $b = \partial\Pi_b \cap \partial\Pi$, $a \neq b$, and when moving from a to b, the half-plane Π lies on the left.

We denote by Π' the closure of the complement of a half-plane Π in $\mathbb{R}^2$, i.e. $\Pi' = \overline{\mathbb{R}^2 \setminus \Pi}$. Let us remark that if a triple $\sigma = (\Pi_a, \Pi, \Pi_b)$ is convex, then the triple $\sigma' = (\Pi'_b, \Pi', \Pi'_a)$ is concave, and vice versa.

A concave triple $\sigma = (\Pi_a, \Pi, \Pi_b)$ is said to be *consistent*, if the triple $\sigma' = (\Pi'_b, \Pi', \Pi'_a)$ is consistent (Fig. 1b).

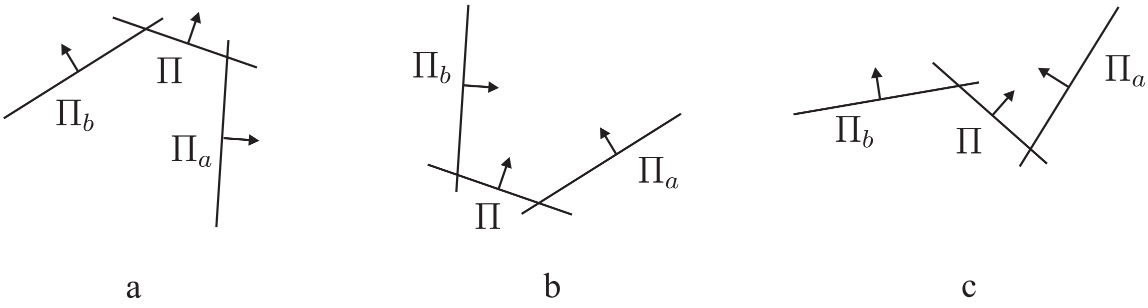

Fig. 1. Consistent convex (a), concave (b), and zigzag (c) triple (Π_a, Π, Π_b) of half-planes

A zigzag triple $\sigma = (\Pi_a, \Pi, \Pi_b)$ is called *consistent*, if in the case of convexity of the pair (Π_a, Π), the convex triple (Π_a, Π, Π'_b) is consistent, and in the case of concavity of the pair (Π_a, Π), the convex triple (Π_b, Π', Π'_a) is consistent (Fig. 1c).

A cyclic list of half-planes is called *consistent* if any successive triple of its elements is consistent. For a consistent list $\mathbf{L}$, we have $[\mathbf{L}]^* \in \mathcal{C}$ and $[[\mathbf{L}]^*]^* = \mathbf{L}$.

III. Solvability sets for convex and concave cases

A. Representation formula for a convex case

The formula

$$G_\vartheta(M) = \big(M - \vartheta P\big) \overset{*}{-} \vartheta Q \tag{2}$$

for the solvability set $G_\vartheta(M)$ is well-known [5] in the case of a closed convex terminal set M. Here, a multiplication by a scalar, the algebraic (Minkowski) sum and the geometric (Minkowski) difference are used:

$$\lambda A = \{\lambda a : \ a \in A\}, \quad \lambda \in \mathbb{R}; \quad A + B = \{a + b : \ a \in A, \ b \in B\};$$

$$A \overset{*}{-} B = \{d : \ d + B \subset A\} = \bigcap_{b \in B} (A - b).$$

If M is a convex polygon, then the set $G_\vartheta(M)$ is also a convex polygon under the conditions on P and Q specified after (1).

Relying on the right-hand side of the formula (2), we define the operator T_τ acting on a set $A \subset \mathbb{R}^2$:

$$T_\tau(A) := \big(A - \tau P\big) \overset{*}{-} \tau Q.$$

We denote by $\Pi_\eta[A]$ the support half-plane to a set $A \subset \mathbb{R}^2$ with the outer normal η:

$$\Pi_\eta[A] = \{x \in \mathbb{R}^2 : \langle x, \eta \rangle \leq \rho(\eta; A) < +\infty\}, \quad \rho(\eta; A) := \sup\{\langle a, \eta \rangle : a \in A\}.$$

In the case of a convex polygon M, it can be shown (see, for example, [6]) that the set $G_\vartheta(M)$ is representable as an intersection of a finite number of supporting half-planes to M shifted by the operator T_ϑ:

$$G_\vartheta(M) = T_\vartheta(M) = \bigcap \big\{ T_\vartheta(\Pi_\eta[M]) : \eta \in \mathcal{N}(M) \cup \mathcal{N}(-P) \big\}. \tag{3}$$

Here, $\mathcal{N}(M)$ and $\mathcal{N}(-P)$ are finite sets of unit outer normals to the edges of the polygons M and $-P$ respectively. If P is a linear segment, then $\mathcal{N}(-P) = \mathcal{N}(P) = \{\nu, -\nu\}$, where ν is a unit normal to P.

Using the definition of the operator T_τ, we calculate the extremal shift of a half-plane Π_η with the outer normal η for a given value $\tau > 0$ by the formula

$$T_\tau(\Pi_\eta) = \Pi_\eta - \tau\big(u_0(\eta) + v_0(\eta)\big),$$

where

$$u_0(\eta) \in \operatorname{Arg}\min_{u\in P}\langle u, \eta\rangle, \quad v_0(\eta) \in \operatorname{Arg}\max_{v\in Q}\langle v, \eta\rangle.$$

Hence, for the convex case, the construction of the solvability set $G_\vartheta(M)$ can be reduced to creating an ordered cyclic list of half-planes Π_η, based on the outer normals of the sets M and $-P$, and to removing those half-planes that are not essential for intersection (3) from the list. One of the algorithms for a convex case is described in [7,8].

B. Base and expanded lists for a convex case

Let M be a convex polygon, $M \sim \mathbf{m} = [m_1, \ldots, m_{n_1}]$ and $\mathbf{L}_M = [\mathbf{m}]^*$. The list $\mathbf{L}_M$ is consistent.

We note that the list $\mathbf{L}_M$ is formed by the half-planes $\Pi_\eta[M]$, $\eta \in \mathcal{N}(M)$, cyclically ordered in such a way that the outer normal η of the half-planes turns counterclockwise when going around the list. In the base list $\mathbf{L}_M$ of a convex polygon M, any triple of adjacent half-planes is convex. A list with this property will be called *convex*.

We have the representation

$$M = \bigcap\big\{\Pi_\eta[M] : \eta \in \mathcal{N}(M)\big\} = \bigcap\big\{\Pi : \Pi \in \mathbf{L}_M\big\}.$$

To apply formula (3), we expand the base list $\mathbf{L}_M$ with additional half-planes $\Pi_\eta[M]$, $\eta \in \mathcal{N}(-P)$, to the list $\mathbf{L}_M^P$. To this end, we insert additional half-planes between the half-planes of the base list in such a way that the outer normal of the half-planes, while traversing the expanded list, is still rotated counterclockwise. The expanded list $\mathbf{L}_M^P$ is also convex. If $\mathbf{L}_M^P \neq \mathbf{L}_M$, then $\mathbf{L}_M^P$ is not consistent.

For $\tau \in [0, \vartheta]$, we write formula (3) as

$$G_\tau(M) = \bigcap\big\{T_\tau(\Pi) : \Pi \in \mathbf{L}_M^P\big\}. \tag{4}$$

We denote by $\mathbf{L}$ the list obtained by removing such half-planes from the list $\mathbf{L}_M^P$ that can be dropped in intersection (4) for $\tau = \vartheta$ without changing the result of the intersection. Then $[\mathbf{L}]^* \sim G_\vartheta(M)$.

C. Representation formula for a concave case

Suppose that the terminal set M is the complement to some convex polygon $M' = \overline{\mathbb{R}^2 \setminus M}$. To describe the solvability set, one can apply the formula for the convex case by swapping the players and swapping the terminal set and its complement. Let us write this formula in the original notation.

In a concave case, the formula for the complement $G'_\vartheta(M) = \overline{\mathbb{R}^2 \setminus G_\vartheta(M)}$ of the solvability set $G_\vartheta(M)$ is

$$G'_\vartheta(M) = \big(M' - \vartheta Q\big) \stackrel{*}{-} \vartheta P \ =: T^*_\vartheta(M').$$

The representation for $G'_\vartheta(M)$, analogous to (3), has the form

$$G'_\vartheta(M) = T^*_\vartheta(M') = \bigcap\Big\{T^*_\vartheta(\Pi_\eta[M']) : \eta \in \mathcal{N}(M') \cup \mathcal{N}(-Q)\Big\}. \tag{5}$$

We note that the equality

$$T^*_\vartheta(\Pi') = \overline{\mathbb{R}^2 \setminus T_\vartheta(\Pi)} \tag{6}$$

holds for a half-plane Π and its complement $\Pi' = \overline{\mathbb{R}^2 \setminus \Pi}$.

We define the set of outer normals to the set M:

$$\mathcal{N}(M) := \{-\eta : \eta \in \mathcal{N}(M')\}.$$

In addition, for a unit vector η, we define the "support" half-plane

$$\Pi_\eta[A] := \overline{\mathbb{R}^2 \setminus \Pi_{-\eta}[A']}. \tag{7}$$

to the set $A \subset \mathbb{R}^2$ with convex complement $A' = \overline{\mathbb{R}^2 \setminus A}$.

By virtue of (6) and (7), we have

$$T^*_\vartheta(\Pi_\eta[M']) = T^*_\vartheta(\overline{\mathbb{R}^2 \setminus \Pi_{-\eta}[M]}) = \overline{\mathbb{R}^2 \setminus T_\vartheta(\Pi_{-\eta}[M])}.$$

Representation (5) implies

$$G_\vartheta(M) = \bigcup \Big\{ T_\vartheta(\Pi_{-\eta}[M]) : \eta \in \mathcal{N}(M') \cup \mathcal{N}(-Q) \Big\}.$$

Since $-\mathcal{N}(M') = \mathcal{N}(M)$ and $-\mathcal{N}(-Q) = \mathcal{N}(Q)$, we find

$$G_\vartheta(M) = \bigcup \Big\{ T_\vartheta(\Pi_\eta[M]) : \eta \in \mathcal{N}(M) \cup \mathcal{N}(Q) \Big\}. \tag{8}$$

D. Base and expanded lists for a concave case

Let M be a set with the convex complement M', $M \sim \mathbf{m} = [m_1, \ldots, m_{n_1}]$ and $\mathbf{L}_M = [\mathbf{m}]^*$. The list $\mathbf{L}_M$ is consistent.

We note that the list $\mathbf{L}_M$ is formed by the half-planes $\Pi_\eta[M]$, $-\eta \in \mathcal{N}(M')$, cyclically ordered in such a way that the outer normal η of the half-planes turns around clockwise. In the base list $\mathbf{L}_M$ for the concave case, any triple of adjacent half-planes is concave. A list with this property will be called *concave*.

We have the representation

$$M = \bigcup \big\{ \Pi_\eta[M] : \eta \in \mathcal{N}(M) \big\} = \bigcup \big\{ \Pi : \Pi \in \mathbf{L}_M \big\}.$$

To apply formula (8), we expand the base list $\mathbf{L}_M$ with additional half-planes $\Pi_\eta[M]$, $\eta \in \mathcal{N}(Q)$, to the list $\mathbf{L}^Q_M$. To this end, we insert additional half-planes between the half-planes of the base list in such a way that the outer normal of the half-planes, while traversing the expanded list, is still rotated clockwise. The expanded list $\mathbf{L}^Q_M$ is also concave. If $\mathbf{L}^Q_M \neq \mathbf{L}_M$, then $\mathbf{L}^Q_M$ is not consistent.

For $\tau \in [0, \vartheta]$, we write formula (8) as

$$G_\tau(M) = \bigcup \{ T_\tau(\Pi) : \Pi \in \mathbf{L}^Q_M \}. \tag{9}$$

We denote by $\mathbf{L}$ the list obtained by removing such half-planes from the list $\mathbf{L}^Q_M$ that can be dropped in union (9) for $\tau = \vartheta$ without changing the result of the union. Then $[\mathbf{L}]^* \sim G_\vartheta(M)$.

IV. Expansion of an ordered list of half-planes

We generalize the operation of expanding the list $\mathbf{L}_M$ to $\mathbf{L}_M^P$ in the convex case and to $\mathbf{L}_M^Q$ in the concave case. To do this, we relate each additional half-plane with the convex or concave pair of adjacent half-planes where it should be inserted.

Let an arbitrary regular list

$$\mathbf{L} = [\Pi_1, \Pi_2, \ldots, \Pi_n]$$

of half-planes be given. The boundaries of adjacent half-planes of $\mathbf{L}$ have a unique intersection point. Any pair (Π_a, Π_b) of adjacent half-planes of $\mathbf{L}$ is either convex or concave. Let η_a and η_b be unit outer normals to the half-planes Π_a and Π_b respectively.

Assume the pair (Π_a, Π_b) is convex. We denote by $I_0(\Pi_a, \Pi_b)$ the supporting half-planes to the angle $\Pi_a \cap \Pi_b$ with outer normals from $\mathcal{N}(-P)$ that lie in the interior of the cone which is spanned by η_a and η_b:

$$I_0(\Pi_a, \Pi_b) = \{\Pi_\eta[\Pi_a \cap \Pi_b] : \langle \eta, \eta_a \rangle > 0,\ \langle \eta, \eta_b \rangle > 0,\ \eta \in \mathcal{N}(-P)\}. \tag{10}$$

If the set $I_0(\Pi_a, \Pi_b)$ is neither empty nor a singleton, we arrange it to be convex, i.e. any two adjacent half-planes form a convex pair.

Assume the pair (Π_a, Π_b) is concave. In this case, the definition analogous to (10) is

$$I_0(\Pi_a, \Pi_b) = \{\Pi_\eta[\Pi_a \cup \Pi_b] : \langle \eta, \eta_a \rangle > 0,\ \langle \eta, \eta_b \rangle > 0,\ \eta \in \mathcal{N}(Q)\}. \tag{11}$$

If the set $I_0(\Pi_a, \Pi_b)$ is neither empty nor a singleton, we arrange it to be concave, i.e. any two adjacent half-planes form a concave pair.

Let us denote by $\mathbf{L}^{PQ}$ the list obtained from a list $\mathbf{L}$ by inserting the ordered set $I_0(\Pi_a, \Pi_b)$ of *additional half-planes* into each adjacent pair (Π_a, Π_b) in the list $\mathbf{L}$.

Further in the paper we use the notation $(\Pi_1, \Pi_2, \ldots, \Pi_m) \subset \mathbf{L}$ for a sequence of m adjacent half-planes of the list $\mathbf{L}$.

V. Weight of an ordered list of half-planes

In convex and concave cases, we suggest removing the half-planes from the list in the order that naturally arises when considering the set $G_\tau(M)$ defined by (4) or (9) with the parameter τ increasing from 0 to ϑ. Such an approach will allow us to generalize the algorithm.

To determine the order of removal of half-planes, a numerical characteristic (weight) is assigned to each half-plane in a regular list $\mathbf{L}$ as follows.

We consider $\Pi \in \mathbf{L}$ and $\sigma = (\Pi_a, \Pi, \Pi_b) \subset \mathbf{L}$. Let us define

$$\Delta(\sigma) := \{\tau_* > 0 :\ T_\tau(\sigma) \in \Sigma,\ \tau \in (0, \tau_*)\}, \tag{12}$$

where $T_\tau(\sigma) := (T_\tau(\Pi_a), T_\tau(\Pi), T_\tau(\Pi_b))$, $\sigma = (\Pi_a, \Pi, \Pi_b)$, $\tau \geq 0$, and Σ denote the class of all consistent triples of half-planes.

The value

$$\Omega_0^{\mathbf{L}}(\Pi) := \begin{cases} 0 & \text{if } \Delta(\sigma) = \varnothing, \\ \sup \Delta(\sigma) & \text{if } \Delta(\sigma) \neq \varnothing. \end{cases}$$

is called *weight of a half-plane* $\Pi \in \sigma \subset \mathbf{L}$ in the list $\mathbf{L}$. We have $\Omega_0^{\mathbf{L}}(\Pi) \in [0, +\infty]$.

The value

$$\Omega_0(\mathbf{L}) = \min\{\, \Omega_0^{\mathbf{L}}(\Pi) :\ \Pi \in \sigma \subset \mathbf{L}\}$$

is called *weight of the list* $\mathbf{L}$.

In the algorithm, the weight will be computed for a list of half-planes obtained by shifting some regular list $\mathbf{L} = [\Pi_1, \Pi_2, \ldots, \Pi_n]$ by the operator T_Θ, $\Theta \geq 0$, that is for the list

$$T_\Theta(\mathbf{L}) := [T_\Theta(\Pi_1), T_\Theta(\Pi_2), \ldots, T_\Theta(\Pi_n)].$$

For brevity, we introduce the notions of weight $\Omega_\Theta^{\mathbf{L}}(\Pi)$ of the half-plane $\Pi \in \sigma \subset \mathbf{L}$ and weight $\Omega_\Theta(\mathbf{L})$ of the list $\mathbf{L}$ with respect to the instant $\Theta \geq 0$:

$$\Omega_\Theta^{\mathbf{L}}(\Pi) := \Theta + \Omega_0^{\mathbf{L}}(T_\Theta(\Pi)), \quad \Omega_\Theta(\mathbf{L}) := \Theta + \Omega_0(T_\Theta(\mathbf{L})).$$

From the definitions of consistent list and weight of the list, we get the following properties.

Proposition 1. 1) *If a list* $\mathbf{L}$ *is consistent, then there exists such* $\varepsilon > 0$ *that the list* $T_\tau(\mathbf{L})$ *is consistent for all* $\tau \in [0, \varepsilon)$.

2) *If* $\Theta \geq 0$ *and* $w = \Omega_\Theta(\mathbf{L}) > 0$ *for a list* $\mathbf{L}$, *then the list* $T_{\Theta+\tau}(\mathbf{L})$ *is consistent for all* $\tau \in (0, w)$.

3) *If* $\Theta \geq 0$ *and* $w = \Omega_\Theta(\mathbf{L}) < +\infty$ *for a list* $\mathbf{L}$, *then the list* $T_{\Theta+w}(\mathbf{L})$ *is not consistent.*

VI. Algorithm for a convex/concave case

We describe the algorithm for a convex polygon M.

1. Consider the list $\mathbf{L} = \mathbf{L}_M^P$, set $\Theta = 0$ and calculate the weight $w_* = \Omega_0(\mathbf{L}_M^P)$, i.e. the value w_* is equal to the minimal weight of the half-planes in the list $\mathbf{L}_M^P$.

2. Assume $w_* \geq \vartheta$. We set $\mathbf{a} = [T_\vartheta(\mathbf{L})]^*$. Then one of the following three cases is possible.

   (a) $\widehat{\mathbf{a}} = [a_1]$. Then $G_\vartheta(M)$ is the point a_1.

   (b) $\widehat{\mathbf{a}} = [a_1, a_2]$. Then $G_\vartheta(M)$ is a linear segment with the ends a_1 and a_2.

   (c) $\widehat{\mathbf{a}} = [a_1, a_2, \ldots, a_n]$, $n > 2$. Then $G_\vartheta(M) \sim \widehat{\mathbf{a}}$, $G_\vartheta(M)$ is a convex polygon.

3. Assume now that $w_* < \vartheta$. Find a maximal sequence $I_* \subset \mathbf{L}$ of adjacent half-planes with the weight w_* and the half-planes Π_a^f and Π_b^f adjacent to I_*, i.e.

   $$I_* = (\Pi_1, \ldots, \Pi_n) \subset \mathbf{L}, \quad (\Pi_a^f, \Pi_1) \subset \mathbf{L}, \quad (\Pi_n, \Pi_b^f) \subset \mathbf{L},$$

   $\Omega_\Theta^{\mathbf{L}}(\Pi_a^f) > w_*$ if $\Pi_a^f \neq \Pi_n$, and $\Omega_\Theta^{\mathbf{L}}(\Pi_b^f) > w_*$ if $\Pi_b^f \neq \Pi_1$.

   One of the following three cases is possible.

   (a) $I_* = \mathbf{L}$. Then $G_{w_*}(M)$ is a point and $G_\vartheta(M) = \varnothing$.

   (b) $\Pi_a^f \parallel \Pi_b^f$. Then $G_{w_*}(M)$ is a linear segment and $G_\vartheta(M) = \varnothing$.

   (c) $\Pi_a^f \nparallel \Pi_b^f$. Then we remove the sequence $I_* \subset \mathbf{L}$ from the list $\mathbf{L}$ and get the list $\mathbf{L}_*$. Set $\mathbf{L} = \mathbf{L}_*$ and calculate the weight $w_* = \Omega_\Theta(\mathbf{L}_*) \in [\Theta, +\infty]$. Redefine $\Theta = w_*$ if $w_* > \Theta$. Go back to step 2.

The algorithm is finite since the initial list $\mathbf{L}_M^P$ is finite and at least one half-plane is removed from the list at each step.

In the case of convexity of the set $M' = \overline{\mathbb{R}^2 \setminus M}$, we initialize $\mathbf{L} = \mathbf{L}_M^Q$ in the first step of the algorithm, and the algorithm remains the same with replacing the interpretation of the results of each step. Namely, in cases 2a, 2b, 3a and 3b, we obtain $G_\vartheta(M) = \mathbb{R}^2$, in the case 2c we have $G_\vartheta(M) \sim \widehat{\mathbf{a}}$, and $G_\vartheta(M)$ is the complement to a convex polygon.

A detailed pseudo-code of the algorithm for a convex/concave case is given below.

Algorithm 1

Input:

Set $M \sim \mathbf{m} = [m_1, m_2, \ldots, m_{n_1}]$, $n_1 \geq 3$: M or M' is a convex polygon

Convex polygon or linear segment $P \sim [u_1, u_2, \ldots, u_{n_2}]$, $n_2 \geq 2$

Convex polygon or linear segment $Q \sim [v_1, v_2, \ldots, v_{n_3}]$, $n_3 \geq 2$

Fixed terminal time $\vartheta > 0$

Output: $\widehat{\mathbf{a}} = \varnothing$ or a finite sequence $\widehat{\mathbf{a}}$ of points that defines a point, a linear segment or a simple polygonal chain

Define the base list $\mathbf{L}_M = [\mathbf{m}]^*$ of half-planes

Expand the list $\mathbf{L}_M$ to the list $\mathbf{L}_M^{PQ}$ by inserting *additional half-planes* $I_0(\Pi_i, \Pi_{i+1})$ between any neighbours Π_i and Π_{i+1} in the list $\mathbf{L}_M$

Initialize the current list $\mathbf{L} = \mathbf{L}_M^{PQ}$ and the current backward time $\Theta = 0$

Calculate the weight $w_* = \Omega_0(\mathbf{L})$

while $w_* < \vartheta$ **do**

  Determine a maximal sequence $I_* \subset \mathbf{L}$ of half-planes with the weight w_*

  if $I_* = \mathbf{L}$ **then**

    return $\widehat{\mathbf{a}} = \varnothing$

  end if

  Determine the half-plane Π_a^f preceding I_*

  Determine the half-plane Π_b^f following I_*

  if Π_a^f and Π_b^f are not parallel **then**

    Remove I_* from $\mathbf{L}$

    Recount $\Omega_\Theta^{\mathbf{L}}(\Pi_a^f)$ and $\Omega_\Theta^{\mathbf{L}}(\Pi_b^f)$

  else ▷ Π_a^f and Π_b^f are parallel

    return $\widehat{\mathbf{a}} = \varnothing$

  end if

  Recalculate the weight $w_* = \Omega_\Theta(\mathbf{L})$

  if $w_* > \Theta$ **then**

    Set $\Theta = w_*$

  end if

end while

Calculate $\mathbf{a} = [\mathbf{L}]^*$ and $\widehat{\mathbf{a}}$

return $\widehat{\mathbf{a}}$

VII. Underlying theory for a general case

A. Basic theorems

Consider a closed set $W \subset \mathbb{R}^3 = \{(t,x)\}$. We introduce the notation

$$W(t) = \{x \in \mathbb{R}^2 : (t,x) \in W\}, \quad t \in \mathbb{R}.$$

Let $t_1 < t_2$. We say that W *belongs to the class* $\mathcal{S}[t_1,t_2]$ if for any $t_* \in [t_1,t_2)$ the following properties hold:

(S1) for any $x_* \in W(t_*)$ and $v \in Q$, there exists a solution $x(\cdot)$ to the differential inclusion $\dot{x} \in P + v$ such that $x(t_*) = x_*$ and $x(t) \in W(t)$, $t \in [t_*, t_2]$;

(S2) for any $x_* \in \overline{\mathbb{R}^2 \setminus W(t_*)}$ and $u \in P$, there exists a solution $x(\cdot)$ to the differential inclusion $\dot{x} \in u + Q$ such that $x(t_*) = x_*$ and $x(t) \in \overline{\mathbb{R}^2 \setminus W(t)}$, $t \in [t_*, t_2]$.

The definition of the class $\mathcal{S}[t_1,t_2]$ includes property (S1) of u-stability of the set W and property (S2) of v-stability of the set $\overline{\mathbb{R}^3 \setminus W}$ [9, §13.1] on the interval $[t_1,t_2]$.

The following two lemmas describe basic examples of sets from the class $\mathcal{S}[t_1,t_2]$ and are immediate consequences of (2) and properties of u- and v-stable sets.

Lemma 1. *Let* Π *be a half-plane in* $\mathbb{R}^2$*. Then*

$$K^{t_2}(\Pi) := \{(t,x) :\ t \le t_2,\ x \in T_{t_2-t}(\Pi)\} \in \mathcal{S}[t_1,t_2].$$

Lemma 2. *Let* (Π_a,Π_b) *be a convex or concave pair of half-planes in* $\mathbb{R}^2$*. We define*

$$\mathbf{A} := \{\Pi_a\} \cup \{\Pi_b\} \cup I_0(\Pi_a,\Pi_b),$$

where the set $I_0(\Pi_a,\Pi_b)$ *of half-planes is given by* (10) *for a convex pair and by* (11) *for a concave pair,*

$$\Psi^{t_2}(t) := \begin{cases} \bigcap_{\Pi\in\mathbf{A}} T_{t_2-t}(\Pi), & \text{if } (\Pi_a,\Pi_b) \text{ is convex,} \\ \bigcup_{\Pi\in\mathbf{A}} T_{t_2-t}(\Pi), & \text{if } (\Pi_a,\Pi_b) \text{ is concave.} \end{cases}$$

Then

$$K^{t_2}(\Pi_a,\Pi_b) := \{(t,x) :\ t \le t_2,\ x \in \Psi^{t_2}(t)\} \in \mathcal{S}[t_1,t_2]. \tag{13}$$

The set $K^{t_2}(\Pi)$ is an intersection of a half-space in $\mathbb{R}^3$ with the half-space $t \le t_2$. The set $K^{t_2}(\Pi_a,\Pi_b)$ is a part of an intersection or union of half-spaces included in the half-space $t \le t_2$.

A cyclic list $\mathbf{L}$ of half-planes is called *complete* if $\Delta(\sigma) = \varnothing$ for any triple $\sigma = (\Pi_a,\Pi,\Pi_b)$, where $(\Pi_a,\Pi_b) \subset \mathbf{L}$ and $\Pi \in I_0(\Pi_a,\Pi_b)$. Here, $\Delta(\sigma)$ is defined by (12).

Let us give a definition of a class $\mathcal{S}_0[t_1,t_2]$. *A set* $W \subset \mathbb{R}^3$ *belongs to the class* $\mathcal{S}_0[t_1,t_2]$ if $W(t_2)$ is a simple polygon and there exists a complete cyclic list $\mathbf{L}$ of half-planes such that for all $\tau \in (0, t_2 - t_1)$ the list $T_\tau(\mathbf{L})$ is consistent and

$$W(t_2 - \tau) \sim [T_\tau(\mathbf{L})]^* \in \mathcal{C}_0. \tag{14}$$

An intersection of a set $W \in \mathcal{S}_0[t_1,t_2]$ with the layer $t_1 \le t \le t_2$ is a polyhedron stretched along the time axis t, whose lateral faces are trapezoids or triangles adjacent along lateral edges.

Let us formulate three basic theorems, which allow us to develop a general algorithm for construction of a solvability set.

Theorem 1.
$$\mathcal{S}_0[t_1, t_2] \subset \mathcal{S}[t_1, t_2]. \tag{15}$$

The idea of the proof of Theorem 1 is as follows. At first, on the basis of Lemma 2, we prove the inclusion $W \in \mathcal{S}[t_1', t_2]$ for any $t_1' \in (t_1, t_2)$. From this, we can obtain the inclusion $W \in \mathcal{S}[t_1, t_2]$ since the sets W, P and Q are closed.

Theorem 2. *Let us suppose* $W \in \mathcal{S}_0[t_1, t_2]$. *Then* $G_{t_2-t_1}(W(t_2)) = W(t_1)$.

The proof of Theorem 2 reduces to verification of maximality of the set W as a set having the u-stability property and the section $W(t_2)$.

We denote by $\operatorname{int} A$ the set of all interior points of the set A.

Theorem 3. *Let us suppose* $t_1 < t_* < t_2$, $W_1 \in \mathcal{S}_0[t_1, t_*]$, $W_2 \in \mathcal{S}_0[t_*, t_2]$, *and* $W_1(t_*) = \overline{\operatorname{int} W_2(t_*)}$. *Then* $G_{t_2-t_1}(W_2(t_2)) = W_1(t_1)$.

This theorem gives conditions of gluing together sets from the classes $\mathcal{S}_0[t_1, t_*]$ and $\mathcal{S}_0[t_*, t_2]$ on two adjacent intervals $[t_1, t_*]$ and $[t_*, t_2]$, which define the corresponding solvability set on the union of the intervals. The proof of Theorem 3 reduces to checking the possibility for the second player to evade the set $W_2(t_*) \setminus \overline{\operatorname{int} W_2(t_*)}$ under any choice of control $u \in P$ by the first player on the interval $[t_* - \varepsilon, t_*]$ for any fixed $\varepsilon \in (0, t_* - t_1)$.

B. Idea of constructions for a general case

The algorithm for a convex/concave case forms a finite sequence

$$0 = \Theta_0 < \Theta_1 < \cdots < \Theta_m < \Theta_{m+1} = \vartheta$$

of those backward time instants when we remove half-planes from the current list $\mathbf{L}$. We denote by $\mathbf{L}_k$ the list of half-planes obtained from the initial list $\mathbf{L}_M^{PQ}$ after removing the corresponding half-planes at the instants $\Theta_0, \ldots, \Theta_k$ of backward time, $k \in \overline{0, m}$.

Let us note that, for the convex/concave case, the set

$$W := \{(t, x) : \ t \in [0, \vartheta], \ x \in G_{\vartheta - t}(M)\}$$

belongs to the class $\mathcal{S}_0[\vartheta - \Theta_{k+1}, \vartheta - \Theta_k]$ for any $k = 0, \ldots, m$, and $W(t) = \overline{\operatorname{int} W(t)}$, $t \in [0, \vartheta]$. Observe that $\Theta_{k+1} - \Theta_k = \Omega_0(T_{\Theta_k}(\mathbf{L}_k))$. The transition from the list $\mathbf{L}_k$ to the list $\mathbf{L}_{k+1}$ means removing half-planes with the weight equal to $\Theta_{k+1} = \Omega_{\Theta_k}(\mathbf{L}_k)$ from the list $\mathbf{L}_k$. Such a transformation of the list restores the consistency of the current list, while maintaining the equality

$$\bigcap \{T_{\Theta_{k+1}}(\Pi) : \Pi \in \mathbf{L}_k\} = \bigcap \{T_{\Theta_{k+1}}(\Pi) : \Pi \in \mathbf{L}_{k+1}\}$$

for the convex case and the equality

$$\bigcup \{T_{\Theta_{k+1}}(\Pi) : \Pi \in \mathbf{L}_k\} = \bigcup \{T_{\Theta_{k+1}}(\Pi) : \Pi \in \mathbf{L}_{k+1}\}$$

for the concave case. In this case, the half-planes are removed only from the convex or concave triples of the half-planes, that preserves the completeness of the list $T_{\Theta_{k+1}}(\mathbf{L}_{k+1})$.

In a general case, when either M or M' is a simple polygon, Algorithm 1 can always form the complete list $\mathbf{L}_0$ based on the initial list $\mathbf{L}_M^{PQ}$, such that $\Omega_0(\mathbf{L}_0) > 0$. Set $\Theta_1 := \min\{\vartheta, \Omega_0(\mathbf{L}_0)\}$. We have $\Omega_0(T_{\Theta_1}(\mathbf{L}_0)) = 0$.

Assume $\Theta_1 = \vartheta$ and $[T_\tau(\mathbf{L}_0)]^* \in \mathcal{C}_0$ for all $\tau \in (0, \vartheta)$. Then the solvability set $G_\vartheta(M)$ is determined on the basis of the list $\mathbf{L}_0$ by restoring the cyclic sequence $\mathbf{a} = [T_\vartheta(\mathbf{L}_0)]^*$ of points, which is transformed into a closed polygonal chain $\widehat{\mathbf{a}}$. By construction, the polygonal chain $\widehat{\mathbf{a}}$ encloses some set A. Given Theorem 2, we obtain $G_\vartheta(M) = A$ if M is a polygon, and $G_\vartheta(M) = \overline{\mathbb{R}^2 \setminus A}$ if M' is a polygon.

Assume now $\Theta_1 < \vartheta$ and $[T_\tau(\mathbf{L}_0)]^* \in \mathcal{C}_0$ for all $\tau \in (0, \Theta_1)$. Then, to continue the construction, we need to convert the list $\mathbf{L}_0$ to a complete list $\mathbf{L}_1$ such that $\Omega_0(T_{\Theta_1}(\mathbf{L}_1)) > 0$. Removing half-planes with the weight $\Theta_1 = \Omega_0(\mathbf{L}_0)$ from the list $\mathbf{L}_0$ implies their possible removal from zigzag triples, which can lead to violation of completeness of the current list. Therefore, after removing the half-planes, we have to insert some other half-planes into the list to provide completeness. In addition, according to Theorem 3, the lists $\mathbf{L}_0$ and $\mathbf{L}_1$ have to correspond to sets A_0 and A_1 connected by the equality $\overline{\operatorname{int} A_0} = A_1$, which means removing all or part of those half-planes from the list $\mathbf{L}_0$ that form overlapped edges of the polygonal chain $\widehat{\mathbf{b}}$, where $\mathbf{b} = [T_{\Theta_1}(\mathbf{L}_0)]^*$.

By analogy with forming the list $\mathbf{L}_1$ on the base of $\mathbf{L}_0$ at the instant Θ_1, one can continue the procedure of constructing the list $\mathbf{L}_k$ on the base of $\mathbf{L}_{k-1}$ at the next step (Θ_{k-1}, Θ_k) of backward time under the conditions $\Theta_k < \vartheta$ and $[T_\tau(\mathbf{L}_{k-1})]^* \in \mathcal{C}_0$, $\tau \in (\Theta_{k-1}, \Theta_k)$, $k = 2, 3, \ldots$.

The condition $[T_\tau(\mathbf{L}_{k-1})]^* \in \mathcal{C}_0$, $\tau \in (\Theta_{k-1}, \Theta_k)$, means that there are no self-intersections of the polygonal chain $[T_\tau(\mathbf{L}_{k-1})]^*$, and its verification is an independent difficult problem. If we abandon the verification, then Algorithm 1 can be generalized by complicating the transition from the list $\mathbf{L}_k$ to the list $\mathbf{L}_{k+1}$.

Suppose that the generalized algorithm have constructed a list $\mathbf{L}_m$ at the last step. Then, based on this list, one can form the polygonal chain $\widehat{\mathbf{a}}$, where $\mathbf{a} = [T_\vartheta(\mathbf{L}_m)]^*$. The polygonal chain $\widehat{\mathbf{a}}$ determines the solvability set $G_\vartheta(M)$ if it is a simple polygonal chain or the limit of simple polygonal chains $[T_{\vartheta-\varepsilon}(\mathbf{L}_m)]^*$ as $\varepsilon \to +0$. In addition, the direction (positive or negative) of passing polygonal chain $\widehat{\mathbf{a}}$ has to coincide with the direction of passing the chain $\mathbf{m} \sim M$, i.e. the corresponding polygonal chain should not be turned inside out during the transformation of the list of half-planes.

VIII. Generalized algorithm

We consider a general case of the terminal set M, where either M or $M' = \overline{\mathbb{R}^2 \setminus M}$ is a simple polygon (convex or non-convex).

A. Algorithm description

Let $M \sim \mathbf{m} = [m_1, \ldots, m_{n_1}]$ and $\mathbf{L}_M = [\mathbf{m}]^*$. The list $\mathbf{L}_M$ is consistent.

We expand the base list $\mathbf{L}_M$ to the complete list $\mathbf{L}_M^{PQ}$ by additional half-planes with outer normals from $\mathcal{N}(-P) \cup \mathcal{N}(Q)$. Any triple of successive half-planes in the list $\mathbf{L}_M^{PQ}$ is either a convex triple, or a concave one, or a zigzag triple. Additional half-planes are contained only in convex or concave triples.

The proposed algorithm for a simple polygon M is as follows.

1. Consider the list $\mathbf{L} = \mathbf{L}_M^{PQ}$, set $\Theta = 0$ and calculate the weight $w_* = \Omega_0(\mathbf{L})$, i.e. the value w_* is equal to the minimal weight of the half-planes in the list $\mathbf{L}$.

2. Assume $w_* \geq \vartheta$. We define the closed polygonal chain $\mathbf{a} = [T_\vartheta(\mathbf{L})]^*$. Then one of the following three cases is possible.

   (a) $\widehat{\mathbf{a}} = [a_1]$. Then $G_\vartheta(M)$ is the point a_1.

   (b) $\widehat{\mathbf{a}} = [a_1, a_2]$. Then $G_\vartheta(M)$ is a linear segment with the ends a_1 and a_2.

   (c) $\widehat{\mathbf{a}} = [a_1, a_2, \dots, a_n]$, $n > 2$.
   If $\widehat{\mathbf{a}} \in \mathcal{C}_0$ or $\widehat{\mathbf{a}}$ is a limit of simple polygonal chains $[T_{\vartheta-\varepsilon}(\mathbf{L}_M)]^*$ as $\varepsilon \to +0$, and the direction of passing polygonal chain $\widehat{\mathbf{a}}$ coincides with the direction of passing the chain $\mathbf{m}$, then $\widehat{\mathbf{a}}$ encloses the solvability set $G_\vartheta(M)$.
   Otherwise, there is a violation of connection or simple connection of the solvability set $G_\tau(M)$ for some $\tau \in (0, \vartheta)$, while the solvability sets are connected and simply connected for smaller terminal times. In this case, the algorithm can compute the solvability set $G_\tau(M)$ in the problem with the fixed terminal time τ. Further calculation requires splitting boundary $\partial G_\tau(M)$ of the resulting set into connected components and applying the algorithm separately to each component.

3. Assume now that $w_* < \vartheta$. Find a maximal sequence $I_* \subset \mathbf{L}$ of adjacent half-planes with the weight w_*, the half-plane Π^f_a preceding I_*, and the half-plane Π^f_b following I_*. One of the options listed below is possible.

   (a) $I_* = \mathbf{L}$. Then $G_{w_*}(M)$ is a point and $G_\vartheta(M) = \varnothing$.

   (b) $\Pi^f_a \parallel \Pi^f_b$. Then the half-planes Π^f_a and Π^f_b, shifted by the operator T_{w_*}, form edges that are partially or completely overlapped. We find the half-plane Π^s_a preceding Π^f_a and the half-plane Π^s_b following Π^f_b in the list $\mathbf{L}$. Set $\mathbf{b} := [T_{w_*}(\mathbf{L})]^*$.

      i. Assume that the half-planes $T_{w_*}(\Pi^s_a)$ and $T_{w_*}(\Pi^s_b)$ form other edges of the polygonal chain $\widehat{\mathbf{b}}$ that are partially or completely overlapped. Let $(\Pi^s_a)^\perp$ be a half-plane obtained by rotating the half-plane Π^s_a by $\pi/2$ counter-clockwise. Remove the collection $I_* \cup \Pi^f_a \cup \Pi^f_b$ from the list $\mathbf{L}$ and insert the half-plane $(\Pi^s_a)^\perp$ between Π^s_a and Π^s_b, assigning the weight w_* to it. This ensures that the overlap of edges corresponding to Π^s_a and Π^s_b will be removed at the next iteration of the algorithm.

      ii. Otherwise, analysing the arrangement of the half-planes
      $$T_{w_*}(\Pi^f_a), \quad T_{w_*}(\Pi^s_a), \quad T_{w_*}(\Pi^s_b), \quad T_{w_*}(\Pi^f_b),$$
      we find a collection $I \supset I_*$ of successive half-planes, the removal of which leads to disappearance of the two overlapping edges of the polygonal chain $\widehat{\mathbf{b}}$ corresponding to the half-planes Π^f_a and Π^f_b, while maintaining the regularity of the list. Find the half-plane Π^I_a preceding the collection I and the half-plane Π^I_b following I in the list $\mathbf{L}$. Remove I from the list. Then the half-planes Π^I_a and Π^I_b form a new convex or concave pair. Insert between Π^I_a and Π^I_b the sequence $I_\Theta(\Pi^I_a, \Pi^I_b)$ of additional half-planes, defined by formula
      $$I_\Theta(\Pi_a, \Pi_b) := \{\Pi : T_\Theta(\Pi) \in I_0(T_\Theta(\Pi_a), T_\Theta(\Pi_b))\}. \tag{16}$$
      If the set $I_\Theta(\Pi_a, \Pi_b)$ is neither empty nor a singleton, we arrange it to be convex (concave), if the pair (Π_a, Π_b) is convex (concave).

(c) $\Pi_a^f \nparallel \Pi_b^f$. Then we remove the sequence I_* from the list $\mathbf{L}$. If the sequence I_* is neither convex nor concave, then we insert additional half-planes $I_\Theta(\Pi_a^f, \Pi_b^f)$, defined by (16), between the half-planes Π_a^f and Π_b^f .

Let us the resulting list is denoted by $\mathbf{L}_*$. Reset $\mathbf{L} = \mathbf{L}_*$ and calculate the weight $w_* = \Omega_\Theta(\mathbf{L}_*) \in [\Theta, +\infty]$. Redefine $\Theta = w_*$ if $w_* > \Theta$. Go back to step 2.

If the set M' is a convex polygon, then the algorithm remains the same with replacing the interpretation of the results of each step. In the case $\widehat{\mathbf{a}} = \varnothing$, we obtain $G_\vartheta = \mathbb{R}^2$. If $\widehat{\mathbf{a}}$ is a simple polygonal chain or the limit of simple polygonal chains, and the direction of passing it coincides with the direction of passing $\mathbf{m}$, then $\widehat{\mathbf{a}}$ encloses the complement $G'_\vartheta(M)$ of the solvability set $G_\vartheta(M)$. Otherwise, there is a violation of connection or simple connection of the complement $G'_\tau(M)$ of the solvability set $G_\tau(M)$ in the problem with fixed terminal instant τ for some $\tau \in (0, \vartheta)$. This requires applying the algorithm separately to connected components of the boundary $\partial G_\tau(M)$.

B. Finiteness of the general algorithm

Let us show that the generalized algorithm is finite.

Consider a cyclic list $\mathbf{L} = \{\Pi_1, \Pi_2, \dots, \Pi_n\}$ of half-planes, $n \geq 2$. Set

$$\alpha(\mathbf{L}) := \sum_{j=1}^{n} |\alpha_j|,$$

where α_j is the angle measured from the outer normal to the half-plane Π_j to the outer normal to the half-plane Π_{j+1} taking into account the direction of rotation, i.e. $\alpha_i > 0$ corresponds to the counterclockwise rotation, and $\alpha_i < 0$ corresponds to the clockwise rotation, $\Pi_{n+1} = \Pi_1$. We always have $\alpha(\mathbf{L}) \geq 2\pi$ for a cyclic list $\mathbf{L}$.

We denote by α_0 the smallest modulus of the angle between the outer normals in the set $\mathcal{N}(M) \cup \mathcal{N}(-P) \cup \mathcal{N}(Q)$, while the coinciding normals are considered to be one normal. We have $0 < |\alpha_0| < 2\pi$.

Now, we initialize $\mathbf{L} = \mathbf{L}_M$, where $\mathbf{L}_M$ is the base list of the terminal set M, and then start to change the list in accordance with the algorithm. We note that insertion of additional half-spaces into the list $\mathbf{L}$ does not change the value $\alpha(\mathbf{L})$. Removing the central half-plane from a convex or concave triple $\sigma \in \mathbf{L}$ also preserves the value $\alpha(\mathbf{L})$.

We show that removing the central half-plane from a zigzag triple reduces $\alpha(\mathbf{L})$ more than by $|\alpha_0|$.

Proposition 2. *Let $\sigma = (\Pi_a, \Pi, \Pi_b) \in \mathbf{L}$ and σ be a zigzag triple. Then*

$$\alpha(\mathbf{L}) - \alpha(\mathbf{L} \setminus \{\Pi\}) > |\alpha_0|.$$

Proof. Let η_a, η, and η_b be the outer normals to the half-planes Π_a, Π, and Π_b, respectively; and β_1, β_2, β_3 be the angles between η_a and η, η and η_b, η_a and η_b, respectively. Then removing the half-plane Π from the list $\mathbf{L}$ means replacing the sum $|\beta_1| + |\beta_2|$ in the definition of $\alpha(\mathbf{L})$ by the value $|\beta_3|$, i.e.

$$\alpha(\mathbf{L} \setminus \{\Pi\}) = \alpha(\mathbf{L}) - (|\beta_1| + |\beta_2|) + |\beta_3|.$$

Let us prove that

$$|\beta_3| < \max\{|\beta_1|, |\beta_2|\}.$$

Assume $|\beta_1| \leq |\beta_2|$. We have $\beta_3 = \beta_1 + \beta_2$ by definition, and $\beta_1\beta_2 < 0$ for a zigzag triple. Then

$$|\beta_3| = |\beta_1 + \beta_2| < |\beta_2| = \max\{|\beta_1|, |\beta_2|\}.$$

In the case of $|\beta_1| > |\beta_2|$, the proof is similar.

Therefore,

$$\alpha(\mathbf{L}) - \alpha(\mathbf{L} \setminus \{\Pi\}) = |\beta_1| + |\beta_2| - |\beta_3| > \min\{|\beta_1|, |\beta_2|\} > |\alpha_0|.$$

□

Let us prove the finiteness of the algorithm by contradiction. An infinite sequence of steps is possible only if, at countably many steps, the list is added no less half-planes, than it is removed. Since half-planes are removed at each step, and are added only after removing a half-plane from a zigzag triple, the value $\alpha(\mathbf{L})$ decreases more than by $|\alpha_0|$ at a countable number of steps starting from the value $\alpha(\mathbf{L}_M)$. By construction of the algorithm, the current list $\mathbf{L}$ has more than one element. This implies $\alpha(\mathbf{L}) \geq 2\pi$, that contradicts reducing the value $\alpha(\mathbf{L}_M)$ more than by $|\alpha_0|$ at a countable number of steps. Thus, we have finiteness of the algorithm.

C. Pseudo-code for a general case

A detailed pseudo-code of the proposed generalized algorithm is given below.

Let $h^\perp$ denote a half-plane obtained by rotating a half-plane h, defined as a vector $\overrightarrow{ab}$, around the point a by $\pi/2$ counterclockwise.

Algorithm 2

Input:
Set $M \sim \mathbf{m} = [m_1, m_2, \ldots, m_{n_1}]$, $n_1 \geq 3$: M or M' is a simple polygon (may be non-convex)
Convex polygon or linear segment $P \sim [u_1, u_2, \ldots, u_{n_2}]$, $n_2 \geq 2$
Convex polygon or linear segment $Q \sim [v_1, v_2, \ldots, v_{n_3}]$, $n_3 \geq 2$
Fixed terminal time $\vartheta > 0$

Output: $\widehat{\mathbf{a}} = \varnothing$ or a finite sequence $\widehat{\mathbf{a}}$ of points that defines a point or a closed polygonal chain

Define the base list $\mathbf{L}_M = [\mathbf{m}]^*$ of half-planes
Expand the list $\mathbf{L}_M$ to the list $\mathbf{L}_M^{PQ}$ by inserting *additional half-planes* $I_0(\Pi_i, \Pi_{i+1})$ between any neighbours Π_i and Π_{i+1} in the list $\mathbf{L}_M$
Initialize the current list $\mathbf{L} = \mathbf{L}_M^{PQ}$ and the current backward time $\Theta = 0$
Calculate the weight $w_* = \Omega_0(\mathbf{L})$
while $w_* < \vartheta$ **do**
    Determine a maximal sequence $I_* \subset \mathbf{L}$ of adjacent half-planes with the weight w_*
    if $I_* = \mathbf{L}$ **then**
        return $\widehat{\mathbf{a}} = \varnothing$
    end if
    Determine the half-plane Π_a^f preceding I_*
    Determine the half-plane Π_b^f following I_*
    if Π_a^f and Π_b^f are not parallel **then**
        Remove I_* from $\mathbf{L}$

Recount $\Omega^{\mathbf{L}}_{\Theta}(\Pi^f_a)$ and $\Omega^{\mathbf{L}}_{\Theta}(\Pi^f_b)$
▷ Start of the 1st insertion into Algorithm 1
if I_* is neither convex nor concave **then**
Insert *additional half-planes* $I_{\Theta}(\Pi^f_a, \Pi^f_b)$ between Π^f_a and Π^f_b
end if
▷ End of the 1st insertion into Algorithm 1
else ▷ Π^f_a and Π^f_b are parallel
▷ Start of the 2nd insertion into Algorithm 1
Determine the half-plane Π^s_a preceding Π^f_a
Determine the half-plane Π^s_b following Π^f_b
Compute $c_1 = \partial T_{w_*}(\Pi^f_a) \cap \partial T_{w_*}(\Pi^s_a)$
Compute $c_2 = \partial T_{w_*}(\Pi^f_b) \cap \partial T_{w_*}(\Pi^s_b)$
if $c_1 \neq c_2$ **then**
if Π^f_a and Π^f_b are collinear **then**
Remove the half-planes $I_* \cup \{\Pi^f_a\}$ from $\mathbf{L}$
else ▷ Π^f_a and Π^f_b are not collinear
if c_1 is “below” c_2 **then**
Remove the half-planes $I_* \cup \{\Pi^f_b\}$ from $\mathbf{L}$
Insert *additional half-planes* $I_{\Theta}(\Pi^f_a, \Pi^s_b)$ between Π^f_a and Π^s_b
Recount $\Omega^{\mathbf{L}}_{\Theta}(\Pi^f_a)$ and $\Omega^{\mathbf{L}}_{\Theta}(\Pi^s_b)$
else ▷ c_1 is “above” c_2
Remove the half-planes $I_* \cup \{\Pi^f_a\}$ from $\mathbf{L}$
Insert *additional half-planes* $I_{\Theta}(\Pi^s_a, \Pi^f_b)$ between Π^s_a and Π^f_b
Recount $\Omega^{\mathbf{L}}_{\Theta}(\Pi^s_a)$ and $\Omega^{\mathbf{L}}_{\Theta}(\Pi^f_b)$
end if
end if
else ▷ $c_1 = c_2$
Set $\mathbf{L}_* = \mathbf{L} \setminus (I_* \cup \{\Pi^s_a, \Pi^f_a, \Pi^f_b, \Pi^s_b\})$
if there are no elements with greater weight in $\mathbf{L}_*$ **then**
return $\widehat{\mathbf{a}} = \varnothing$
end if
if Π^s_a and Π^s_b are not parallel **then**
Remove the half-planes $I_* \cup \{\Pi^f_a, \Pi^f_b\}$ from $\mathbf{L}$
Insert *additional half-planes* $I_{\Theta}(\Pi^s_a, \Pi^s_b)$ between Π^s_a and Π^s_b
Recount $\Omega^{\mathbf{L}}_{\Theta}(\Pi^s_a)$ and $\Omega^{\mathbf{L}}_{\Theta}(\Pi^s_b)$
else ▷ Π^s_a and Π^s_b are parallel
if Π^s_a and Π^s_b are collinear **then**
Remove the half-planes $I_* \cup \{\Pi^s_a, \Pi^f_a, \Pi^f_b\}$ from $\mathbf{L}$
else ▷ Π^s_a and Π^s_b are not collinear
Remove the half-planes $I_* \cup \{\Pi^f_a, \Pi^f_b\}$ from $\mathbf{L}$
Insert the half-plane $(\Pi^s_a)^{\perp}$ between Π^s_a and Π^s_b
Put the weight w_* to the half-plane $(\Pi^s_a)^{\perp}$
end if
end if
end if
▷ End of the 2nd insertion into Algorithm 1
end if
Recalculate the weight $w_* = \Omega_{\Theta}(\mathbf{L})$

    if $w_* > \Theta$ **then**
        Set $\Theta = w_*$
    end if
end while
Calculate $\mathbf{a} = [\mathbf{L}]^*$ and $\widehat{\mathbf{a}}$
return $\widehat{\mathbf{a}}$

IX. Examples

Four examples of numerical computations of solvability sets for non-convex terminal polygons are calculated. The sets M, P, Q and $G_\vartheta(M)$ are given in Figs. 2–5. Dashed lines show restored sets at instants of reconstruction of the current list of half-planes. In Examples 1 and 2, additional half-planes take part in the solution. Examples 3 and 4 illustrate removing overlap edges of intermediate polygonal chains. In Example 4, we get empty set for $\vartheta > 3$.

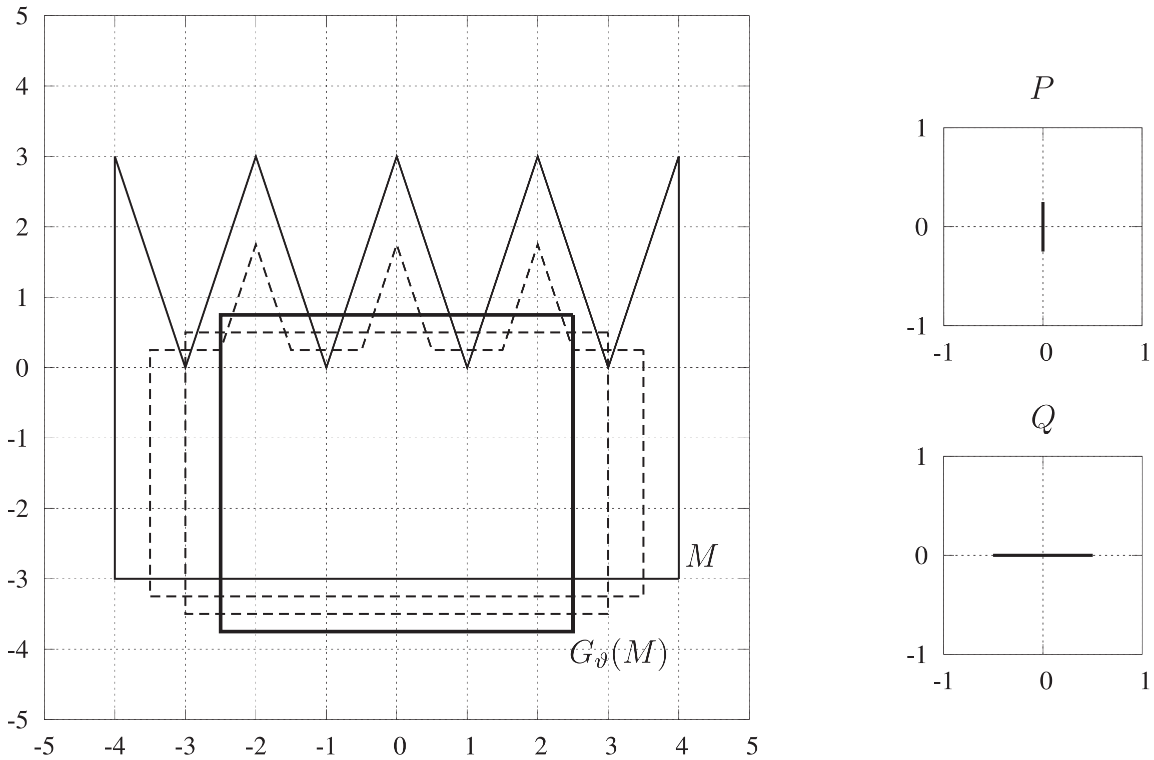

Fig. 2. Example 1 showing the terminal polygon M (solid line) and the boundary of $G_\vartheta(M)$ (solid bold line) computed for $\vartheta = 3.0$

Conclusion

Three main specifications of the problem are essential in the paper: the control system is described by simple motion in the plane; terminal time of the game is fixed; the terminal set and constraints of the players' controls are polygonal. An important, but difficult to verify, assumption for proper applying the developed algorithm is that the current polygonal chain is not rigidly self-crossing in the interval of calculations. Upgrading of the algorithm to remove this assumption seems to be an independent and difficult problem.

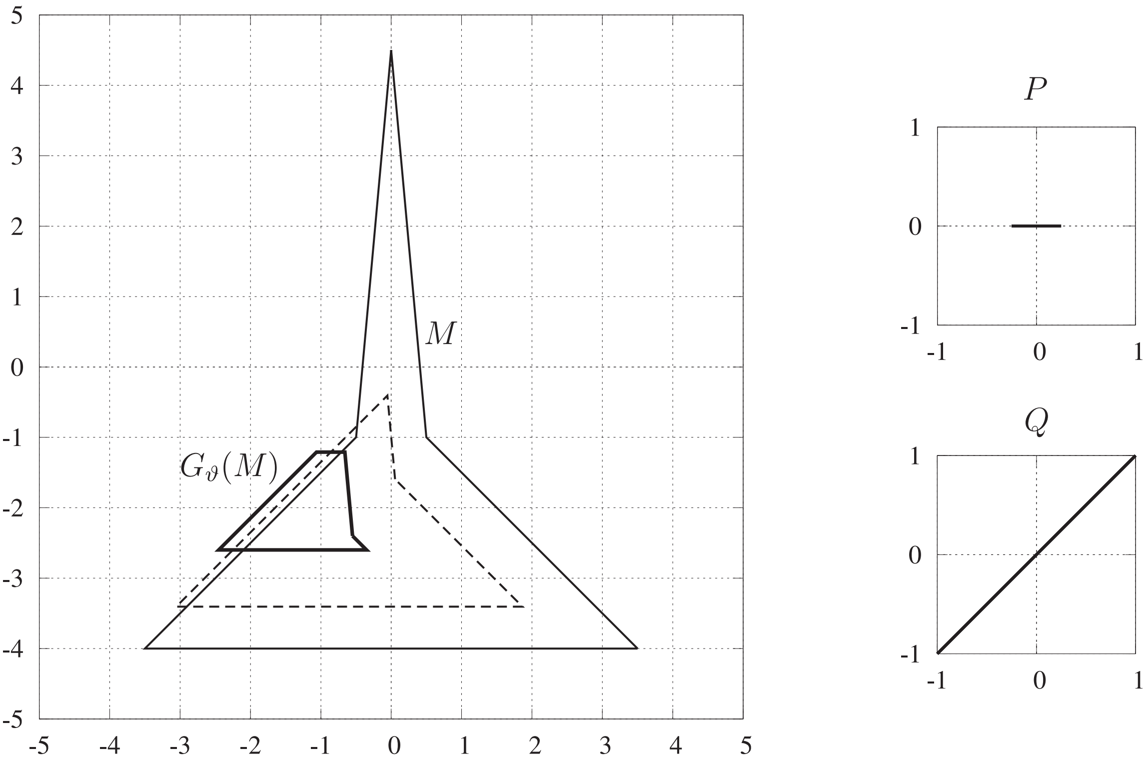

Fig. 3. Example 2 showing the terminal polygon M (solid line) and the boundary of $G_\vartheta(M)$ (solid bold line) computed for $\vartheta = 1.4$

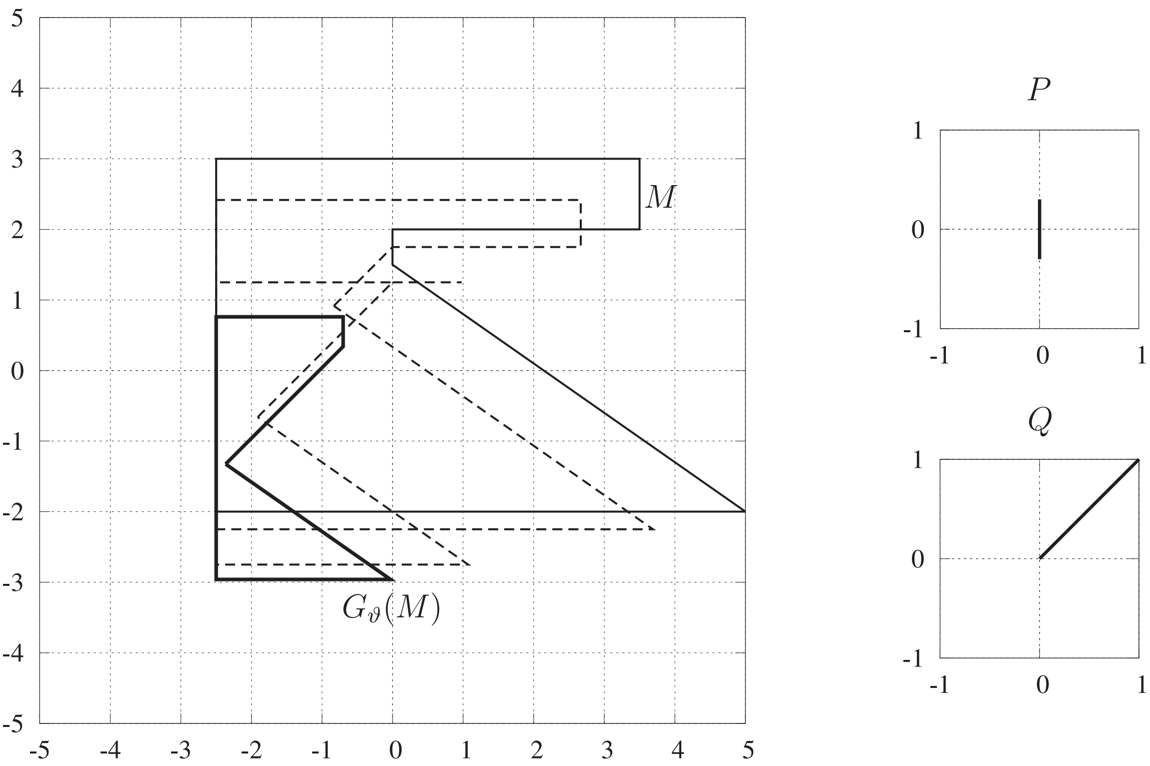

Fig. 4. Example 3 showing the terminal polygon M (solid line) and the boundary of $G_\vartheta(M)$ (solid bold line) computed for $\vartheta = 3.2$

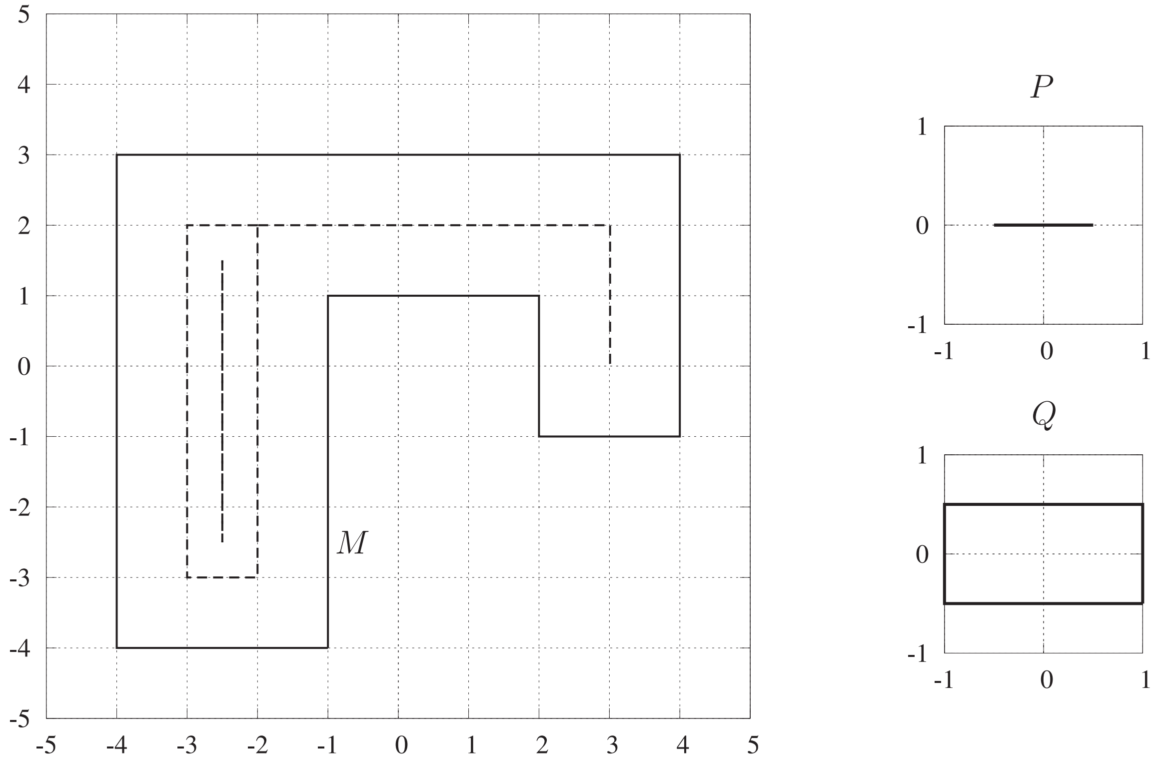

Fig. 5. Example 4 showing the terminal polygon M (solid line) and intermediate polygonal chains. The algorithm returns $G_\vartheta(M) = \varnothing$ for $\vartheta = 3.5$

Acknowledgments

The author thanks V.S. Patsko for the discussion of the work and useful comments. This work was supported by the Russian Foundation for Basic Research under project no. 18-01-00410.

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