Reachable set for Dubins car and its application to observation problem with incomplete information

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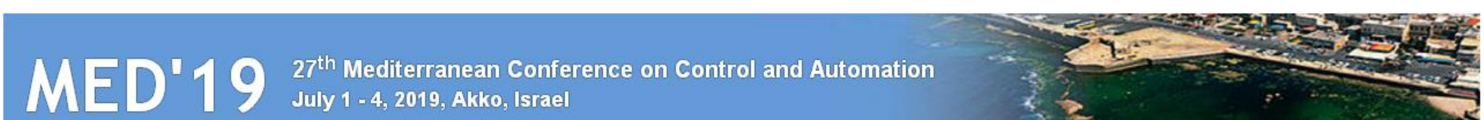


Slide 1

The Dubins car is a very popular model in the nowadays control theory. For example, in the Technion student laboratory (founded by T. Shima) about 60% of scientific topics on the posters are concerned with this model.

Our aim is to investigate three-dimensional reachable sets for Dubins car. We will also discuss very shortly the application of the reachable sets to observation problem with incomplete information.

The terms “reachable set” and “attainability set” are the same.

Nonlinear control system of car motion (Dubins Car)

Markov, A. A. (1889). Some examples of the solution of a special kind of problem on greatest and least quantities, Soobscenija Charkovskogo matematiceskogo obscestva, Vol. 2-1 (No. 5,6), 250–276 (in Russian).

Нѣсколько примѣровъ рѣшенія особаго рода задачъ о наибольшихъ и наименьшихъ величинахъ.

А. А. Маркова.

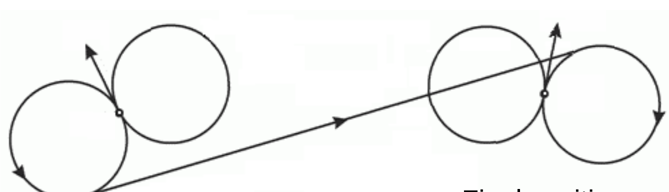
Isaacs, R. (1951). Games of pursuit, Scientific report of the RAND Corporation, Santa Monica.

AMERICAN
JOURNAL OF MATHEMATICS
Volume LXXIX, Number 3 JULY, 1957, pp 497 – 515

ON CURVES OF MINIMAL LENGTH WITH A CONSTRAINT ON AVERAGE CURVATURE, AND WITH PRESCRIBED INITIAL AND TERMINAL POSITIONS AND TANGENTS.*

By L. E. Dubins.

* Received April 28, 1956; revised January 3, 1957.



Final position

Initial position

Slide 2

Several history hints. In 1889, A.A. Markov published a paper with formulations and solutions of four model problems on optimal curves with bounded radius of curvature. These problems were connected with projecting the railroads.

In 1951, R. Isaacs was the first one who called the "car" a mathematical object that moves with a constraint on the radius of turn.

In 1957, a paper by L. Dubins was published in a mathematical journal. He proved that, in the bunch of curves with the constrained radius of turn for given initial and terminal geometric states and also with given directions of output and enter, a curve of the minimal length consists of not more than three standard parts. These are: the turn with the minimal radius to one side, the turn with the minimal radius to the other side, and the linear-direction part. Dubins had pointed out all six variants, which are sufficient to take into account in investigations of the shortest curves.

Dubins Car: 3D-Reachable set at instant t_f

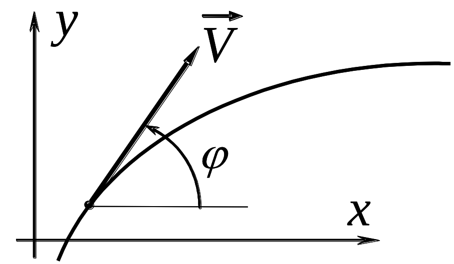
$$\dot{x} = \cos\varphi, \qquad u_1 = -1 \quad \text{(symmetric case)}$$

$$\dot{y} = \sin\varphi, \qquad u_1 \in (-1, 0) \quad \text{(asymmetric case)}$$

$$\dot{\varphi} = u; \qquad u_1 = 0 \quad \text{(one-sided case)}$$

$$u \in [u_1, u_2]. \qquad u_1 \in (0, 1) \quad \text{(strictly one-sided case)}$$

$u_2 = 1,\quad u_1$ is a parameter of problem



We consider $\varphi \in (-\infty, +\infty)$

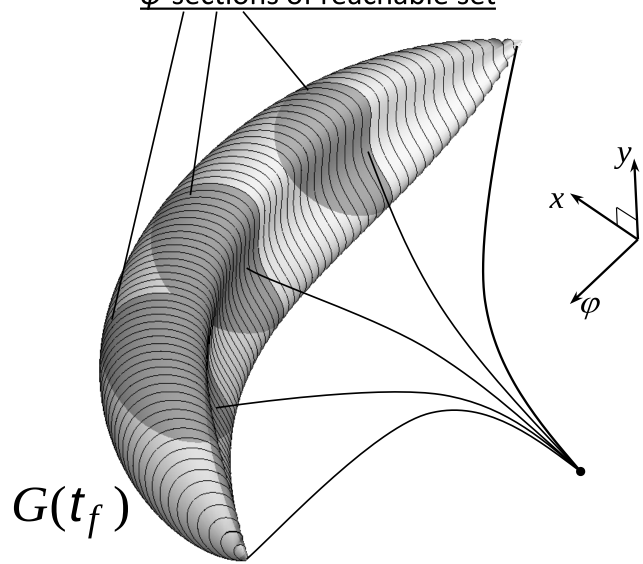
Reachable set at instant t_f:

$$G(t_f) = \bigcup_{u(\cdot)} \begin{pmatrix} x(t_f) \\ y(t_f) \\ \varphi(t_f) \end{pmatrix}$$

~~Reachable set up to instant t_f:~~

$$\cancel{G^*(t_f) = \bigcup_{0 \le t \le t_f} G(t)}$$

φ-sections of reachable set



$t_0 = 0, \quad x(t_0) = y(t_0) = \varphi(t_0) = 0$

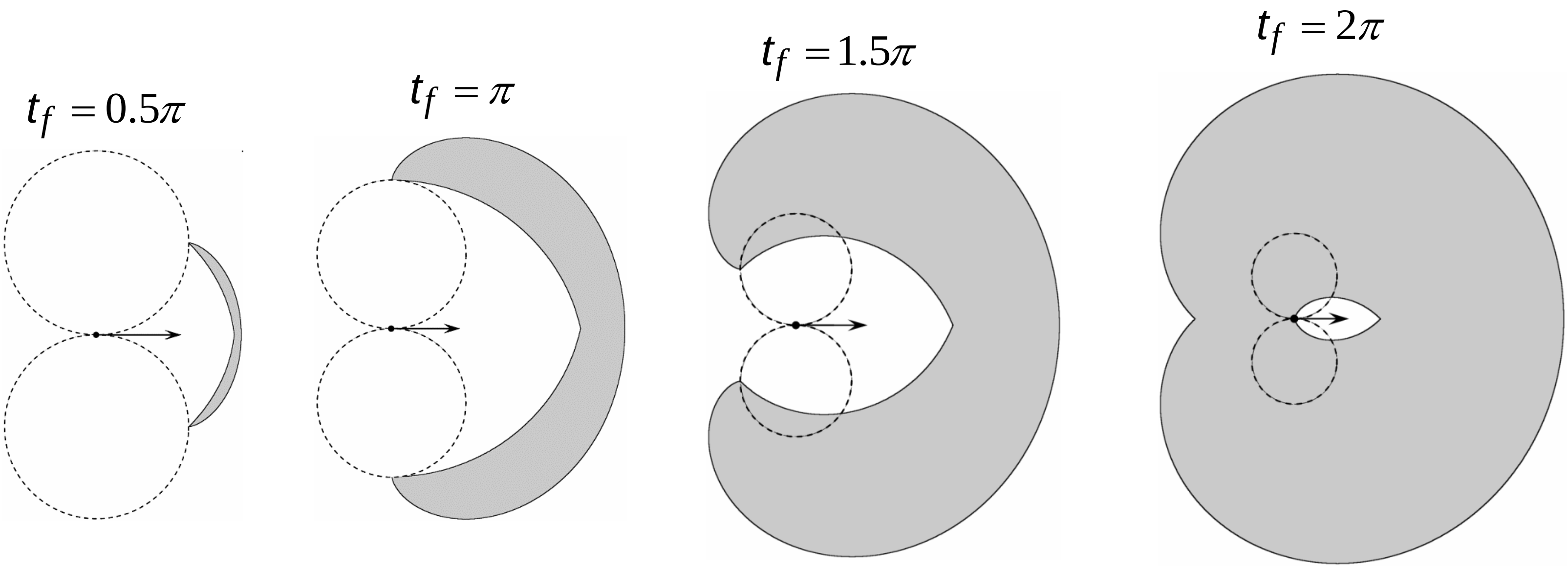
Slide 3

Dynamics of Dubins car comprises geometric coordinates x, y, and the angle φ of the linear velocity vector. The value u_1 in the description of the constraint on the control u is the problem parameter. We distinguish four cases. If $u_1 = -1$, then it is a symmetric case. If $u_1 \in (-1, 0)$, then it is an asymmetric case. In the third one, we have $u_1 = 0$. Here, the turn is permitted only to one side. The fourth case $u_1 \in (0, 1)$ has strictly one-sided turn, when a motion along a straight line is prohibited.

We are interested in the three-dimensional reachable set $G(t_f)$ "at instant" t_f. To avoid misunderstanding, we underline the distinctions between the reachable sets "at instant" and "up to instant". In our work, we speak only about the reachable sets "at instant".

Note also that we consider the angle φ in the interval $(-\infty, +\infty)$, *i.e.*, we do not identify the angles by "modulo" 2π.

Reachable sets in projection onto a geometric plane, symmetric case



SIAM J. CONTROL
Vol. 13, No. 1, January 1975

PLANE MOTION OF A PARTICLE SUBJECT TO CURVATURE CONSTRAINTS*

E. J. COCKAYNE AND G. W. C. HALL†

Abstract. A particle P moves in the plane with constant speed and subject to an upper bound on the curvature of its path. This paper studies the classes of trajectories by which P can reach a given point in a given direction and obtains, for all t, the set $R(t)$ of all possible positions for P at time t, thus extending the results of several recent authors.

Yu.I.Berdyshev

Nelineinye zadachi posledovatel'nogo upravleniya i ikh prilozhenie. [Nonlinear Problems in Sequential Control and Their Application]. Ekaterinburg: IMM UB RAS, 2015, 193 p.

Slide 4

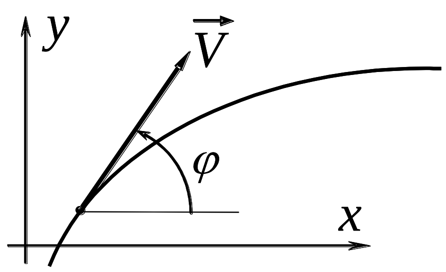
What is known about reachable sets "at instant" from works of other authors? It is only a description of two-dimensional reachable sets in the plane of the geometric coordinates for the symmetric case that appeared in the work by E.J. Cockayne and G.W.C. Hall in 1975. At our Institute, such a description was applied to various problems by Yu.I. Berdyshev.

Pontryagin Maximum Principle

It is known [Lee, E.B., Markus, L.] that controls that carry a system onto the reachable set boundary satisfy the Pontryagin Maximum Principle (PMP).

$$\begin{cases} \dot{x} = \cos\varphi, \\ \dot{y} = \sin\varphi, \\ \dot{\varphi} = u; \end{cases}$$

Dynamic description
of Dubins car
in normalized coordinates



$$u \in [u_1, 1].$$

Let $u^*(\cdot)$ be some admissible control and $(x^*(\cdot), y^*(\cdot), \varphi^*(\cdot))^{\mathrm{T}}$ be the corresponding motion of Dubins car on the interval $[t_0, t_f]$.

Differential equations for the adjoint system:

$$\begin{cases} \dot{\psi}_1 = 0, \\ \dot{\psi}_2 = 0, \\ \dot{\psi}_3 = \psi_1 \sin\varphi^* - \psi_2 \cos\varphi^*. \end{cases}$$

Lee, E.B., Markus, L. *Foundations of Optimal Control Theory*, Wiley and Sons, New York, 1967

Slide 5

It is natural that in our investigation of reachable sets, we are based on the Pontryagin Maximum Principle (PMP). It could be said that the PMP is the property that a controlled motion possesses if it leads onto the boundary of the reachable set. The corresponding statement is in the book by E.B Lee and L. Markus.

In the problem under investigation, the conjugated system is simple and has the form shown in the slide.

Pontryagin Maximum Principle Condition

The PMP means that a nonzero solution $(\psi_1^*(\cdot), \psi_2^*(\cdot), \psi_3^*(\cdot))^{\mathrm{T}}$ of the adjoint system exists, for which almost everywhere (a.e.) on the interval $[t_0, t_f]$ the following condition is satisfied :

$$\psi_1^*(t)\cos\varphi^*(t) + \psi_2^*(t)\cos\varphi^*(t) + \psi_3^*(t)u^*(t) = \max_{u\in[u_1,\,1]} \left[\psi_1^*(t)\cos\varphi^*(t) + \psi_2^*(t)\cos\varphi^*(t) + \psi_3^*(t)u\right]$$

$$\Rightarrow \qquad \psi_3^*(t)u^*(t) = \max_{u\in[u_1,\,1]} \left[\psi_3^*(t)u\right], \quad \text{a.e. } t\in[t_0, t_f]$$

The functions $\psi_1^*(\cdot)$ and $\psi_2^*(\cdot)$ are constant.

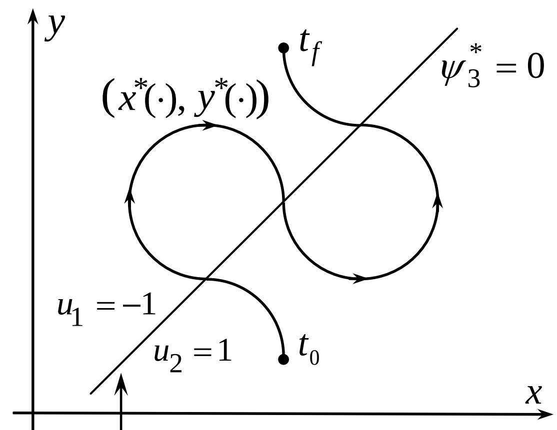
If $\psi_1^* = 0$ and $\psi_2^* = 0$, then $\psi_3^*(\cdot) = \mathrm{const} \neq 0$ on the interval $[t_0, t_f]$.
Therefore, we have $u^*(t) = u_1$ or $u^*(t) = 1$ a.e.

Let at least one of the numbers $\psi_1^*(\cdot)$ and $\psi_2^*(\cdot)$ be non-zero. Using the equations of dynamics and adjoint system equations, one can write

$$\psi_3^*(t) = \psi_1^* y^*(t) - \psi_2^* x^*(t) + C\,.$$

Therefore, $\psi_3^*(t) = 0$ iff the point $(x^*(t), y^*(t))^{\mathrm{T}}$ of the geometric position at the instant t obeys the straight line equation

$$\psi_1^* y - \psi_2^* x + C = 0.$$

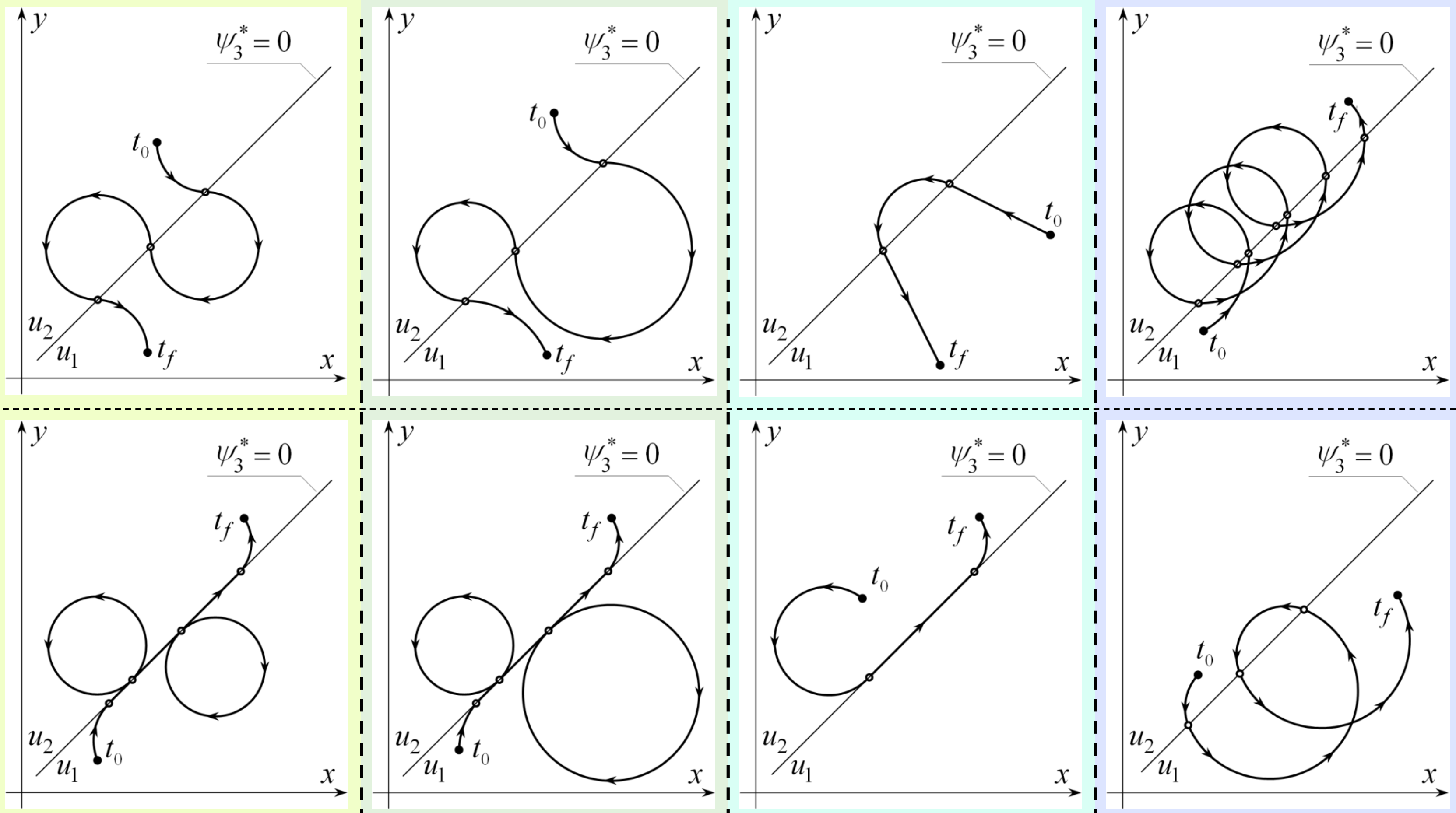


Switch line

Slide 6

Here, the PMP formula is presented. A variant of an extreme motion is shown for the symmetric case. Each movement satisfying the PMP and having two or more switchings corresponds to its own switch line.

Types of motions (trajectories $(x^*(\cdot), y^*(\cdot))$)



symmetric case
$(u_1 = -1)$

an example with $u_1 = -0.5$

asymmetric case
$(-1 < u_1 < 0)$

one - sided case
$(u_1 = 0)$

an example with $u_1 = 0.5$

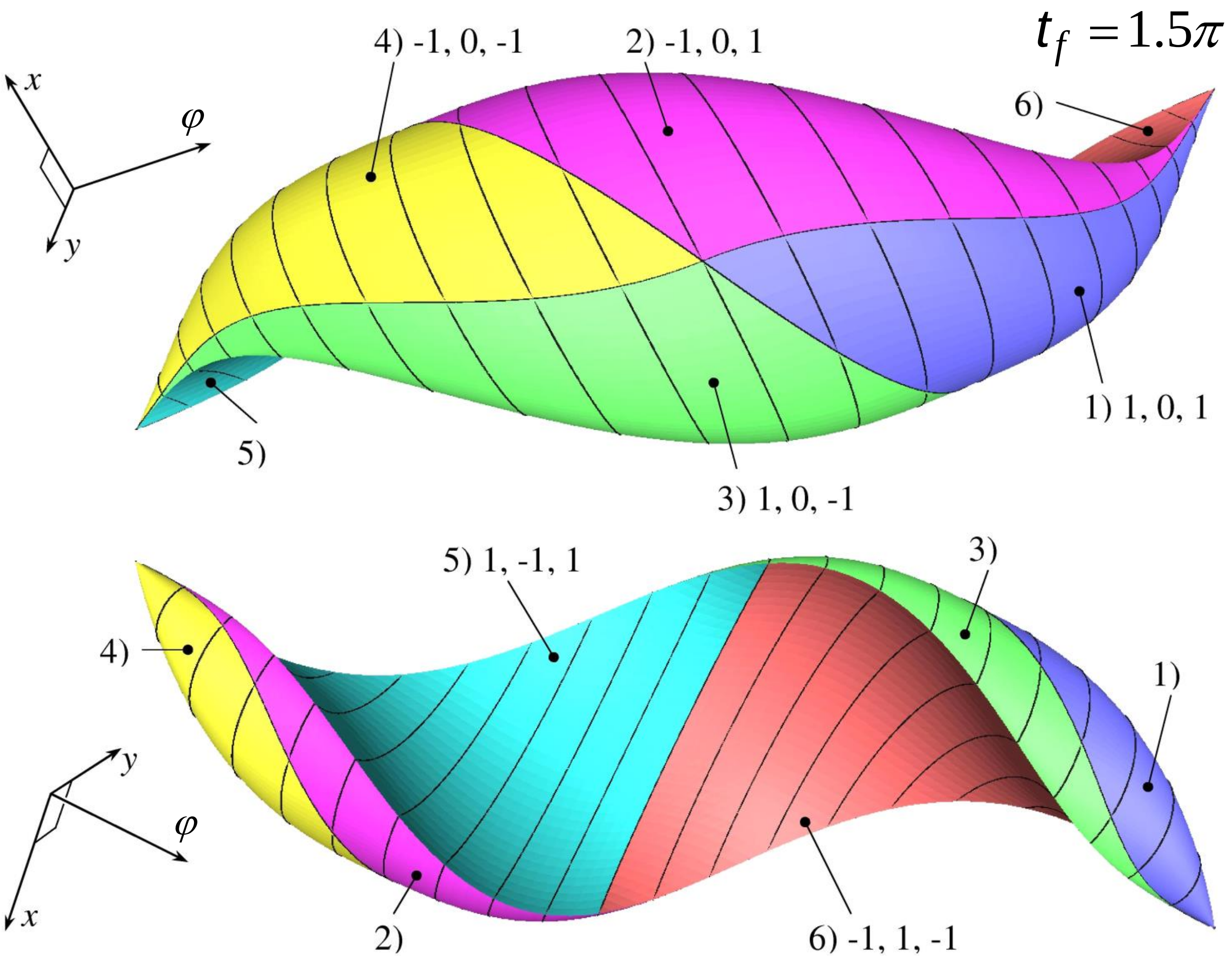
strictly one - sided case
$(0 < u_1 < 1)$

Slide 7

On this slide, some variants of motion are shown for each of the mentioned cases; these motions satisfy the PMP.

But the PMP is only the first step of investigation of the reachable sets.

Reachable set G(t_f) for the symmetric case



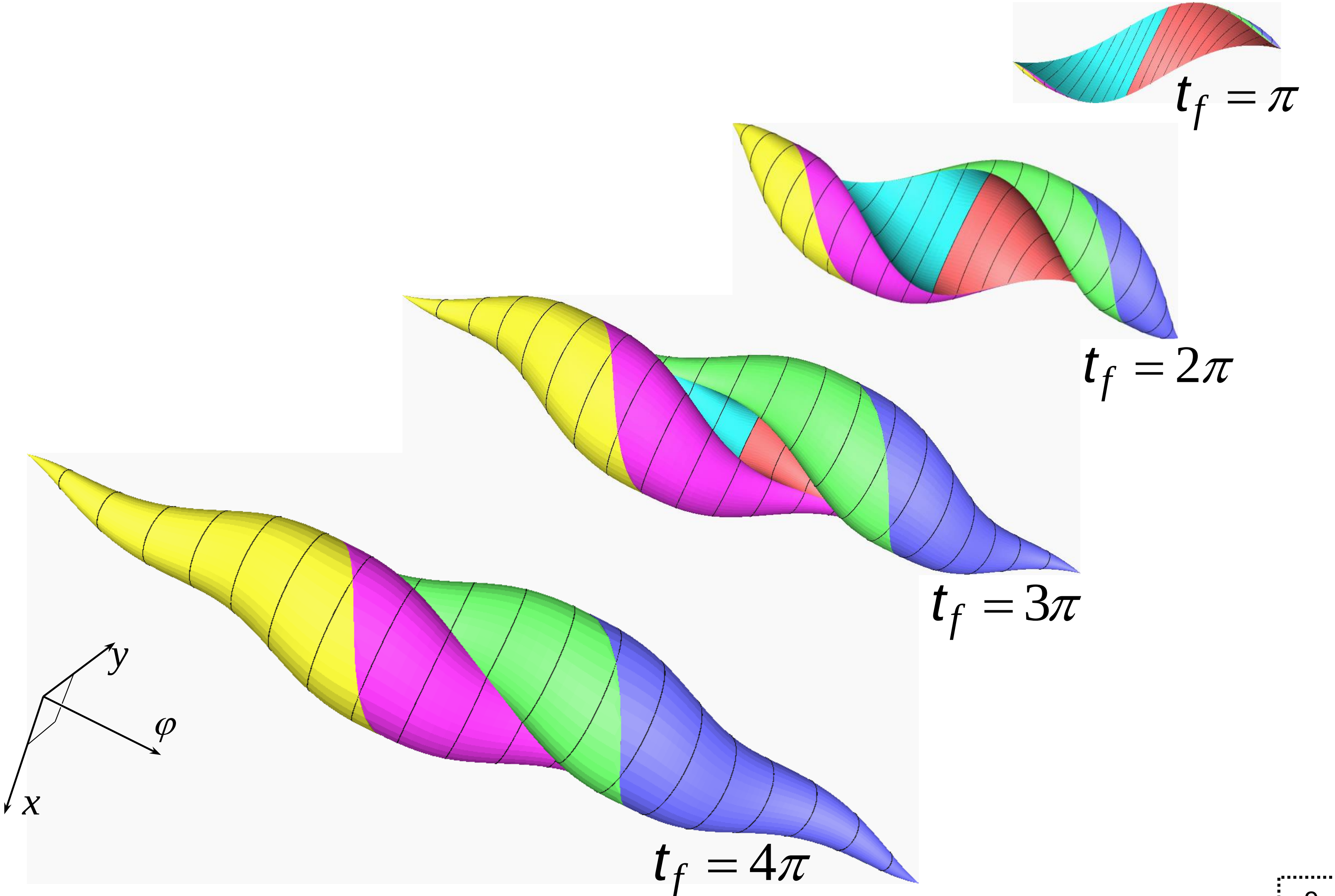
Patsko V.S., Pyatko S.G., Fedotov A.A. (2003) *Three-dimensional reachability set for a nonlinear control system*. Journal of Computer and Systems Sciences International. Vol. 42, No. 3, pp. 320–328

Slide 8

For the symmetric case, we had found that a motion that leads to any point on the boundary of the reachable set $G(t_f)$ has not more than two switches. Under this, the whole boundary can be divided into six parts (cells), and each of them has its own character of extreme motions. For example, the sequence of controls -1, 0, $+1$ corresponds to cell 2. So, there exists the first time interval with the control -1; further, the linear-direction motion goes with the control $u = 0$; in the third time interval, the control is equal to $+1$. But in cell 5, the control sequence is $+1$, -1, $+1$. The shown six variants of controls are the same as in the famous Dubins theorem.

On the slide, the thin lines mark the reachable set sections by the angular coordinate (φ-sections).

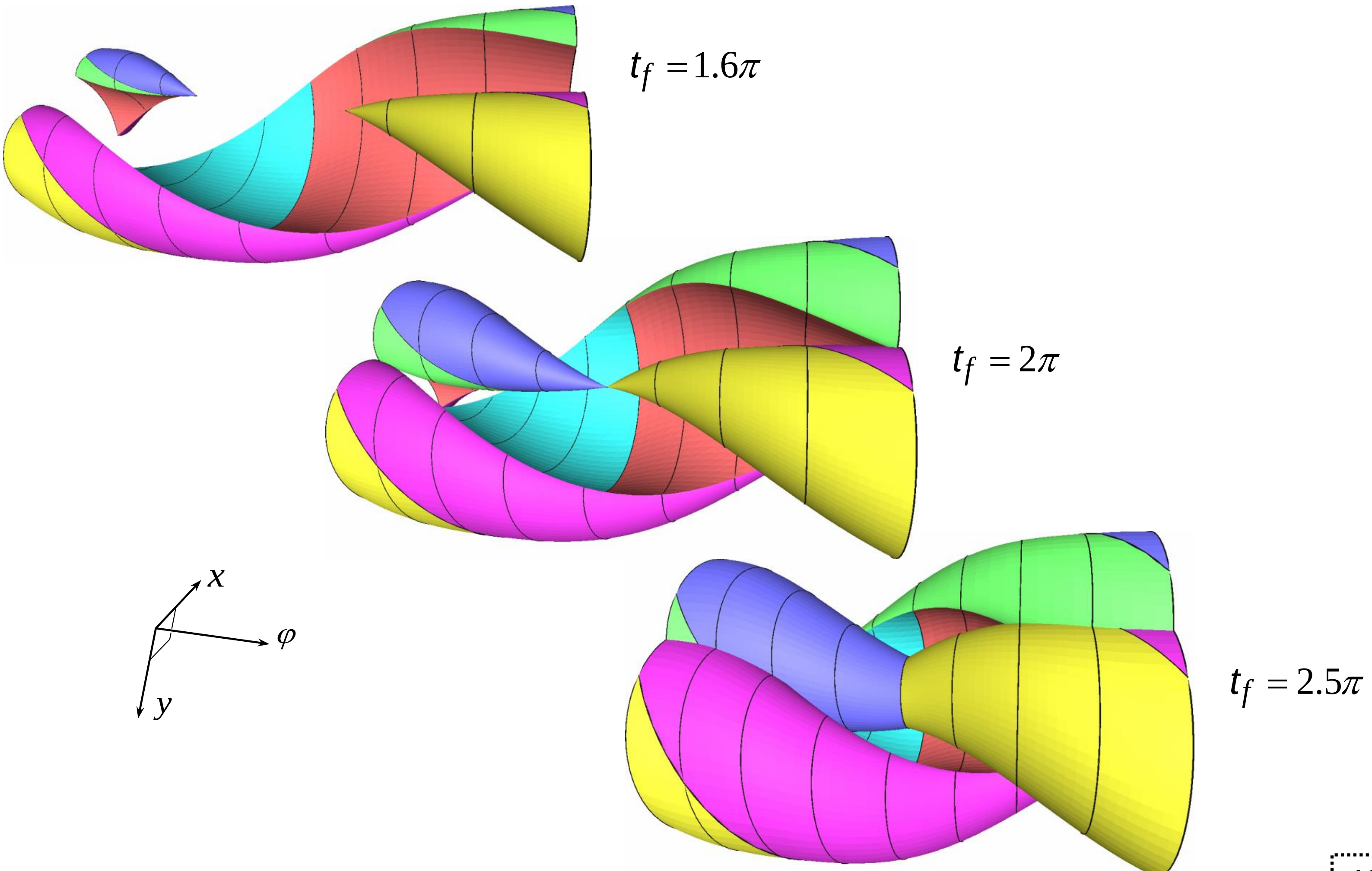
Evolution of reachable set in the symmetric case



Slide 9

Such a picture of developing the reachable set in time has been shown at various Conferences. It is rather similar to growth of shell on a snail.

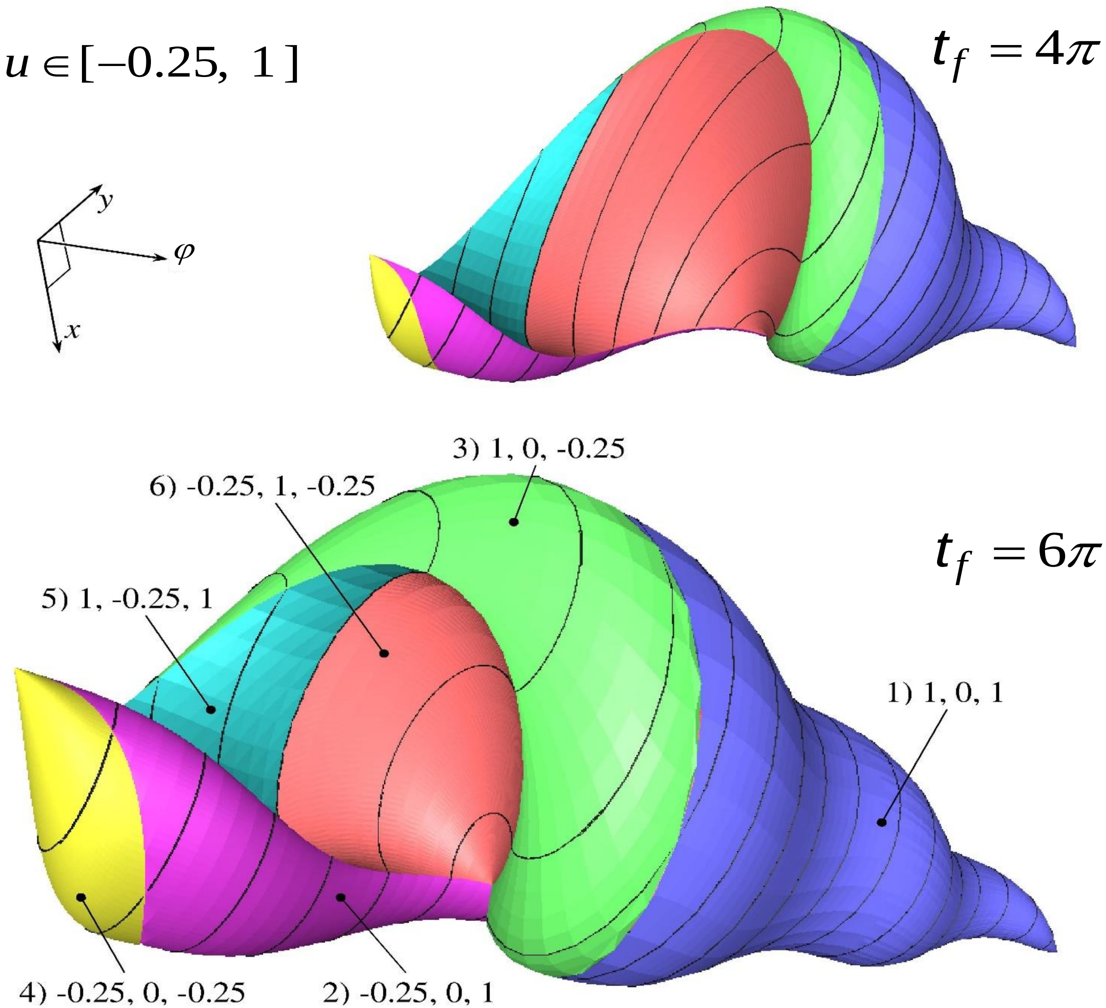
Reachable sets with φ computed by modulo 2π in the symmetric case



Slide 10

If the angle φ is calculated by modulo 2π, the pictures become difficult for comprehension. Here, we see reachable sets for three instants. Due to these complicated configurations, almost nobody (except us) deals with the three-dimensional reachable sets for Dubins car.

Reachable sets in the asymmetric case

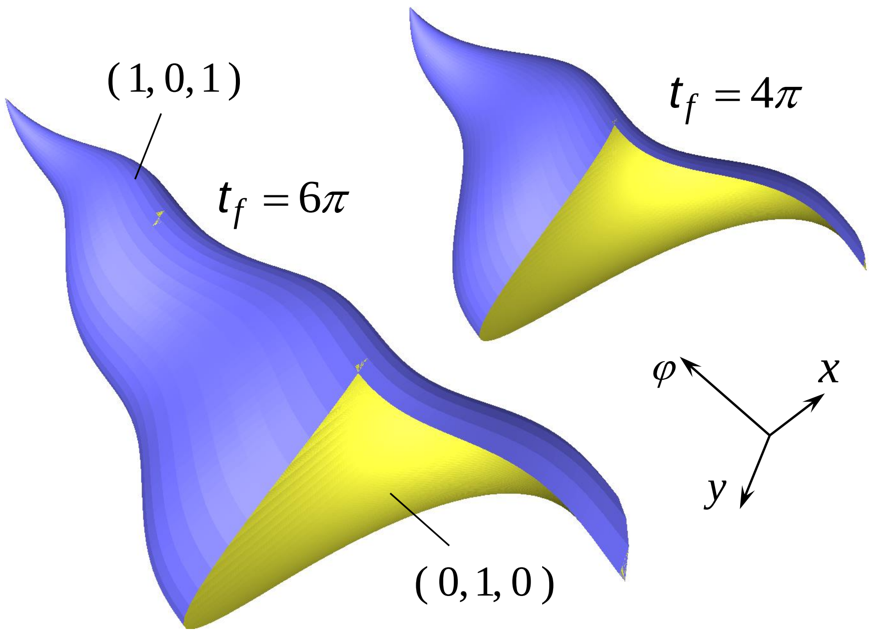


Fedotov A. A., Patsko V. S., Turova V. L. (2011) *Reachable sets for simple models of car motion*. Ed.by A.V. Topalov. Rijeka: InTech Open Access Publisher, pp. 147–172

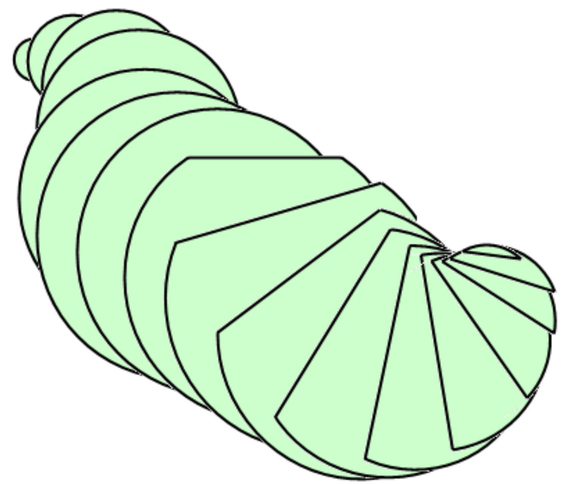
Slide 11

Here, three-dimensional reachable sets are shown for the asymmetric case.
Till now, we have no analytical description of the φ-sections for the symmetric and asymmetric cases.

One-sided case u_1=0 (it is allowed to move in a straight line)



Two variants of controls carrying the motion onto the boundary



Convexity of φ-sections

We have a description of the φ-sections of the reachable set that actually represents either a circular segment (for $\varphi < 2\pi$) or an entire circle (for $\varphi \geq 2\pi$).

Slide 12

For the case $u_1 = 0$, an analytical description of the φ-sections is obtained. Here, each φ-section is either a circle or a circular segment. Thus, under $u_1 = 0$, the φ-sections are convex. It was found that in investigation of motions leading onto the boundary, it is sufficient to deal with only two sequences of the control; which are $+1$, 0, $+1$ and 0, $+1$, 0.

Strictly one-sided case $u_1 > 0$.
Types of motions which lead onto the boundary

0 Motions with constant (in time) control **:**

$u \equiv u_1$ or $u \equiv u_2$

(two extreme points of reachable set)

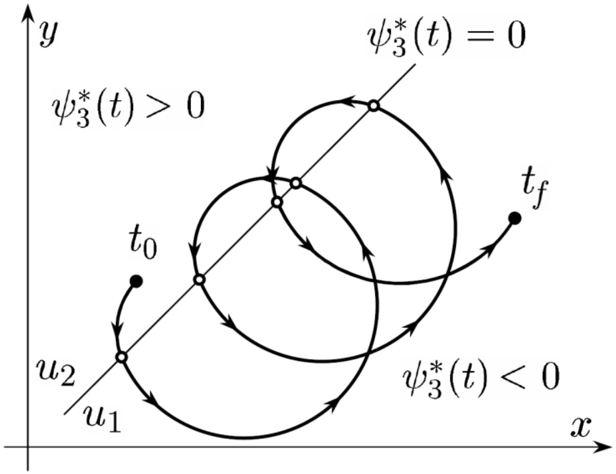
----

1 Motions BS: start with control $u = u_1$
and finish with control $u = u_2$;

2 Motions BB: start with control $u = u_1$
and finish with control $u = u_1$;

3 Motions SB: start with control $u = u_2$
and finish with control $u = u_1$;

4 Motions SS: start with control $u = u_2$
and finish with control $u = u_2$;



Example of SB-type motion

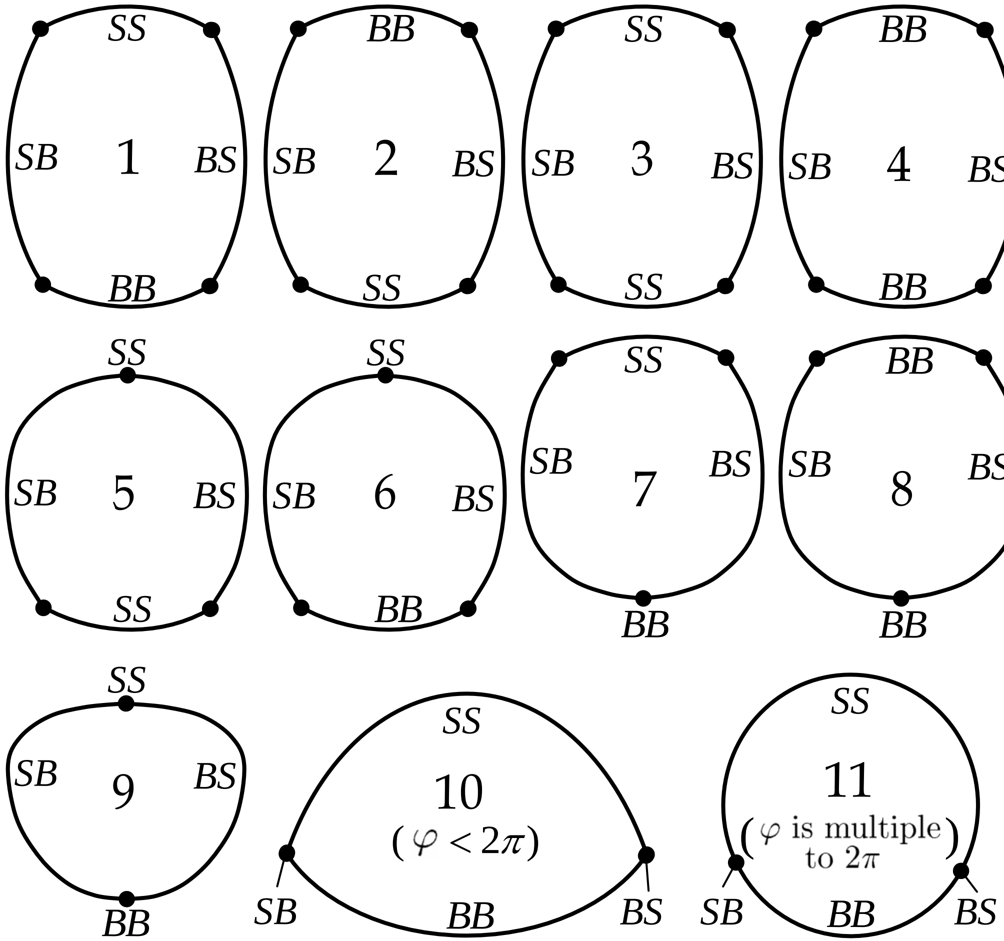
Slide 13

Our new recent result is in obtaining an analytical description of the φ-sections for the case of strictly one-sided turn. The strict convexity of the φ-sections is proved. Under this, the boundary of any φ-section comprises not more than four smooth arcs.

Motions of type SB are those, in which the control $u = 1$ acts in the first time interval. The corresponding trajectory in the plane of the geometric coordinates is an arc of the small radius. But at the final time interval, the control u_1 acts and the arc of the large (big) radius is implemented. Between the initial and final time intervals, the sequence of arcs of the large and small radius appears (several cycles).

The motions of types BS, BB, and SS have some similar interpretations.

Strictly one-sided case $u_1 > 0$.
Variants of φ-sections



4 arcs with smooth connection

3 arcs with smooth connection

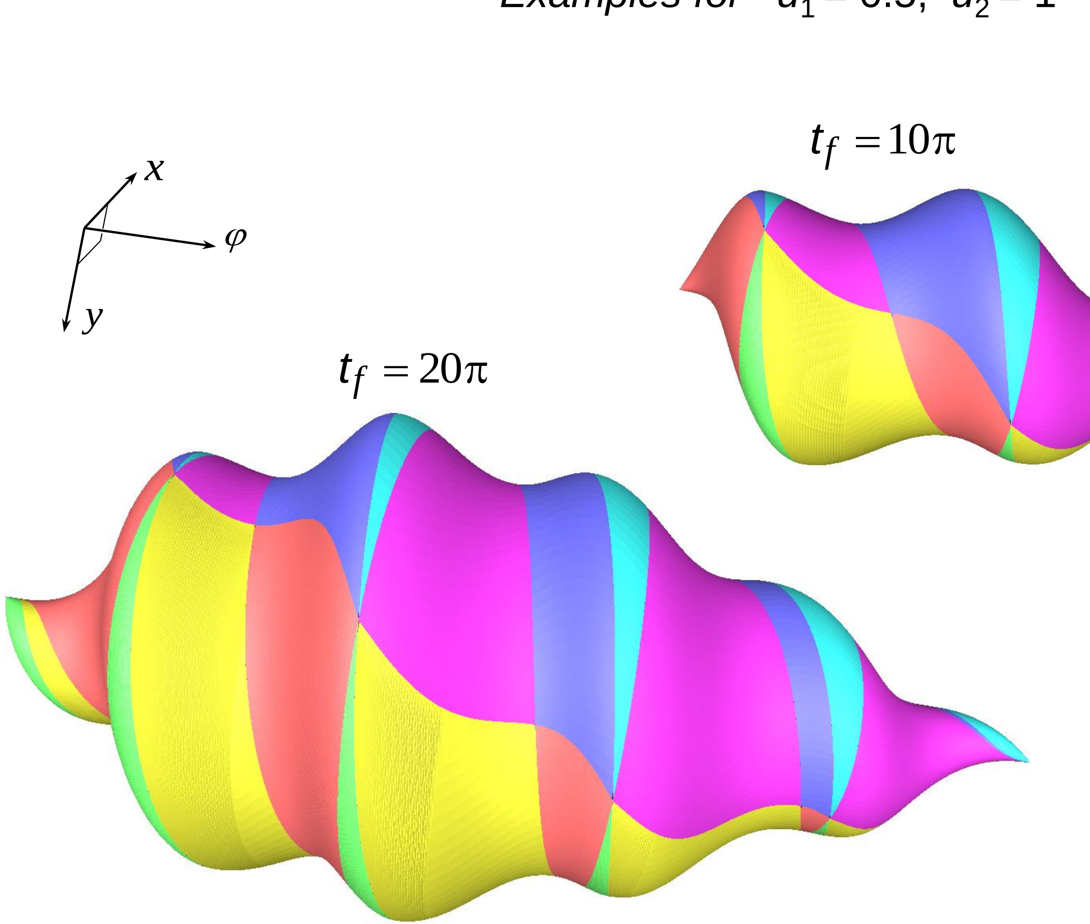
2 arcs with smooth connection or with nonsmooth connection

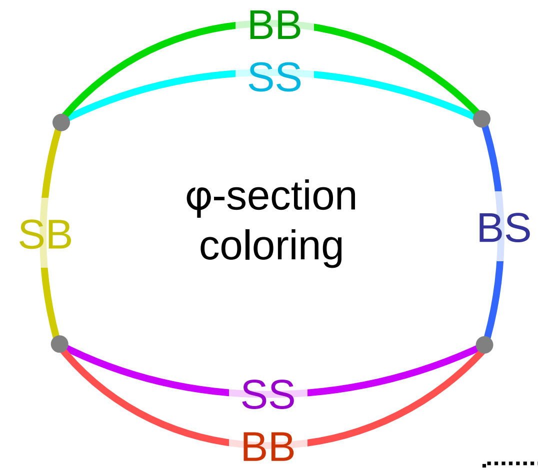
Slide 14

On this slide, all possible variants of the φ-sections are presented for the case of the strictly one-sided turn. There are 11 variants. Here, SB is an arc on the φ-section boundary, where SB-type motions come. The analytical descriptions of the φ-sections are not so simple as for the case $u_1 = 0$; but it has been obtained.

Reachable sets for the case $u_1 > 0$ (it is not allowed to move in a straight line)

Examples for $u_1 = 0.5, \; u_2 = 1$





Slide 15

Here, three-dimensional reachable sets are shown for the case of the strictly one-sided turn. Those parts of the boundary that have the same type of the arcs on the boundary of the φ-sections are marked by the same color. There are only six variants. The arcs BB and SS can be included in a φ-section, each in some two standard variants, which differ by the number of intermediate cycles in description of the extreme motions. The arcs SB and BS for each φ are given by the only way.

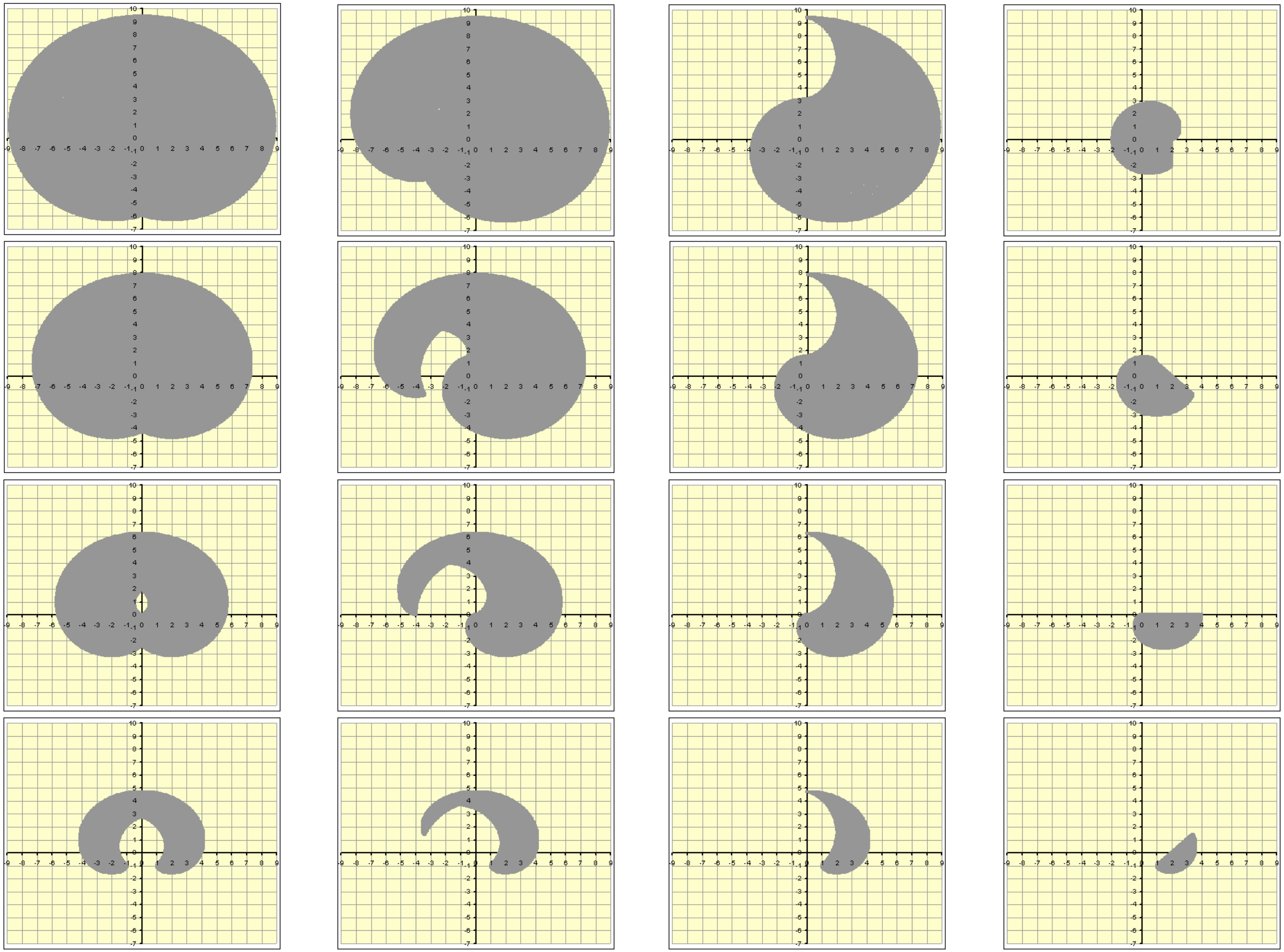
Pontryagin maximum principle, φ-sections of the reachable set, and controls leading onto the boundary

<table>
<tr><th></th><th>Pontryagin maximum principle</th><th>φ-sections of the reachable set</th><th>Controls leading onto the boundary</th></tr>
<tr><td>$u_1 = -1$
symmetric case</td><td rowspan="2">Only necessary condition</td><td rowspan="2">Non-convex</td><td rowspan="3">Non-uniqueness in the class of piecewise constant controls</td></tr>
<tr><td>$u_1 \in (-1, 0)$
asymmetric case</td></tr>
<tr><td>$u_1 = 0$
one-sided case</td><td rowspan="2">Necessary and sufficient condition</td><td>Convex</td></tr>
<tr><td>$u_1 \in (0, 1)$
strictly one-sided case</td><td>Strict convex</td><td>Uniqueness in the class of piecewise constant controls</td></tr>
</table>

Slide 16

There exists certain dependence between the character of the φ-sections and the PMP properties. In the symmetric and asymmetric cases, the PMP is only necessary condition for transfer onto the boundary of the reachable set. Generally speaking, the φ-sections are not convex here. But in the cases of the one-sided and strictly one-sided turns, the PMP is the necessary and sufficient condition for transfer onto the boundary. Here, the φ-sections are convex. Moreover, in the case of the strictly one-sided turn, the φ-sections are strictly convex sets. This fact corresponds to the uniqueness of motions leading to each point on the boundary.

Reachable sets in the projection onto a geometric plane



$t_f = 3\pi$

$t_f = 2.5\pi$

$t_f = 2\pi$

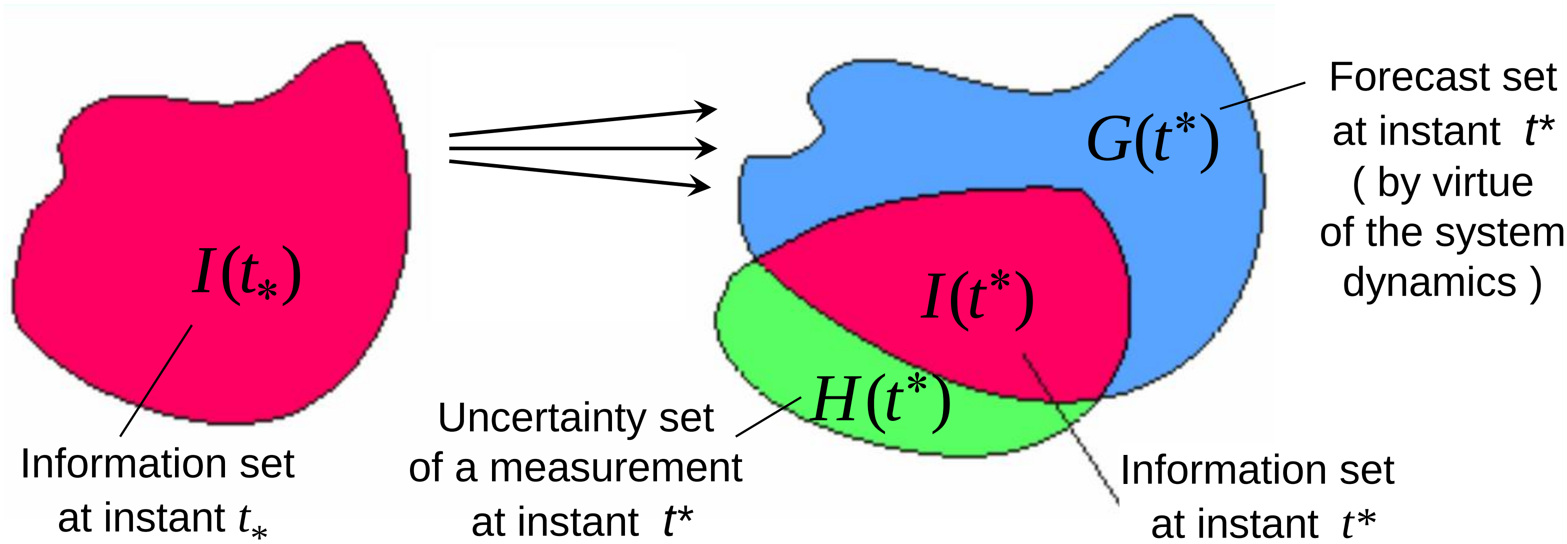
$t_f = 1.5\pi$

$u_1 = -1$	$u_1 = -0.5$	$u_1 = 0$	$u_1 = +0.5$
symmetric case	asymmetric case	one-sided case	strictly one-sided case

Slide 17

On this slide, the projections of reachable sets onto the plane of the geometric coordinates are shown for each of the mentioned four cases.

Transformation of information sets



Information set at a current instant is a totality of all phase states consistent with description of the dynamics, constraints on measurement errors, and history of the observation process.

Terms equivalent to the term **"information set"** are "feasible set", "membership set", "likelihood set".

The approach is often called the "set membership estimation" or "unknown but bounded error description (UBB approach)"

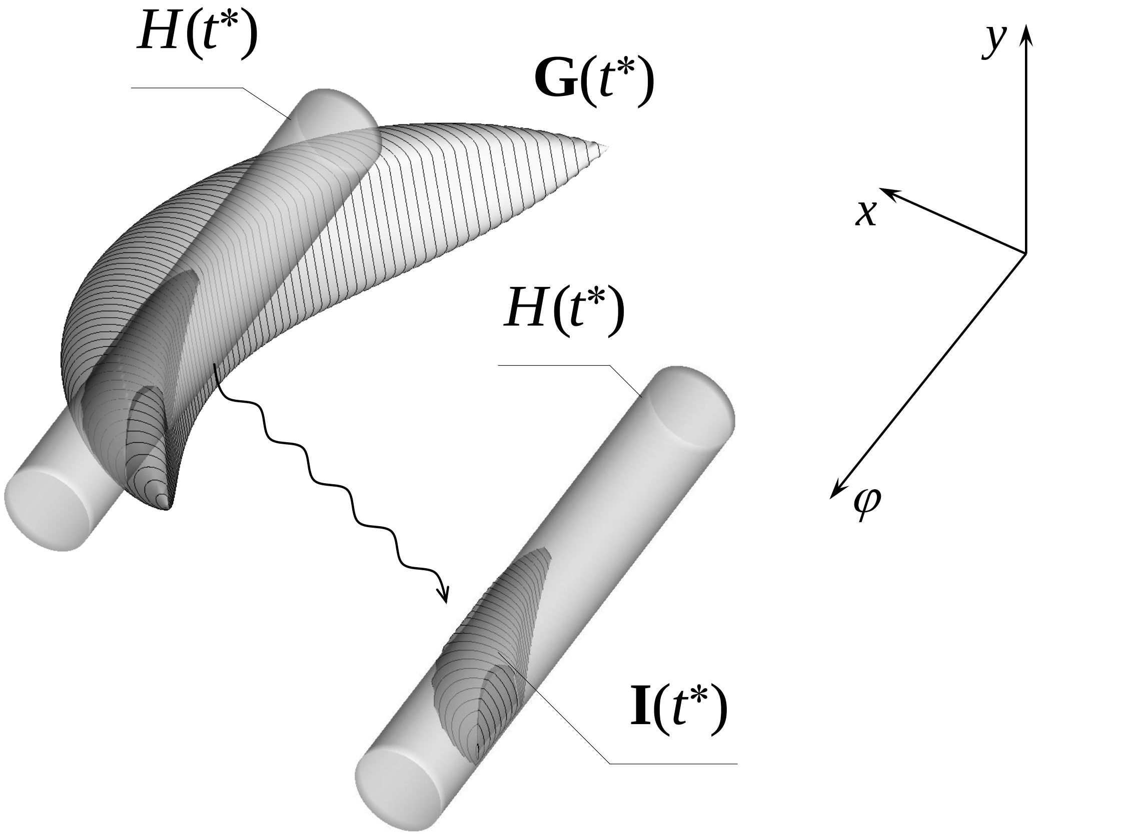
Slide 18

Now, let us move to a possible application of three-dimensional reachable sets to problems of observation with incomplete information. Let us observe an object motion with the dynamics of Dubins car. Measurements of geometric positions are provided at some discrete instants. The measurements are inexact, but the bounds of errors are known. Each measurement gives an "uncertainty set" $H(t)$. The set has some concrete form in the plane x, y and is cylindrical on the coordinate φ if this coordinate is not measured.

The information set $I(t)$ is the totality of all three-dimensional phase states compatible with the observation history. Having the information set $I(t_*)$ at some instant t_*, we build the forecast three-dimensional set $G(t^*)$ for the instant t^*, when the next measurement comes. The set $G(t^*)$ is the reachable set not from some initial point, but from the initial three-dimensional set $I(t_*)$. The new information set $I(t^*)$ is the intersection of the sets $G(t^*)$ and $H(t^*)$.

Here, some variants of now existing terminology are written out.

Intersection of an approximating forecast set with a measurement uncertainty set



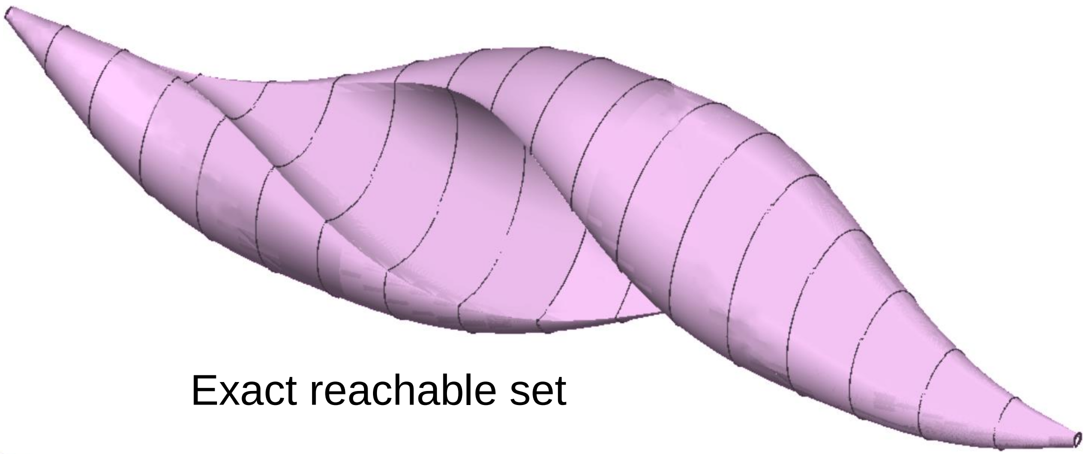
Convexity of φ-sections allows to construct fast procedures for intersection

Slide 19

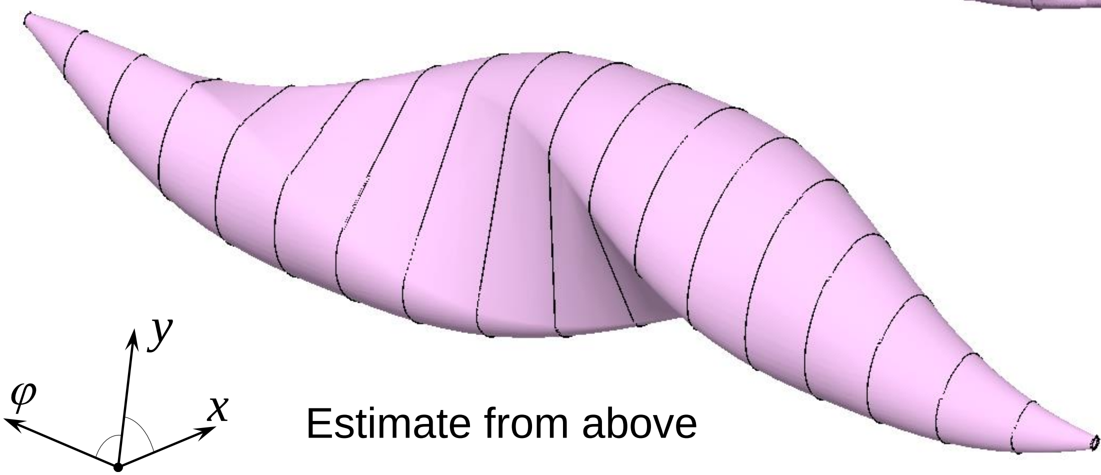
The uncertainty set $H(t)$ of each measurement is naturally regarded to be convex. If the φ-sections of the forecast set $G(t)$ were convex, then it would essentially simplify the intersection procedure.

Approximation from above for the reachable set

(at the instant $t = 2\pi$)



Exact reachable set



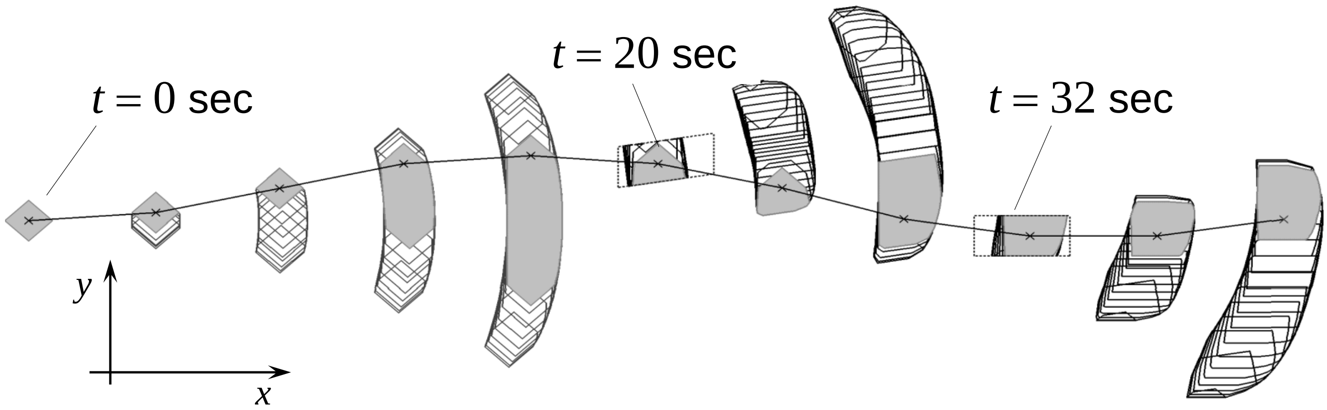
Estimate from above

Slide 20

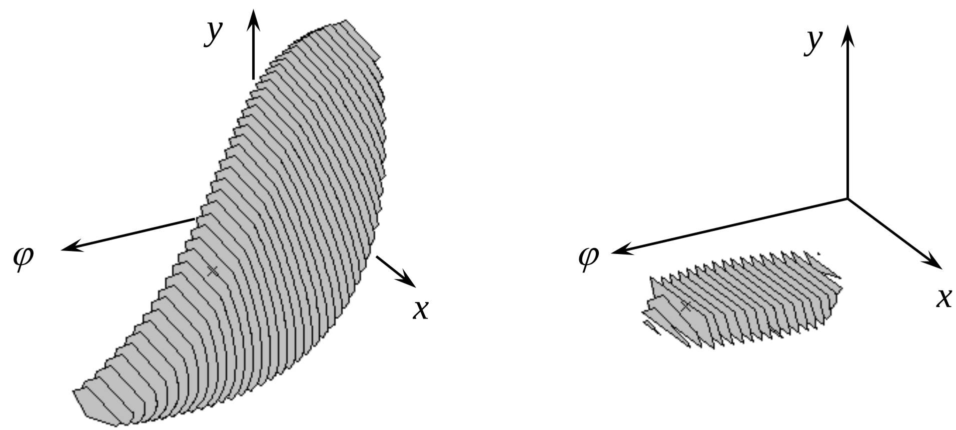
We have elaborated a special procedure for the convexification of the φ-sections of the forecast set $G(t)$. Under this, the “true” non-convex φ-sections are unknown. Nevertheless, we build an approximating set $\mathbf{G}(t)$, which has φ-sections that are convex hulls of the true φ-sections. As a result, we obtain an “economic from above” estimation $\mathbf{G}(t)$ of the true forecast set.

Here, such a convexification φ-sections of is shown for a one-point initial set.

Motion of an information set, example 1



Measurement instants: 0, 20, 32 sec



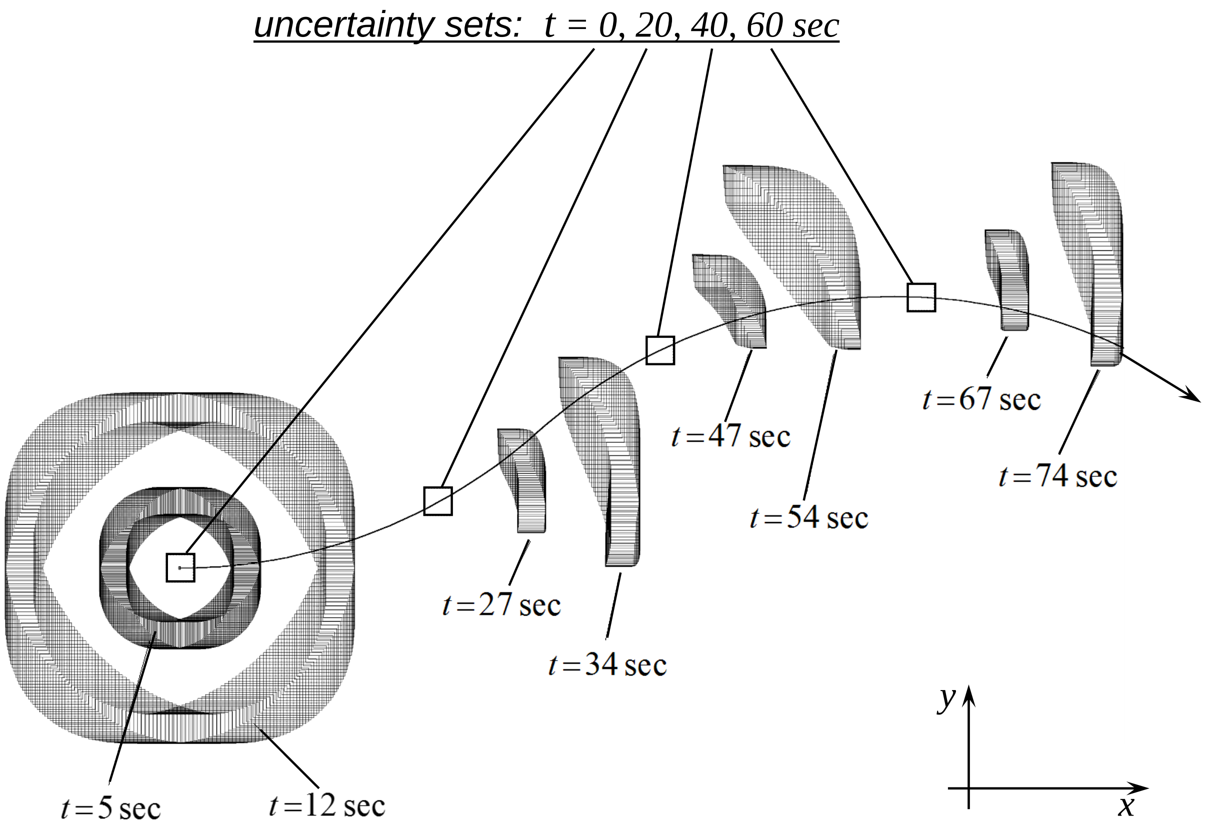
Structure of the information set at t = 20 sec:
Before the measurement (at the left) and after (at the right)

Slide 21

On this slide, a motion of an information set is shown. The set is represented as a collection of its φ-sections. Let us consider the case when we have only one φ-section at the initial instant. So, we assume that the initial angle is known. The measurement comes at the instant $t = 20\,$sec. Till this instant, the information set grows. It coincides with the current approximating forecast set and consists of convex φ-sections. The forecast set, which is built for the instant $20\,$sec, is intersected with the obtained uncertainty set. We perform this by intersection of the corresponding φ-sections. Further, the current information set grows again. Its sharp narrowing takes place at the instant $t = 32\,$sec, when the next measurement comes.

Motion of an information set, example 2

No initial angular coordinate information is available



Case with the φ-sections approximation by rectangles

Slide 22

The main difference from the previous slide is in the fact that here, for convexification of the φ-sections, we used only four directions forth and back along the x, y axes. There are no measurements in the interval $(0, 20)$ sec. The information set grows. The next measurements come at the instants 20, 40, and 60 sec.

References

Dubins car :

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E. J. Cockayne, G. W. C. Hall, T. Pecsvaradi, Yu. I. Berdyshev,
J.-P. Laumond, P. Souéres, S.M. LaValle, T. Shima, P. Tsiotras, E. Bakolas, M. Pachter,
H. Choi, R.W. Beard, T.W. McLain, J. Campbell, M. Moll, L.E. Kavraki
Z. Chen, R. Takei, R. Tsai, …

Information sets :

N.N. Krasovskii, A.I. Subbotin, A.B.Kurzhanski, M.I. Gusev, Yu.S. Osipov,
A.V. Kryazhimskiy, F.L. Chernousko, B.T. Polyak, V.M. Kuntsevich

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M. Milanese, E. Walter, …

Recent papers of the authors :

A.A. Fedotov, V.S. Patsko, Investigation of reachable set at instant for the Dubins’ car, Proceedings of the 58th Israel Annual Conference on Aerospace Sciences, Tel-Aviv & Haifa, Israel, pp. 1655–1669, 2018.

V.S. Patsko, A.A. Fedotov, Attainability set at instant for one-side turning Dubins car, Proceedings of 17th IFAC Workshop on Control Applications of Optimization, Yekaterinburg, Russia, pp. 201–206, 2018.

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The literature related to the Dubins model is huge. On the slide, we listed only some of the authors. The information set theory was seriously developed in the USSR and not only there. At the bottom of the slide, two our recent works are mentioned, which are concerned with the reachable sets for Dubins car.