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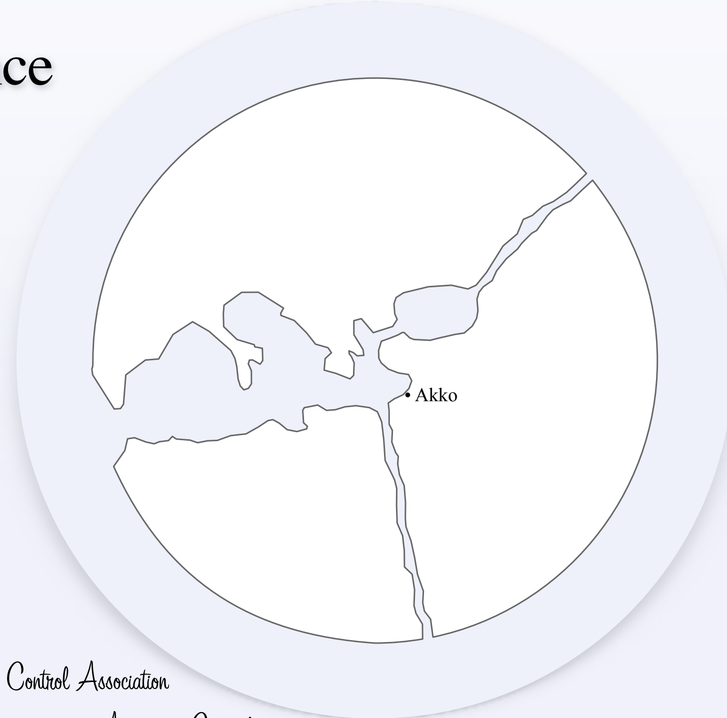

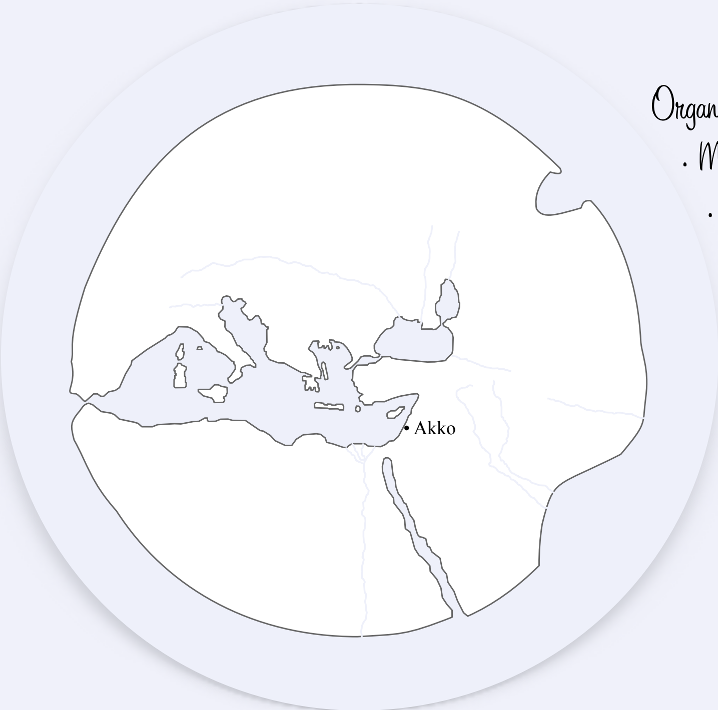

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Reachable set for Dubins car and its application to observation problem with incomplete information

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***Abstract*— A three-dimensional reachable set "at the instant" for a controlled object called the Dubins car is considered. The turn angular velocity of the linear velocity vector is a control variable. Along with the case when, according to the statement of the problem, rotation is possible in both directions, the cases of one-sided rotation are studied. Three-dimensional images of reachable sets are presented. Sufficiency of the Pontryagin maximum principle conditions for the control leading to the boundary of the reachable set is analyzed. Possibility of using reachable sets in the problem of motion observation under conditions of inaccurate measurements of the geometric position is considered.**

I. INTRODUCTION AND PROBLEM FORMULATION

Reachable set is one of the central concepts of the contemporary mathematical control theory [1], [2]. The reachable set $G(t_f)$ "at an instant" t_f for a given initial phase state x_0, y_0, φ_0 can be defined as a collection of all phase states of system (1) that can be reached by means of admissible controls at the instant t_f. In order to avoid misunderstandings, we emphasize the difference between the reachable set at the instant (which is considered in this paper) from the reachable set "up to the instant". The latter is the union of reachable sets "at the instant" on the interval $[t_0, t_f]$.

It is remarkable that any admissible open-loop control leading to the boundary of the reachable set satisfies the Pontryagin maximum principle [2]. This "extremality" property is used in the paper to find the reachable set. Effective description of reachable sets, in turn, can be used in a variety of optimal control problems, motion observation, and conflict dynamic tasks.

Among the models of controlled motion most commonly used in robotics and applied aviation works the "Dubins car" is very popular [3] – [12]. In this model, which describes the motion of a point object on a plane, the value of the linear velocity is constant, and the value of the instantaneous angular velocity is limited both from below and from above. The latter is equivalent to a limit on the instantaneous turning radius.

Dynamics of the Dubins car is described by the third order system of differential equations

$$\begin{aligned} \dot{x} &= \cos\varphi, \\ \dot{y} &= \sin\varphi, \\ \dot{\varphi} &= u, \qquad u \in [u_1, u_2]. \end{aligned} \tag{1}$$

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Here, x, y are the coordinates of the object geometric position, φ is the velocity direction angle counted counterclockwise from the axis x (Fig. 1), u is a scalar control. The value of the linear velocity equals unit. The magnitude u_1 is a parameter of the problem and satisfies the inequality

$$-u_2 \le u_1 < u_2. \tag{2}$$

We assume that $u_2 = 1$.

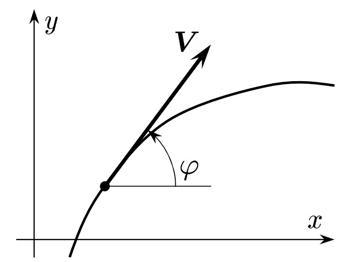

Fig. 1. The coordinate system, $\boldsymbol{V} = (\dot{x}, \dot{y})^{\mathsf{T}}$

Any controlled system of the third order that describes a motion in the plane with non-zero constant linear velocity and some given range of the turn angular velocity can be reduced to representation (1), (2) with $u_2 = 1$. To do this, one needs to rescale the geometric coordinates and the time.

As feasible controls $u(\cdot)$, we consider measurable functions depending on time and having their values $u(t)$ from the segment $[u_1, u_2]$. It is assumed that the angular coordinate φ takes its values in the interval $(-\infty, \infty)$.

When studying the reachable set $G(t_f)$ at the instant t_f for a one-point initial phase state, without loss of generality, we consider it zero at the initial instant $t_0 = 0$: $x_0 = 0$, $y_0 = 0$, $\varphi_0 = 0$.

We consider the following cases:

a) $u_1 = -1$, $u_2 = 1$; b) $-1 < u_1 < 0$, $u_2 = 1$;
c) $u_1 = 0$, $u_2 = 1$; d) $0 < u_1 < u_2 = 1$.

In [13], the projection of the reachable set into the plane x, y is described for the case a). Some results of investigation and construction of three-dimensional reachable sets for cases a) – d) have been considered in the previous papers [14] – [18].

Investigation of reachable sets up to the instant is associated with time-optimal problems. For the cases of a) and b), the construction of such sets was studied in [4], [19]. There are works (see, for example, [20]), in which images of the three-dimensional reachable sets up to the instant were obtained by using numerical methods developed for the Hamilton – Jacobi type equations.

A new result of this work is the analysis of the convexity property of sections by the angular coordinate (φ-sections) for the three-dimensional reachable set $G(t_f)$. This property is specific for the cases c) and d), but it is absent for the cases a) and b). The convexity property of φ-sections is related to the sufficiency of the Pontryagin maximum principle conditions for controls leading to the boundary of the reachable set $G(t_f)$. Corresponding facts are also formulated in this paper.

As a natural application of reachable sets, the paper considers the construction of information sets [21] in the problem of motion observation.

Let measurements of the object current state in the plane x, y come with some time step. The maximal magnitude of the measurement error is given. The third coordinate, which is the current course angle, is unobservable.

As the information set $I(t)$ at an instant t, one consider a collection of all three-dimensional phase states of system (1) consistent with all measurements got up to the instant t. The exact construction of information sets is hardly possible. But, reasonably applying the convexification procedure, one can construct a satisfactory upper estimate in the form of sets $\mathbf{I}(t_i) \supset I(t_i)$ with convex sections at the angular coordinate φ.

The suggested recurrent procedure for constructing sets $\mathbf{I}(t_{i+1})$ for a discrete measurement sample is the following. Let the set $\mathbf{I}(t_i)$ is known for some instant t_i. For the instant t_{i+1}, a three-dimensional forecast set $\mathbf{G}(t_{i+1}, \mathbf{I}(t_i))$ is constructed by means of phase states of system (1) reachable from the given (at the instant t_i) set $\mathbf{I}(t_i)$. On the basis of the measurement that comes at the instant t_{i+1}, the uncertainty set $H(t_{i+1})$ is constructed of all states consistent with this measurement (with its value and maximal error magnitude). The new set $\mathbf{I}(t_{i+1})$ is the intersection $\mathbf{G}(t_{i+1}, \mathbf{I}(t_i)) \bigcap H(t_{i+1})$.

The described procedure for constructing the information sets implements the classic predictor/corrector scheme in the theory of observation with discrete input of measurements.

A practical construction of the information sets for system (1) is based on their representation as a collection of two-dimensional sections by planes orthogonal to the angular coordinate. To compute numerically such reachable sets one needs a very good idea about their geometric structure in the case of one-point initial set.

The paper is organized as follows. In Section II, we consider types of motions, which go onto the boundary of the reachable sets. Section III deals with a description of the boundary of reachable sets including description of boundaries of the angular sections. In Section IV, we suggest a method for constructing an upper estimate of information sets, which is based on simple operations with convex sets in the plane. An example of numerical construction of a forecast set is given.

II. CONTROLS LEADING THE SYSTEM ONTO THE BOUNDARY OF REACHABLE SET

We will use the fact that the controls that lead the system onto the boundary of a reachable set $G(t_f)$ at the instant t_f obey the Pontryagin maximum principle (PMP).

Let $u^*(\cdot)$ be some admissible control and $(x^*(\cdot), y^*(\cdot), \varphi^*(\cdot))$ be the motion of system (1) generated by this control in the interval $[t_0, t_f]$. The adjoint differential equations [4], [14], [19] are

$$\begin{aligned} \dot{\psi}_1 &= 0, \\ \dot{\psi}_2 &= 0, \\ \dot{\psi}_3 &= \psi_1 \sin\varphi^*(t) - \psi_2 \cos\varphi^*(t). \end{aligned} \tag{3}$$

PMP means that there is a *non-zero* solution $(\psi_1^*(\cdot), \psi_2^*(\cdot), \psi_3^*(\cdot))$ to system (3), for which the condition

$$\begin{aligned} &\psi_1^*(t)\cos\varphi^*(t) + \psi_2^*(t)\sin\varphi^*(t) + \psi_3^*(t)u^*(t) \\ &= \max_{u\in[u_1,u_2]} [\psi_1^*(t)\cos\varphi^*(t) + \psi_2^*(t)\sin\varphi^*(t) + \psi_3^*(t)u] \end{aligned}$$

holds almost everywhere (a.e.) in the interval $[t_0, t_f]$, or, what is the same,

$$\psi_3^*(t)u^*(t) = \max_{u\in[u_1,u_2]} \psi_3^*(t)u \quad \text{a.e. in } [t_0, t_f]. \tag{4}$$

The functions $\psi_1^*(\cdot)$ and $\psi_2^*(\cdot)$ are constants. Denote them by ψ_1^* and ψ_2^*.

If $\psi_1^* = 0$ and $\psi_2^* = 0$, then $\psi_3^*(t) = \text{const} \neq 0$ in the entire interval $[t_0, t_f]$. In this case, one has $u^*(t) = u_1$ a.e. in $[t_0, t_f]$ or $u^*(t) = u_2$ a.e. in $[t_0, t_f]$.

Let us assume that at least one of the numbers ψ_1^*, ψ_2^* does not equal zero. Using the equations of dynamics (1) and adjoint equations (3), one can write a relation for $\psi_3^*(t)$:

$$\psi_3^*(t) = \psi_1^* y^*(t) - \psi_2^* x^*(t) + C.$$

Hence, $\psi_3^*(t) = 0$ if and only if the point $(x^*(t), y^*(t))^\mathsf{T}$ of the geometric position at the instant t obeys the linear equation

$$\psi_1^* y - \psi_2^* x + C = 0. \tag{5}$$

So, $\psi_3^*(t) > 0$ in the half plane $\psi_1^* y - \psi_2^* x + C > 0$, and $\psi_3^*(t) < 0$ in the half plane $\psi_1^* y - \psi_2^* x + C < 0$.

Since change of the sign of $\psi_3^*(\cdot)$ implies change of the control from one extremal value to the another, the line defined by (5) is often called *the switching line*.

Due to relation (4), if $\psi_3^*(t) > 0$ in some interval, then $u^*(t) = u_2$ a.e. in this interval. The projection of the corresponding motion into the plane x, y goes counterclockwise along a circular arc of radius $1/u_2$. If $\psi_3^*(t) < 0$, then $u^*(t) = u_1$. The projection of the corresponding motion goes clockwise along a circular arc of radius $1/|u_1|$ when $u_1 < 0$, counterclockwise when $u_1 > 0$, and is rectilinear when $u_1 = 0$.

If $\psi_3^*(t) = 0$ in some interval, then the motion $(x^*(\cdot), y^*(\cdot))$ in this interval goes along the switching line (5). With that, $u^*(t) = 0$ a.e. in the interval. Such a case is impossible when $u_1 > 0$.

Typical variants of motions on the x, y plane with the corresponding switching lines for $u_1 \in (-1,\, 1)$ are given in [17].

Thus for system (1), any motion obeying PMP has a projection into the plane x, y, which consists of circular arcs and straight line segments. Within each of the arcs or segments, the control can be considered as a constant. So, during analysis of controls obeying PMP, one can consider only *piecewise constant* controls (assuming the right-continuity at points of discontinuity). The number of switchings in the interval $[t_0, t_f]$ is finite.

For the cases a) and b) of the problem formulation, the following theorem is valid.

T h e o r e m 1 [14], [17]. It is possible to lead system (1) to any point of the boundary of the reachable set by means of a control having no more than two switchings. With that, in the case of exactly two switching, one can consider only six variants of the control sequences:

$$\begin{array}{lll} 1)\ u_2, 0, u_2; & 2)\ u_1, u_2, u_1; & 3)\ u_2, u_1, u_2; \\ 4)\ u_1, 0, u_1; & 5)\ u_1, 0, u_2; & 6)\ u_2, 0, u_1. \end{array} \tag{6}$$

Let us note that options (6) coincide with the optimal controls pointed out for a time-optimal control problem in [3]. Each of the intervals, on which the constant control operates, can degenerate.

T h e o r e m 2 [16], [17]. Let $u_1 = 0$. Then any point on the boundary of the reachable set of the system (1) can be attained by using a piecewise constant control $u^*(\cdot)$ that takes the values $u_1 = 0$ and $u_2 = 1$ with no more than two switchings. There are two possible sequences of controls here:

$$1)\ 1, 0, 1; \quad 2)\ 0, 1, 0.$$

Let $u_1 > 0$. Then the possible number of the control switchings grows with increasing t_f; however, there is a finite number of switchings for any finite t_f. Investigation of this case is given in [18].

As a result, for all four considered variants of the value u_1, we have descriptions of the control functions $u(\cdot)$ leading the system onto the boundary of the reachable set. These descriptions are constructive and allow one to obtain the boundary of a reachable set in the three-dimensional space as a finite collection of pieces of smooth surfaces. Corresponding parts of the boundary are represented as two-parametrical families of points [14].

In general, the boundary of the three-dimensional reachable set is not smooth. However, its particular pieces formed by similar controls are smooth. In some cases (but not always), the conjugation of such pieces is also smooth.

Pictures of the obtained reachable sets for the instant $t_f = 3\pi$ and values $u_1 = -0.25,\, 0.0,\, 0.25$ are shown in Fig. 2. Enumeration of the colours of the boundary parts corresponds to the list of controls (6). The lowest point of all three sets is the same and corresponds to a motion with the constant control $u(t) \equiv u_2 = 1$.

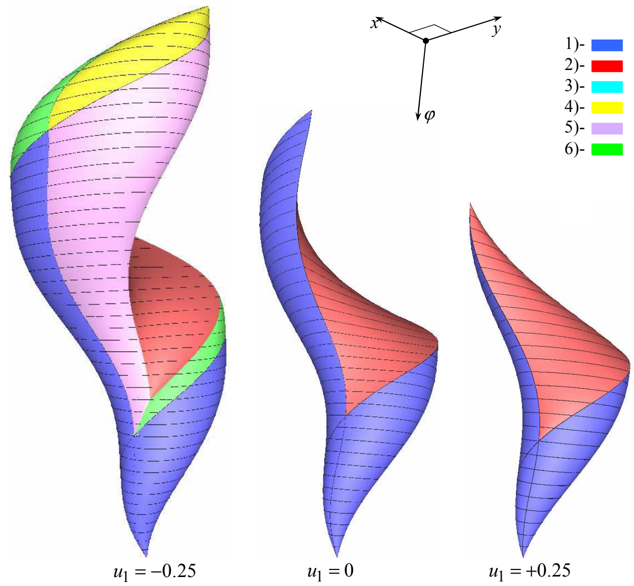

Fig. 2. Reachable sets for $t_f = 3\pi$ with different values of u_1 and the same value of $u_2 = 1$

III. ANGULAR SECTIONS OF REACHABLE SET

We describe sections of a reachable set $G(t_f)$ by plane orthogonal to the angular axis φ (angular sections or φ-sections) for different cases.

$*$ Let $0 < u_1 < u_2 = 1$ (the case d). In this case, a number of switchings of controls leading the system onto the boundary of a reachable set grows with increasing t_f. But with that, the φ-sections are strictly convex. The convexity property of the φ-sections was noticed at the first time during a numerical simulation [17] and theoretically proved later [18]. Also, we have established that the boundary of any φ-section consists of four types of arcs: SB, BB, SS, BS. Each of the types has an analytical description. Possible variants of their conjunction are shown schematically in Fig. 3.

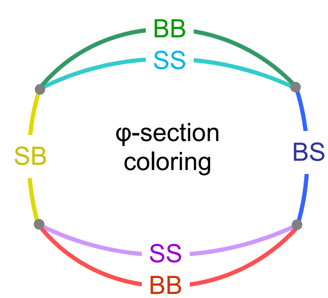

Fig. 3. Possible variants of the structure of an angular section when $u_1 \in (0, 1)$, $u_2 = 1$

An arc of the type SB is generated by means of the piecewise constant controls having the value u_2 in the first time interval and the value u_1 in the final interval. A control producing points from an arc of the type BB has the value u_1

in both first and final time intervals. In the same way, changing u_1 by u_2 and vice versa, we obtain arcs of the types BS and SS.

The number of switchings of a control leading to the same arc depends on the instant t_f and the chosen value of φ. For fixed t_f and φ, the number of switchings for the arcs of the types BS and SB is the same. For the arcs of the types BB and SS, it is either the same or differs by 1.

With chosen direction of the boundary (clockwise or counterclockwise), four variants of the arc sequence are possible:

SB, BB, BS, SS;   SB, BB, BS, BB;
SB, SS, BS, SS;   SB, SS, BS, BB.

Depending on t_f and φ, some arcs can degenerate. With that, the arcs of the types BS and SB degenerate simultaneously. In work [18], it is shown that there are 11 types of the φ-sections.

In Fig. 4, one can see some examples of the reachable sets $G(t_f)$ for $u_1 = 0.5$, $u_2 = 1$, and three instants $t_f = 6\pi$, 10π, 20π. The colors of parts of the boundary correspond to the ones in Fig. 3. The same color may occur several times because the number of switchings changes with changing φ.

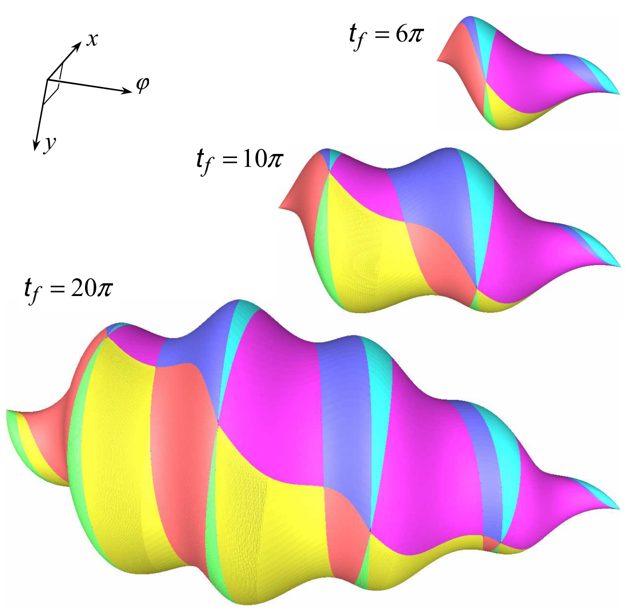

Fig. 4. Reachable sets for the case of one-sided turn when $u_1 = 0.5$, $u_2 = 1$

$**$ Let $u_1 = 0$ and $u_2 = 1$ (the case c). This case is the simplest one. Each φ-section is either a circle (when $\varphi \geq 2\pi$ regardless of t_f), or a circular segment (a circle cut by a chord) if $\varphi < 2\pi$ [16], [17]. Thus, in this case, the φ-sections are convex.

$***$ Let $u_1 = -1$ or $-1 < u_1 < 0$; $u_2 = 1$ (the cases a and b). In this situation, the boundary of a φ-section is generated by six control types mentioned in Section II of the paper. The φ-sections can be non-convex or even non-simply connected. In Fig. 5, examples of the φ-sections are shown for $u_1 = -1$, $t_f = 3.564$. The section is non-simply connected when $\varphi = 0.3$. But for $\varphi = 0$ and $\varphi = 0.15$, the sections are simply connected. The colors correspond to the ones in Fig. 2.

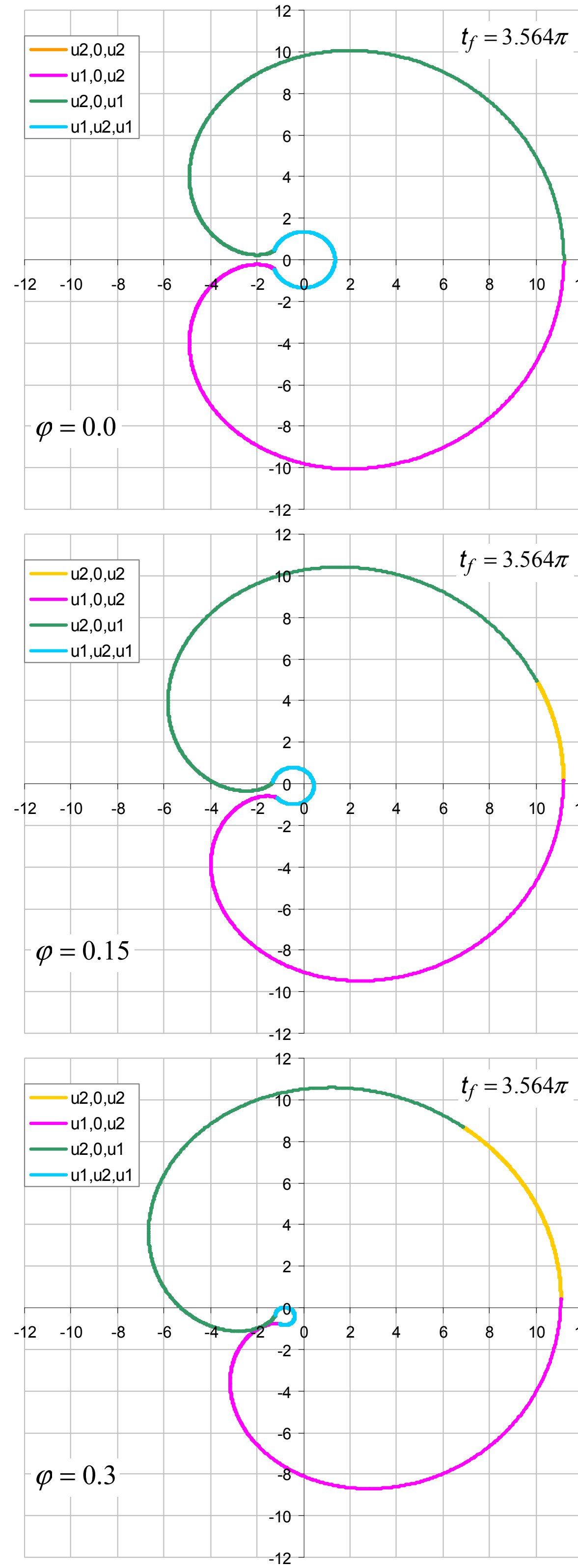

Fig. 5. Two-dimensional φ-sections of the reachable set for $u_1 = -1$, $u_2 = 1$, $t_f = 3.564$

Validity of PMP is a necessary condition for controls leading onto the boundary of a reachable set. Generally speaking, this condition is not sufficient for the Dubins car. It is proved that, in each of the cases a) and b), there is a piecewise constant open-loop control obeying PMP, but the corresponding trajectory is in the interior of the set $G(t_f)$ at the instant t_f, *i.e.*, in the interior of a φ-section. For the cases c) and d), it is verified that PMP is the sufficient condition for a control to lead the motion onto the boundary. It stands out that the sufficiency is connected with convexity of the φ-sections. For the case d), the φ-sections are strictly convex. It is established that in this case a piecewise constant open-loop control satisfying PMP determines a single motion leading to the corresponding point on the boundary of the set $G(t_f)$. The connection between PMP and the convexity property of φ-sections is shown in Fig. 6.

Convexity and, even more so, strict convexity of the φ-sections, can be used in analyzing the sufficiency of the PMP in various optimization problems, in which the motion of an object is described by system (1), (2) with constraints of the form c) or d) on control u. Such tasks include, for example, the time-optimal problem where the control goal is the shortest transition of the system onto a convex closed set in the geometric coordinates. Under this, at the instant of transition, the angle φ must take the specified value.

<table>
<tr><th></th><th>Pontryagin Maximum Principle</th><th>φ-sections of the reachable set</th><th>Controls leading onto the boundary</th></tr>
<tr><td>$u_1 = -1$
symmetric case</td><td rowspan="2">Only necessary condition</td><td rowspan="2">Non-convex</td><td rowspan="3">Non-uniqueness in the class of piecewise constant controls</td></tr>
<tr><td>$u_1 \in (-1, 0)$
asymmetric case</td></tr>
<tr><td>$u_1 = 0$
case of one - sided turn</td><td rowspan="2">Necessary and sufficient condition</td><td>Convex</td></tr>
<tr><td>$u_1 \in (0, 1)$
case of strictly one - sided turn</td><td>Strict convex</td><td>Uniqueness in the class of piecewise constant controls</td></tr>
</table>

Fig. 6. The Pontryagin maximum principle and the convexity property of the φ-sections of the reachable set

IV. APPLICATION TO OBSERVATION PROBLEM

In Sections II and III, the initial set was a point in the three-dimensional space. But solving the problem of observation, one needs to construct a reachable set for an arbitrary initial set. Analogously as it was made in [22], we use an approximation of a reachable set by a collection of plane convex approximations of the φ-sections. An angular section of a reachable set G corresponding to the value φ is denoted by G_φ. The collection of all non-empty φ-sections of the set $G(t)$ is denoted by $\{G_\varphi(t)\}$. The same denotations are used also for the angular sections of upper estimate information sets. For example, $\mathbf{I}_\varphi(t)$ is the angular section of the set $\mathbf{I}(t)$ corresponding to the value φ. The convex hull of a set A in the plane x, y is denoted by $\mathrm{conv}(A)$.

Let an upper estimate information set $\mathbf{I}(t_i)$ be given at an instant t_i. We assume that it has convex φ-sections. Below, a procedure for constructing the upper estimate $\mathbf{G}(t_{i+1}, \mathbf{I}(t_i))$ for the reachable set $G(t_{i+1}, \mathbf{I}(t_i))$ at an instant $t_{i+1} > t_i$ is described. The set $\mathbf{G}(t_{i+1}, \mathbf{I}(t_i))$ is called the forecast set.

Taking into account the ideas from Section III, we suppose that we have description of entire collection $\{G_\varphi(t_f)\}$ of all non-empty φ-sections $G_\varphi(t_f)$ of the reachable set for one-point initial set ($x_0 = 0$, $y_0 = 0$, $\varphi_0 = 0$). Let us denote by $G_\varphi(t_f, \varphi_0)$ the corresponding φ-section of the reachable set at the instant t_f for the initial state $x(t_0) = 0$, $y(t_0) = 0$, $\varphi(t_0) = \varphi_0$. The set $G_\varphi(t_f, \varphi_0)$ is the set $G_\varphi(t_f)$ turned by the angle φ_0 around the origin.

Let us take the instant t_i as the initial one, and the instant t_{i+1} as t_f. Consider the states of the system in the plane x, y at the instant t_i as the section $\mathbf{I}_{\varphi_*}(t_i)$ of the set $\mathbf{I}(t_i)$ for some φ_*. Then for any φ such that

$$G_\varphi(t_{i+1}, \varphi_*) \neq \varnothing,$$

the collection of all possible states at the instant t_{i+1} can be written as an algebraic sum (the Minkowski sum)

$$\mathbf{I}_{\varphi_*}(t_i) + G_\varphi(t_{i+1}, \varphi_*).$$

It is possible due to absence of the variables x, y in the right-hand side of system (1). Further, let us take the union over all φ_* such that $\mathbf{I}_{\varphi_*}(t_i) \neq \varnothing$, $G_\varphi(t_{i+1}, \varphi_*) \neq \varnothing$, and convexify it:

$$\mathbf{G}_\varphi(t_{i+1}, \mathbf{I}(t_i)) = \mathrm{conv} \bigcup_{\varphi_* \in \{\varphi_*(t_i)\}} \left[\mathbf{I}_{\varphi_*}(t_i) + G_\varphi(t_{i+1}, \varphi_*)\right].$$

The forecast set $\mathbf{G}_\varphi(t_{i+1}, \mathbf{I}(t_i))$ defined by its φ-sections gives an upper approximation for the collection of phase states of the system at the instant t_{i+1} consistent with dynamics (1) and the given set $\mathbf{I}(t_i)$. The angular sections of the upper estimate information set $\mathbf{I}(t_{i+1})$ are obtained by intersection of the φ-sections of the set $\mathbf{G}_\varphi(t_{i+1}, \mathbf{I}(t_i))$ with the uncertainty set $H(t_{i+1})$ of the next measurement in the plane x, y:

$$\mathbf{I}_\varphi(t_{i+1}) = \mathbf{G}_\varphi(t_{i+1}, \mathbf{I}(t_i)) \bigcap H(t_{i+1}).$$

At the instant t_0, we assume that the convex uncertainty set $H(t_0)$ is known. For reasonable causes, we fix a set $\{\varphi(t_0)\}$ of values for the coordinate φ. Let $\mathbf{I}(t_0) = H(t_0) \times \{\varphi(t_0)\}$.

Assuming convexity of the sets $H(t_{i+1})$, we obtain convex sections of the set $\mathbf{I}(t_{i+1})$.

In a practical implementation of this procedure, we use some fixed grid in the angular coordinate φ. The exact convex hull can be replaced with an upper approximation in the plane x, y by polygons having a given collection of the outer normal vectors to their edges.

Figure 7 shows the forecast set $\mathbf{G}(t_1, \mathbf{I}(t_0))$ in three views, computed for the case a) when $u_1 = -1$. The initial set $\mathbf{I}(t_0)$ was taken with only one φ-section in the form of a square $H(t_0)$, which sides are parallel to the axes x, y and are equal to $2/3$. The forecast instant t_1 equals π. Some collection of φ-sections $\mathbf{G}_\varphi(t_1, \mathbf{I}(t_0))$ of the set $\mathbf{G}(t_1, \mathbf{I}(t_0))$ is highlighted with black lines.

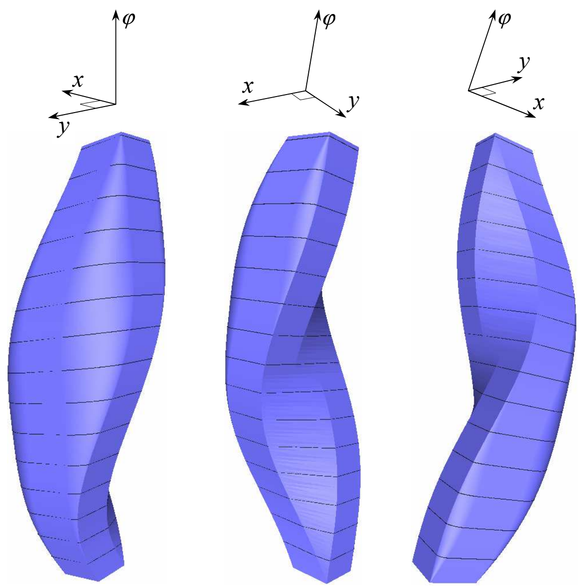

Fig. 7. An example of the forecast set: three views

V. CONCLUSIONS

In a mathematical model of controlled motion called the Dubins car, two coordinates have the meaning of the geometric position of a point object in a plane, and the third coordinate is interpreted as the direction angle φ of the linear velocity vector. Rate of change of the angle φ is a control variable. The paper has analysed φ-sections of the three-dimensional reachable set at the instant for the Dubins car. Cases when the φ-sections are convex are distinguished. Computer images of reachable sets at the instant are presented. In the future, it is planned to develop an algorithm for constructing information sets in the problem of observing the Dubins car with inexact measurements of the geometric position.

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