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Attacker–Defender–Target Problem in the Framework of Space Intercept

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The paper considers a zero-sum linear differential game of attacker–defender–target type in the case when all three objects move in the straight line. The attacker tries to be out the capture radius of the defender at some given instant and to capture the target inside its own capture radius at some other later instant. The authors applying numerical algorithms investigate the structure of solvability sets of such a game for some variants of the objects' dynamics.

I. Problem Formulation

Let us consider a situation of a space intercept with three objects: the attacker A, the defender D, and the target T (see Fig. 1). Usual assumptions are supposed to be true:

- vectors $(V_A)_{\mathrm{nom}}$, $(V_D)_{\mathrm{nom}}$, and $(V_T)_{\mathrm{nom}}$ of objects' nominal velocities as well as the vectors of actual velocities during motion belong to one plane;
- initial lines-of-sight attacker–defender and attacker–target are almost parallel to the objects' velocities (angles $(\chi_A)_{\mathrm{nom}}$, $(\chi_D)_{\mathrm{nom}}$, and $(\chi_T)_{\mathrm{nom}}$ are close to 0° or 180°);
- the objects' can maneuver applying lateral accelerations orthogonal to their velocities; but the accelerations are relatively small, thus in general the longitudinal magnitudes of velocities stay almost the same during the entire process.

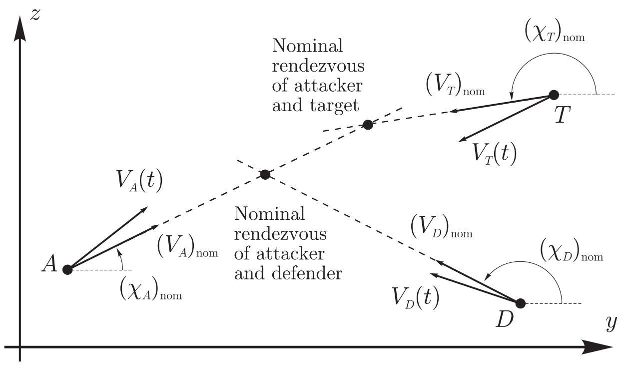

Figure 1. Scheme of a space intercept, planar engagement

All these assumptions allow one to linearize motions along the nominal trajectories. After linearization, the longitudinal motions (along the axis y) become uniform and only define the instants of nominal rendezvous of attacker with defender and target. Neglecting

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the longitudinal motions, one can consider only lateral one-dimensional motions (along the axis z) of the objects, which are described by linear dynamics:

$$\dot{\mathbf{z}}_\sigma = A_\sigma \mathbf{z}_\sigma + B_\sigma u_\sigma, \quad t \geq t_0, \ \mathbf{z}_\sigma \in R^{n_\sigma}, \ u_\sigma \in P_\sigma \subset R^{p_\sigma}, \quad \sigma \in \{A, D, T\}, \tag{1}$$

Here, $\mathbf{z}_A$, $\mathbf{z}_D$, and $\mathbf{z}_T$ are phase vectors of the attacker, defender, and target, respectively. The objects' controls u_A, u_D, u_T are constrained by convex compact sets P_A, P_D, P_T in their own spaces. The matrices A_A, A_D, and A_T are square; B_A, B_D, and B_T are, generally speaking, rectangular matrices; if some object has a scalar control, then the corresponding matrix is a column.

Denote by z_A, z_D, and z_T the first components of the vectors $\mathbf{z}_A$, $\mathbf{z}_D$, and $\mathbf{z}_T$. Assume that they are the lateral geometric coordinates of the objects.

Fix two instants t_D and t_T that are the instants of the nominal rendezvous of attacker with defender and target. Naturally assume $t_T > t_D$. The interest of the attacker is not to be intercepted by the defender at the instant t_D and to intercept the target at the instant t_T. Denoting by d_D the capture radius of the defender and by d_T the capture radius of the attacker (the radius, within which the target can be captured), we can formally define the objective of the attacker:

$$\bigl|z_A(t_D) - z_D(t_D)\bigr| \geq d_D, \quad \bigl|z_A(t_T) - z_T(t_T)\bigr| \leq d_T. \tag{2}$$

Consider the following zero-sum differential game (called the *ADT game* or *ADT problem*): the first player, the attacker, using the control u_A tries to guide system (1) to target set (2); the second player that joined the defender and the target, tries to hinder this by its controls u_D and u_T. We assume that during the game both players know exact values of all phase coordinates. It is necessary to construct the solvability set of the game, that is the set, wherefrom the first player guarantees achievement of its objective.

The considered problem arises when studying a pursuit in upper atmosphere layers. The scheme of the pursuit is given in Fig. 1 and is taken from works by J. Shinar.

Originally, problem of this type have been outlined in the famous book [1] by R. Isaacs. In the formulation by R. Isaacs, the target is an immobile set (from other point of view, this game was named a *game with a lifeline*). But J. Shinar is one of the first authors constructively explored problems of ADT type (see, for example, [2]). Publications on ADT problems are various. Let us mention some of them. Isaacs' formulation but with a attacker–defender capture in a neighborhood of the defender (instead of exact coincidence of positions) is considered in [3]. ADT type games with many attackers and/or defenders are considered, in particular, by I. Rusnak in [4,5]. Work [6] by the same author considers an ADT game without constraints on the objects' controls, but with an integro-terminal payoff (the integral part of the payoff contains a penalty for large valued controls). In [7], a situation with attacker and defender with simple motion dynamics and passive defenders in the plane is studied: there are moving obstacles, which cannot be passed through by the attacker. The obstacle motion law is known. The authors suggest a numerical algorithm for constructing solvability sets for games of this type. Work [8] considers an ADT problem with objects having simple motion dynamics in the plane. In the paper, the authors suggest a suboptimal solution of the problem.

II. Zero-Effort Miss Coordinates

Consider new coordinates x_1 and x_2 that are the values of relative coordinates $z_A - z_D$ and $z_A - z_T$, forecasted to the corresponding instants t_D and t_T, respectively, under zero players controls:

$$x_1(t) = X^1_A(t_D,t)z_A(t) - X^1_D(t_D,t)z_D(t), \quad x_2(t) = X^1_A(t_T,t)z_A(t) - X^1_T(t_T,t)z_T(t). \quad (3)$$

Here, $X^1_\sigma(t,\theta)$, $\sigma \in \{A,D,T\}$, are the first rows of the fundamental Cauchy matrices $X_\sigma(t,\theta)$ that corresponds to linear differential equations $\dot{\mathbf{z}}_\sigma = A_\sigma \mathbf{z}_\sigma$. Often, the variables x_1, x_2 are named *zero-effort miss coordinates* because they are the forecast values of the corresponding misses at the corresponding instants under zero players' controls.

Differentiating the values $x_i(t)$ by t, we obtain the dynamics in the new coordinates:

$$\begin{aligned}
&\dot{x}_1 = X^1_A(t_D,t)B_A u_A - X^1_D(t_D,t)B_D u_D, \\
&\dot{x}_2 = X^1_A(t_T,t)B_A u_A - X^1_T(t_T,t)B_T u_T, \\
&t \in [t_0,t_T], \quad u_A \in P_A,\ u_T \in P_T,\ u_D \in P_D, \\
&\big|x_1(t_D)\big| \geq d_D, \quad \big|x_2(t_T)\big| \leq d_T.
\end{aligned} \qquad (4)$$

From results of the differential game theory, it follows (see, for instance, [9–11]) that the differential game (4) is equivalent to the differential game with dynamics (1) and target set (2). The equivalence means that the first player can successfully finish game (1), (2) from some initial position $\big(t, \mathbf{z}_A(t), \mathbf{z}_D(t), \mathbf{z}_T(t)\big)$ iff the first player can successfully finish game (4) from the position $\big(t, x_1(t), x_2(t)\big)$, where $x_1(t)$ and $x_2(t)$ are computed by formula (3). Computations with dynamics (4) are more convenient since the dimension of the phase vector $x = (x_1,x_2)^\top$ equals two and the phase vector x is absent in the right-hand part of system (4).

III. Evolution of Solvability Set

At the instant t_T the inequality $\big|x_2(t_T)\big| \leq d_T$ (see (4)) defines an infinite strip along the axis x_1 — Fig. 2a. Further in the backward time before the instant t_D, the strip changes its width according the dynamics of the variable x_2. Since the strip is horizontally

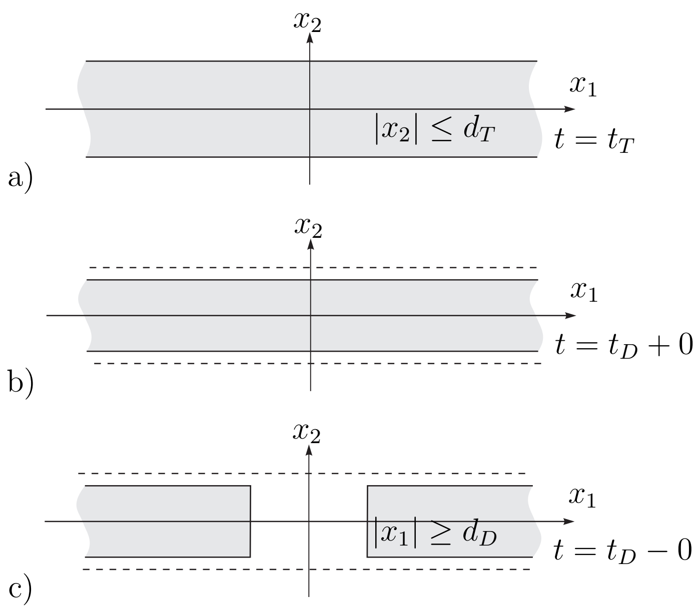

Figure 2. Backward evolution of the solvability set: a) at the instant t_T; b) before the instant t_D; c) after the instant t_D

infinite, the dynamics of the variable x_1 does not matter. If the target is quite "stronger" than the attacker, the width of the strip can become zero and the strip can vanish. In this situation, the successful capture is impossible.

If the target is not stronger too much, then before the instant t_D, we have a strip with some width — Fig. 2b; it can be wider or narrower than the initial strip depending on the relation of dynamic capabilities of the target and attacker. At the instant t_D, there is the inequality $\left|x_1(t_D)\right| \geq d_D$, that cuts off a part of the strip near the origin — Fig. 2c. After this instant (in the backward time), the solvability set becomes disconnected, and just this fact makes the problem interesting for a mathematical investigation. There are a number of scenarios of further solvability set evolution that depends on relation of dynamics capabilities of the attacker and target, and of the attacker and defender.

If the target is now stronger than the attacker, then in the backward time the strip can vanish. If the target is weaker than the attacker, but the defender is stronger, then vertical size of the set increases in the backward time, and the central hole grows too. Growing of the central hole means that simultaneous tracking the target and avoiding the defender is difficult for the attacker and it should make some reserve in distance to the defender. A similar effect can appear even if the attacker is slightly stronger than the defender and the target individually: it has no enough capabilities to overcome both opponents together.

If the attacker is stronger than the defender and target in total, then the central hole can diminish. But with that, it changes its shape: the vertical parts of its boundary obtain some slope zones. Therefore, in this case, if the vertical parts hit each other, the boundary of the new simple connected section is not just a horizontal line, but has some "teeth" directed inside the set.

If the attacker's dynamic advantage varies in time, there can be a very complicated evolution of time sections (t-sections) of the solvability set. In the next section, some examples of solvability sets are given.

IV. Examples of Solvability Sets

A. Dynamics of Objects

In the following examples, all objects have dynamics of the first order link:

$$\dot{z}_1 = z_2,\ \dot{z}_2 = z_3,\ \dot{z}_3 = (u - z_3)/l,\ |u| \leq \mu.$$

Here, z_1 is the coordinate of the object, z_2 is its velocity, z_3 is the acceleration. The dynamics of z_3 is the simplest model of inertial servomechanism that converts the *command signal* u to the acceleration. After setting some level of the command signal, the acceleration reaches it in time approximately three time greater than the *time constant* l.

If two objects have such a dynamics, the pursuer with constants l_1 and μ_1, and the evader with constants l_2 and μ_2, then we distinguish 4 variants depending on the values η and $\varepsilon\eta$, where $\eta = \mu_1/\mu_2$ and $\varepsilon = l_2/l_1$. The difference is how the solvability set in such a one-to-one game (1×1 game) changes in time (Fig. 3). In the figure, the symbol τ denotes the backward time.

In a 2×1 or 1×2 game, the solvability set evolution depends not only on the dynamic advantage of the objects, but on the shape of the t-sections of the set too.

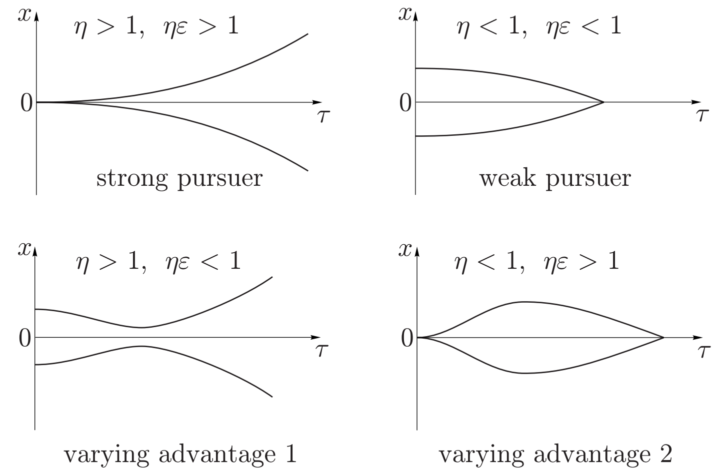

Figure 3. Evolution of the solvability set in a one-to-one game in different situations of the pursuer's dynamic advantage. The symbol τ denotes the backward time

B. Example 1. Strong Attacker

Consider the following parameters of the game:

$$\mu_A = 5.0, \; l_A = 1.0, \quad \mu_D = 0.25, \; l_D = 5.0, \quad \mu_T = 0.5, \; l_T = 1.5,$$
$$t_T = 3.0, \; t_D = 2.0, \quad d_T = 1.0, \; d_D = 1.0.$$

These parameters define the attacker to be stronger than both the defender and target.

Two views of the solvability set are given in Figs. 4 and 5. The set is assumed to be infinite along the axis x_1. During numerical computations, the infinite strip has been replaced by a rectangle very long in this direction (sufficiently long to be sure that no shape changes would reach its side edges). During visualization, the obtained body is cut again at some far distance (but less far than the rectangle edges are).

In Fig. 4, one can see both results of the dynamic advantage of the attacker: in the backward time the vertical size of the set increases, that is the target can be intercepted from a quite wide area; also the central hole arising at the instant t_D collapses. The latter means that the attacker is so strong that can choose at what side it goes round the defender even if at the initial instant they are quite close. Also, the "jags" are seen that remain after disappearing the central hole.

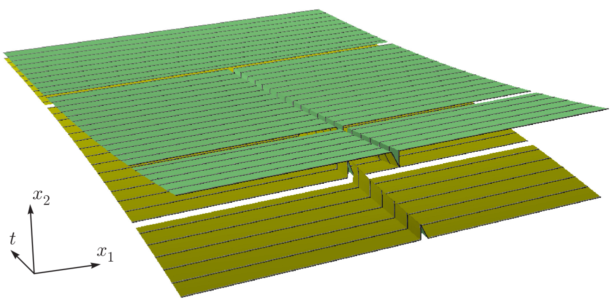

Figure 4. Example 1: a 3D-view of the solvability set

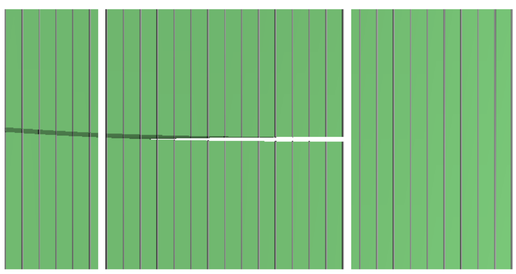

Figure 5. Example 1: a look along the axis x_2

Empty strips denote places where the solvability set t-sections change connectivity: at first (in the backward time) such a strip is at the instant t_D when the t-section become disconnected; the second strip is at the instant when the central hole disappears and the t-sections become connected again.

In Fig. 5, one can see that the central hole during its evolution in the backward time obtains slope sides such that they overlaps when seeing along the axis x_2.

C. Example 2. Pass around Pillar

Consider the following parameters of the game:

$$\mu_A = 5.0,\ l_A = 1.0, \quad \mu_D = 0.0,\ l_D = 5.0, \quad \mu_T = 5.0,\ l_T = 1.0,$$
$$t_T = 3.0,\ t_D = 2.0, \quad d_T = 1.0,\ d_D = 1.0.$$

The peculiarity of this example is that the defender has zero control only. Since at the beginning all velocities and accelerations supposed to be zero, than the defender is fixed and cannot move at all. So, it is reasonable to call it "pillar". Also, it is seen that the attacker and target have the same dynamic capabilities, no one of them has dynamic advantage.

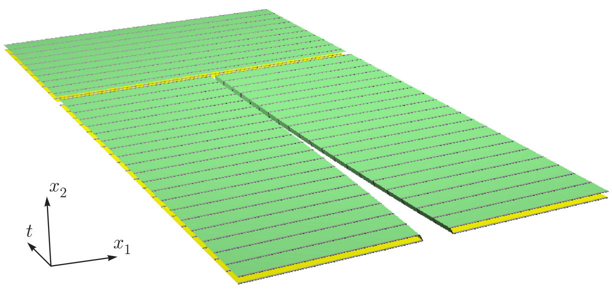

Figure 6. Example 2: a general view of the solvability set

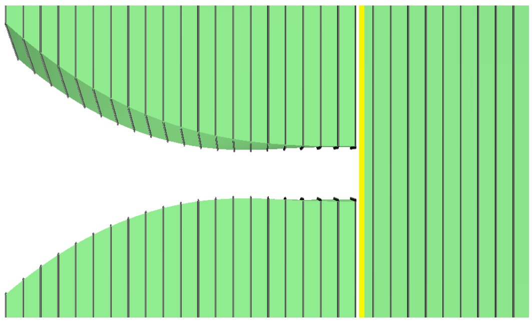

Figure 7. Example 2: growth of the central hole despite of absolute advantage of the attacker over the defender

A general view of the solvability set is given in Fig. 6. Since dynamic capabilities of the attacker and the target are the same, the set does not neither grow, nor contract vertically. At the same time, since all capabilities of the attacker are spent for tracking the target, it has no dynamics resources to maneuver well to go round the pillar. Therefore, the zone grows where the target can control in such a way that the parrying control of the attacker leads it to the pillar; that is the central hole grows in the backward time. Figure 7 (which has a scale and an aspect differing from Fig. 6) shows that the hole contracts a bit during a small period after the instant t_D when it appears, but further contraction is changed by expanding. In Fig. 7, again one can see slope side walls of the central hole.

D. Example 3. Varying Advantage of Attacker

Now, let the parameters of the game be the following:

$$\mu_A = 0.35,\ l_A = 0.2, \quad \mu_D = 0.0,\ l_D = 5.0, \quad \mu_T = 0.4,\ l_T = 1.0,$$
$$t_T = 17.0,\ t_D = 16.5, \quad d_T = 2.0,\ d_D = 0.82.$$

The defender is immovable again. But now the mutual dynamics advantage of the attacker and target changes in time. Namely, this is the case shown in Fig. 3 at the right lower

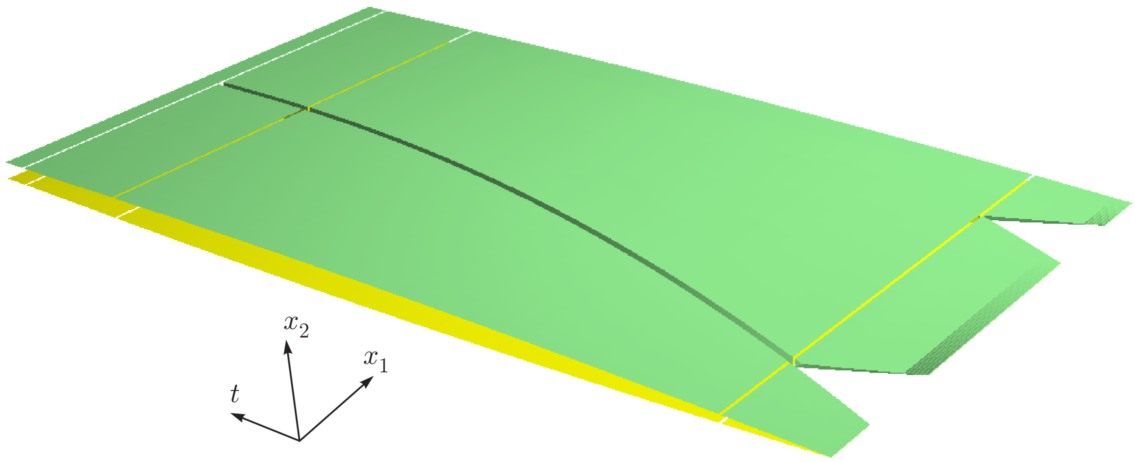

Figure 8. Example 3: the solvability set in the case of varying advantage of the attacker over the target

subfigure: at the beginning of the backward time, the attacker is stronger than the target, the solvability set grows in the vertical direction. Later in the backward time, the target becomes stronger, and the solvability set starts to diminish vertically.

The solvability set is shown in Fig. 8. One can see a quite complicated geometric structure. At first, there can bee seen how dynamic advantage passes from the attacker to the target and how the solvability set height changes in time: growth, then contraction until vanishing. With that, the central hole that appears at the instant t_D further collapses leaving to "teeth" directed inside the set: one on the upper boundary of the set, another on the lower one. The appearing of the hole and changing its sides can be seen in Fig. 9a. In this figure, the solvability set is cut off at some instant less than t_D (that is after appearing the central hole), but later than the instant when the sides of the hole meet each other.

Figure 9b shows the interior of the solvability set up to some instant after closing (in the backward time) the central hole, but the attacker still has dynamics advantage: the set still grows vertically. An interesting fact is that the inner teeth start to go aside; even if at this time period the attacker is weaker and the set contracts vertically, the central hole cannot reappear after joining these teeth because they cannot meet each other.

In Fig. 9c, a later (in the backward time) cut of the solvability set is given. At this instant, the attacker is already weaker and the set is diminishing vertically. The teeth go farer from the middle of the set.

Further in the backward time due to closing the upper and lower boundaries of the set, the upper tooth meets the lower boundary, and the lower tooth meets the upper boundary. At this instant, two holes appear, and the set disjoins into three disconnected parts: two side infinite parts and a central bounded one (see Fig. 8). (Some dark spots on the boundary of the set near this instant are because of some faults of the algorithm that constructs the boundary of the solvability set on the basis of separate time sections produced by the computational program.)

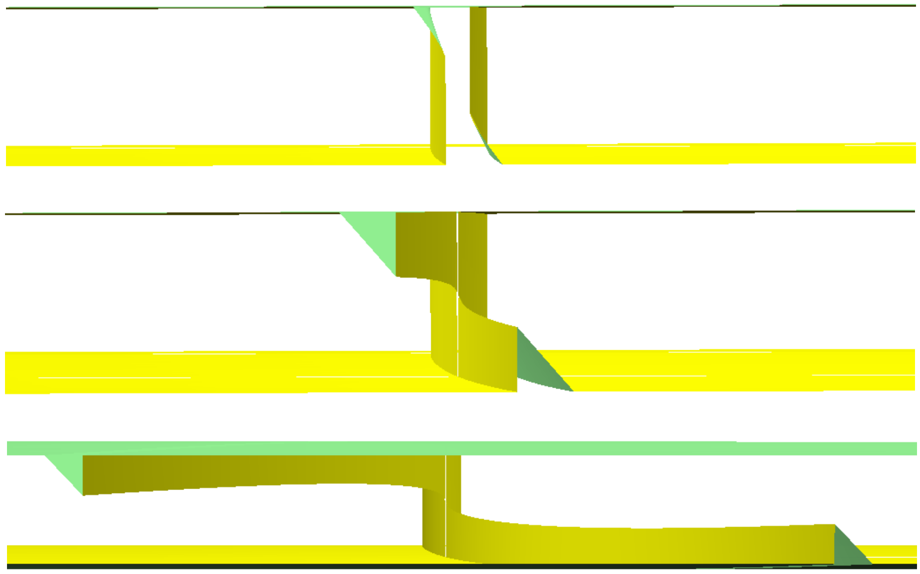

Figure 9. Example 3: evolution of the solvability set, view from the positive direction of the backward time axis

Since the attacker is weaker than the target, the set continues to decrease its vertical size and vanishes. But if the objects would have some oscillating dynamics, then the contraction of the solvability set would change by expansion, and these two holes would disappear and produce two new teeth each. That would lead to a much more complicated shape of the t-sections of the solvability set.

E. Discussion of Results

The visual investigation of the obtained solvability sets shows that all difficulties of the game solution are concentrated near the central hole appearing due to the presence of the defender. If the system position is such that the state of equivalent game is far from the hole and/or teeth of the solvability set, then we have a regular situation and optimal controls of all objects are easy to define: the defender directs itself towards the attacker, but cannot reach it; the attacker puts all its efforts to track the target ignoring the defender.

But if the equivalent state is on the side edge of the hole or a tooth, then the behavior of the attacker is more complicated: it should both track the target and simultaneously avoid the defender. Evidently, there are some singular lines (in the sense of R. Isaacs) on the side edges of the hole and teeth: dispersal or even equivocal. The attacker should control very accurately in such a situation because its rough control can lead the trajectory of the system outside the solvability set and, therefore, either the attacker will be intercepted by the defender, or the target will not be captured by the attacker.

Also, an investigation of the singular lines would be interesting. The initial problem is formulated with a target set, not a payoff function: therefore, optimal trajectories of the system should be investigated in the surface of the solvability set boundary.

V. Conclusion

The results given in the paper are obtained by the authors during the last year and are new. Of course, they are connected with model mathematical formulations in the framework of the theory of differential games. But these formulations appear in real problems of space pursuit. The results show that the solvability sets in the attacker–defender–target problem can rarely be constructed analytically. Therefore, to construct them one needs effective numerical algorithms and programs including programs for visualization. Solvability sets in quite typical situations can have a quite complicated structure. From the authors' point of view, its exotic peculiarities are important in actual engineering practice.

Acknowledgments

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