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CONTROL THEORY

# Semigroup Property of the Program Absorption Operator in Games with Simple Motions on the Plane

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**Abstract**—The program absorption operator takes the terminal set given at the terminal time to some set defined at the initial time. For differential games with simple motions on the plane, we obtain sufficient conditions under which the semigroup property also holds in the case of a nonconvex terminal set.

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## 1. INTRODUCTION

The simplest model description of dynamics in the theory of differential games is given by a dynamics of the form

$$\dot{x} = p + q, \qquad p \in P, \qquad q \in Q.$$

Here the right-hand side does not contain the state variable $x$, and the state velocity $\dot{x}$ is defined only by the controls $p$ and $q$ of the first and second players; moreover, the constraints $P$ and $Q$ are independent of time. In the monograph [1, pp. 22, 45 of the Russian translation], games with such dynamics were referred to as *games with simple motions*.

In numerical methods of the theory of differential games, the dynamics of simple motions naturally appears for the local approximation of a linear or nonlinear dynamics in the case where the possibilities of the players are "frozen" with respect to time or space variables. The actions corresponding to the next step of an iterative procedure used for finding the value

function of a game are performed in the framework of the dynamics of simple motions.

For example, one important class of differential games consists of games with linear dynamics, fixed terminal time, and continuous terminal payoff function. For such games, there exists a passage to new coordinates [2, pp. 159–161; 3, pp. 89–91], which has the meaning of the prediction of the state variable for the terminal time in view of the "free" motion of the system under zero control influences of the players. The passage is performed with the use of the Cauchy matrix of the original system. The right-hand side of the new system does not contain the state variable, but the controls of the players have time-dependent coefficients.

For the numerical construction of level sets of the value function, the part of the time axis lying to the left of the terminal time is divided with some increment, and the coefficients of the dynamics are frozen on each partition interval [4, 5]. The dynamics of simple motions is thereby obtained at each step. By determining a level set of the value function and by moving backwards in time from the terminal time, one recomputes the resulting set on each partition interval with the use of a game with simple motions. Then one passes to the limit by letting the partition increment tend to zero. For an appropriately chosen recomputation operator (at one step), the limit set coincides with the level set (the Lebesgue set) of the value function.

For an efficient implementation of the described scheme, it is important to appropriately choose the operator that is used for the passage in the framework of a single step of the backward procedure. It is desired that the operator has the semigroup property: for the dynamics frozen on a selected interval, the introduction of additional partition points does not change the result.

In the present paper, we consider the operator that is referred to in the theory of differential games as the program absorption operator [2, p. 122]. In games with simple motion, its semigroup property was earlier proved [6] for the case in which the operator is applied to a convex set. In the present paper, for problems on the plane, we state and prove sufficient conditions under which the semigroup property also holds in the nonconvex case.

The obtained results can be used for the development and justification of numerical methods in the theory of differential games.

## 2. STATEMENT OF THE PROBLEM

Consider the conflict-control system with simple motions [1, pp. 22, 45 of the Russian translation]

$$\frac{dx}{dt} = p + q, \qquad p \in P, \qquad q \in Q, \qquad x \in \mathbb{R}^n, \tag{1}$$

where $t$ is time and $P$ and $Q$ are convex compact sets in $\mathbb{R}^n$. Let $M$ be a compact set in $\mathbb{R}^n$.

We introduce the program absorption operator [2, p. 122; 6; 7]

$$T_\varepsilon(M) := (M - \varepsilon P) \overset{*}{-} \varepsilon Q, \qquad \varepsilon > 0, \qquad M \subset \mathbb{R}^n.$$

Here we have used the operations of algebraic sum

$$A + B = \{d : \ d = a + b, \ a \in A, \ b \in B\}$$

and the geometric difference (the Minkowski difference) [7; 8, p. 203]

$$A \overset{*}{-} B := \{d : \ d + B \subseteq A\}.$$

In addition, we define the operator with multiple recomputation [6]

$$\widetilde{T}_\varepsilon(M) := \bigcap_{\substack{\omega=(\varepsilon_1,\ldots,\varepsilon_m)\\ |\omega|=\varepsilon}} T_{\varepsilon_1}(T_{\varepsilon_2}(\ldots T_{\varepsilon_m}(M)\ldots)).$$

Here $\omega$ is the symbol of the partition of the interval $[0, \varepsilon]$. The intersection is taken over all finite partitions.

The action of the operators $T_\varepsilon$ and $\widetilde{T}_\varepsilon$ on a set $M$ coincides if the operator $T_\varepsilon$ has the *semigroup property* with respect to the set $M$, i.e., if the relation

$$T_{\varepsilon_1+\varepsilon_2}(M) = T_{\varepsilon_1}(T_{\varepsilon_2}(M)) \tag{2}$$

holds for all $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon_1 + \varepsilon_2 \le \varepsilon$.

The problem is to impose conditions on the sets $M$, $P$, and $Q$ and the range of $\varepsilon_1$ and $\varepsilon_2$ ensuring the validity of relation (2).

The semigroup property of the operator $T_\varepsilon$ was analyzed in [6]. It was shown that the semigroup property holds for a convex set $M$.

One application of the semigroup property of the operator $T_\varepsilon$ is the following. Consider a two-person zero-sum game with dynamics (1), fixed terminal time $\vartheta$, and payoff function $J(x(\cdot)) = \varphi(x(\vartheta))$, where $\varphi$ is a continuous function in $\mathbb{R}^n$. Let the first player minimize the value of $J$, and let the second player maximize it. Then the set of states at the initial time $t = 0$ for which the game value does not exceed $c$ coincides [2, pp. 76, 77; 6; 9, p. 209] with the set $\widetilde{T}_\vartheta(M_c)$, where

$$M_c = \{x \in \mathbb{R}^n : \ \varphi(x) \le c\}.$$

In the case of the semigroup property, we have $T_\vartheta(M_c) = \widetilde{T}_\vartheta(M_c)$.

## 3. ANALYSIS OF THE OPERATOR $T_\varepsilon$

Note the following three obvious properties.

1. $T_\varepsilon(M) = \bigcap_{q\in Q}(M - \varepsilon(P+q))$.
2. The inclusion $x \in T_\varepsilon(M)$ is equivalent to the relation $(x+\varepsilon(P+q)) \cap M \neq \varnothing$, $q \in Q$.
3. The condition $x \notin T_\varepsilon(M)$ is equivalent to the existence of a $q \in Q$ such that

$$(x+\varepsilon(P+q)) \cap M = \varnothing.$$

The following two assertions were proved in [6].

**Lemma 1.** *The inclusion*

$$T_{\varepsilon_1}(T_{\varepsilon_2}(M)) \subseteq T_{\varepsilon_1+\varepsilon_2}(M) \tag{3}$$

*holds for all* $\varepsilon_1, \varepsilon_2 > 0$.

**Proof.** Let $x \in T_{\varepsilon_1}(T_{\varepsilon_2}(M))$. Consequently, $x + \varepsilon_1 Q \subseteq T_{\varepsilon_2}(M) - \varepsilon_1 P$; i.e., for each $q \in Q$, there exists a $p_1 \in P$ such that $x + \varepsilon_1 q + \varepsilon_1 p_1 \in T_{\varepsilon_2}(M)$.

By taking into account the definition of the set $T_{\varepsilon_2}(M)$, we find a $p_2 \in P$ such that

$$z := (x + \varepsilon_1 q + \varepsilon_1 p_1) + \varepsilon_2 q + \varepsilon_2 p_2 \in M.$$

Set $p_* = (\varepsilon_1 p_1 + \varepsilon_2 p_2)/(\varepsilon_1 + \varepsilon_2)$. By virtue of the convexity of $P$, the inclusion $p_* \in P$ holds. We have

$$x + (\varepsilon_1+\varepsilon_2)(p_* + q) = z \in M.$$

Therefore, for each $q \in Q$, there exists a $p_* \in P$ such that $x+(\varepsilon_1+\varepsilon_2)(p_*+q) \in M$. Consequently, $x \in T_{\varepsilon_1+\varepsilon_2}(M)$. The proof of the lemma is complete.

**Lemma 2.** *Let $M$ be a convex set. Then for each $\varepsilon > 0$, the operator $T_\varepsilon$ has the semigroup property with respect to the set $M$; i.e., relation* (2) *holds for all $\varepsilon_1, \varepsilon_2 > 0$, $\varepsilon_1 + \varepsilon_2 \leq \varepsilon$.*

**Proof.** By Lemma 1, it remains to prove the inclusion

$$T_{\varepsilon_1+\varepsilon_2}(M) \subseteq T_{\varepsilon_1}(T_{\varepsilon_2}(M)).$$

Let $x \in T_{\varepsilon_1+\varepsilon_2}(M)$. Then $x + (\varepsilon_1+\varepsilon_2)Q \subseteq M - (\varepsilon_1+\varepsilon_2)P$; i.e., for each $q_1 \in Q$, there exists a $p_1 \in P$ such that $z_1 := x + (\varepsilon_1+\varepsilon_2)(p_1+q_1) \in M$.

Let us prove the inclusion

$$x + \varepsilon_1(p_1+q_1) \in T_{\varepsilon_2}(M). \tag{4}$$

Indeed, for each $q_2 \in Q$, there exists a $p_2 \in P$ such that

$$z_2 := x + (\varepsilon_1+\varepsilon_2)(p_2+q_2) \in M.$$

By taking into account the convexity of the set $M$, we have

$$x + \varepsilon_1(p_1+q_1) + \varepsilon_2(p_2+q_2) = (\varepsilon_1 z_1 + \varepsilon_2 z_2)/(\varepsilon_1+\varepsilon_2) \in M.$$

Therefore, the inclusion (4) holds. Consequently, $x \in T_{\varepsilon_1}(T_{\varepsilon_2}(M))$. The proof of the lemma is complete.

In addition, we need three lemmas stated and proved below. Lemma 3 justifies inequality (6), which should be necessarily true, in particular, if relation (2) holds. A similar inequality is used in Theorems 1 and 2 as one of the assumptions. In Lemma 4, the case of a convex set $M$ is considered; therefore (by Lemma 2), the above-mentioned property is necessarily satisfied. The proof is based on Lemma 3. Lemma 3 is also used in the proof of Lemma 5, which, in turn, is required for the proof of Theorem 3.

Let $\varrho(\cdot\,; A)$ be the support function of a compact set $A \subset \mathbb{R}^n$; i.e.,

$$\varrho(\eta; A) = \max\{\langle \eta, x\rangle :\ x \in A\}, \qquad \eta \in \mathbb{R}^n.$$

**Lemma 3.** *Let $\varepsilon_1, \varepsilon_2 > 0$, let $T_{\varepsilon_2}(M)$ and $T_{\varepsilon_1}(T_{\varepsilon_2}(M))$ be nonempty sets, let $\eta \in \mathbb{R}^n$, and let the following relation be satisfied*:

$$\varrho(\eta; T_{\varepsilon_1+\varepsilon_2}(M)) = \varrho(\eta; T_{\varepsilon_1}(T_{\varepsilon_2}(M))). \tag{5}$$

*Then*

$$\varrho(\eta; T_{\varepsilon_1+\varepsilon_2}(M)) + \varepsilon_1(\min_{p\in P}\langle p, \eta\rangle + \max_{q\in Q}\langle q, \eta\rangle) \le \varrho(\eta; T_{\varepsilon_2}(M)). \tag{6}$$

**Proof.** Since $T_{\varepsilon_1}(T_{\varepsilon_2}(M)) \subseteq T_{\varepsilon_1+\varepsilon_2}(M)$, it follows that $T_{\varepsilon_1+\varepsilon_2}(M) \ne \varnothing$. From the definition of the set $T_{\varepsilon_1}(T_{\varepsilon_2}(M))$, we have the inclusion

$$T_{\varepsilon_1}(T_{\varepsilon_2}(M)) + \varepsilon_1 Q \subseteq T_{\varepsilon_2}(M) - \varepsilon_1 P.$$

Consequently,

$$\varrho(\eta; T_{\varepsilon_1}(T_{\varepsilon_2}(M))) + \varepsilon_1 \max_{q\in Q}\langle q, \eta\rangle \le \varrho(\eta; T_{\varepsilon_2}(M)) + \varepsilon_1 \max_{p\in P}\langle -p, \eta\rangle.$$

Hence we obtain inequality (6) with regard of relation (5) and the relation $\max_{p\in P}\langle -p, \eta\rangle = -\min_{p\in P}\langle p, \eta\rangle$. The proof of the lemma is complete.

**Lemma 4.** *Let $M$ be a convex set, let $\varepsilon_1, \varepsilon_2 > 0$, and let $T_{\varepsilon_2}(M)$ and $T_{\varepsilon_1+\varepsilon_2}(M)$ be nonempty sets. Then inequality* (6) *is satisfied for each $\eta \in \mathbb{R}^n$.*

**Proof.** Since $M$ is a convex set, it follows from Lemma 2 that the sets $T_{\varepsilon_1+\varepsilon_2}(M)$ and $T_{\varepsilon_1}(T_{\varepsilon_2}(M))$ coincide. Consequently, relation (5) holds. By taking into account Lemma 3, we obtain inequality (6). The proof of the lemma is complete.

**Lemma 5.** *Suppose that $\eta \in \mathbb{R}^n$, $\eta \ne 0$, and there exists a $z_* \in M$ such that the intersection $M \cap \Pi_*$ of the set $M$ with the half-space $\Pi_* = \{x \in \mathbb{R}^n : \langle x - z_*, \eta\rangle \le 0\}$ is convex and has a nonempty interior.*

*Then there exists an $\varepsilon_* > 0$ such that the set $T_\varepsilon(M)$ is nonempty set for each $\varepsilon \in [0, \varepsilon_*]$ and the relation*

$$\varrho(-\eta; T_{\varepsilon_1+\varepsilon_2}(M)) + \varepsilon_1(\min_{p\in P}\langle p, -\eta\rangle + \max_{q\in Q}\langle q, -\eta\rangle) \le \varrho(-\eta; T_{\varepsilon_2}(M)) \tag{7}$$

*holds for arbitrary $\varepsilon_1, \varepsilon_2 > 0$, $\varepsilon_1 + \varepsilon_2 \le \varepsilon_*$.*

**Proof.** 1. Let $\mu_* := \langle z_*, -\eta\rangle$. We have $\mu_* < \varrho(-\eta, M)$. Take an arbitrary $\mu \in (\mu_*, \ \varrho(-\eta, M))$ and set $\Pi_\mu := \Pi_* - (\mu - \mu_*)\eta/\|\eta\|$.

Since the interior of the intersection $M \cap \Pi_\mu$ is nonempty (because $M \cap \Pi_*$ is a convex set with a nonempty interior), we have

$$T_\varepsilon(M) \cap \Pi_\mu \ne \varnothing \tag{8}$$

for sufficiently small $\varepsilon > 0$. Therefore, there exists an $\varepsilon_1^* > 0$ such that relation (8) holds for all $\varepsilon \in [0, \varepsilon_1^*]$.

Let $\alpha := \min_{p\in P} \min_{q\in Q}\langle p + q, -\eta\rangle$. Take a number $\varepsilon_2^* > 0$ such that $\varepsilon\alpha \ge \mu_* - \mu$, $\varepsilon \in (0, \varepsilon_2^*]$. Since $\mu_* - \mu < 0$, it follows that each $\varepsilon_2^* > 0$ can be chosen for $\alpha \ge 0$; otherwise, we choose a sufficiently small $\varepsilon_2^* > 0$.

Set $\varepsilon_* = \min\{\varepsilon_1^*, \varepsilon_2^*\}$. (Therefore, $\varepsilon_*$ depends on the choice of $\mu$.)

2. Let $\varepsilon \in [0, \varepsilon_*]$. Let us show that

$$T_\varepsilon(M) \cap \Pi_\mu \subseteq T_\varepsilon(M \cap \Pi_*). \tag{9}$$

Let $x \in T_\varepsilon(M) \cap \Pi_\mu$, $q \in Q$. We have $x + \varepsilon q \in M - \varepsilon P$ and $\langle x, -\eta\rangle \ge \mu$.

It follows from the left inclusion that there exists a $p_* \in P$ such that $x+\varepsilon q+\varepsilon p_* \in M$. By virtue of the right inequality, we have

$$-\langle x, -\eta\rangle + \mu_* \le -\mu + \mu_* \le \varepsilon\alpha \le \varepsilon\langle q + p_*, -\eta\rangle.$$

Hence it follows that $\langle x + \varepsilon q + \varepsilon p_*, -\eta\rangle \ge \mu_*$; i.e., $x + \varepsilon q + \varepsilon p_* \in \Pi_*$. Therefore,

$$x + \varepsilon q + \varepsilon p_* \in M \cap \Pi_*$$

and consequently, $x + \varepsilon q \in (M \cap \Pi_*) - \varepsilon P$. Since $q \in Q$ has been chosen arbitrarily, we have

$$x \in \bigcap_{q\in Q}((M \cap \Pi_*) - \varepsilon(P + q)) = T_\varepsilon(M \cap \Pi_*).$$

The proof of the inclusion (9) is complete.

3. Let $\varepsilon_1 + \varepsilon_2 \in [0, \varepsilon_*]$. By virtue of relation (8), $T_{\varepsilon_2}(M)$ and $T_{\varepsilon_1+\varepsilon_2}(M) \cap \Pi_\mu$ are nonempty sets. By taking into account the inclusion (9), the convexity of the set $M \cap \Pi_*$, Lemma 2, and the monotonicity of the operator $T_\varepsilon$, we obtain the relations

$$T_{\varepsilon_1+\varepsilon_2}(M) \cap \Pi_\mu \subseteq T_{\varepsilon_1+\varepsilon_2}(M \cap \Pi_*) = T_{\varepsilon_1}(T_{\varepsilon_2}(M \cap \Pi_*)) \subseteq T_{\varepsilon_1}(T_{\varepsilon_2}(M)).$$

Hence it follows that $T_{\varepsilon_1}(T_{\varepsilon_2}(M)) \ne \varnothing$ and

$$\varrho(-\eta; T_{\varepsilon_1+\varepsilon_2}(M) \cap \Pi_\mu) \le \varrho(-\eta; T_{\varepsilon_1}(T_{\varepsilon_2}(M))). \tag{10}$$

Since $\Pi_\mu$ is a half-space with the outward normal $\eta$ and the intersection $T_{\varepsilon_1+\varepsilon_2}(M) \cap \Pi_\mu$ is nonempty, we have $\varrho(-\eta; T_{\varepsilon_1+\varepsilon_2}(M)) = \varrho(-\eta; T_{\varepsilon_1+\varepsilon_2}(M) \cap \Pi_\mu)$. By taking into account inequality (10), we obtain

$$\varrho(-\eta; T_{\varepsilon_1+\varepsilon_2}(M)) \le \varrho(-\eta; T_{\varepsilon_1}(T_{\varepsilon_2}(M))).$$

On the other hand, by Lemma 1, we have the opposite inequality. Therefore,

$$\varrho(-\eta; T_{\varepsilon_1}(T_{\varepsilon_2}(M))) = \varrho(-\eta; T_{\varepsilon_1+\varepsilon_2}(M)).$$

By using Lemma 3, hence we obtain the desired inequality (7). The proof of the lemma is complete.

## 4. SEMIGROUP PROPERTY OF THE OPERATOR $T_\varepsilon$ FOR NONCONVEX SETS ON THE PLANE

We consider the case of the plane $\mathbb{R}^2$. A polygon is defined as a part of the plane bounded by a closed polyline without self-intersections. We assume that the polyline has finitely many links.

We assume that the set $P$ is either a nondegenerate segment or a convex polygon. The sets $M$ and $Q$ are compact sets on a plane; in addition, $Q$ is a convex set.

Let $\mathcal{V}$ be the set of unit inward normals to the sides of $P$. If $P$ is a segment, then the set $\mathcal{V}$ is formed by two oppositely directed vectors orthogonal to the segment $P$.

For a compact set $A \subset \mathbb{R}^2$, we define the intersection of half-planes

$$\widetilde{\Pi}(A) := \bigcap_{\nu\in\mathcal{V}}\{x \in \mathbb{R}^2 : \ \langle x, -\nu\rangle \le \varrho(-\nu; A)\}.$$

We have $A \subset \widetilde{\Pi}(A)$. If $P$ is a segment, then $\widetilde{\Pi}(A)$ is a closed strip; but if $P$ is a polygon, then $\widetilde{\Pi}(A)$ is a convex compact set.

Let $\mathcal{P}$ be the set of vertices of a segment or polygon $P$. For a vertex $p \in \mathcal{P}$, we define the pencil of unit vectors

$$\mathcal{N}(p) := \{(p - x)/\|p - x\| : \ x \in P\backslash\{p\}\}.$$

If $P$ is a segment, then it has two vertices, and the set $\mathcal{N}(p)$ is a singleton.

For the ray with origin at a point $a \in \mathbb{R}^2$ and direction along a vector $\eta \in \mathbb{R}^2$, we set $l(a, \eta) := \{a + \alpha\eta : \alpha \ge 0\}$.

**Lemma 6.** *Let $\varepsilon_1, \varepsilon_2 > 0$ be numbers such that the sets $T_{\varepsilon_2}(M)$ and $T_{\varepsilon_1+\varepsilon_2}(M)$ are nonempty and the following conditions are satisfied.*

$(A_1)$ *If $x \in \mathbb{R}^2$, $\varepsilon > 0$, $(x+\varepsilon P)\cap T_{\varepsilon_2}(M) = \varnothing$, and $(x+\varepsilon P)\cap \widetilde{\Pi}(T_{\varepsilon_2}(M)) \neq \varnothing$, then there exists a $p_* \in \mathcal{P}$ such that $l(x+\varepsilon p_*, \eta)\cap T_{\varepsilon_2}(M) \neq \varnothing$, $\eta \in \mathcal{N}(p_*)$.*

$(A_2)$ *If $x \in \mathbb{R}^2$, $\varepsilon > 0$, $p \in \mathcal{P}$, $(x+\varepsilon P)\cap M = \varnothing$, and for each $\eta \in \mathcal{N}(p)$, there exists an $\alpha_\eta > 0$ such that $(x+\varepsilon P + \alpha_\eta \eta)\cap M \neq \varnothing$, then $l(x+\varepsilon p, -\eta)\cap M = \varnothing$, $\eta \in \mathcal{N}(p)$.*

$(A_3)$ *The inequality*

$$\varrho(-\nu; T_{\varepsilon_1+\varepsilon_2}(M)) + \varepsilon_1\big(\min_{p\in P}\langle p, -\nu\rangle + \max_{q\in Q}\langle q, -\nu\rangle\big) \le \varrho(-\nu; T_{\varepsilon_2}(M)) \tag{11}$$

*holds for each $\nu \in \mathcal{V}$.*

*Then relation* (2) *is satisfied for the considered values $\varepsilon_1$ and $\varepsilon_2$.*

**Proof.** Suppose that relation (2) fails. Then, by taking into account Lemma 1, we have $Y := T_{\varepsilon_1+\varepsilon_2}(M)\backslash T_{\varepsilon_1}(T_{\varepsilon_2}(M)) \neq \varnothing$.

Take a $y \in Y$. By virtue of the property 3 of the operator $T_\varepsilon$, there exists a $q_1 \in Q$ such that

$$(y+\varepsilon_1(P+q_1))\cap T_{\varepsilon_2}(M) = \varnothing. \tag{12}$$

The set $T_{\varepsilon_2}(M)$ is compact. Let $\widetilde{\Pi} := \widetilde{\Pi}(T_{\varepsilon_2}(M))$ and $G_1 := y+\varepsilon_1(P+q_1)$.

I. Suppose that

$$G_1 \cap \widetilde{\Pi} \neq \varnothing. \tag{13}$$

(a) We prove the existence of a $q_2 \in Q$ such that

$$(G_1+\varepsilon_2(P+q_2))\cap M = \varnothing. \tag{14}$$

By relations (12) and (13) and condition $(A_1)$ (for $x = y+\varepsilon_1 q_1$ and $\varepsilon = \varepsilon_1$), there exists a $p_* \in \mathcal{P}$ ensuring the validity of the relation

$$l(y+\varepsilon_1(p_*+q_1), \eta)\cap T_{\varepsilon_2}(M) \neq \varnothing, \qquad \eta \in \mathcal{N}(p_*). \tag{15}$$

For brevity, set $a_* = y+\varepsilon_1(p_*+q_1)$. Since $a_* \in G_1$ and relation (12) is valid, we have $a_* \notin T_{\varepsilon_2}(M)$, and there exists a control $q_2 \in Q$ such that

$$(a_*+\varepsilon_2(P+q_2))\cap M = \varnothing. \tag{16}$$

By virtue of relation (15), for each $\eta \in \mathcal{N}(p_*)$, there exists an $\alpha_\eta > 0$ such that $a_*+\alpha_\eta\eta \in T_{\varepsilon_2}(M)$. This, together with property 2 of the operator $T_\varepsilon$, implies that, for each $\eta \in \mathcal{N}(p_*)$, there exists an $\alpha_\eta > 0$ such that

$$(a_*+\alpha_\eta\eta+\varepsilon_2(P+q_2))\cap M \neq \varnothing. \tag{17}$$

By virtue of relations (16) and (17) and condition $(A_2)$ (for $x = a_*+\varepsilon_2 q_2$, $\varepsilon = \varepsilon_2$, and $p = p_*$), we have

$$l(a_*+\varepsilon_2 q_2+\varepsilon_2 p_*, -\eta)\cap M = \varnothing, \qquad \eta \in \mathcal{N}(p_*). \tag{18}$$

Since the representation

$$z+\varepsilon_2(p+q_2) = a_*+\varepsilon_2(p_*+q_2)-\eta_*, \quad \eta_* := (a_*-z)+\varepsilon_2(p_*-p), \qquad \eta_*/\|\eta_*\| \in \mathcal{N}(p_*),$$

holds for arbitrary $z \in G_1$ and $p \in P$, it follows from relation (18) that $(z+\varepsilon_2(P+q_2))\cap M = \varnothing$ for each $z \in G_1$. Consequently, condition (14) is satisfied.

(b) Let $\widetilde{q} = (\varepsilon_1 q_1+\varepsilon_2 q_2)/(\varepsilon_1+\varepsilon_2)$. Then

$$y+(\varepsilon_1+\varepsilon_2)(P+\widetilde{q}) = y+\varepsilon_1(P+q_1)+\varepsilon_2(P+q_2) = G_1+\varepsilon_2(P+q_2).$$

By virtue of relation (14), we have $(y + (\varepsilon_1 + \varepsilon_2)(P + \widetilde{q})) \cap M = \varnothing$. By using property 3 of the operator $T_\varepsilon$, we obtain $y \notin T_{\varepsilon_1+\varepsilon_2}(M)$, and this contradicts the choice of $y$ in the set $Y \subseteq T_{\varepsilon_1+\varepsilon_2}(M)$.

Therefore, assumption (13) fails.

II. Now suppose that $G_1 \cap \widetilde{\Pi} = \varnothing$. We have

$$G_1 \subseteq \mathbb{R}^2 \backslash \widetilde{\Pi} = \bigcup_{\nu \in \mathcal{V}} \{x \in \mathbb{R}^2 : \ \langle x, -\nu \rangle > \varrho(-\nu; T_{\varepsilon_2}(M))\}.$$

Since $G_1$ is a polygon such that $\mathcal{V}$ is the set of outward normals to its sides, it follows that there exists a $\nu_0 \in \mathcal{V}$ such that

$$\langle z, -\nu_0 \rangle > \varrho(-\nu_0; T_{\varepsilon_2}(M)), \qquad z \in G_1. \tag{19}$$

In addition, by virtue of the inclusion $y \in T_{\varepsilon_1+\varepsilon_2}(M)$, we have the inequality

$$\langle y, -\nu_0 \rangle \le \varrho(-\nu_0; T_{\varepsilon_1+\varepsilon_2}(M)). \tag{20}$$

Let

$$p_0 \in \operatorname{Arg}\max_{p \in P} \langle p, \nu_0 \rangle, \qquad z_0 = y + \varepsilon_1(p_0 + q_1).$$

Since $z_0 \in G_1$, it follows from inequalities (19) and (20) and the relations

$$\langle p_0, -\nu_0 \rangle = \min_{p \in P} \langle p, -\nu_0 \rangle, \qquad \langle q_1, -\nu_0 \rangle \le \max_{q \in Q} \langle q, -\nu_0 \rangle$$

that

$$\begin{aligned} \varrho(-\nu_0; T_{\varepsilon_2}(M)) &< \langle z_0, -\nu_0 \rangle = \langle y, -\nu_0 \rangle + \varepsilon_1 \langle p_0 + q_1, -\nu_0 \rangle \\ &\le \varrho(-\nu_0; T_{\varepsilon_1+\varepsilon_2}(M)) + \varepsilon_1 (\min_{p \in P} \langle p, -\nu_0 \rangle + \max_{q \in Q} \langle q, -\nu_0 \rangle), \end{aligned}$$

which contradicts inequality (11).

Therefore, the assumption on the failure of relation (2) leads to a contradiction. The proof of the lemma is complete.

A set $A$ is said to be *arcwise connected* [10, vol. 1, p. 92] (or, briefly, *connected*) if two arbitrary point of it can be joined by a continuous curve lying in the set $A$.

In the case in which $P$ is a nondegenerate segment, by $l_P(x)$ we denote the line passing through the point $x \in \mathbb{R}^2$ and parallel to the segment $P$.

**Theorem 1.** *Let $P$ be a nondegenerate segment, and let the following conditions be satisfied.*

$(T_{1.1})$ *The intersection $l_P(x) \cap M$ (if it is nonempty) is a segment for any $x \in \mathbb{R}^2$.*

$(T_{1.2})$ *$\varepsilon_1, \varepsilon_2 > 0$ are numbers such that the sets $T_{\varepsilon_2}(M)$ and $T_{\varepsilon_1+\varepsilon_2}(M)$ are nonempty, the set $T_{\varepsilon_2}(M)$ is connected, and*

$$\varrho(\pm\nu; T_{\varepsilon_1+\varepsilon_2}(M)) + \varepsilon_1(\langle p_0, \pm\nu \rangle + \max_{q \in Q} \langle q, \pm\nu \rangle) \le \varrho(\pm\nu; T_{\varepsilon_2}(M)), \tag{21}$$

*where $p_0 \in P$ is an arbitrarily chosen point and $\nu$ is a nonzero vector orthogonal to the segment $P$.*

*Then relation (2) holds for the considered numbers $\varepsilon_1$ and $\varepsilon_2$.*

**Proof.** Let us verify assumptions $(A_1)$–$(A_3)$ of Lemma 6.

Note that the set $\mathcal{P}$ consists of two elements (the ends of the segment $P$), and the set $\mathcal{N}(p)$ is a singleton for any $p \in \mathcal{P}$.

$(A_1)$ Let $x \in \mathbb{R}^2$, $\varepsilon > 0$, $P_1 = x + \varepsilon P$, $P_1 \cap T_{\varepsilon_2}(M) = \varnothing$, and $P_1 \cap \widetilde{\Pi}(T_{\varepsilon_2}(M)) \neq \varnothing$. Since the set $P_1$ is a segment parallel to $P$ and the set $\widetilde{\Pi}(T_{\varepsilon_2}(M))$ is a strip parallel to $P$, we have the inclusion $P_1 \subset \widetilde{\Pi}(T_{\varepsilon_2}(M))$. Since $T_{\varepsilon_2}(M)$ is a connected set, it follows that there exists a $p_* \in \mathcal{P}$ such that $l(x + \varepsilon p_*, \eta_*) \cap T_{\varepsilon_2}(M) \neq \varnothing$ and $\mathcal{N}(p_*) = \{\eta_*\}$. Consequently, condition $(A_1)$ is satisfied.

$(A_2)$ Suppose that $x \in \mathbb{R}^2$, $\varepsilon > 0$, $P_1 = x + \varepsilon P$, $p \in \mathcal{P}$, $\mathcal{N}(p) = \{\eta\}$, $P_1 \cap M = \varnothing$, and there exists an $\alpha_\eta > 0$ such that $(P_1 + \alpha_\eta \eta) \cap M \neq \varnothing$. Then $l(x + \varepsilon p, \eta) \cap M \neq \varnothing$. We have

$$l_P(x + \varepsilon p) = l(x + \varepsilon p, \eta) \cup l(x + \varepsilon p, -\eta), \qquad P_1 \subset l(x + \varepsilon p, -\eta).$$

Since the intersection $l_P(x+\varepsilon p)\cap M$ is connected, we have $l(x+\varepsilon p, -\eta)\cap M = \varnothing$; i.e., condition $(A_2)$ is satisfied as well.

Condition $(A_3)$ in Lemma 6 is satisfied as well by virtue of inequality (21). The proof of the theorem is complete.

A set $A \subset \mathbb{R}^2$ is said to be *simply connected* [10, vol. 2, p. 281] if any continuous mapping of a circle into $A$ is homotopic to some point in $A$. From the geometric viewpoint, this implies the absence of "holes" in the set $A$.

Set $\Pi_M(x, \nu) := \{z \in M : \langle z - x, \nu\rangle \leq 0\}$, $x \in \mathbb{R}^2$, $\nu \in \mathcal{V}$.

**Theorem 2.** *Let $P$ be a convex polygon, and let the following conditions be satisfied.*

$(T_{2.1})$ *$M$ is a simply connected set, and $\Pi_M(x, \nu)$ is a connected set for arbitrary $x \in \mathbb{R}^2$ and $\nu \in \mathcal{V}$.*

$(T_{2.2})$ *$\varepsilon_1, \varepsilon_2 > 0$ are numbers such that $T_{\varepsilon_2}(M)$ and $T_{\varepsilon_1+\varepsilon_2}(M)$ are nonempty sets, $T_{\varepsilon_2}(M)$ is a connected set, and inequality* (11) *is satisfied for each $\nu \in \mathcal{V}$.*

*Then relation* (2) *holds for the considered numbers $\varepsilon_1$ and $\varepsilon_2$.*

**Proof.** Let us verify assumptions $(A_1)$–$(A_3)$ of Lemma 6.

$(A_1)$ Let $x \in \mathbb{R}^2$, $\varepsilon > 0$, $P_1 = x + \varepsilon P$, $P_1 \cap T_{\varepsilon_2}(M) = \varnothing$, and

$$P_1 \cap \widetilde{\Pi}(T_{\varepsilon_2}(M)) \neq \varnothing. \tag{22}$$

Let us prove the existence of a $p_* \in \mathcal{P}$ such that

$$l(x + \varepsilon p_*, \eta) \cap T_{\varepsilon_2}(M) \neq \varnothing, \qquad \eta \in \mathcal{N}(p_*). \tag{23}$$

Suppose the contrary: for each $p \in \mathcal{P}$, there exists an $\eta \in \mathcal{N}(p)$ such that

$$l(x + \varepsilon p, \eta) \cap T_{\varepsilon_2}(M) = \varnothing. \tag{24}$$

The set $P_1$ is a convex polygon. Therefore,

$$P_1 = \bigcap_{\nu\in\mathcal{V}} \Pi_1(\nu), \qquad \Pi_1(\nu) := \{z \in \mathbb{R}^2 : \langle z, \nu\rangle \leq \varrho(\nu, P_1)\}.$$

By virtue of relation (24) and the connectedness of the set $T_{\varepsilon_2}(M)$, there exists a $\nu_* \in \mathcal{V}$ such that

$$\Pi_1(\nu_*) \cap T_{\varepsilon_2}(M) = \varnothing \tag{25}$$

(see Fig. 1). Let $\Pi_2 := \{z \in \mathbb{R}^2 : \langle z, -\nu_*\rangle \leq \varrho(-\nu_*; T_{\varepsilon_2}(M))\}$. By taking into account relation (25), we obtain $\Pi_1(\nu_*) \cap \Pi_2 = \varnothing$. Since $P_1 \subset \Pi_1(\nu_*)$ and $\widetilde{\Pi}(T_{\varepsilon_2}(M)) \subset \Pi_2$, it follows that $P_1 \cap \widetilde{\Pi}(T_{\varepsilon_2}(M)) = \varnothing$, which contradicts assumption (22). Therefore, the proof of relation (23) is complete.

$(A_2)$ Suppose that $x \in \mathbb{R}^2$, $\varepsilon > 0$, $P_1 = x + \varepsilon P$, $p \in \mathcal{P}$,

$$P_1 \cap M = \varnothing, \tag{26}$$

and for each $\eta \in \mathcal{N}(p)$, there exists an $\alpha_\eta > 0$ such that

$$(P_1 + \alpha_\eta \eta) \cap M \neq \varnothing. \tag{27}$$

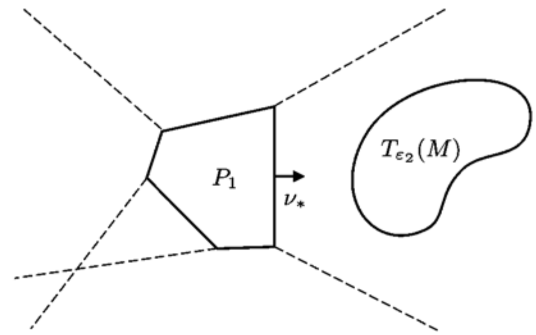


**Fig. 1.**

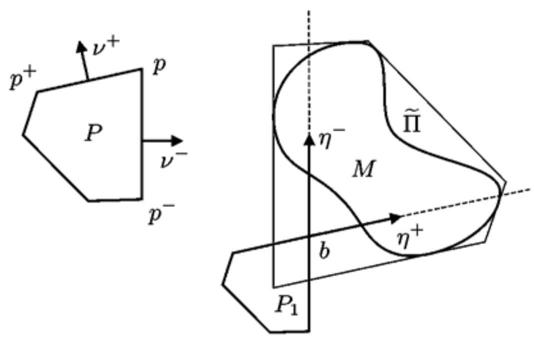


**Fig. 2.**

Let us show that

$$l(x+\varepsilon p, -\eta) \cap M = \varnothing, \qquad \eta \in \mathcal{N}(p). \tag{28}$$

Let $p^{\pm}$ be the vertices of the polygon $P$ adjacent to the vertex $p$ (see Fig. 2). We have $p^+ \neq p^-$. To be definite, assume that the vertices $p^-$, $p$, and $p^+$ are passed in counterclockwise direction. Set $b = x + \varepsilon p$ and $\eta^{\pm} = p - p^{\pm}$, and let $\nu^{\pm}$ be the outward normals to the sides of $P$ with endpoints $p$ and $p^{\pm}$.

We organize the proof of relation (28) into two stages. At the first stage, we prove an auxiliary assertion. At the second stage, we assume that relation (28) fails and obtain a contradiction with the simple connectedness of the set $M$.

**Stage I.** Let us show that there exists a continuous curve $\gamma$ (Fig. 3) joining some points $e^+ \in l(b, \eta^+)$ and $e^- \in l(b, \eta^-)$ and such that

$$\gamma \subset (b + K) \cap M, \tag{29}$$

where $K := \{\alpha\eta : \ \eta \in \mathcal{N}(p), \ \alpha \geq 0\}$.

Let $B^{\pm} = \{z + \alpha\eta^{\pm} : \ \alpha > 0, \ z \in P_1\} \backslash P_1$. By using relation (27) for $\eta = \eta^{\pm}/\|\eta^{\pm}\|$, we obtain $B^{\pm} \cap M \neq \varnothing$. Take arbitrary points $b^{\pm} \in B^{\pm} \cap M$. Since $B^+ \cap B^- = \varnothing$, we have $b^+ \neq b^-$.

(a) Suppose that

$$\operatorname{Arg}\min_{z \in P_1} \langle z, \nu^+ \rangle = \operatorname{Arg}\min_{z \in P_1} \langle z, \nu^- \rangle =: E.$$

In this case, since $P_1$ is a convex set, it follows that $E$ is a singleton; i.e., $E = \{\widetilde{e}\,\}$, and $\widetilde{e}$ is a vertex of the polygon $P_1$ (see Fig. 3 a).

Let $\widetilde{\nu}^{\pm}$ be the outward normals of the sides of the polygon $P_1$ adjacent to the vertex $\widetilde{e}$; moreover, the rotation from $\widetilde{\nu}^-$ to $\widetilde{\nu}^+$ by an angle less than $\pi$ is counterclockwise. Set

$$\widetilde{K} := \{\alpha(z - \widetilde{e}\,) : \ \alpha \geq 0, \ z \in P_1\}$$

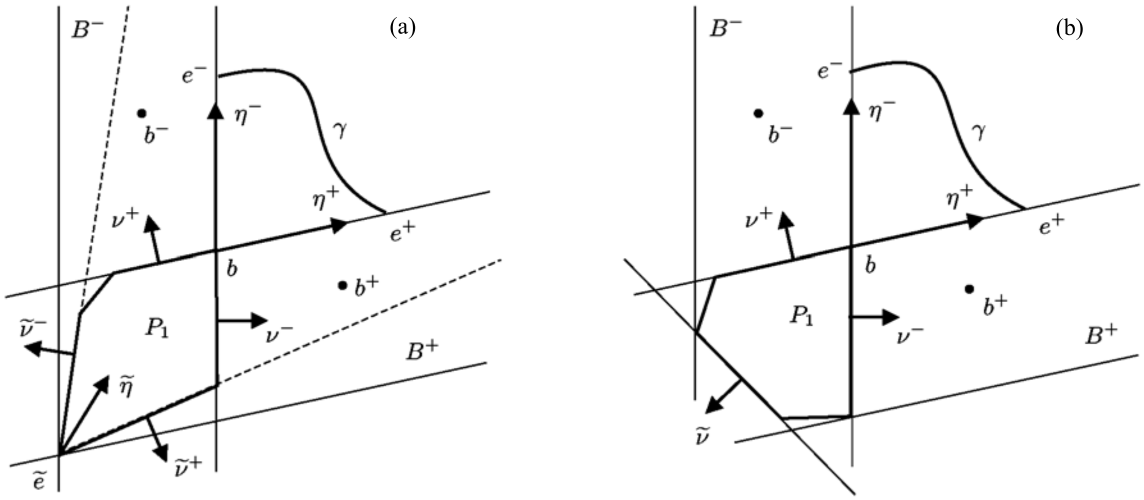


**Fig. 3.**

and note that $\widetilde{K} \subset K$, $\partial\widetilde{K} \cap \partial K = \{0\}$, and

$$b^{\pm} \in \Pi_M(\widetilde{e}, \widetilde{\nu}^{\mp}). \tag{30}$$

In addition, by using condition $(T_{2.1})$, we find that the set $\Pi_M(\widetilde{e}, \widetilde{\nu}^{\pm})$ is connected.

Let $\widetilde{\eta} \in \widetilde{K}$. By using relation (27) for $\eta = \widetilde{\eta}/\|\widetilde{\eta}\|$, we find that there exists an $\widetilde{\alpha} > 0$ such that $(P_1 + \widetilde{\alpha}\widetilde{\eta}) \cap M \neq \varnothing$. Take a $c \in (P_1 + \widetilde{\alpha}\widetilde{\eta}) \cap M$. Note that

$$c \in \Pi_M(\widetilde{e}, \widetilde{\nu}^{\pm}). \tag{31}$$

The following cases are possible: (i) $c \in (B^+ \cup B^-)$; (ii) $c \notin (B^+ \cup B^-)$.

(i) Let $c \in B^{\pm}$. By taking into account the inclusions (30) and (31) and the connectedness of the set $\Pi_M(\widetilde{e}, \widetilde{\nu}^{\pm})$, we find that there exists a continuous curve $\gamma_1 \subset \Pi_M(\widetilde{e}, \widetilde{\nu}^{\pm})$ joining the points $c$ and $b^{\mp}$. The points $c$ and $b^{\mp}$ in the set $\Pi_M(\widetilde{e}, \widetilde{\nu}^{\pm})$ are separated by the set $P_1 \cup (b + K)$. By virtue of relation (26), from the curve $\gamma_1$, one can extract the desired continuous curve $\gamma$ without self-intersections, which lies in the set $b + K$ and joins some points $e^+ \in l(b, \eta^+)$ and $e^- \in l(b, \eta^-)$.

(ii) Let $c \notin (B^+ \cup B^-)$. In this case, the point $c$ belongs to the interior of the set $b + K$. We have $c, b^+ \in \Pi_M(\widetilde{e}, \widetilde{\nu}^-)$ and $c, b^- \in \Pi_M(\widetilde{e}, \widetilde{\nu}^+)$. By taking into account the connectedness of the set $\Pi_M(\widetilde{e}, \widetilde{\nu}^{\pm})$, we find that there exists a continuous curve $\gamma_1 \subset \Pi_M(\widetilde{e}, \widetilde{\nu}^+)$ joining the points $c$ and $b^+$ and a continuous curve $\gamma_2 \subset \Pi_M(\widetilde{e}, \widetilde{\nu}^-)$ joining the points $b^-$ and $c$. By virtue of relation (26), from the composite curve $\gamma_1\gamma_2$, one can extract the desired continuous curve $\gamma$ without self-intersections that lies in the set $b + K$ and joins some points $e^+ \in l(b, \eta^+)$ and $e^- \in l(b, \eta^-)$.

(b) It remains to consider the case in which

$$\operatorname{Arg}\min_{z \in P_1} \langle z, \nu^+ \rangle \neq \operatorname{Arg}\min_{z \in P_1} \langle z, \nu^- \rangle.$$

In this case (see Fig. 3 b), there exists a $\widetilde{\nu} \in \mathcal{V}$ such that $B^{\pm} \subset \Pi_3 := \{z \in \mathbb{R}^2 : \langle z, \widetilde{\nu} \rangle \leq \varrho(\widetilde{\nu}; P_1)\}$. Since $b^+, b^- \in \Pi_3$, it follows from condition $(T_{2.1})$ that there exists a continuous curve $\gamma_1 \subset \Pi_3 \cap M$ joining the points $b^+$ and $b^-$. The points $b^+$ and $b^-$ in the half-plane $\Pi_3$ are separated by the set $P_1 \cup (b + K)$. By virtue of relation (26), from the curve $\gamma_1$ one can single out the desired continuous curve $\gamma$ without self-intersections lying in the set $b + K$ and joining some points $e^+ \in l(b, \eta^+)$ and $e^- \in l(b, \eta^-)$.

We have thereby proved the existence of a curve $\gamma$ with the desired properties.

**Stage II.** Suppose that relation (28) fails; i.e., there exists an $\eta_0 \in \mathcal{N}(p)$ such that $l(b, -\eta_0) \cap M \neq \varnothing$. Let $b_0 \in l(b, -\eta_0) \cap M$ (see Fig. 4).

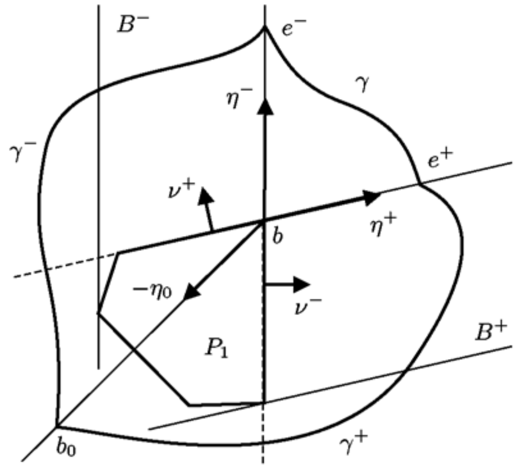


**Fig. 4.**

Let us construct a continuous closed curve without self-intersections that lies in the set $M$ and surrounds the set $P_1$. We have $e^+, b_0 \in \Pi_M(b,\nu^+)$ and $e^-, b_0 \in \Pi_M(b,\nu^-)$. By assumption $(T_{2.1})$, there exists a continuous curve $\gamma^+ \subset \Pi_M(b,\nu^+)$ joining the points $b_0$ and $e^+$ and a continuous curve $\gamma^- \subset \Pi_M(b,\nu^-)$ joining the points $e^-$ and $b_0$. Without loss of generality, one can assume that the $\gamma^\pm$ are curves without self-intersections.

The composite curve $\gamma^+\gamma\gamma^- \subset M$ is continuous and closed, has no self-intersections, and surrounds the set $P_1$. By taking into account relation (26), we obtain a contradiction with the simple connectedness of the set $M$.

Assumption $(A_3)$ of Lemma 6 holds by virtue of inequality (11). The proof of the theorem is complete.

**Remark 1.** Theorem 1 is stated for the case in which the set $P$ is a nondegenerate segment, and Theorem 2 is formulated for the case in which the set $P$ is a convex polygon. Conditions $(T_{1.1})$ and $(T_{2.1})$ in those theorems are imposed only on the sets $M$ and $P$, have a geometric nature, and can readily be verified. Note also that condition $(T_{2.1})$ is also well posed in the case of a segment $P$. One can readily see that if $P$ is a segment and $M$ is connected, then assumption $(T_{1.1})$ is equivalent to condition $(T_{2.1})$.

**Remark 2.** In the assumptions of Theorems 1 and 2, $\varepsilon_1$, $\varepsilon_2$ are fixed numbers. To say that the operators $T_\varepsilon$ and $\widetilde{T}_\varepsilon$ are equal on some interval of length $\varepsilon$ for a given set $M$ (with specific geometric properties with respect to the set $P$), one should require the validity of condition $(T_{1.2})$ in Theorem 1 [respectively, condition $(T_{2.2})$ of Theorem 2] for arbitrary $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon_1 + \varepsilon_2 \le \varepsilon$.

**Theorem 3.** *Let $M$ be a polygon, let $P$ be a nondegenerate segment (respectively, a convex polygon), and let condition $(T_{1.1})$ [respectively, condition $(T_{2.1})$] be satisfied. Then there exists an $\bar\varepsilon > 0$ such that the operator $T_{\bar\varepsilon}$ has the semigroup property; i.e., relation (2) holds for arbitrary $\varepsilon_1, \varepsilon_2 > 0$, $\varepsilon_1 + \varepsilon_2 \le \bar\varepsilon$.*

**Proof.** Note that $T_\varepsilon(M)$ is a connected set for small values of $\varepsilon > 0$.

Let us separately consider the case of a segment $P$ and the case of a convex polygon $P$.

1. Let $P$ be a nondegenerate segment, and let $\nu$ be a nonzero vector orthogonal to the segment $P$. By virtue of condition $(T_{1.1})$, $\Pi_M(x, \pm\nu)$ is a connected set for arbitrary $x \in \mathbb{R}^2$. Since $M$ is a polygon, it follows that, for $\eta = \nu$ and $\eta = -\nu$, one can choose a $z_* \in M$ such that the set $\Pi_M(z_*, \eta)$ is either a triangle or a trapezoid. Then the assumptions of Lemma 5 for $\eta = \pm\nu$ hold. Consequently, there exists an $\bar\varepsilon > 0$ such that the assumptions of Theorem 1 providing the desired semigroup property hold for $\varepsilon_1$ and $\varepsilon_2$ satisfying the relation $\varepsilon_1 + \varepsilon_2 \in (0, \bar\varepsilon]$.

2. Let $P$ be a convex polygon, and let $\mathcal{V}$ be the set of unit outward normals to the sides of $P$. By virtue of condition $(T_{2.1})$, the set $\Pi_M(x,\nu)$ is connected for arbitrary $x \in \mathbb{R}^2$ and $\nu \in \mathcal{V}$. Since $M$ is a polygon, it follows that there exists a $z_* \in M$ such that the set $\Pi_M(z_*,\nu)$ is either a triangle or a trapezoid. Then the assumptions of Lemma 5 for $\eta = \nu$ are satisfied. Consequently, there exists an $\bar{\varepsilon} > 0$ such that the assumptions of Theorem 2 ensuring the desired semigroup property hold for $\varepsilon_1$ and $\varepsilon_2$ satisfying the relation $\varepsilon_1 + \varepsilon_2 \in (0, \bar{\varepsilon}\,]$. The proof of the theorem is complete.

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