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Mathematics in Computer Science

Level Sets of the Value Function in Differential Games with Two Pursuers and One Evader. Interval Analysis Interpretation

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Abstract An algorithm for numerical constructing level sets of the value function is shortly described for one class of linear differential games with fixed termination instant. Some model interception problems with one target and two interceptors are considered; all objects are weak maneuverable.

Keywords Group differential pursuit-evasion games · Linear dynamics · Value function · Level sets · Numerical construction

Mathematics Subject Classification Primary 49N70; Secondary 49N90

1 Introduction

The standard formulation of a zero-sum differential game (see, for example, [6–8]) includes definition of the dynamics of the system and a scalar *payoff function* that describes the magnitude of the game result on certain realizations of system motions. For a wide class of games, the best guaranteed results of the minimizing and maximizing players in feedback controls are equal and called the *value function*.

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Construction of the value function is one of the main problems in solving zero-sum differential games. For example, the function can be found in the form of a collection of its level sets. Let $V(t, x)$ be a value of the value function at the position (t, x), where t is the time and x is the phase vector. Then $W_c = \{(t, x) : V(t, x) \le c\}$ is the level set of the value function that corresponds to a number c.

The set W_c coincides with the set of all initial positions, from which the player minimizing the payoff guarantees himself the result not exceeding the value c. The problem of numerical construction of the sets W_c for different values of c is very important. If we have a numerical instrument for constructing sets W_c, then, using computer simulation, we can investigate dependence of properties of these sets on the value c and, moreover, on parameters of the players' dynamics.

In this paper, we present results of constructing the level sets of the value function for the following differential game. Three inertial points (two pursuers P_1 and P_2 and one evader E) move along a straight line. At the instant T given in advance, the distances between P_1 and E and between P_2 and E are measured. The payoff φ is the minimum of these two distances. The first player that unites two pursuers P_1 and P_2 minimizes the value of the payoff. The second player interpreted as the evader E tries to maximize it. The case of linear dynamics of each of the objects is considered.

In spite of a model character, the problem under investigation has a significant applied meaning. It is connected with an interception problem in the upper atmosphere layers when there are two pursuing objects and one evader. The instant T fixed in advance corresponds to the termination time on the nominal objects' trajectories. The linear dynamics is stipulated by linearization of the nonlinear dynamics with respect to the nominal straight line motions of the objects.

In the case of a one-to-one game (*i.e.*, one pursuer–one evader), some versions of the linear dynamics (reasonable in an engineering sense) were described in [1–3]. In these works, properties of level sets of the value function had been investigated analytically. In the case with two pursuers, the analytical investigation becomes extremely complicated, and numerical procedures are needed for constructing the level sets of the value function.

Introducing the difference geometric coordinates of positions of P_1 and E and, also, of P_2 and E, we obtain that the payoff φ (or, in other words, the payoff function) depends only on two coordinates of the common phase vector of the system at the instant T. This allows one to use standard techniques known in the control theory and the differential game theory. Namely, we can consider an equivalent differential game with two-dimensional phase variable x. The component x_1 of this variable has the sense of the miss (with taking into account its sign) between P_1 and E that is forecasted onto the instant T under zero controls of the players. Similarly, the component x_2 is the forecasted miss between P_2 and E. Such phase coordinates are often called the *zero effort miss* coordinates [4].

At the instant T, we have $V(T, x) = \varphi(x)$. To produce the set $W_c = \{(t, x) : V(t, x) \le c\}$ in the space (t, x), we use the procedure of the backward construction of t-sections $W_c(t)$ of this set. Assuming $t_k = T - (N - k)\Delta$, $k = 0, \ldots, N$, $t_N = T$, $t_0 = \bar{t}$, we divide the time interval $[\bar{t}, T]$ of the game by the step Δ to the left from the instant T. We suppose $W_c(T) = \{x : \varphi(x) \le c\}$. On the basis of the set $W_c(T)$, the set $W_c(t_{N-1})$ is built and, further, using $W_c(t_{N-1})$, the set $W_c(t_{N-2})$ is calculated, and so on. Actually, we implement a procedure of the dynamic programming that takes into account peculiarities of the differential game theory. The limit set obtained as $\Delta \to 0$ gives the ideal result, and a set corresponding to some finite Δ is an approximation.

Choosing an algorithm of passage from the section $W_c(t_{k+1})$ to the section $W_c(t_k)$, we aim to a simple calculation scheme that works well for objects with dynamics of general type including ones described in [1–3,5]. The main operation is the algebraic sum (Minkowski sum) of two sets in the plane when one of the sets is a polygon and the second one is a segment. Nowadays, such an operation is typical for the computational geometry in the plane. Any segment in the plane is an image of an interval in the real axis. Thus, the situation can be considered in the framework of interval analysis. In particular, we can construct upper and lower estimates of the Minkowski sum.

The paper has the following structure. Sect. 2 is devoted to the problem formulation. In Sect. 3, the equivalent differential game is described. Algorithm for numerical constructing the level sets of the value function is given in Sect. 4. Several examples of constructing the level sets are given in Sect. 5. The paper ends by a conclusion.

2 Problem Formulation

Let motions of the pursuers P_1, P_2 and the evader E be described in the vector form by the relations

$$\begin{aligned}
&\dot{\mathbf{z}}_{P_i} = A_{P_i}\mathbf{z}_{P_i} + B_{P_i}u_{P_i},\\
&u_{P_i} = (u^1_{P_i}, u^2_{P_i})^\top, \quad |u^1_{P_i}| \le \mu^1_{P_i}, \quad |u^2_{P_i}| \le \mu^2_{P_i}, \quad \mathbf{z}_{P_i} \in R^{n_{P_i}},\ i = 1,\ 2;\\
&\dot{\mathbf{z}}_E = A_E\mathbf{z}_E + B_E u_E,\\
&|u_E| \le \mu_E, \quad \mathbf{z}_E \in R^{n_E}.
\end{aligned} \tag{2.1}$$

Here, A_{P_1}, A_{P_2}, and A_E are square matrices of corresponding dimensions; B_{P_1}, B_{P_2} are matrices of sizes $n_{P_1} \times 2$ and $n_{P_2} \times 2$, and B_E is a column matrix. The scalar controls $u^1_{P_i}$, $u^2_{P_i}$, $i = 1, 2$, and u_E are restricted by the geometric constraints.

Denote by z_{P_i}, z_E the first components of the vectors $\mathbf{z}_{P_i}$, $i = 1, 2$, and $\mathbf{z}_E$. These components represent the geometric coordinates of the objects on the line.

Let us fix an instant T and introduce the payoff in the form

$$\varphi = \min\big\{\big|z_{P_1}(T) - z_E(T)\big|, \big|z_{P_2}(T) - z_E(T)\big|\big\}. \tag{2.2}$$

Consider the following zero-sum differential game. The first player using the controls $u^1_{P_i}$, $u^2_{P_i}$, $i = 1, 2$, and having the dynamics (2.1) minimizes the payoff (2.2). The second player applying the control u_E maximizes the payoff. We suppose that during the motion both players know the exact values of all phase coordinates of all objects. It is necessary to suggest a method for computing level sets of the value function (*i.e.*, the solvability sets for the problem).

3 Two-Dimensional Equivalent Game

Denote by the symbol $x_i(t)$, $i = 1, 2$, the value of the difference $z_E - z_{P_i}$ forecasted from the current instant t and current states $\mathbf{z}_E(t)$, $\mathbf{z}_{P_i}(t)$ onto the termination instant T under the condition that the players' zero controls are applied in system (2.1). We have

$$x_i(t) = X^1_E(T, t)\mathbf{z}_E(t) - X^1_{P_i}(T, t)\mathbf{z}_{P_i}(t), \quad i = 1,\ 2. \tag{3.1}$$

In this relation, the upper index 1 means the first rows of the Cauchy fundamental matrices $X_{P_i}(T, t)$, $X_E(T, t)$ that correspond to the matrices A_{P_i}, A_E and are written for the termination instant T and the current instant t.

For a differential equation $\dot{\mathbf{z}} = A(t)\mathbf{z}$ with a vector variable $\mathbf{z}$, the *fundamental Cauchy matrix* $X(T, t)$ [that corresponds to the matrix $A(t)$] is defined (see, for example, [6–8]) as the solution of the matrix differential equation $\frac{\partial X}{\partial t}(T, t) = -X(T, t)A(t)$ with the boundary condition $X(T, T) = I$. Here, I is a square unit matrix of the corresponding dimension. Respectively, the first row $X^1(T, t)$ is the solution of the differential equation $\frac{\partial X^1}{\partial t}(T, t) = -X^1(T, t)A(t)$ with the boundary condition $X^1(T, T) = (1,\ 0,\ \ldots,\ 0)$.

Since in our case, the matrices A_{P_i}, A_E do not depend on the time t, the matrices $X_{P_i}(T, t)$, $X_E(T, t)$ depend only on the difference $T - t$. Underline that $x_i(T) = z_E(T) - z_{P_i}(T)$.

By differentiation of relations (3.1) on time t, we obtain

$$\dot{x}_i(t) = X^1_E(T, t)B_E u_E - X^1_{P_i}(T, t)B_{1,P_i}u^1_{P_i} - X^1_{P_i}(T, t)B_{2,P_i}u^2_{P_i}. \tag{3.2}$$

Here, the same as in (2.1): $\big|u^1_{P_i}\big| \le \mu^1_{P_i}$, $\big|u^2_{P_i}\big| \le \mu^2_{P_i}$, $|u_E| \le \mu_E$, $t \le T$, $i = 1, 2$. The symbols B_{1,P_i} and B_{2,P_i} denote the first and second columns of the matrix B_{P_i}.

From results of the differential game theory (see, for example, [6–8]), it follows that the differential game with dynamics (3.2) and the payoff

$$\varphi = \min\big\{\big|x_1(T)\big|, \big|x_2(T)\big|\big\} \tag{3.3}$$

is equivalent (in the sense of the value of the value function) to the differential game (2.1) with payoff (2.2). The reformulation of dynamic system (2.1) into equation (3.2) simplifies numerical computations. Indeed, the dimension of the phase vector $x = (x_1, x_2)^\top$ is only 2 and x does not enter to the right-hand side of equation (3.2).

Let the vector $\mathbf{z}$ is composed of the vectors $\mathbf{z}_{P_1}$, $\mathbf{z}_{P_2}$, $\mathbf{z}_E$. Consider a pair of positions $(t, \mathbf{z})$ in game (2.1), (2.2) and (t, x) in game (3.2), (3.3) such that relation (3.1) is true. Then the value functions of these games are equal in these positions: $\mathbf{V}(t, \mathbf{z}) = V(t, x)$.

When analyzing the form of system (3.2), note that the controls $u^1_{P_i}$, $u^2_{P_i}$ affect the coordinate x_i only. But at the same time, the velocity of both coordinates x_1, x_2 depend on the control u_E. With that, the first summands are the same in the expressions for $\dot{x}_1(t)$ and $\dot{x}_2(t)$. Dynamics (3.2) is symmetric with respect to the origin of the plane x_1, x_2. The bounds for the players' controls and the level sets (the Lebesgue sets) of the payoff function are also symmetric with respect to the zeros of their spaces. As a consequence, the t-sections $W_c(t)$ of the level sets $W_c = \{(t, x) : V(t, x) \le c\}$, $c \ge 0$, of the value function are symmetric with respect to the origin. During numerical constructions of the t-sections $W_c(t)$, the property of symmetry is not used, but we take it into account when check the correctness of the result.

Suppose

$$
\begin{aligned}
&D_{j,P_1}(t) = \big(X^1_{P_1}(T, t)B_{j,P_1}, 0\big)^\top, \quad D_{j,P_2}(t) = \big(0, X^1_{P_2}(T, t)B_{j,P_2}\big)^\top, \quad j = 1,\ 2;\\
&\quad D_E(t) = \big(X^1_E(T, t)B_E, X^1_E(T, t)B_E\big)^\top.
\end{aligned}
$$

Then system (3.2) has the following vector form:

$$
\begin{aligned}
&\dot{x}(t) = D_E(t)u_E - \sum\nolimits_{j,i} D_{j,P_i}(t)u^j_{P_i},\\
&|u_E| \le \mu_E, \quad |u^j_{P_i}| \le \mu^j_{P_i}, \quad i,\ j = 1,\ 2.
\end{aligned} \tag{3.4}
$$

Here, all controls are scalar, and the matrix coefficients are column vectors. Under this, the vector $D_E(t)$ is directed along the bisectrix of the first and third quadrants of the plane x_1, x_2; vectors $D_{1,P_1}(t)$, $D_{2,P_1}(t)$ are directed horizontally, and vectors $D_{1,P_2}(t)$, $D_{2,P_2}(t)$ are directed vertically. Indices j and i at the sum symbol mean summation on $j = 1, 2$ and $i = 1, 2$.

4 Numerical Construction of Level Sets

System (3.4) is a particular case of the linear system of the form

$$
\dot{x}(t) = \mathcal{D}_E(t)u_E + \mathcal{D}_P(t)u_P, \quad t \in [\bar{t}, T],\ x \in R^2,\ u_E \in U_E,\ u_P \in U_P,
$$

where $D_E(t)$, $D_P(t)$ are matrices, u_E, u_P are vectors, U_E, U_P are convex compact sets in the corresponding spaces. The instant T is fixed, and the continuous payoff function $\Phi\big(x(T)\big)$ is given. Its value is minimized by the first player (the player P) and maximized by the second player (the player E).

We suppose that the time interval $[\bar{t}, T]$ of the game is divided by the instants $t_N = T > t_{N-1} > \ldots > t_{k+1} > t_k > \ldots > t_0 = \bar{t}$ into the semi-intervals $[t_k, t_{k+1})$ of the same length Δ.

Usually, two versions of approximate constructing the sections $W_c(t_k)$ of the level set W_c of the value function are considered. Let $M_c = \{x : \Phi(x) \le c\}$ be a level set of the payoff function.

(1) Now, we describe the idea of the first version. We assume $W^{(1)}_c(t_N) = M_c$. Let the backward procedure be performed up to the instant t_{k+1}, and we have the set $W^{(1)}_c(t_{k+1})$. Having fixed the vector $u_E \in U_E$, we construct the attainability set for the first player at the instant t_k:

$$
G^{(1)}\big(t_k; t_{k+1}, W^{(1)}_c(t_{k+1}), u_E\big) = \bigcup_{u_P(\cdot)} \int_{t_{k+1}}^{t_k} \mathcal{D}_P(t)u_P(t)\,dt + \int_{t_{k+1}}^{t_k} \mathcal{D}_E(t)u_E\,dt + W^{(1)}_c(t_{k+1}).
$$

Here, $u_P(\cdot)$ means a measurable function given on $[t_k, t_{k+1}]$ and such that $u_P(t) \in U_P$ for any t. Further, we perform the intersection

$$W_c^{(1)}(t_k) = \bigcap_{u_E \in U_E} G^{(1)}\big(t_k; t_{k+1}, W_c^{(1)}(t_{k+1}), u_E\big).$$

For the ideal section $W_c(t_k)$, we have the inclusion

$$W_c(t_k) \subset W_c^{(1)}(t_k).$$

(2) In the second version of construction, we change the roles of the first and second players and instead of the set M_c we use closure of its supplement cl M_c'. We assume $W_c^{(2)}(t_N) = \mathrm{cl}\, M_c'$. Fixing the vector $u_P \in U_P$, we take

$$G^{(2)}\big(t_k; t_{k+1}, W_c^{(2)}(t_{k+1}), u_P\big) = \bigcup_{u_E(\cdot)} \int_{t_{k+1}}^{t_k} \mathcal{D}_E(t) u_E(t)\, dt + \int_{t_{k+1}}^{t_k} \mathcal{D}_P(t) u_P\, dt + W_c^{(2)}(t_{k+1}).$$

Here, $u_E(\cdot)$ means a measurable function given on $[t_k, t_{k+1}]$ and such that $u_E(t) \in U_E$ for any t. Further, we use the intersection

$$W_c^{(2)}(t_k) = \bigcap_{u_P \in U_P} G^{(2)}\big(t_k; t_{k+1}, W_c^{(2)}(t_{k+1}), u_P\big).$$

For the ideal section $W_c(t_k)$, we have the inclusion

$$\mathrm{cl}\big(W_c(t_k)\big)' \subset W_c^{(2)}(t_k).$$

So,

$$\mathrm{cl}\big(W_c^{(2)}(t_k)\big)' \subset W_c(t_k).$$

Both mentioned versions reflect the idea of estimation from above and below (in the sense of inclusion) of the ideal object $W_c(t_k)$. In their concrete performing, we must approximate the terminal set M_c, the attainability sets $G^{(1)}$ and $G^{(2)}$, and so on.

(3) Now, take into account the peculiarity of our dynamics (3.4). Suppose

$$d_{j,P_i}^{\max}(k) = \max\big\{\big|D_{j,P_i}(t)\big| : t \in [t_k, t_{k+1}]\big\}, \qquad d_E^{\min}(k) = \min\big\{\big|D_E(t)\big| : t \in [t_k, t_{k+1}]\big\}.$$

Let F_{j,P_i} be a centered (with respect to the origin) segment placed on the axis x_i of the half-length $\mu_{P_i}^j$. Suppose

$$S_{j,P_i}^{\max}(k) = d_{j,P_i}^{\max}(k) \cdot F_{j,P_i}.$$

Let us denote by the symbol F_E a segment of the half-length μ_E centered with respect to the origin and placed on the bisectrix of the first and third quadrants. Suppose

$$S_E^{\min}(k) = d_E^{\min}(k) \cdot F_E.$$

Having fixed the vector $s_E \in S_E^{\min}(k)$, consider the set

$$\mathcal{G}^{(1)}\big(t_k; t_{k+1}, \mathcal{W}_c^{(1)}(t_{k+1}), s_E\big) = \mathcal{W}_c^{(1)}(t_{k+1}) + \sum_{i,j} S_{j,P_i}^{\max}(k) \cdot \Delta - s_E \cdot \Delta. \tag{4.1}$$

We perform the intersection

$$\mathcal{W}_c^{(1)}(t_k) = \bigcap_{s_E \in S_E^{\min}(k)} \mathcal{G}^{(1)}\big(t_k; t_{k+1}, \mathcal{W}_c^{(1)}(t_{k+1}), s_E\big). \tag{4.2}$$

Formulas (4.1), (4.2) define the backward procedure for constructing the sets $\mathcal{W}_c^{(1)}(t_k)$ under the initial set $\mathcal{W}_c^{(1)}(t_N) = M_c$. By using the supplementation operation, relation (4.2) can be rewritten in the form

$$\mathcal{W}_c^{(1)}(t_k) = \Bigg(\bigcup_{s_E \in S_E^{\min}(k)} \mathcal{G}(1)\big(t_k; t_{k+1}, \mathcal{W}_c^{(1)}(t_{k+1}), s_E\big)'\Bigg)'$$
$$= \Bigg(\Big(\mathcal{W}_c^{(1)}(t_{k+1}) + \sum_{i,j} S_{j,P_i}^{\max}(k) \cdot \Delta\Big)' + S_E^{\min}(k) \cdot \Delta\Bigg)'. \tag{4.3}$$

Here, the symmetry (with respect to the origin) of the summand segments is taken into account. Thus, finding the set $\mathcal{W}_c^{(1)}(t_k)$ is reduced to the algebraic sums.

We have

$$W_c(t_k) \subset W_c^{(1)}(t_k) \subset \mathcal{W}_c^{(1)}(t_k). \tag{4.4}$$

(4) Introducing similarly the segments $S_E^{\max}(k)$, $S_{j,P_i}^{\min}(k)$ and fixing a vector $s_P \in \sum_{i,j} S_{j,P_i}^{\min}(k)$, we obtain

$$\mathcal{G}^{(2)}\big(t_k; t_{k+1}, \mathcal{W}_c^{(2)}(t_{k+1}), s_P\big) = \mathcal{W}_c^{(2)}(t_{k+1}) + S_E^{\max} \cdot \Delta - s_P \cdot \Delta.$$

Let

$$\mathcal{W}_c^{(2)}(t_k) = \bigcap_{s_P \in \sum_{i,j} S_{j,P_i}^{\min}(k)} \mathcal{G}^{(2)}\big(t_k; t_{k+1}, \mathcal{W}_c^{(2)}(t_{k+1}), s_P\big).$$

We suppose that $\mathcal{W}_c^{(2)}(t_N) = \operatorname{cl} M_c'$.

Applying the supplementation operation, we have

$$\mathcal{W}_c^{(2)}(t_k) = \Bigg(\bigcup_{s_P \in \sum_{i,j} S_{j,P_i}^{\min}(k)} \mathcal{G}^{(2)}\big(t_k; t_{k+1}, \mathcal{W}_c^{(2)}(t_{k+1}), s_P\big)'\Bigg)'$$
$$= \Bigg(\Big(\mathcal{W}_c^{(2)}(t_{k+1}) + S_E^{\max}(k) \cdot \Delta\Big)' + \sum_{i,j} S_{j,P_i}^{\min}(k) \cdot \Delta\Bigg)'.$$

Since

$$\operatorname{cl}\big(W_c(t_k)\big)' \subset W_c^{(2)}(t_k) \subset \mathcal{W}_c^{(2)}(t_k),$$

the following inclusion holds:

$$W_c(t_k) \supset \operatorname{cl}\big(W_c^{(2)}(t_k)\big)' \supset \operatorname{cl}\big(\mathcal{W}_c^{(2)}(t_k)\big)'. \tag{4.5}$$

Uniting relations (4.4) and (4.5), we get, as a result,

$$\operatorname{cl}\big(\mathcal{W}_c^{(2)}(t_k)\big)' \subset \operatorname{cl}\big(W_c^{(2)}(t_k)\big)' \subset W_c(t_k) \subset W_c^{(1)}(t_k) \subset \mathcal{W}_c^{(1)}(t_k). \tag{4.6}$$

(5) Thus, for obtaining the outer and internal estimations of the set $W_c(t_k)$, we use the sets $\mathcal{W}_c^{(1)}(t_k)$ and $\operatorname{cl}\big(\mathcal{W}_c^{(2)}(t_k)\big)'$, and each of these sets is a result of sequential application of algebraic sums. Under this, the summation is performed with the segments symmetric with respect to the origin. Orientation of the segments does not change in time. To get the set $\mathcal{W}_c^{(1)}(t_k)$, estimating from above the ideal set $W_c(t_k)$, we increase the segments of the first player (the increasing is implemented by coefficients $d_{j,P_i}^{\max}(k)$, $j = 1, 2$, $i = 1, 2$) and decrease the segment of the second player [using the coefficient $d_E^{\min}(k)$]. For obtaining the estimation $\operatorname{cl}\big(\mathcal{W}_c^{(2)}(t_k)\big)'$ from below of the set $W_c(t_k)$, we, in contrast, increase the opportunities of the second player [by introducing the coefficient $d_E^{\max}(k)$] and decrease opportunities of the first player (using the coefficients $d_{j,P_i}^{\min}(k)$, $j = 1, 2$, $i = 1, 2$).

According to the works of Pontryagin and his pupils [9–12], the results of summation described above are the upper and lower alternating sums. The constructing of the alternating sums under fixed directions of segments of the first and second players brings together the effective backward constructions of differential game theory and methods of interval analysis (see, for example, [13,14]).

(6) In practical computations, the step Δ for dividing the time axis is taken rather small in a way to make small the Hausdorff distance between the sets $\mathcal{W}_c^{(1)}(t_k)$ and $\mathrm{cl}\big(\mathcal{W}_c^{(2)}(t_k)\big)'$. Of course, we talk about the Hausdorff distance only in the case when the set $\mathrm{cl}\big(\mathcal{W}_c^{(2)}(t_k)\big)'$ is non-empty and $\mathcal{W}_c^{(1)}(t_k) = \mathrm{cl}\,\mathrm{int}\,\mathcal{W}_c^{(1)}(t_k)$. The latter means that the set $\mathcal{W}_c^{(1)}(t_k)$ has no interiorless "sprouts". Here, the symbol int denotes the operation of taking interior of a set.

So, under a small step Δ of the backward procedure, instead of finding the accurate values of $d_E^{\max}(k)$, $d_E^{\min}(k)$, $d_{j,P_i}^{\max}(k)$, and $d_{j,P_i}^{\min}(k)$, we usually take the "frozen" vectors $\widetilde{D}_E(t)$, $\widetilde{D}_{j,P_i}(t)$, $j = 1, 2$, $i = 1, 2$, on each interval $[t_k, t_{k+1})$ of the chosen time partition. For example, it is possible to take $\widetilde{D}_E(t) = D_E(t_k)$, $\widetilde{D}_{j,P_i}(t) = D_{j,P_i}(t_k)$, $t \in [t_k, t_{k+1})$. This corresponds to the following choice of the coefficients:

$$d_E(k) = \big|D_E(t_k)\big| \in \big[d_E^{\min}(k), d_E^{\max}(k)\big], \quad d_{j,P_i}(k) = \big|D_{j,P_i}(t_k)\big| \in \big[d_{j,P_i}^{\min}(k), d_{j,P_i}^{\max}(k)\big].$$

In each time step of the backward procedure, computations of the set $\mathcal{W}_c(t_k)$ analogous to the set $\mathcal{W}_c^{(1)}(t_k)$ in formula (4.3) are made, but on the basis of the coefficients $d_E(k)$, $d_{j,P_i}(k)$ chosen in the shown way. We work (in each algebraic sum) with the present boundary of the current set and correct it. The obtained set $\mathcal{W}_c(t_k)$ obeys inclusions

$$\mathrm{cl}\big(\mathcal{W}_c^{(2)}(t_k)\big)' \subset \mathcal{W}_c(t_k) \subset \mathcal{W}_c^{(1)}(t_k).$$

Due to (4.6), we get that for some small Δ the set $\mathcal{W}_c(t_k)$ constructed numerically and the ideal set $W_c(t_k)$ are close. An algorithm based on this choice of the coefficients $d_E(k)$, $d_{j,P_i}(k)$ is described in the works [15,16], where it is used to investigate problems with two pursuers and one evader for dynamics of some particular type.

Theoretically, the set M_c is the cross with infinite strips parallel to the coordinate axes of the forecasted miss. In the computational algorithm, we cut off the strips on a finite but sufficiently large distance from the coordinate origin.

Some difficulties can occur at the instants of bifurcations when the set, which we construct in the backward procedure, disjoins into two separate subsets that can degenerate or, conversely, join again under increasing the backward time. In the problem under consideration, such instants of bifurcations are either absent, or their number is not large. This depends on concrete parameters of the problem. At the instants of bifurcations, we change the description of the constructed set. In the case of disjoining, we begin to process independently two polygons. In the case of gathering, the boundary of their union is calculated, which is computed further.

(7) The entire algorithm, which corresponds to the item (6), can be described as follows:

Step 1. Read input data:

- the dimensions n_{P_i}, n_E;
- the matrices A_{P_i}, A_E, B_{P_i}, B_E of objects' dynamics;
- the constraints $\mu_{P_i}^j$, μ_E for the players' controls;
- the information on the time grid: $\bar{t}$, T, Δ $(N = (T - \bar{t})/\Delta)$;
- the information on the grid of values c of the payoff function: $c_{\min}$, $c_{\max}$, Δc.

Step 2. For instant $t_k = T - (N - k)\Delta$, $k = 0, \ldots, N$, integrate the first rows of the fundamental Cauchy matrices (by means of the Euler method):

$$X_{P_i}^1(T, t_N) = X_{P_i}^1(T, T) = I^1, \quad X_{P_i}^1(T, t_k) = X_{P_i}^1(T, t_{k+1}) + X_{P_i}^1(T, t_{k+1}) A_{P_i} \Delta,$$
$$X_E^1(T, t_N) = X_E^1(T, T) = I^1, \quad X_E^1(T, t_k) = X_E^1(T, t_{k+1}) + X_E^1(T, t_{k+1}) A_E \Delta.$$

Here, $I^1 = (1,\ 0,\ \ldots,\ 0)$ are row vectors of the corresponding sizes.

Step 3. Loop on the values of c from $c_{\min}$ to $c_{\max}$ with the step Δc.

Step 4. Take

$$\mathcal{W}_c(t_N) = \mathcal{W}_c(T) = M_c = \big\{(x_1, x_2) : |x_1| \le c, |x_2| \le \mathcal{M}\big\} \bigcup \big\{(x_1, x_2) : |x_2| \le c, |x_1| \le \mathcal{M}\big\}.$$

Here, $\mathcal{M}$ is some large constant used for "cutting" the infinite "cross" of ideal level set of the payoff function.

Step 5. Loop on time instants with the index k that changes from N down to 0.

Step 6. Compute values, which define the pursuers' and evader's vectograms at the current instant t_k:

$$d_{j,P_1}(k) = \left|X^1_{P_1}(T,t_k)B_{j,P_1}\right|,\ d_{j,P_2}(k) = \left|X^1_{P_2}(T,t_k)B_{j,P_2}\right|,\ j = 1,\ 2;$$
$$d_E(k) = \left|X^1_E(T,t_k)B_E\right|.$$

Step 7. Using geometric algorithms, compute the new time section according to formula

$$\mathcal{W}_c(t_k) = \left(\left(\mathcal{W}_c(t_{k+1}) + \sum_{i,j}\Delta\cdot d_{j,P_i}(k)\cdot F_{j,P_i}\right)' + \Delta\cdot d_E(k)\cdot F_E\right)'.$$

As it was told above, the procedure for computing the algebraic sum "+" of a polygon and a segment deals with the boundary contour of the polygon. During the computations, we control degeneration of the set $\mathcal{W}_c(t_k)$, loss of connectivity and joining of its separate components.

Step 8. If the new section $\mathcal{W}_c(t_k)$ is degenerated (that is, it is empty), the loop on time breaks.

Step 9. End of the loop on time.

Step 10. The computed sections $\mathcal{W}_c(t_k)$ are written to the output file.

Step 11. End of the loop on the values of c.

The geometric procedures used at step 7 are quite non-trivial, and their description is too large to be included to this paper.

5 Results of Numerical Constructions

All computations, which results are given in this Section, have the time step $\Delta = 0.05$.

5.1 Pursuers with Dual Control

Let each pursuer have the following dynamics:

$$\dot z_1 = z_2,$$
$$\dot z_2 = z_3 + d_c u_c + d_t u_t,$$
$$\dot z_3 = \big((1-d_c)u_c + (1-d_t)u_t - z_3\big)/l_P.$$

This dynamics describes an interceptor missile with a canard control located at the head part of the missile (the control u_c) and a tail control located at the rear part of the missile (the control u_t). The coefficients d_c (which is positive) and d_t (which is negative) have the absolute values showing how far from the center of mass of the missile the controls are located. The value l_P is the *time constant* showing the inertiality of the control servomechanisms. There is a total control limit $a_{P,\max}$, which is divided between the canard and tail controls: $|u_c| \le \alpha\cdot a_{P,\max}$, $|u_t| \le \beta\cdot a_{P,\max}$; $\alpha,\beta \ge 0$, $\alpha+\beta = 1$. One-to-one problems with the pursuer having such a dynamics are studied in [2].

Consider an evader that has the dynamics

$$\dot z_1 = z_2,$$
$$\dot z_2 = z_3,$$
$$\dot z_3 = (v - z_3)/l_E, \qquad |v| \le a_{E,\max}. \tag{5.1}$$

Take the parameters of the game, which assume equal pursuers:

$$a_{P_1,\max} = a_{P_2,\max} = 1.0,\ l_{P_1} = l_{P_2} = 1/0.9,\ d_{c,1} = d_{c,2} = 0.5,$$
$$d_{t,1} = d_{t,2} = 0.5,\ \alpha_1 = \alpha_2 = 0.5,\ a_{E,\max} = 1.0,\ l_E = 1.0,\ T = 12.$$

The level set W_0 for this problem is shown in Fig. 1. This set corresponds to $c = 0$, that is, there is the exact capture of the evader if the initial position of the system is inside the set. In this and following figures, t denotes the time axis, and x_1, x_2 are the coordinates of system (3.2). In each figure, the point of view is chosen in such a way that all patterns of evolution in time of the level sets are well distinguishable. In Fig. 1, one can see that the size of the sections $W_c(t)$ grows with the increasing of the backward time $T - t$. This defines the situation of strong pursuers.

Now let us take the pursuers with different maximal values of the control. Namely,

$$a_{P_1,\max} = 1.0,\ a_{P_2,\max} = 0.9,\ l_{P_1} = l_{P_2} = 1/0.9,\ d_{c,1} = d_{c,2} = 0.5,$$
$$d_{t,1} = d_{t,2} = 0.5,\ \alpha_1 = \alpha_2 = 0.5,\ a_{E,\max} = 1.0,\ l_E = 1.0,\ T = 15.$$

We show the level set W_0 in Fig. 2. In this case of parameters, the pursuer P_1 dynamics capabilities exceed the ones of the evader E. The pursuer P_2 is stronger than the evader only in some small initial period of the backward time. When the backward time grows, it becomes weaker than the evader. Emphasize that change of the dynamic advantage in time does not mean a change of the parameters of the game and occurs just under their fixed values due to the certain fundamental Cauchy matrices that defines dynamics (3.2) of the game.

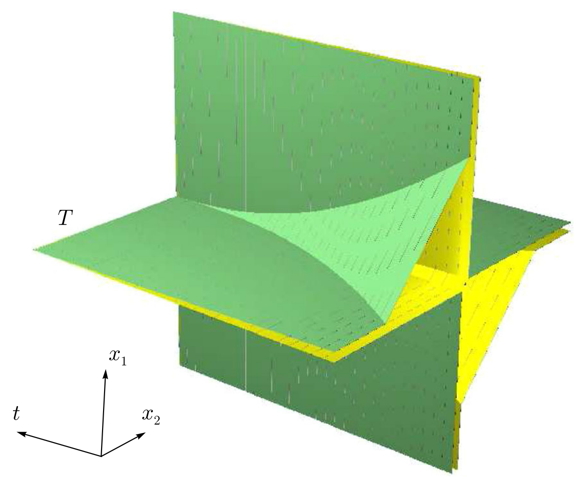

Fig. 1 The case of two strong pursuers

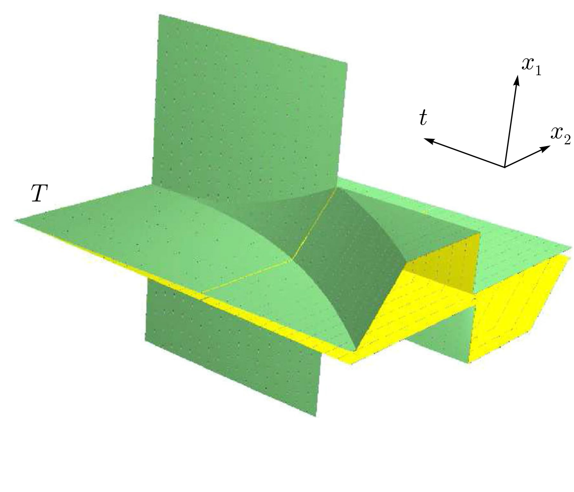

Fig. 2 The case of one strong pursuer and one pursuer with varying advantage

Consider a situation, when both pursuers have the maximal control (acceleration) level less than the evader:

$$a_{P_1,\max} = 0.8,\ a_{P_2,\max} = 0.9,\ l_{P_1} = l_{P_2} = 1/0.9,\ d_{c,1} = d_{c,2} = 0.5,$$
$$d_{t,1} = d_{t,2} = 0.5,\ \alpha_1 = \alpha_2 = 0.5,\ a_{E,\max} = 1.0,\ l_E = 1.0,\ T = 15.$$

The level set W_0 for this variant is given in Fig. 3. Here, the set W_0 disjoins into two separate sets at some instant. These sets contract with the increasing of the backward time and finally degenerate (this instant is outside the interval of the computations).

In the fourth example, let the dynamics of the evader be a one-controlled tail/canard scheme [1]:

$$\dot z_1 = z_2,$$
$$\dot z_2 = z_3 + dv,$$
$$\dot z_3 = \big((1-d)v - z_3\big)/l_E, \quad |v| \le a_{E,\max}.$$

The parameter d defines the position of the control: $d > 0$ corresponds to the canard scheme, $d < 0$ corresponds to the tail scheme.

Game parameters are

$$a_{P_1,\max} = a_{P_2,\max} = 1.12,\ l_{P_1} = l_{P_2} = 1/0.18807,\ d_{c,1} = d_{c,2} = 0.521431,$$
$$d_{t,1} = d_{t,2} = -0.5,\ \alpha_1 = \alpha_2 = 0.9,\ a_{E,\max} = 1.0,\ d_E = 0.305845, l_E = 1.0,\ T = 15.$$

Again, we deal with equal pursuers. Fig. 4 contains a view of the set W_0. Unlike the previous example, the parts, to which the t-section $W_0(t)$ disjoins, start to grow and join back with the increasing of the backward time.

Concluding consideration of the situation with pursuers having dual control, let us show the result for pursuers with different dynamics capabilities:

$$a_{P_1,\max} = 1.12,\ a_{P_2,\max} = 1.21,\ l_{P_1} = l_{P_2} = 1/0.18807,\ d_{c,1} = d_{c,2} = 0.605,$$
$$d_{t,1} = d_{t,2} = -0.5,\ \alpha_1 = 0.9,\ \alpha_2 = 0.8,\ a_{E,\max} = 1.0,\ d_E = 0.157980,\ l_E = 1.0,\ T = 25.$$

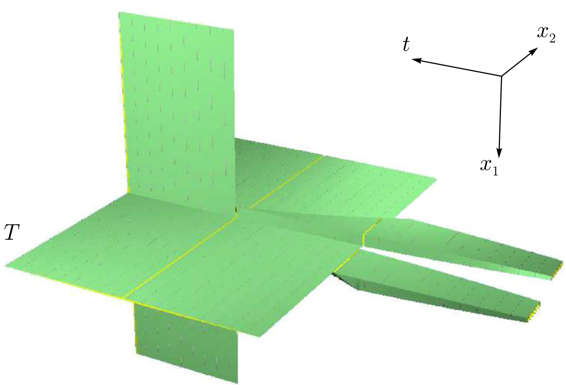

Fig. 3 The case of two pursuers with varying advantage; they are stronger than the evader at the beginning of the backward time, then they become weaker

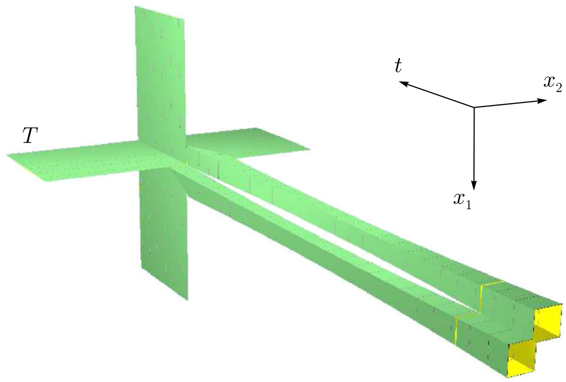

Fig. 4 The case of equal pursuers with varying advantage: initially the pursuers are stronger than the evader, then they become weaker, and finally they are stronger again

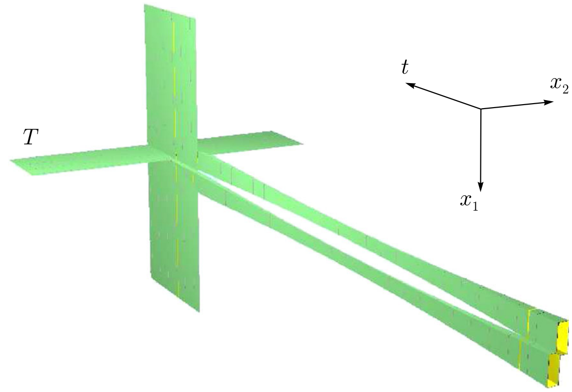

Fig. 5 The case of different pursuers with varying advantage: initially the pursuers are stronger than the evader, then they become weaker, and finally they are stronger again

The set W_0 can be seen in Fig. 5.

5.2 Pursuers with Damped Oscillating Control Link

Consider a situation with the pursuers each having one scalar control only. It can be formally obtained by setting the constraint for the corresponding control equal to zero: $\mu^2_{P_i} = 0$. Let the evader have dynamics (5.1). Both pursuers have the same dynamics that describes a servomechanism, which works out the command signal, as a damped oscillator:

$$\begin{aligned}
\dot z_1 &= z_2,\\
\dot z_2 &= z_3,\\
\dot z_3 &= z_4,\\
\dot z_4 &= -\omega^2 z_3 - \zeta z_4 + u, \quad |u| \le a_{P,\max}.
\end{aligned}$$

Here, ω is the fundamental frequency of the servomechanism, ζ is the damping factor. Such a dynamics of the pursuer in a one-to-one game is considered in [3].

Let us take the following parameters of the game:

$$a_{P_1,\max} = a_{P_2,\max} = 0.3,\ \omega_1 = \omega_2 = 0.5,\ \zeta_1 = \zeta_2 = 0.0025,$$
$$a_{E,\max} = 1.3,\ l_E = 1.0,\ T = 30.$$

In general, the pursuers are weaker than the evader. This leads to emptiness of all time sections $W_c(t)$, $t < T$, of the set W_0: it degenerates just at the termination instant. Thus, in Fig. 6, we show the set W_c that corresponds to $c = 1.6$. One can see two places of narrow throats of the set.

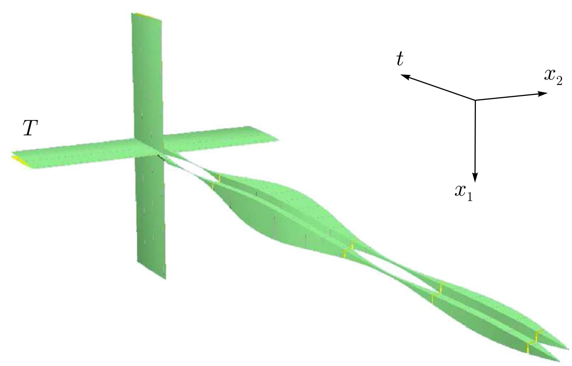

Fig. 6 The case of equal one-controlled pursuers; three instants of disjoining

6 Conclusion

Investigations of the sets of initial states, from which interception is guaranteed with the prescribed level of miss, are very important in engineering practice. Such sets are often called the solvability ones. Numerical methods in differential game theory (together with corresponding visualization tools) allow one to construct and investigate the solvability sets in the form of the level sets of the value function.

The paper presents an algorithm for constructing the level sets. The algorithm is oriented to problems with linear dynamics and fixed termination instant. It is assumed that the terminal payoff function is described by two components of the phase vector. Under this, its level sets have the crosswise form that is characteristic for problems with two pursuers and one evader.

Our experiments of constructing level sets of the value function (together with the results on one-to-one interception problems known in the literature) show that in spite of various dynamics descriptions, the variety of qualitatively different types of structures of the solvability sets is not too large. An accurate description of typical variants of structure with distinguishing "exotic" examples can be a subject for further investigation.

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