Convergence of Numerical Method for Time-Optimal Differential Games with Lifeline

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1 Introduction

This paper discusses time-optimal differential games with lifeline and numerical scheme constructing the value function for such games. In games of this type, the first player tends to lead the system to a prescribed closed target set while keeping the trajectory inside some open set where the game takes place. The second player hinders this, because it wins as soon as either the trajectory of the system leaves this open set not touching the target one, or it succeeds in keeping the system infinitely inside this open set.

Apparently, the first, who formulated a problem with lifeline, was R. Isaacs in his book [20]. In his definitions, the *lifeline* is a set, after the reaching of which the second player wins unconditionally. Significant contribution into research of games with lifeline was made by L.A. Petrosyan (see e.g., [28]). However, the authors do not know works, which would consider exhaustively games of this sort: L.A. Petrosyan researched mostly problems with simple motion dynamics, that is, the problems where the players' controls are the velocities of the objects. In books [21, 22] of N.N. Krasovskii and A.I. Subbotin, games with lifeline are analyzed as problems with state constraints: the first player is not supposed to lead the system outside a prescribed set. Also, problems with state constraints have been studied by many authors (see, for example, [3, 10, 11, 19, 29]).

Problems very close to games with lifeline have been studied by French authors P. Cardaliaguet, M. Quincampoix, P. Saint-Pierre [12–15]. For controlled systems on

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the basis of the set-valued analysis, the theory of differential inclusions, and the theory of viability, they analyze the sets where the controller is able to keep the system infinitely (viability kernels). Passing to games, the authors consider a situation with two target sets for the first and second players, respectively, to which the players try to guide the system avoiding the target of the opposite player. Another variant considered in these works is games with state constraints for the first player. In these situations, the main objectives are to study victory domains of the players, that is, the sets wherefrom the corresponding player can reach its target without hitting the target of the opposite player (or state constraints). Also, in the terms of viability, the upper value function of such games (the guaranteed result of the first player) is characterized as a function, which epigraph is a viability set of the first player. Grid-geometric algorithms are suggested for approximation of viability kernels and, therefore, for approximation of the upper value. However, we have not found papers of these authors where existence of the value function is proved for games of this type and/or its coincidence with generalized solution of the corresponding boundary value problem of a HJE is justified (although such a connection is mentioned).

The main boost that stimulated the authors to study time-optimal games with lifeline is the investigation of questions connected with numerical methods for solving classic time-optimal games. In particular, in works [1, 2], Italian mathematicians M. Bardi and M. Falcone together with their colleagues suggested a theoretic numerical method for constructing the value function of a time-optimal game (without lifeline) as a generalized (viscosity) solution of the corresponding boundary value problem for HJE. The suggested procedure is of a grid type, and its proof is made in assumption that the grid is infinite and covers the entire game space. But practical computer realization, apparently, deals with a finite grid, which covers only a bounded part of the game space. So, the problem arises what boundary condition to set on the outer boundary of the domain covered by the grid. M. Bardi and M. Falcone suggest to set these conditions to plus infinity, that is, actually declaring that the second player wins when reaching the outer boundary of this domain. Therefore, the practical realization of the procedure solves a game with lifeline. That is why the authors decided to fill this gap connected to the problems with lifeline in a very general formulation.

Also, there is one more grid method for solving time-optimal problems suggested by authors from Germany. In works by N. Botkin, K.-H. Hoffmann, V. Turova, and their colleagues, a numerical procedure is suggested, which is based on a so-called *upwind operator* involving approximations for left and right partial derivatives of the value function in a node (see, for example, [7–9]). This algorithm is applicable to problems with state constraints for the first player, which can be treated as problems with lifeline.

This paper provides a numerical method for constructing the value function of a time-optimal game with lifeline as a viscosity solution of the corresponding boundary value problem for HJE. A pointwise convergence of the numerical method to the value function is proved. The method is just the one suggested by the Italian mathematicians, but its convergence should be proved anew in the framework of the new formulation. Also, theorems on coincidence of the value functions of time-optimal problems with and without lifeline are proved under a very important assumption that the value function is continuous in the domain where the game takes place. The coincidence of the limit of discrete numerical solutions with the value function needs such a continuity. The continuity can be derived, in particular, from the assumptions of the local dynamic advantage of one player over another near their sets: if the system position is close to the target set, then the first player can guide the system to this set; vice versa, if the system is close to the lifeline, then the second player can push it to the lifeline. These assumptions have been taken for the proof of existence of the generalized solution justified in other papers [24–26] by the authors.

The structure of this paper is as follows. In Sect. 2, the formulation of the problem is given. Section 3 deals with the formulation of the numerical scheme and the convergence of computations performed according to it. In Sect. 4, a proof of convergence of the functions obtained as a result of the computations to the viscosity solution of the corresponding boundary value problem for the HJE coincides with the value function of the original game. Section 5 contains discussion on coincidence of the value function of time-optimal differential games with and without lifeline. In Sect. 6, one can see results of numerical computations performed by the realization of the numerical procedure. The paper is finalized by a conclusion.

2 Problem Formulation

Let us consider a conflict controlled system

$$\dot{x} = f(x, a, b), \qquad t \geq 0,\ a \in A,\ b \in B, \tag{1}$$

where $x \in \mathbb{R}^n$ is the phase vector of the system; a and b are the controls of the first and second players constrained by the compact sets A and B in their Euclidean spaces. We are given a compact set $\mathscr{T}$ and an open set $\mathscr{W} \subset \mathbb{R}^n$ such that $\mathscr{T} \subset \mathscr{W}$ and the boundary $\partial\mathscr{W}$ is bounded. Denote $\mathscr{G} := \mathscr{W} \setminus \mathscr{T}$ and $\mathscr{F} := \mathbb{R}^n \setminus \mathscr{W}$ (see Fig. 1). The game takes place in the set $\mathscr{G}$; the objective of the first player is to guide the system to the set $\mathscr{T}$ as soon as possible keeping the trajectory outside the set $\mathscr{F}$; the objective of the second player is to guide the system to the set $\mathscr{F}$, or if it is impossible, to keep the trajectory inside the set $\mathscr{G}$ forever, or if the latter is impossible too, to postpone reaching the set $\mathscr{T}$ as long as he can.

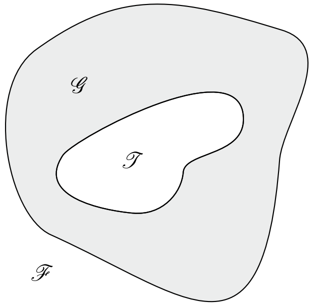
Such a game can be called a *game with lifeline*; the boundary $\partial\mathscr{F}$ of the set $\mathscr{F}$ is the lifeline where the second player wins unconditionally.

We assume that the following conditions are fulfilled:

C.1 The function $f : \mathbb{R}^n \times A \times B \to \mathbb{R}^n$ is continuous in all variables and Lipschitz continuous in the variable x: for all $x^{(1)}, x^{(2)} \in \mathbb{R}^n$, $a \in A$, $b \in B$

$$\big\| f(x^{(1)}, a, b) - f(x^{(2)}, a, b)\big\| \leq L\|x^{(1)} - x^{(2)}\|; \tag{2}$$

Fig. 1 Sets $\mathscr{T}$, $\mathscr{F}$, and $\mathscr{G}$



moreover, it satisfies Isaacs' condition:

$$\min_{a\in A}\max_{b\in B}\big\langle p,\, f(x,a,b)\big\rangle = \max_{b\in B}\min_{a\in A}\big\langle p,\, f(x,a,b)\big\rangle \quad \forall p \in \mathbb{R}^n. \tag{3}$$

Here and below, the symbol $\langle\cdot,\cdot\rangle$ stands for the scalar product.

C.2 The boundary $\partial\mathscr{G}$ of the set $\mathscr{G}$ (that is the boundaries $\partial\mathscr{T}$ and $\partial\mathscr{F}$) is compact, smooth, and has a bounded curvature.
Remark. In our previous paper [26], we do not demand the boundedness of the curvature of $\mathscr{G}$. When that paper was written, we thought that a sufficient smoothness of the boundary provides the boundedness of its curvature. It is necessary to prove existence of a generalized solution of the corresponding boundary problem of a Hamilton–Jacobi equation. However, after consultations with specialists in topology, it turned out that even infinitely smooth bounded curve in the plane can have an unbounded curvature. So, this demand should be formulated explicitly.

C.3 One can find a constant $c > 0$ and a bounded uniformly continuous function $\eta : \mathrm{cl}\,\mathscr{G} \cap O(\partial\mathscr{G}, c) \to \mathbb{R}^n$ such that the embedding $O\big(x + t\eta(x), ct\big) \subseteq \mathscr{G}$ is true for all $x \in \mathrm{cl}\,\mathscr{G} \cap O(\partial\mathscr{G}, c)$ and $0 < t \le c$. Here and below, $O(y, r)$ is an open ball of the radius r with the center at the point y, $O(X, r) := \big\{x : \mathrm{dist}(x, X) < r\big\}$ and $O(\varnothing, R) = \varnothing$.
Remark. It seems to us that the latter condition C.3 follows from the previous one C.2, but now we have no proof of this implication. So, we explicitly demand existence of the function η, which is called the generalized normal.

The players' aims of the mentioned kind can be formalized in the following way. Let the function $x(\cdot; x_0)$ be a trajectory of the system emanated from the initial point $x(0) = x_0$. We consider two instants

$$t_* = t_*\big(x(\cdot, x_0)\big) = \min\big\{t \ge 0 : x(t; x_0) \in \mathscr{T}\big\},$$
$$t^* = t^*\big(x(\cdot, x_0)\big) = \min\big\{t \ge 0 : x(t; x_0) \in \mathscr{F}\big\},$$

which are the instants when the trajectory $x(\cdot; x_0)$ hits for the first time the sets $\mathscr{T}$ and $\mathscr{F}$, respectively. If the trajectory doesn't arrive at the set $\mathscr{T}$ ($\mathscr{F}$), then the value t_* (t^*) is equal to $+\infty$.

To say what is a system trajectory, one can use either the formalization with nonanticipating strategies, or the positional formalization of N.N. Krasovskii and A.I. Subbotin [21, 22]. In the latter case, the feedback strategies of the first and the second player are functions $a(\cdot) : \mathbb{R}^n \to A$ and $b(\cdot) : \mathbb{R}^n \to B$, respectively.

We define the result of the game on the trajectory $x(\cdot; x_0)$ as

$$\tau\big(x(\cdot; x_0)\big) = \begin{cases} +\infty, & \text{if } t_* = +\infty \text{ or } t^* < t_*, \\ t_*, & \text{otherwise.} \end{cases} \tag{4}$$

In [23], the authors prove that a time-optimal problem with lifeline has the value function $T(x)$.

The unboundedness of the value function and cost functional can cause some uneasiness of a numerical research of game (1), (4). For this reason, one often substitutes the unbounded cost functional with a bounded one by means of the *Kruzhkov's transform*:

$$J\big(x(\cdot, x_0)\big) = \begin{cases} 1 - \exp\big(-\tau\big(x(\cdot; x_0)\big)\big), & \text{if } \tau\big(x(\cdot; x_0)\big) < +\infty, \\ 1, & \text{otherwise.} \end{cases} \tag{5}$$

In such a case, the value function $v(x)$ also becomes bounded and its magnitude belongs to the range from zero to one:

$$v(x) = \begin{cases} 1 - \exp\big(- T(x)\big), & \text{if } T(x) < +\infty, \\ 1, & \text{otherwise.} \end{cases} \tag{6}$$

3 Numerical Scheme

In general, the numerical scheme construction and justification of its convergence are analogous to the ones in paper [2] where the numerical scheme for the classic time-optimal problem is constructed and its convergence is proved. Herewith, the value function is characterized as the unique generalized (viscosity) solution of the corresponding boundary value problem for HJE.

3.1 Discrete Scheme

Let us replace the continuous dynamics with a discrete one by the time step $h > 0$:

$$x_n = x_{n-1} + hf(x_{n-1}, a_{n-1}, b_{n-1}), \; n = 1, \ldots, N, \quad x_0 \text{ is given},$$

where $a_n \in A$ and $b_n \in B$.

By the discrete Dynamic Programming Principle, one can get the following characterization for the value function $w_h(\cdot)$ of the discrete time problem:

$$w_h(x) = \begin{cases} \gamma \max\limits_{b\in B} \min\limits_{a\in A} w_h\big(z(x,a,b)\big) + 1 - \gamma, & \text{if } x \in \mathscr{G}, \\ 0, & \text{if } x \in \mathscr{T}, \\ 1, & \text{if } x \in \mathscr{F}. \end{cases}$$

Here, $\gamma = e^{-h}$, $z(x,a,b) = x + hf(x,a,b)$.

Further, let us describe the space discretization. Let us consider a grid $\mathscr{L}$ with the step k, which covers the entire space $\mathbb{R}^n$ and consists of nodes $q_{i_1,\ldots,i_n} = (x_{i_1},\ldots,x_{i_n})$, $i_1,\ldots,i_n \in \mathbb{Z}$, $x_{i_j} = ki_j$. (Generally speaking, steps along different axes can differ, but this fact doesn't change the main idea of the numerical scheme construction.) Here and below, mostly, a linear indexation q_ν, $\nu \in \mathbb{Z}$, for the nodes of the grid $\mathscr{L}$ is used. The symbol $\mathscr{L}_\mathscr{T}$ stands for the set of those nodes of the grid $\mathscr{L}$, which belong to the set $\mathscr{T}$; the symbol $\mathscr{L}_\mathscr{G}$ denotes the collection of nodes falling into the set $\mathscr{G}$; and the symbol $\mathscr{L}_\mathscr{F}$ stands for the set of nodes from the set $\mathscr{F}$. In theoretical constructions, the grid is assumed infinite.

For every point $x \in \mathbb{R}^n$, one can find a simplex $S(x)$ with vertices $\big\{q_l(x)\big\}$ from $\mathscr{L}$ such that the point x belongs to the simplex $S(x)$ and $S(x)$ does not contain other nodes of the grid. It is assumed that with choosing the grid $\mathscr{L}$, we also choose a separation of the game space to simplices with their vertices at nodes of the grid. On the basis of $S(x)$, one can obtain the *barycentric (local) coordinates* $\lambda_l(x)$ of the point x with respect to the vertices $q_l(x)$ of the simplex $S(x)$:

$$x = \sum_{l=1}^{n+1} \lambda_l(x) q_l(x), \qquad \lambda_l(x) \geq 0,\ \sum_{l=1}^{n+1} \lambda_l(x) = 1.$$

Sometimes, the arguments of the coefficients λ and vertices q will be omitted if they are clear from the context.

Let us substitute the function $w_h(\cdot)$ with a new one $w(\cdot)$, which magnitudes $w(q_\nu)$ at the nodes q_ν of the grid $\mathscr{L}$ form an infinite vector $W = \big(w(q_\nu)\big)_{\nu\in\mathbb{Z}}$. The magnitude $w(x)$ at some point x, which is not a node of the grid, can be reconstructed by means of the following piecewise-linear approximation based on the local coordinates of the point x:

$$w_{loc}(x, W) = \sum_{l=1}^{n+1} \lambda_l(x)\, w\big(q_l(x)\big). \tag{7}$$

Hereby, the characterization of the value function of a fully discrete problem is obtained:

$$w(q_\nu) = \begin{cases} \gamma \max\limits_{b\in B} \min\limits_{a\in A} w_{loc}\big(z(q_\nu, a, b), W\big) + 1 - \gamma, & \text{if } q_\nu \in \mathscr{L}_{\mathscr{G}}, \\ 0, & \text{if } q_\nu \in \mathscr{L}_{\mathscr{T}}, \\ 1, & \text{if } q_\nu \in \mathscr{L}_{\mathscr{F}}. \end{cases}$$

This characterization is of a recursive kind, because the magnitude $w(q_\nu)$ at some node q_ν depends on the magnitude of the local reconstruction w_{loc}. Note that the latter in its turn depends on the magnitudes of the function $w(\cdot)$ at nodes of the grid, which may include the node q_ν. Such kind of relations obtained is typical for the dynamic programming principle. In the following, on the basis of this formula, an iterative numerical method for construction of the vector W and function w is proposed. Moreover, from the definition of $w(\cdot)$, one can see that in a practical realization of the numerical method, it is necessary to remember values of this function only at the nodes from $\mathscr{L}_{\mathscr{G}}$. If the set $\mathscr{G}$ is bounded, then $\mathscr{L}_{\mathscr{G}}$ contains only finite number of nodes and can be represented in a computer.

For the chosen grid $\mathscr{L} = \{q_\nu\}_{\nu\in\mathbb{Z}}$, we denote by $\mathscr{M}$ the set of infinite vectors with the elements $W = \big(w(q_\nu)\big)_{\nu\in\mathbb{Z}}$. We denote by $\mathscr{M}_1$ those vectors in the set $\mathscr{M}$, which elements $w(q_\nu)$ satisfy the inequality $0 \le w(q_\nu) \le 1$. For every $s \in \mathbb{Z}$, we define an operator $F_s : \mathscr{M} \to \mathbb{R}$ using a vector $W = \big(w(q_\nu)\big)_{\nu\in\mathbb{Z}}$ in the following way:

$$F_s(W) = \begin{cases} \gamma \max\limits_{b\in B} \min\limits_{a\in A} w_{loc}\big(z(q_s, a, b), W\big) + 1 - \gamma, & \text{if } q_s \in \mathscr{L}_{\mathscr{G}}, \\ 0, & \text{if } q_s \in \mathscr{L}_{\mathscr{T}}, \\ 1, & \text{if } q_s \in \mathscr{L}_{\mathscr{F}}. \end{cases}$$

Here, $w_{loc} : \mathbb{R}^n \times \mathscr{M} \to \mathbb{R}$ is the local reconstruction (7) of the function $w(\cdot)$ corresponding to the vector W. The manifold of values of the operators F_s over all indices s (that is, over all nodes q_s) defines an operator $F : \mathscr{M} \to \mathscr{M}$.

A partial order can be defined in the set $\mathscr{M}$ using the elementwise comparison: $W_1 \le W_2 \Leftrightarrow \forall \nu \in \mathbb{Z}\ w_1(q_\nu) \le w_2(q_\nu)$. Also, in $\mathscr{M}_1$, one can reasonably introduce the norm $\|W\|_\infty = \sup\big\{w(q_\nu) : \nu \in \mathbb{Z}\big\}$.

Let us prove the following lemma on properties of the operator F analogous to the one from paper [2, pp. 124–125, Proposition 2.1].

Lemma 1 *The operator $F : \mathscr{M} \to \mathscr{M}$ has the following properties:*

1. *$F(\mathscr{M}_1) \subseteq \mathscr{M}_1$;*
2. *F is monotone with respect to the partial order in $\mathscr{M}$;*
3. *F is a contraction map in $\mathscr{M}_1$ with respect to the norm $\|\cdot\|_\infty$.*

Proof Basically, the proof repeats the analogous one in [2, pp. 124–125].

1. Let $W \in \mathscr{M}_1$ and $q_s \in \mathscr{L}_{\mathscr{G}}$. Then

$$F_s(W) = \gamma \max_{b\in B} \min_{a\in A} \sum_{m=1}^{n+1} \lambda_m\big(z(q_s, a, b)\big) W_m\big(z(q_s, a, b)\big) + 1 - \gamma.$$

Here, $W_m(z)$ is the element of the vector W corresponding to the node, which is the mth vertex of the simplex $S\big(z(q_s, a, b)\big)$.
Since, $\lambda_m\big(z(q_s, a, b)\big) \geq 0$, $\sum \lambda_m\big(z(q_s, a, b)\big) = 1$, and $0 \leq W_m \leq 1$, we have

$$0 \leq F_s(W) \leq \gamma \max_{b \in B} \min_{a \in A} \sum_{m=1}^{n+1} \lambda_m\big(z(q_s, a, b)\big) + 1 - \gamma = \gamma + 1 - \gamma = 1.$$

If $q_s \notin \mathscr{L}_{\mathscr{G}}$, then $F_s(W) = 0$ or $F_s(W) = 1$. Hence, it appears that $F : \mathscr{M}_1 \to \mathscr{M}_1$.

2. Let $U,\ V \in \mathscr{M}$ and $U \geq V$. If $q_s \in \mathscr{L}_{\mathscr{G}}$, then

$$\begin{aligned} F_s(V) - F_s(U) = \gamma \max_{b \in B} \min_{a \in A} \sum_{m=1}^{n+1} \lambda_m\big(z(q_s, a, b)\big) V_m\big(z(q_s, a, b)\big) \\ - \gamma \max_{b \in B} \min_{a \in A} \sum_{m=1}^{n+1} \lambda_m\big(z(q_s, a, b)\big) U_m\big(z(q_s, a, b)\big). \end{aligned}$$

Let us choose the control $\overline{a}(b)$ of the first player attaining the minimum in $F_s(U)$ for a fixed b. Then the minuend in the inequality increases, because $\overline{a}(b)$ not necessarily attains the minimum in $F_s(V)$, while the subtrahend keeps its value. We get

$$\begin{aligned} &\gamma \max_{b \in B} \min_{a \in A} \sum_{m=1}^{n+1} \lambda_m\big(z(q_s, a, b)\big) V_m\big(z(q_s, a, b)\big) \\ &\qquad - \gamma \max_{b \in B} \min_{a \in A} \sum_{m=1}^{n+1} \lambda_m\big(z(q_s, a, b)\big) U_m\big(z(q_s, a, b)\big) \\ &\qquad \leq \gamma \max_{b \in B} \sum_{m=1}^{n+1} \lambda_m\Big(z\big(q_s, \overline{a}(b), b\big)\Big) V_m\Big(z\big(q_s, \overline{a}(b), b\big)\Big) \\ &\qquad\qquad - \gamma \max_{b \in B} \sum_{m=1}^{n+1} \lambda_m\Big(z\big(q_s, \overline{a}(b), b\big)\Big) U_m\Big(z\big(q_s, \overline{a}(b), b\big)\Big). \end{aligned}$$

Now, let us consider the control $\overline{b}$ of the second player attaining the maximum in the expression for the minuend, that is,

$$\overline{b} \in \operatorname{Arg}\max_{b \in B} \left[\gamma \sum_{m=1}^{n+1} \lambda_m\Big(z\big(q_s, \overline{a}(b), b\big)\Big) V_m\Big(z\big(q_s, \overline{a}(b), b\big)\Big) \right].$$

It follows that

$$\gamma \max_{b\in B} \sum_{m=1}^{n+1} \lambda_m\big(z\big(q_s, \overline{a}(b), b\big)\big) V_m\big(z\big(q_s, \overline{a}(b), b\big)\big)$$
$$- \gamma \max_{b\in B} \sum_{m=1}^{n+1} \lambda_m\big(z\big(q_s, \overline{a}(b), b\big)\big) U_m\big(z\big(q_s, \overline{a}(b), b\big)\big)$$
$$\leq \gamma \sum_{m=1}^{n+1} \lambda_m\big(z\big(q_s, \overline{a}(\overline{b}), \overline{b}\big)\big)\Big(V_m\big(z\big(q_s, \overline{a}(\overline{b}), \overline{b}\big)\big)$$
$$- U_m\big(z\big(q_s, \overline{a}(\overline{b}), \overline{b}\big)\big)\Big) \leq 0.$$

If $q_s \in \mathscr{L}_{\mathscr{T}}$ or $q_s \in \mathscr{L}_{\mathscr{F}}$, then $F_s(V) - F_s(U) = 0$. Hence, F is the monotone operator.

3. Let $U,\ V \in \mathscr{M}_1$. If $q_s \in \mathscr{L}_{\mathscr{G}}$, then

$$\big|F_s(V) - F_s(U)\big| \leq \gamma \sum_{m=1}^{n+1} \lambda_m\big(z\big(q_s, \overline{a}(\overline{b}), \overline{b}\big)\big)$$
$$\times \big|V_m\big(z\big(q_s, \overline{a}(\overline{b}), \overline{b}\big)\big) - U_m\big(z\big(q_s, \overline{a}(\overline{b}), \overline{b}\big)\big)\big|$$
$$\leq \gamma \max_m \big|V_m\big(z\big(q_s, \overline{a}(\overline{b}), \overline{b}\big)\big) - U_m\big(z\big(q_s, \overline{a}(\overline{b}), \overline{b}\big)\big)\big|$$
$$\times \sum_{m=1}^{n+1} \lambda_m\big(z\big(q_s, \overline{a}(\overline{b}), \overline{b}\big)\big) \leq \gamma \|V - U\|_\infty.$$

It holds for every $s \in \mathbb{Z}$.
If $q_s \in \mathscr{L}_{\mathscr{T}}$ or $q_s \in \mathscr{L}_{\mathscr{F}}$, then $F_s(V) - F_s(U) = 0$. So, it immediately follows that the function F is a contraction map, since $\gamma = e^{-h} < 1$.

As a consequence from this lemma, one can obtain that there exists a unique fixed point $\mathbf{W} \in \mathscr{M}_1$ of the operator F, which determines a function $\mathbf{w}(\cdot)$ in $\mathbb{R}^n$, $\mathbf{w}(x) \in [0, 1]$. This function depends on the time h and space k discretization steps of the original problem:

$$\mathbf{w}(x) = \begin{cases} \sum\limits_m \lambda_m \mathbf{w}(q_m), & \text{if } x \notin \mathscr{L} \text{ and } x = \sum\limits_m \lambda_m q_m, \\ \gamma \max\limits_{b\in B} \min\limits_{a\in A} \mathbf{w}_{loc}\big(z(q_s, a, b), \mathbf{W}\big) + 1 - \gamma, & \text{if } q_s \in \mathscr{L}_{\mathscr{G}}, \\ 0, & \text{if } q_s \in \mathscr{L}_{\mathscr{T}}, \\ 1, & \text{if } q_s \in \mathscr{L}_{\mathscr{F}}. \end{cases} \tag{8}$$

3.2 Viscosity Solution of Boundary Problem for HJE

Let us consider the following boundary value problem for HJE:

$$\begin{aligned} &z + H(x, Dz) = 0, x \in \mathscr{G}, \\ &z(x) = 0 \text{ if } x \in \partial\mathscr{T}, \\ &z(x) = 1 \text{ if } x \in \partial\mathscr{F}. \end{aligned} \tag{9}$$

Here and below, the symbol Dz denoted the gradient of the function z. The function H is called the *Hamiltonian* and in the case of dynamics (1) is defined as follows:

$$H(x, p) = \max_{a\in A} \min_{b\in B} \big\langle p, -f(x, a, b)\big\rangle - 1, \quad x, p \in \mathbb{R}^n. \tag{10}$$

Equations of this type can have no classical solution. That is why we use the notion of the *generalized viscosity solution* introduced in [17] to deal with this problem. In book [30], an alternative method of obtaining a generalized solution of HJE was introduced. It is called the *generalized minimax solution*. Also in book [30], it is proved that viscosity and minimax solutions coincide at the points of continuity.

In [24, 25], the authors prove that the value function of game (1), (5) is a viscosity solution of problem (9). The proof demands smoothness of boundaries $\partial\mathscr{T}$ and $\partial\mathscr{F}$, the boundedness of these boundaries curvature. It was performed under the assumption of the dynamical advantage of each player on the boundaries of the corresponding sets:

$$\begin{aligned} &\forall x \in \partial\mathscr{T} \;\; \min_{a\in A} \max_{b\in B} \big\langle n_{\mathscr{T}}(x),\, f(x, a, b)\big\rangle < 0, \\ &\forall x \in \partial\mathscr{F} \;\; \max_{b\in B} \min_{a\in A} \big\langle n_{\mathscr{F}}(x),\, f(x, a, b)\big\rangle < 0. \end{aligned} \tag{11}$$

Here, $n_{\mathscr{T}}(x)$ $\big(n_{\mathscr{F}}(x)\big)$ is a normal vector to the boundary $\partial\mathscr{T}$ $(\partial\mathscr{F})$ of the set $\mathscr{T}$ $(\mathscr{F})$ at the point x directed outward the corresponding set or (what is the same) inward the set $\mathscr{G}$. The sense of these relations is that if the system is at the boundary of the set $\mathscr{T}$ $(\mathscr{F})$, then the first (second) player can guarantee leading the trajectory of the system inside the corresponding set despite the action of the opponent. Combination of these assumptions results in the continuity of the value function inside the set $\mathscr{G}$. Indeed, from the results of paper [26], it follows that under these assumptions an upper generalized solution exists, which is continuous in $\operatorname{cl}\mathscr{G}$. Then, the statements in [30, Sect. 18.6, pp. 224–225] imply that a generalized solution exists, which is continuous in $\mathscr{G}$. Moreover, since the value function coincides with the generalized solution, it is continuous too (the coincidence is proved in [26]).

Definition 1 ([2], *p. 112, Definition 1.3*) For some domain Ω, an upper semicontinuous function $u(\cdot)$ is called a *viscosity subsolution* of Eq. (9) in the domain Ω if for all $\varphi \in C^1(\Omega)$ and for any local maximum point $y \in \Omega$ for $u - \varphi$, the inequality $u(x) + H\big(x, D\varphi(x)\big) \leqslant 0$ holds.

Definition 2 ([2], *p. 112, Definition 1.3*) For some domain Ω, a lower semicontinuous function $u(\cdot)$ is called a *viscosity supersolution* of Eq. (9) in the domain Ω if for all $\varphi \in C^1(\Omega)$ and for any local minimum point $y \in \Omega$ for $u - \varphi$, the inequality $u(x) + H\big(x, D\varphi(x)\big) \geqslant 0$ holds.

Definition 3 Let us consider two sequences of real numbers $h_n > 0$ and $k_n > 0$ (which are time and space discretization steps). We will refer to them as *admissible sequences* if $h_n \to 0$ and $k_n/h_n \to 0$ as $n \to \infty$.

Let us consider admissible sequences of real numbers $h_n > 0$, $k_n > 0$, and a sequence of the solutions $\mathbf{w}_n$ of problem (8) corresponding to these admissible sequences.

The proof of the facts given in the next section is based on the notion of the *weak limit in the viscosity sense* introduced in [1, 6]. An upper and a lower limit of the functional sequence $\mathbf{w}_n$ in the viscosity sense are defined as follows:

$$\begin{aligned} \limsup_{(y,n)\to(x,\infty)} \mathbf{w}_n(y) &:= \lim_{\delta\to 0+} \sup\big\{\mathbf{w}_n(y) : |x-y| \le \delta, n \ge 1/\delta\big\}, \\ \liminf_{(y,n)\to(x,\infty)} \mathbf{w}_n(y) &:= \lim_{\delta\to 0+} \inf\big\{\mathbf{w}_n(y) : |x-y| \le \delta, n \ge 1/\delta\big\}. \end{aligned} \tag{12}$$

Note that these limits exist if the functional sequence $\mathbf{w}_n$ is locally uniformly bounded [1, p. 288, Definition 1.4].

Definition 4 For some domain Ω, an upper semicontinuous function $u : \mathrm{cl}\,\Omega \to \mathbb{R}$ satisfies the boundary condition $u + H(x, Du) \le 0$ at the boundary $\partial\Omega$ *in the viscosity sense* if for all $\varphi \in C^1(\mathrm{cl}\,\Omega)$ and a point $x \in \partial\Omega$ such that the function $u - \varphi$ has a local maximum at x, the inequality $u(x) + H(x, D\varphi(x)) \le 0$ holds.

Definition 5 For some domain Ω, a lower semicontinuous function $u : \mathrm{cl}\,\Omega \to \mathbb{R}$ satisfies the boundary condition $u + H(x, Du) \ge 0$ at the boundary $\partial\Omega$ *in the viscosity sense* if for all $\varphi \in C^1(\mathrm{cl}\,\Omega)$ and a point $x \in \partial\Omega$ such that the function $u - \varphi$ has a local minimum at x, the inequality $u(x) + H(x, D\varphi(x)) \ge 0$ holds.

4 Numerical Scheme Convergence

Let us formulate and prove a lemma for a time-optimal game with lifeline analogous to [2, p. 127, Lemma 2.2]. Some derivations in the original lemma were omitted. For example, the proof for an upper solution was absent, proof of the inequalities analogous to (19) and (20) from this paper was not completely performed, and some essential remarks were missed (e.g., in the original lemma the function φ is defined on the closure of the set of the game but is used in a such a way that it is defined on the whole $\mathbb{R}^n$).

Lemma 2 *Let us consider admissible sequences of real numbers* $h_n > 0$ *and* $k_n > 0$, *and let* $\mathbf{w}_n$ *be the corresponding sequence of solutions* (8). *Denote*

$$\overline{v}(x) := \limsup_{(y,n)\to(x,\infty)} \mathbf{w}_n(y), \quad \underline{v}(x) := \liminf_{(y,n)\to(x,\infty)} \mathbf{w}_n(y). \tag{13}$$

Then the functions $\overline{v}$ and $\underline{v}$ are, respectively, a viscosity subsolution and supersolution of the boundary value problem (9) *with the boundary conditions*

$$\underline{v} \geq 0 \text{ on } \partial\mathscr{T}, \tag{14}$$
$$\overline{v} \leq 0 \text{ or } \overline{v} + H\big(x, D\overline{v}(x)\big) \leq 0 \text{ on } \partial\mathscr{T}, \tag{15}$$
$$\underline{v} \geq 1 \text{ or } \underline{v} + H\big(x, D\underline{v}(x)\big) \geq 0 \text{ on } \partial\mathscr{F}, \tag{16}$$
$$\overline{v} \leq 1 \text{ on } \partial\mathscr{F}. \tag{17}$$

The second inequalities in (15) *and* (16) *are understood in the viscosity sense.*

Proof Proofs of the facts that the boundary conditions (14), (15) are fulfilled and that $\overline{v}$ is a viscosity subsolution are similar to those from [2, pp.127–129]. The fulfilment of the last boundary condition (17) is obvious from the construction of the function $\overline{v}$. Therefore, it is necessary to show only that the function $\underline{v}$ is a viscosity supersolution and that the boundary condition (16) holds. Let us prove these facts simultaneously (in (16), we prove the second inequality).

Choose a function $\varphi \in C^1(\mathbb{R}^n)$ and a point $y \in \mathrm{cl}\,\mathscr{G}$ such that the function $\underline{v} - \varphi$ attains the local strict minimum at the point y. Although, the function φ in the definition of the viscosity solution is considered only at the set $\mathrm{cl}\,\mathscr{G}$, we define it in the whole space $\mathbb{R}^n$, because we shall need it henceforth; restriction of the function φ to the set $\mathrm{cl}\,\mathscr{G}$ is smooth. As far as the property of the point y doesn't change under adding a constant to the function φ, we consider that $\varphi(y) = \underline{v}(y)$. The point y can belong to the set $\mathscr{G}$ or to the boundary $\partial\mathscr{F}$. The case when the point y belongs to the boundary $\partial\mathscr{T}$ does not require consideration, because it is taken into account in condition (14). If $y \in \partial\mathscr{F}$ and $\underline{v}(y) \geq 1$, then inequality (16) holds. Thus hereafter, we shall assume that $\underline{v}(y) < 1$ if $y \in \partial\mathscr{F}$ and $\underline{v}(y) \leq 1$ if $y \in \mathscr{G}$.

It has to be shown that $\underline{v}(y) + H\big(y, D\varphi(y)\big) \geq 0$. Let us choose a sequence of points x_n such that

$$\min_{\mathrm{cl}\left(\mathscr{G}\cap B(y,1)\right)} (\mathbf{w}_n - \varphi) = (\mathbf{w}_n - \varphi)(x_n).$$

The basic property of weak limits in the viscosity sense [1, 5, 18] is the existence of a subsequence (we suppose that it is the sequence x_n itself) such that $x_n \to y$ and $\mathbf{w}_n(x_n) \to \underline{v}(y)$ as $n \to \infty$. It means that one can choose such a number $\varepsilon > 0$ that $B(y, \varepsilon) \subset \mathscr{G}$ if $y \in \mathscr{G}$ or $\varphi(y') < 1 - \varepsilon$ for every $y' \in B(y, \varepsilon)$ if $y \in \partial\mathscr{F}$. It can always be achieved by means of decreasing ε, because if $y \in \partial\mathscr{F}$, then $\varphi(y) = \underline{v}(y) < 1$. Moreover, one can choose such a sufficiently big number n that the following inequalities hold

(a) $x_n \in B(y, \varepsilon/3)$ holds, because x_n converges to the point y as $n \to \infty$;
(b) $\big|h_n f(x_n, a, b)\big| \leq \varepsilon/3$ holds, because h_n tends to 0;

(c) $k_n \cdot \max\left\{2+\sigma, \sqrt{d}\right\} \le \varepsilon/3$ holds, because the sequence k_n tends to 0; here, $\sigma = \max\left\{\left|D\varphi(z)\right| : z \in B(y,1)\right\}$;

(d) $\varphi(x_n) - \mathbf{w}_n(x_n) > -\varepsilon$ holds, because we assume that $\varphi(y) = \underline{v}(y)$; hence, $\varphi(x_n) < \mathbf{w}_n(x_n)$ (as $\varphi(y') < \underline{v}(y')$ and $\underline{v}(y') \le \mathbf{w}_n(y')$ for all y' in some sufficiently small neighborhood of the point y; the points x_n belong to this neighborhood for indices n starting from some sufficiently large index).

The following calculations are made for n fixed, so we temporarily omit the subscript in h_n, k_n, $\mathbf{w}_n$, x_n, $\gamma_n = e^{-h_n}$.

1. Let $y \in \mathscr{G}$. Let us write the local coordinates of the point x via the vertices q_s of the corresponding simplex: $x = \sum_s \lambda_s q_s$. Note that $q_s \in B(y,\varepsilon)$, because $x \in B(y,\varepsilon/3)$ and $q_s \in B(x,\varepsilon/3)$ (the latter is true due to $k\sqrt{d} \le \varepsilon/3$). So, $q_s \in \mathscr{G}$, whence it follows that for $\mathbf{w}(q_s)$ the following representation holds

$$\mathbf{w}(q_s) = \gamma \max_{b\in B} \min_{a\in A} \mathbf{w}_{loc}\big(z(q_s,a,b),\mathbf{W}\big) + 1 - \gamma.$$

2. Let $y \in \partial\mathscr{F}$. Then $-\varepsilon < \varphi(x) - \mathbf{w}(x) < 1 - \varepsilon - \mathbf{w}(x) \Rightarrow \mathbf{w}(x) < 1$. So, if $x = \sum_s \lambda_s q_s$, then there exists a node q_s such that $\lambda_s \ne 0$ and $\mathbf{w}(q_s) < 1$. Then again for $\mathbf{w}(q_s)$, the following representation holds

$$\mathbf{w}(q_s) = \gamma \max_{b\in B} \min_{a\in A} \mathbf{w}_{loc}\big(z(q_s,a,b),\mathbf{W}\big) + 1 - \gamma.$$

Let us note that

$$\begin{aligned}\mathbf{w}(q_s) = \gamma \max_{b\in B} \min_{a\in A} \mathbf{w}_{loc}\big(z(q_s,a,b),\mathbf{W}\big) + 1 - \gamma \\ \ge \gamma \min_{a\in A} \mathbf{w}_{loc}\big(z(q_s,a,b),\mathbf{W}\big) + 1 - \gamma\end{aligned}$$

for every $b \in B$. Moreover, for every $\rho > 0$, there exists a value $a_s(\rho)$ (for example, the one attaining the minimum) such that the following inequality holds

$$\gamma \min_{a\in A} \mathbf{w}_{loc}\big(z(q_s,a,b),\mathbf{W}\big) + 1 - \gamma > \gamma \mathbf{w}_{loc}\big(z\big(q_s,a_s(\rho),b\big),\mathbf{W}\big) + 1 - \gamma - \rho h.$$

We denote by $z_s(\rho,b) = z\big(q_s,a_s(\rho),b\big) = q_s + hf\big(q_s,a_s(\rho),b\big)$. Whence it follows that for every $\rho > 0$ the relation holds

$$\mathbf{w}(q_s) - \gamma\mathbf{w}_{loc}\big(z_s(\rho,b),\mathbf{W}\big) - (1-\gamma) > -\rho h \quad \forall b \in B. \tag{18}$$

Let $z_s = \sum_p \mu_p q_p$ and b is arbitrary. Now, let us prove that

$$\mathbf{w}(x) - \varphi(x) \le \mathbf{w}_{loc}\big(z_s(\rho,b),\mathbf{W}\big) - \varphi\big(z_s(\rho,b)\big) + \sigma k\sqrt{d} + o_1, \tag{19}$$

where $o_1 = o(\big|z_s(\rho, b) - q_{p^\star}\big|)$ and $q_{p^\star}$ is such a vertex of the simplex $S\big(z_s(\rho, b)\big)$ that $\varphi(q_{p^\star})$ is the minimum magnitude of φ over the vertices of this simplex. Here and below, all o-variables are considered as $n \to \infty$.

If $z_s(\rho, b) \in \operatorname{cl}\mathscr{G}$, then, in virtue of condition (c), we obtain $z_s(\rho, b) \in B(q_s, \varepsilon/3)$. Since $q_s \in B(x, \varepsilon/3)$, one has $z_s(\rho, b) \in B(x, 2\varepsilon/3) \subset B(x, \varepsilon)$. In this case, inequality (19) holds, because x is the point of a local minimum of the function $\mathbf{w} - \varphi$.

Now, let $z_s(\rho, b) \notin \operatorname{cl}\mathscr{G}$. Two cases are possible

1. There is a term in the representation of z_s such that $\mu_p \neq 0$ and $q_p \in \operatorname{cl}\mathscr{G}$. Then, similarly, we get $q_p \in B\big(z_s(\rho, b), \varepsilon/3\big)$, $z_s(\rho, b) \in B(q_s, \varepsilon/3)$, and $q_s \in B(x, \varepsilon/3)$. Hence, $q_p \in B(x, \varepsilon)$. From this, it follows that $\mathbf{w}(x) - \varphi(x) \le \mathbf{w}(q_p) - \varphi(q_p)$, because x is the point of a local minimum of the function $\mathbf{w} - \varphi$.
2. For all p such that $\mu_p \neq 0$, one has that $q_p \notin \operatorname{cl}\mathscr{G}$. Recall that the function φ is defined on the whole space $\mathbb{R}^n$ and that for every $y' \in B(y, \varepsilon)$ the condition $\varphi(y') < 1 - \varepsilon$ holds. Then, in virtue of condition (d), we get

$$\mathbf{w}(x) - \varphi(x) < \varepsilon < 1 - \varphi(q_p) = \mathbf{w}(q_p) - \varphi(q_p),$$

because the function $\mathbf{w}(q_p) = 1$ at the node $q_p \in \mathscr{F}$.

Then

$$\begin{aligned}\mathbf{w}(x) - \varphi(x) &\le \sum_p \mu_p\big(\mathbf{w}(q_p) - \varphi(q_p)\big) = \sum_p \mu_p \mathbf{w}(q_p) - \sum_p \mu_p \varphi(q_p)\\ &\le \mathbf{w}_{loc}\big(z_s(\rho, b), \mathbf{W}\big) - \sum_p \mu_p \varphi(q_{p^\star}) = \mathbf{w}_{loc}\big(z_s(\rho, b), \mathbf{W}\big) - \varphi(q_{p^\star}),\end{aligned}$$

where the index $p^\star$ is as defined above.

Note that

$$\begin{aligned}\big|\varphi\big(z_s(\rho, b)\big) - \varphi(q_{p^\star})\big| &\le \sigma\big|z_s(\rho, b) - q_{p^\star}\big| + o(\big|z_s(\rho, b) - q_{p^\star}\big|)\\ &< \sigma k\sqrt{d} + o(\big|z_s(\rho, b) - q_{p^\star}\big|).\end{aligned}$$

Then $-\varphi(q_{p^\star}) \le -\varphi\big(z_s(\rho, b)\big) + \sigma k\sqrt{d} + o_1$. Hence, we obtain inequality (19).

Now, let us show that $\big|\mathbf{w}(x) - \mathbf{w}(q_s)\big| \le \sigma k\sqrt{d}$.

Since x, q_s belong to one simplex S, then $\mathbf{w}$ is affine in the segment $X = [x, q_s]$. As function $(\mathbf{w} - \varphi)\big|_X$ attains minimum at the point x, we get

$$\frac{|\mathbf{w}(x) - \mathbf{w}(q_s)|}{k\sqrt{d}} \le \frac{|\mathbf{w}(x) - \mathbf{w}(q_s)|}{|x - q_s|} = |D_X\mathbf{w}| = |D_X\varphi| \leqslant \sigma.$$

We denote by $D_X g$ a derivative of the restriction of a function g to the set X as a derivative of a function of one variable.

Also, let us note that

$$\begin{aligned}\big|\varphi\big(z_s(\rho,b)\big)-\varphi\big(x+hf(x,a_s(\rho),b)\big)\big| &\le \sigma\big|z_s(\rho,b)-x-hf(x,a_s(\rho),b)\big|\\ &= \sigma\big|q_s+hf(q_s,a_s(\rho),b)-x-hf(x,a_s(\rho),b)\big|\\ &\le \sigma\big(|q_s-x|+h\big|f(q_s,a_s(\rho),b)-f(x,a_s(\rho),b)\big|\big)\le \sigma(k\sqrt{d}+hLk).\end{aligned} \tag{20}$$

Now, let us apply the educed inequalities to (18) for any $b\in B$:

$$\begin{aligned}-\rho h &< \mathbf{w}(q_s)-\gamma\mathbf{w}_{loc}\big(z_s(\rho,b),\mathbf{W}\big)-(1-\gamma)\\ &\le \mathbf{w}(x)-\gamma\mathbf{w}_{loc}\big(z_s(\rho,b),\mathbf{W}\big)-(1-\gamma)+\sigma k\sqrt{d}\\ &= (1-\gamma)\mathbf{w}(x)+\gamma\Big(\mathbf{w}(x)-\mathbf{w}_{loc}\big(z_s(\rho,b),\mathbf{W}\big)\Big)-(1-\gamma)+\sigma k\sqrt{d}\\ &\le (1-\gamma)\mathbf{w}(x)+\gamma\Big(\varphi(x)-\varphi\big(z_s(\rho,b)\big)\Big)-(1-\gamma)+(1+\gamma)\sigma k\sqrt{d}+\gamma o_1\\ &\le (1-\gamma)\mathbf{w}(x)+\gamma\Big(\varphi(x)-\varphi\big(x+hf(x,a_s,b)\big)\Big)\\ &\quad -(1-\gamma)+(1+2\gamma+\gamma hL)\sigma k\sqrt{d}+\gamma o_1,\end{aligned}$$

where L is the Lipschitz constant for the function f from condition (2).

Since ρ is arbitrary, it holds

$$\begin{aligned}0 \le \frac{1-\gamma_n}{h_n}\mathbf{w}_n(x^n) &+ \min_{b\in B}\max_{a\in A}\left\{\gamma_n\frac{\varphi(x^n)-\varphi\big(x^n+h_nf\big(x^n,a,b\big)\big)}{h_n}-\frac{1-\gamma_n}{h_n}\right\}\\ &+\sigma\frac{k_n}{h_n}\sqrt{d}(1+2\gamma_n+\gamma_nh_nL)+\gamma o_1.\end{aligned}$$

Passing to the limit in n to the infinity, we obtain $0\le \underline{v}(y)+H\big(y,D\varphi(y)\big)$. That establishes relation (16) as far as the fact that $\overline{v}$ and $\underline{v}$ are viscosity subsolution and supersolution of problem (9) with the boundary conditions (14)–(17) in the viscosity sense.

Now, we can prove a theorem on the convergence of the proposed numerical scheme analogous to [2, pp. 125–129, Theorem 2.3]. Firstly, it should be noted that the proof of the auxiliary theorem for a time-optimal problem with lifeline corresponding to [2, pp. 117–118, Theorem 1.10] can be conducted in an analogous way with the set Ω substituted by the set $\mathscr{G}$ and is not given here.

Theorem 1 *Assume that Conditions C.1, C.2, and C.3 hold. Also, suppose that the value function v (6) of game (1), (5) is continuous on the set* $\operatorname{cl}\mathscr{G}$. *Then the sequence $\{\mathbf{w}_n\}$ converges to the function $v=\overline{v}=\underline{v}$ as $n\to\infty$ uniformly on every compact set $\mathscr{K}\subset\operatorname{cl}\mathscr{G}$.*

Note that conditions (11) are crucial for all constructions and argument carried out by the authors, in particular, in the framework of this paper. Theorem 1 is proved under continuity of the function v, which follows from these assumptions (as it was said in Sect. 3.2).

Proof By Lemma 2, function $\overline{v}$ (13) is a viscosity subsolution of the boundary value problem (9) and the function v is a viscosity supersolution by virtue of [2, pp. 115–116, Theorem 1.6], which is common for the boundary value problems for the HJE. Applying Theorem 1.1 from [4, pp. 23–27], we get that for function $\overline{v}$ (13), the inequality $\overline{v} \le v$ holds on $\operatorname{cl}\mathscr{G}$. In the same manner, it is proved that $v \le \underline{v}$. So, $\overline{v} \le \underline{v}$ in $\operatorname{cl}\mathscr{G}$. By definition of $\underline{v}$ and $\overline{v}$ (as lim inf and lim sup of $\mathbf{w}_n$), one has $\underline{v} \le \overline{v}$. From these two inequalities, we obtain $\underline{v} = \overline{v} = v$.

Let us show that the sequence $\{\mathbf{w}_n\}$ converges to the function v uniformly on compact sets. Suppose by contradiction that there exist $\varepsilon > 0$, $n_m \to \infty$, and $x_m \in \mathscr{K}$ such that $x_m \to x$ and $\left|\mathbf{w}_{n_m}(x_m) - v(x_m)\right| > \varepsilon$. This implies that the sequences can be chosen in such a way that either $\mathbf{w}_{n_m}(x_m) > v(x_m) + \varepsilon$, or $\mathbf{w}_{n_m}(x_m) < v(x_m) - \varepsilon$. Passing to the limit over m and using the definition of $\overline{v}$ and $\underline{v}$ and the continuity of v, we obtain either $\overline{v}(x) \ge v(x) + \varepsilon$, or $\underline{v}(x) \le v(x) - \varepsilon$ what contradicts to coincidence of either v and $\overline{v}$, or v and $\underline{v}$.

5 Connection Between Value Functions of Problems with and Without Lifeline

In this section, we consider the problem of coincidence of the value functions (not processed by Kruzhkov's transform, that is, representing the time of the optimal motion) for the problems with and without lifeline. Let us consider a classic time-optimal problem with dynamics (1), the constraints A and B for the players' controls, and the terminal set $\mathscr{T}$. The result of such a game on a trajectory $x(\cdot; x_0)$ emanated form the initial point x_0 is determined by the payoff functional

$$\widetilde{\tau}\big(x(\cdot; x_0)\big) = \begin{cases} \min\big\{t : x(t; x_0) \in \mathscr{T}\big\}, & \\ +\infty, & \text{if } \forall t\ x(t; x_0) \notin \mathscr{T}. \end{cases}$$

Here and below, notations with a tilde stand for the classic time-optimal game (without lifeline).

Let us introduce the guaranteed results of the players and the value function as it is described in books [21, 22]. We define a functional

$$\widetilde{\tau}_\varepsilon\big(x(\cdot)\big) := \min\big\{t \in \mathbb{R}^+ : x(t) \in \mathscr{T}_\varepsilon\big\},$$

where $\mathscr{T}_\varepsilon$ is the ε-neighborhood of the terminal set $\mathscr{T}$: $\mathscr{T}_\varepsilon := \mathscr{T} + B(\mathbf{0}, \varepsilon)$, the symbol $\mathbf{0}$ denotes the origin in the corresponding space. The sign $+$ here stands for the Minkowski sum.

Let $\bar{x} \in B(x_0, \varepsilon)$. Denote by $\mathbb{X}(\bar{x}, \mathscr{A}, \Delta)$ the set of stepwise motions of the first player emanated under its strategy $\mathscr{A}$ from the point $\bar{x}$ in the discrete control scheme [21, 22] with the time step Δ. Also, denote by $\mathbb{X}(x_0, \mathscr{A})$ the set of constructive motions emanated from the point x_0 [21, 22] under the strategy $\mathscr{A}$. The guaranteed result $\widetilde{T}_1^0(x_0)$ of the first player at the point x_0 is defined as follows:

$$\widetilde{T}_1^\varepsilon(x_0, \mathscr{A}) := \sup \Big\{ \widetilde{\tau}_\varepsilon\big(x(\cdot)\big) : x(\cdot) \in \mathbb{X}(x_0, \mathscr{A}) \Big\},$$
$$\widetilde{T}_1^\varepsilon(x_0) := \inf_{\mathscr{A} \in \mathbb{A}} \widetilde{T}_1^\varepsilon(x_0, \mathscr{A}), \quad \widetilde{T}_1^0(x_0) := \lim_{\varepsilon \downarrow 0} \widetilde{T}_1^\varepsilon(x_0).$$

The guaranteed result $\widetilde{T}_2^0(x_0)$ of the second player at the point x_0 is defined in a similar way:

$$\widetilde{T}_2^\varepsilon(x_0, \mathscr{B}) := \inf \Big\{ \widetilde{\tau}_\varepsilon\big(x(\cdot)\big) : x(\cdot) \in \mathbb{X}(x_0, \mathscr{B}) \Big\},$$
$$\widetilde{T}_2^\varepsilon(x_0) := \sup_{\mathscr{B} \in \mathbb{B}} T_2^\varepsilon(x_0, \mathscr{B}), \quad \widetilde{T}_2^0(x_0) := \lim_{\varepsilon \downarrow 0} \widetilde{T}_2^\varepsilon(x_0),$$

where $\mathbb{X}(x_0, \mathscr{B})$ is the set of constructive motions of the second player emanated from the point x_0 under its strategy $\mathscr{B}$.

It is known that under the assumptions made above, the value function $\widetilde{T}$ of a classic time-optimal problem exists. So, the following equality holds [22]:

$$\widetilde{T}(x_0) := \widetilde{T}_1^0(x_0) = \widetilde{T}_2^0(x_0).$$

Now, let us consider a classic time-optimal problem and a time-optimal problem with lifeline with the same dynamics and sets A, B, and $\mathscr{T}$. We choose a point $x_0 \in \mathbb{R}^n \setminus \mathscr{T}$. Let the magnitude of the value function of classic time-optimal problem be $\widetilde{T}(x_0) = \theta$.

By Condition C.1, the function f is continuous and satisfies the condition of the sublinear growth, that is, there exists a number $\alpha > 0$ such that for every $x \in \mathbb{R}^n$, $a \in A$, and $b \in B$ the following inequality holds

$$\big\| f(x, a, b) \big\| \leq \alpha\big(1 + \|x\|\big).$$

It follows from the global Lipschitz condition. Let us consider a function

$$M\big(x\big) := \max_{a \in A,\ b \in B} \big\| f(x, a, b) \big\|,$$

which provides an upper estimate for the magnitude of possible velocities of the system at the point x. This function also is continuous and satisfies the condition of the sublinear growth with the same constant α; the maximum is attained, because the sets A and B are compact. Let us choose measurable realizations $a(\cdot)$ and $b(\cdot)$ of controls of the first and second players defined for $t \geqslant 0$. They generate a trajectory $x(\cdot) =$

$x(\cdot; x_0)$ of the system emerged from the point x_0. Using the standard reasoning involving the Grönwall's lemma, one can obtain the following estimate: for any trajectory $x(\cdot)$ emanated from a point x_0 under some admissible controls $a(\cdot)$ and $b(\cdot)$ of the players, it is true that $M\big(x\big(t; x_0, a(\cdot), b(\cdot)\big)\big) \leq \alpha\big(1 + \|x_0\|\big)e^{\alpha\theta}$ for any $t \in [0, \theta]$.

Let us choose the constant $\widetilde{M}$ such that $\widetilde{M} \geq \alpha\big(1 + \|x_0\|\big)e^{\alpha\theta}$.

Firstly, we consider a classic time-optimal problem. Let us denote an optimal strategy of the first player as $\mathscr{A}^*$. We choose a point $\bar{x} \in B(x_0, \varepsilon)$ and a time partition Δ with the diameter less than ε. Since the strategy $\mathscr{A}^*$ is optimal, for every stepwise motion $x(\cdot) \in \mathbb{X}(\bar{x}, \mathscr{A}^*, \Delta)$ of the system, the inequality $\widetilde{\tau}\big(x(\cdot)\big) \leq \theta + \varepsilon$ holds. Hence, $\big\{x(t) : t \in [0, \theta + \varepsilon)\big\} \subset B(x_0, \theta\widetilde{M})$. Passing to the limit $\varepsilon \to 0$, we obtain that for every constructive motion $x(\cdot) \in \mathbb{X}(x_0, \mathscr{A}^*)$, the embedding $\big\{x(t) : t \in [0, \theta]\big\} \subset B(x_0, \theta\widetilde{M})$ holds.

Now, let us consider a time-optimal game with lifeline; the guaranteed results of the first and the second players at the point x_0 are $T_1(x_0)$ and $T_2(x_0)$. As the game set $\mathscr{G}$, we take a set such that $B(x_0, \theta\widetilde{M}) \subset \mathscr{G} \cup \mathscr{T} = \mathscr{W}$. In the game with lifeline, the same strategy $\mathscr{A}^*$ guarantees the same result for the first player. In other words, under the strategy $\mathscr{A}^*$ for every stepwise motion $x(\cdot) \in \mathbb{X}(\bar{x}, \mathscr{A}^*, \Delta)$, the inequality $\tau\big(x(\cdot)\big) \leq \theta$ holds. It is true, because all the trajectories are embedded into the set $\mathscr{W}$; as a result, the second player does not get any advantage connected to the existence of the lifeline. Hence, $T_1(x_0) \leq \theta$.

Let us conduct similar considerations from the point of view of the second player. Let us take an optimal strategy $\mathscr{B}^*$ of the second player in the classic time-optimal problem and construct a set of stepwise motions $\mathbb{X}(\bar{x}, \mathscr{B}^*, \Delta)$. For every stepwise motion $x(\cdot) \in \mathbb{X}(\bar{x}, \mathscr{B}^*, \Delta)$, the inequality $\widetilde{\tau}\big(x(\cdot)\big) \geq \theta + \varepsilon$ holds. Hence, $\big\{x(t) : t \in [0, \theta + \varepsilon)\big\} \subset B(x_0, \theta\widetilde{M})$. Passing to the limit $\varepsilon \to 0$, we get that the set $\mathscr{G}$ is such that all constructive motions $x(\cdot)$ from the set $\mathbb{X}(x_0, \mathscr{B}^*)$ are embedded into $\mathscr{W}$. Thus, the inequality $\tau\big(x(\cdot)\big) \geq \theta$ holds also in the time-optimal problem with lifeline, and $T_2(x_0) \geq \theta$. So, $T_2(x_0) \geq \theta \geq T_1(x_0)$. For the time-optimal problem with lifeline, the classic inequality $T_2(x_0) \leq \theta \leq T_1(x_0)$ also holds. Hence, $T_2(x_0) = \theta = T_1(x_0)$.

Then, we get that if we choose the set $\mathscr{G}$ such that $B(x_0, \theta\widetilde{M}) \subset \mathscr{W}$, then the value function of the classic time-optimal problem coincides with the value function of the corresponding time-optimal problem with lifeline at the point x_0.

So, we have proved the following

Theorem 2 *Assume that Condition C.1 holds. Let the value function of a classic time-optimal problem $\widetilde{T}(x_0)$ at a point x_0 be equal to θ. Then there exists such a constant $\widetilde{M} \geq \alpha\big(1 + \|x_0\|\big)e^{\alpha\theta}$ that if a closed ball $B(x_0, \widetilde{M}\theta) \subset \mathscr{W}$, then the magnitude of the value function of the corresponding time-optimal problem with lifeline $T(x_0)$ at the point x_0 is also equal to θ.*

Moreover, an opposite theorem also holds (since the value function of a time-optimal problem with lifeline is always not less than the value function of the corresponding classic time-optimal problem):

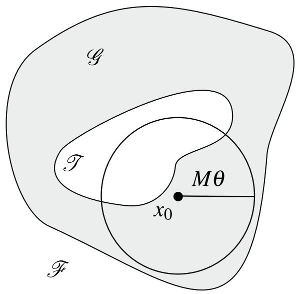


Fig. 2 Illustration to Theorem 3

Theorem 3 *Assume that Condition C.1 holds. Let the function* $T(x_0)$ *of a time-optimal problem with lifeline at the point* x_0 *is equal to* θ. *Then there exists such a constant* $M \geq \alpha\big(1 + \|x_0\|\big)e^{\alpha\theta}$ *that if a closed ball* $B(x_0, M\theta) \subset \mathscr{W}$ *(see Fig. 2), then the magnitude of the value function of the classic time-optimal problem* $\widetilde{T}(x_0)$ *at the point* x_0 *is equal to* θ.

6 Numerical Examples

The numerical procedure described in Sects. 3 and 4 is constructive except the fact that the set $\mathscr{G}$ is not restricted to be bounded. If the set $\mathscr{G}$ is unbounded, then the grid $\mathscr{L}_{\mathscr{G}}$ covering it is infinite and cannot be represented in computer. However, in the opposite case, if the set $\mathscr{G}$ is bounded, then the straightforward computer realization of the proposed procedure is possible.

For the given time step h and space step k, the computer procedure starts with the initial vector W_0, which consists only of 0 and 1: if a node belongs to the set $\mathscr{G}$, then the magnitude at this node is equal to 1, and if the node belongs to the set $\mathscr{T}$, then the magnitude is equal to 0. The computer procedure iteratively recomputes magnitudes at the nodes of the grid $\mathscr{L}_{\mathscr{G}}$ by the consequent application of the operator F to the initial vector. The procedure stops if the desired number of iterations is achieved.

We have an own cross-platform realization of this numerical method written using the environment .NetCore 3.0 and language C# of version 6.0 or later. A single - threaded program was written and then, by means of the capabilities of C#, it was made multi-threaded in order to compute faster on multi-core processors.

The best probation for the program would be comparison of some results computed by it with some value functions obtained theoretically. However, time-optimal games are extremely hard to solve analytically, so, nowadays, there is no non-trivial problems solved completely. The collection of problems that could be solved analytically includes problems with the simple motion dynamics and problems with one-type objects, which can be reduced to control problems. Problems of these types were used to debug the program and optimize its performance. But for other problems, we can compare our results only with the numerical ones obtained by other authors. Below, in several subsections, such examples are shown.

6.1 Homicidal Chauffeur Game

In the homicidal chauffeur game [20], a pursuing object, which represents a car with a bounded turn radius, tries to catch up an evading one with dynamics of simple motions, which is treated as a pedestrian.

The original dynamics describing separately both the car and the pedestrian are

$$\begin{aligned} \dot{x}_p &= w_1 \cos\psi, \\ \dot{y}_p &= w_1 \sin\psi, \\ \dot{\psi} &= \frac{w_1}{R} a, \end{aligned} \qquad \begin{aligned} \dot{x}_e &= b_1, \\ \dot{y}_e &= b_2. \end{aligned}$$

Here, (x_p, y_p) and (x_e, y_e) are the geometric positions of the pursuer and the evader in the plane; ψ is the course angle of the car's velocity; w_1 is the magnitude of the linear velocity of the car; the value R/w_1 describes the minimal turn radius of the car. The control $a \in [-1, +1]$ of the pursuer shows how sharply the car turns: the value $a = -1$ corresponds to the maximally sharp right turn, the value $a = +1$ corresponds to the maximally sharp left turn, and $a = 0$ corresponds to the instantaneous rectilinear motion. The control (b_1, b_2) of the pedestrian obeys the constraint $\|(b_1, b_2)\| \le w_2$. The terminal set can be chosen in different ways reflecting one or another model.

A strong disadvantage of this representation of the dynamics is that it has a quite high dimension, namely, 5. However, it permits a reduction of the dimension of the phase vector in the following way. Superpose the origin and the position of the pursuer. Direct the ordinate axis along the current vector of the pursuer's velocity. So, the new state position (x, y) of the system is two-dimensional and its dynamics are the following:

$$\begin{aligned} \dot{x} &= -\frac{w_1}{R} ya + w_2 \sin b, \\ \dot{y} &= \frac{w_1}{R} xa - w_1 + w_2 \cos b. \end{aligned}$$

Here, $b \in [-\pi, \pi]$ is a newly introduced control of the evader.

Two following examples have been taken from work [27]. It is necessary to note that the value function is discontinuous in these examples, so, formally the algorithm is not meant to solve problems of this type. However, as one can see, there is good coincidence of results obtained by us and the other authors. Of course, the coincidence is considered in the areas where the lifeline does not affect the behavior of the players.

The computations have been performed on a computer with the CPU Intel i7 of the 8th generation, which has 6 kernels with HyperThreading. The volume of RAM is 16 GB (however, it is not critical, since in the examples shown below, the program takes less than 1 Gb for keeping the grid information). The three-dimensional graphs of the value function have been reconstructed from the grid data by means of an algorithm suggested by the authors. Visualization of these graphs was made by a free system MeshLab.

6.1.1 Homicidal Chauffeur Game, Example 1

For the first example, the following parameters have been taken: $w_1 = 3$, $w_2 = 1$, $R = 3$. The terminal set $\mathscr{T}$ is a circle with the center at the origin and the radius equal to 1.0. The set $\mathscr{W} = [-20, 20] \times [-10, 20]$. The time step $h = 0.1$, the spatial step $k = 0.1$. The number of iterations equals 150. The total time of computation was about 2.5 h.

A three-dimensional view of the value function graph is given in Fig. 3. It is restricted to a disk with the center at the point (0, 5) and the radius equal to 15. The magenta-purple area corresponds to the terminal set and small magnitudes of the value function, the yellow color marks places with large times to reach the terminal set. In Fig. 4, one can see contour lines of the value function from 0 to 15 with the step 0.2. The black thick "lines" correspond to the barriers where the value function is discontinuous. This figure and other figures with contour lines have been prepared by means of the system GNU Plot, whose algorithms are oriented to continuous functions, so, near the discontinuities, the picture of contours can be inaccurately restored.

Figure 5 again shows level sets of the value function, not by contours, but by a color gradient filling, which corresponds to the colors in Fig. 3. The red areas stand for the infinite magnitude of the value function, which have been cut off in Fig. 3. These areas appear just due to presence of the lifeline: trajectories leading the system to the terminal set from these areas leave the set $\mathscr{W}$. Also, near the terminal set, one

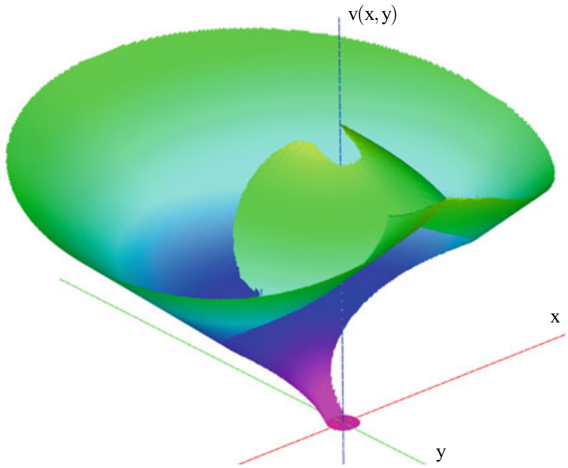


Fig. 3 Homicidal chauffeur, Example 1, a three-dimensional view of the value function graph

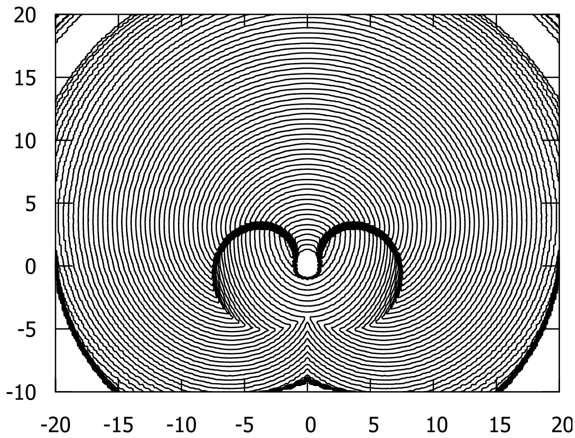


Fig. 4 Homicidal chauffeur, Example 1, contour lines of the value function

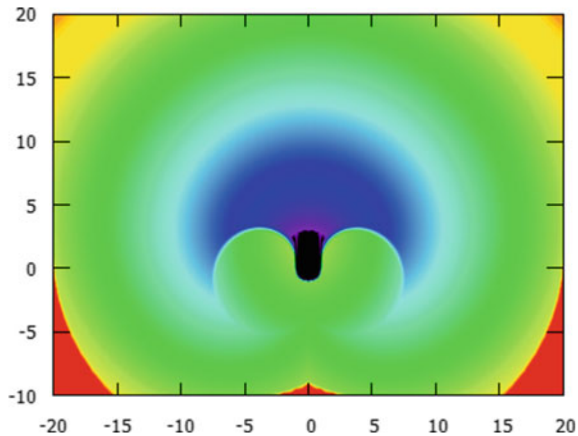


Fig. 5 Homicidal chauffeur, Example 1, the area of the guaranteed coincidence

can see a black spot, which marks the area where the value function of the Homicidal chauffeur game with lifeline coincides with the classic one by Theorem 3. The area is not too large, because the theorem considers all motions of the system including "silly" ones, which go not to the terminal set, but to the lifeline.

6.1.2 Homicidal Chauffeur Game, Example 2

This example uses the same dynamics with the parameters $w_1 = 2$, $w_2 = 0.6$, $R = 0.2$. The terminal set $\mathscr{T}$ is a circle with the center at the point $(0.2, 0.3)$ and the radius is equal to 0.015. The set $\mathscr{W} = [-1.5, 1.5] \times [-1, 1.5]$. The time step $h = 0.001$, the spatial step $k = 0.005$. The number of iterations equals 200. The total time of computation was 7 h and 51 min. A three-dimensional view of the value function graph is given in Fig. 6. It is restricted to a disk with the center at the point $(0, 0.25)$

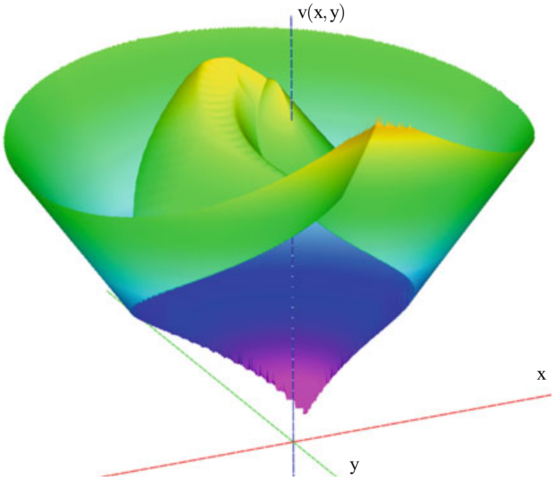


Fig. 6 Homicidal chauffeur, Example 2, a three-dimensional view of the value function graph

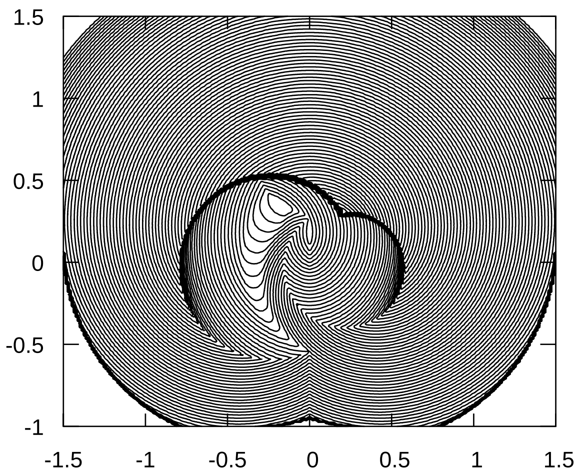
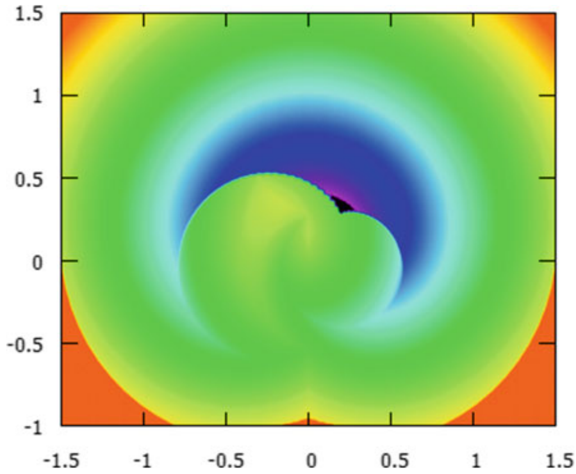


Fig. 7 Homicidal chauffeur, Example 2, contour lines of the value function

and the radius equal to 1.25. The magenta-purple area corresponds to the terminal set and small magnitudes of the value function, the yellow color marks places with large times to reach the terminal set. In Fig. 7, one can see contour lines of the value function from 0 to 1.25 with the step 0.015. The black thick "lines" corresponds to the barriers where the value function is discontinuous. In Fig. 8, a black spot marks the area where the value function of the Homicidal chauffeur game with lifeline certainly coincides with the classic one.

Fig. 8 Homicidal chauffeur, Example 2, the area of the guaranteed coincidence



6.2 Dubins' Car

The (reduced) two-dimensional dynamics of this classic model system are the following:

$$\dot{x} = -ya, \quad \dot{y} = xa - 1.$$

Here, $a \in [-1, 1]$. The time step $h = 0.05$, the spatial discretization step $k = 0.01$. The target set $\mathscr{T} = \{(x, y) \in \mathbb{R}^2 : \max\{|x|, |y|\} \le 0.1\}$. The set $\mathscr{W}$ is a square with the center at the origin and sides of length 6. The number of iterations is 150. Actually, the Dubins' car is an optimal control problem, however, we consider this problem as a differential game with the fictitious second player, which does not affect the dynamics and has its control constrained by a one-pointed set coinciding with the origin. The total time of computation was 13 min.

A three-dimensional view of the value function graph is given in Fig. 9. The magenta-purple area corresponds to the terminal set and small magnitudes of the value function, the yellow and orange colors mark places with large times to reach the terminal set. In Fig. 10, one can see contour lines of the value function from 0 to 7 with the step 0.01. The black thick "lines" corresponds to the barriers where the value function is discontinuous. In Fig. 11, a black spot marks the area where the value function of Dubins' car problem with lifeline certainly coincides with the classic one.

Comparison of these results was made with the ones in paper [16] where an analytical study of reachable sets for this problem is set forth. That work studies reachable sets *at instant*, or in other words a problem with a fixed termination instant is considered. Nevertheless, for control problems, situations *at instant* and *upto instant* are connected very tightly (in contrast to differential games). Thus, we compare

Fig. 9 Dubins' car, a three-dimensional view of the value function graph

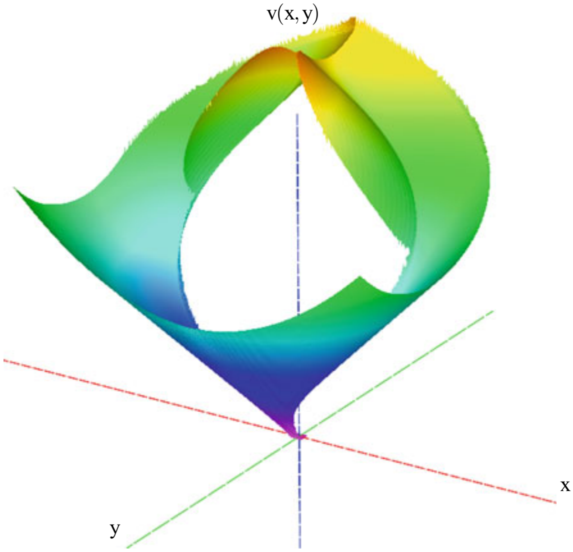
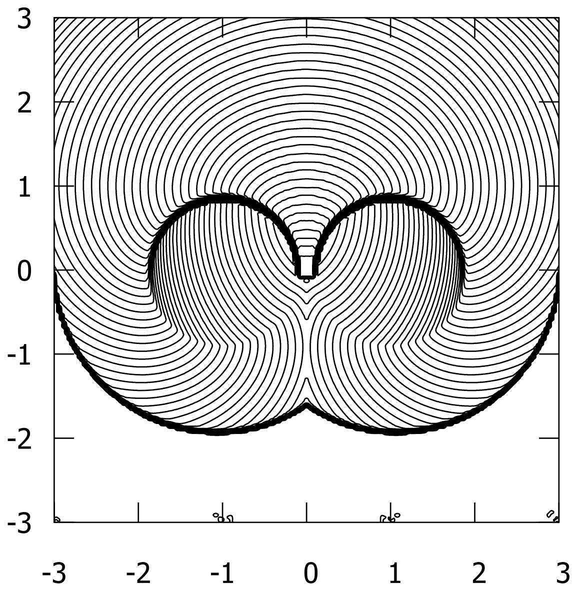


Fig. 10 Dubins' car, contour lines of the value function



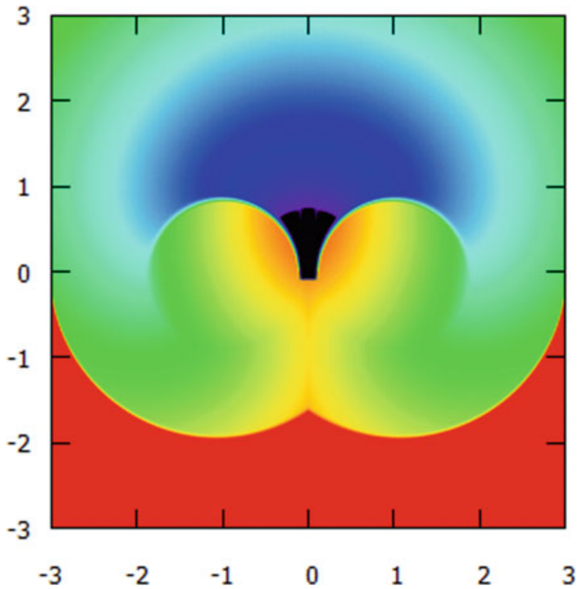


Fig. 11 Dubins' car, the area of the guaranteed coincidence

boundaries of the level sets of the value function for a time-optimal problem and the front parts of the boundaries of the reachable sets *at instant*. The coincidence seems to be good enough.

6.3 *Material Point with Shifted Target*

Dynamics of the system are the following:

$$\dot{x} = y, \quad \dot{y} = a,$$

where $a \in [-1, 1]$. The target set $\mathscr{T}$ is a square with the center at $(0, 1)$ and sides with length of 0.4. The set $\mathscr{W}$ is a square, the length of sides is equal to 8. The number of iterations is 150. The time step $h = 0.05$, the spatial step $k = 0.01$. A three-dimensional view of the value function graph is given in Fig. 12. The magenta-purple area corresponds to the terminal set and small magnitudes of the value function, the yellow and orange colors mark places with large times to reach the terminal set. In Fig. 13, one can see contour lines of the value function from 0 to 9 with the step 0.01. The black thick "lines" corresponds to the barriers where the value function is discontinuous. In Fig. 14, a black spot marks the area where the value function of the material point problem with lifeline certainly coincides with the classic one.

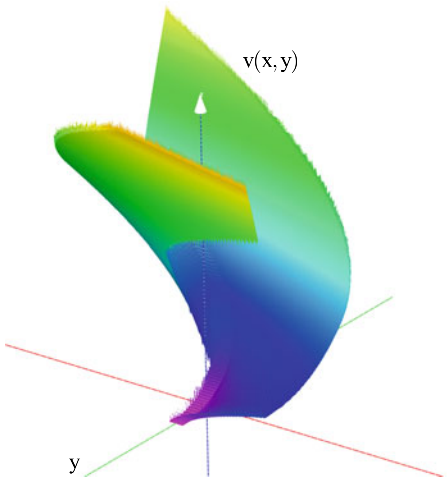


Fig. 12 Material point with shifted target, a three-dimensional view of the value function graph

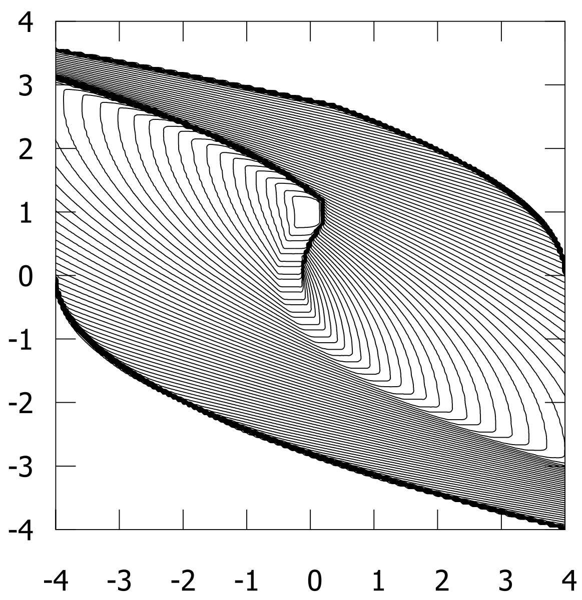


Fig. 13 Material point with shifted target, contour lines of the value function

This control problem is classic and well studied. The boundary of the value function level sets can be obtained by direct integration of trajectories of the system, which can be easily performed due to linearity of the dynamics. There is a good coincidence of theoretical and numerical results.

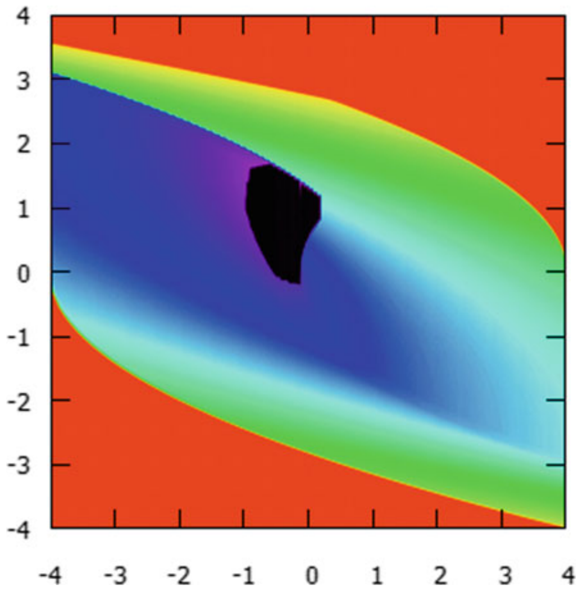


Fig. 14 Material point with shifted target, the area of the guaranteed coincidence

7 Conclusion

The paper discusses proposed numerical method, which constructs the value function of a time-optimal differential game with lifeline as a generalized (viscosity) solution of the corresponding boundary value problem for HJE. Convergence of this method is proved. Previously, authors have proved existence of the generalized solution and its coincidence with the value function of the corresponding time-optimal problem with lifeline under strong conditions (11) of dynamical advantage of each player on the boundary of the corresponding set. It is known that simultaneous accomplishment of these two inequalities results in continuity of the value function. The convergence of the numerical method is proved under the same assumptions. Further, it is planned to prove existence of the generalized solution and its coincidence with the value function under some weaker assumptions. Also, it would be useful to prove convergence of the numerical method to the discontinuous value function of time-optimal problem with lifeline.

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