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Antony Merz and His Works

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Abstract The paper is devoted to the memory of Antony Willits Merz who solved the homicidal chauffeur problem and was very active in differential games in the 1970s and 1980s. A description of his main works, together with his biography, is presented.

Keywords Differential games · homicidal chauffeur game · pursuit-evasion

1 Introduction

The first report [Isaacs, 1951] by R. Isaacs on differential games was published by the RAND Corporation (Santa Monica, USA) in 1951. Already at this time Isaacs formulated the homicidal chauffeur problem. In this problem, a "car" moving in the plane pursues a "pedestrian" and strives to capture him in a given neighborhood of his own state as soon as possible. The dynamics of the car motion, along with the constant magnitude of the linear velocity and the prescribed range of possible turn rates, are described by a system of three differential equations. Two phase variables specify the geometric location of the object in the plane, and the third one is the velocity heading. Scalar control defines the current angular velocity of rotation of the velocity direction

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or, which is equivalent, the instantaneous turn radius. Values of the control parameter are chosen from a closed interval. The pedestrian (evader) is a point non-inertia object, which can instantaneously change the direction of his velocity. The magnitude of the evader's velocity is fixed.

Considering such a model problem, Isaacs most likely had in mind a guided torpedo and an evading small ship as a real prototype [Breitner, 2005]. But this was his genius – to formulate a model problem with a real prototype behind it and very interesting from a mathematical point of view.

In his book "Differential games" [Isaacs, 1965], Isaacs introduced basic notions of the theory: dynamics of objects, problem formulation in a class of feedback controls, and a "main equation" in partial derivatives that the Value function of the game must satisfy. As the basis of the study of differential games, Isaacs considered finding singular lines and surfaces in the game space on which optimal trajectories merge, have breakpoints, etc., so that they do not constitute a regular field of trajectories.

In general, Isaacs preached a retrograde solution to differential games, as when, by moving back from a terminal manifold and using terminal values of the payoff function, one fills out the whole game space with optimal trajectories. Since optimal trajectories carry the value of optimal result, in the end we obtain the Value function defined in the whole game space.

Isaacs clearly realized that many of his hypotheses required careful verification and development. In his book there are many questions that he formulated for this or that occasion. This is also related to the homicidal chauffeur problem. Isaacs presented the solution to this problem only for a certain range of parameters.

One of the first among US mathematicians who recognized the Isaacs method was J. V. Breakwell, who worked at Stanford University at the time. During these and subsequent years he was surrounded by remarkable students, such as Antony Merz, Pierre Bernhard, Joseph Lewin, and Geert Jan Olsder (see photographs in Fig. 1), each of whom later made a significant contribution to the development of differential game theory. Among them A. Merz was the oldest. P. Bernhard in his essay "Isaacs, Breakwell, and their sons" [Bernhard, 2015] wrote the following about him: "Tony had remained a close friend of him [Breakwell], going for long mountain hikes with him, and served as a link among the former students of John, who were all his friends."

It was Merz who began to investigate the homicidal chauffeur problem. Merz noted in his PhD thesis that the complexity of this differential game had been underestimated for some time. Quoting from the thesis, page 3: "... early in the study of this game, it was mistakenly believed that little remained to be learned from it. Attention was consequently directed to generalizing the dynamical model, the third order "game of two cars" being the most natural generalization. Certain difficulties in this third-order problem raised doubts about our understanding of the simpler homicidal chauffeur game. These doubts proved to be well founded, and led to the decision to find the complete solution to the present game for all values of parameters."

This article is written in memory of Merz. We give his biography and describe his thesis on the homicidal chauffeur game. We also briefly mention his main works on differential games and control theory. The bibliographic list includes all the works of A. Merz which were found on the Internet.

A. Merz' family kindly provided us with his autobiography, which we used.

Fig. 1: a: J. V. Breakwell at a Stanford graduation; b: A. W. Merz; c: P. Bernhard; d: J. Lewin; e: G. J. Olsder.

2 The biography

2.1 Basic biographical information

Antony Merz was born on November 25, 1932, in Zambia. His father, Russel Merz, was a mining engineer, and his mother, Virginia Willitz Merz, held an English degree from Cornell University and occasionally worked for small newspapers.

Antony attended British school in Zambia. When he was 6 years old, the family moved to the United States, settling in Texas. In 1945, before starting high school, the family went with his father to his new job in Peru, where Antony learned Spanish. After finishing high school at the age of 16, he came back to the United States, to Kansas city, where Antony attended Kansas University from 1949 to 1953, transferring to Massachusetts Institute of Technology (MIT) as a junior.

While at MIT, an astronautics professor encouraged him to apply for a Fulbright Fellowship in Paris, and he attended the Sorbonne for one year. His studies at MIT resulted in two degrees – the Bachelor and Master of Sciences, with the Master Thesis entitled "An approximation to the transient response for systems having slowly-varying parameters."

In 1957, Merz returned to Kansas, and his employer convinced him to study applied mathematics at Texas Christian University. In 1960, he received his master's degree in applied mathematics.

In 1960, Merz moved to New York, where he attended MIT and met his future wife, Peggy, in 1962. He realized that the PhD program in Aero-Astronautic Engineering had changed since his earlier studies. Living in Boston, Antony was preparing for his oral examination, which he did not pass. He contacted Stanford University and was admitted for a PhD program in the department of aeronautics and astronautics in 1967. His scientific adviser was professor John Valentine Breakwell.

The dissertation was completed in 1971.

After defending the dissertation, Merz worked until 1979 in small research companies near Stanford University. He still had a connection to Breakwell.

From 1979 until his retirement in 1996, Merz worked for the Lockheed Research Laboratory where he participated, particularly, in the project on Global Positioning System. In his autobiography, he wrote: "The most memorable job I concluded at Lockheed was a solo task typical in size, which can be described in qualitative terms. This job, worked on for over a year, produced a computer program which modeled the 18 satellites forming at that time the Global Positioning System. Its purpose was the tracking of moving targets on or near the earth, and updating the estimates of the three position and velocity figures which model the target trajectories."

Antony Merz died on October 19, 2017. He is survived by his wife, Peggy Merz, his daughters Rebecca Stephens and Alison Merz, and his brother Joy J. Merz of Tucson.

2.2 Hobbies, humanity, and sport

Since his student years at the University of Kansas, Antony had been fond of playing the tenor banjo. While living in Cambridge, he took lessons from the professional banjoist William Bradford Keith ("Bill Keith"). In the early 60s, in New York, Antony continued his passion for music, studying banjo with the famous musician Roger Sprung, about whom he wrote: "...he was plenty good enough for me, then and later, and I had a lot of fun getting over to his place in New York during the week, and hearing him talk about the two albums he had recorded to that time."

All who knew Merz noted his delicate sense of humor and love of puns. His daughter Alison says (from the letter to the authors of 04/03/2019): "Antony had a silly side too, and was an avid fan of British comedy troupe Monty Python. Like those comedians, he himself had a very dry wit and was known for his straight-faced delivery. He was also very clever with the English language and was a master of puns, wordplay, and double entendres. His family and friends enjoyed his witty banter and remember him for nearly always being in good spirits."

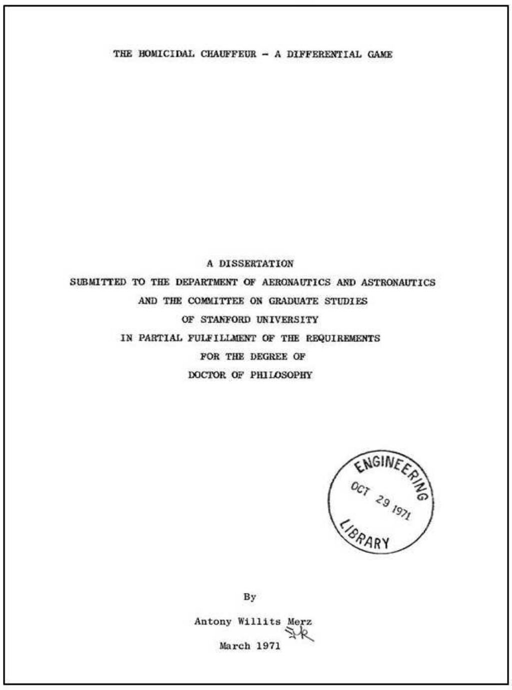
These qualities of Antony played a happy role in his meeting his future wife Peggy. In the summer of 1962, together with his friend Jordan Bonfante, he participated in an alternating weekend house share on Fire Island, a sand bar accessible by ferry from Long Island, New York. Once, Antony got confused about his calendar and mistakenly visited the beach two weekends in a row, resulting in a chance meeting with Peggy who was there on her scheduled weekend. Antonys banjo playing and their mutual sense of humor brought them closer, so they began to meet often and were married in 1964.

From his school years, Antony had been fond of playing softball. The official newspaper of the undergraduates of MIT called him "the top all-around player in the league" (No. 26, May 20, 1955). A year earlier in the same newspaper: "Merz has been touted as the fastest hurler in the intramurals" (No. 19, April 30,1954). In his mid-40s, he joined a softball league in Palo Alto and pitched for a team sponsored by a neighborhood restaurant near Stanford University.

3 Merz' thesis "The Homicidal Chauffeur – a Differential Game"

The Merz' thesis can be found in the library of Stanford University. In Figure 2 the title page of the thesis is shown.

We will begin the story about Merz' dissertation by quoting from his acknowledgements, page iii: "I express my most profound gratitude to Professor Breakwell for his direction, patience, unflagging enthusiasm during the course of this research: Many of the results of the work would certainly not have come into light without his extraordinary mathematical perception. I also thank Professor Bryson and Professor Franklin for agreeing to serve as readers and for the interest and understanding they have shown in this work. Fellow doctorate candidates Pierre Bernhard and John Dixon were frequently helpful during the study, and I thank them both. Further, I wish to record my gratitude to my wife, Peggy, who typed several versions of the manuscript, and who provided encouragement and solace when they were needed. Mrs. Diana Shull typed the final version of this document with perseverance and skill, for which I am grateful."



THE HOMICIDAL CHAUFFEUR - A DIFFERENTIAL GAME

A DISSERTATION
SUBMITTED TO THE DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS
AND THE COMMITTEE ON GRADUATE STUDIES
OF STANFORD UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

By

Antony Willits Merz

March 1971

Fig. 2: Title page of the PhD thesis by A. W. Merz.

3.1 Dynamics of the game

The motions of the car and the pedestrian are described in the original coordinates by the following system of equations, which is explained in Fig. 3,

$$\begin{aligned}
\dot{X}_p &= V_p \sin\theta_p & \dot{X}_e &= V_e \sin\theta_e \\
\dot{Y}_p &= V_p \cos\theta_p & \dot{Y}_e &= V_e \cos\theta_e \\
\dot{\theta}_p &= \frac{V_p}{R}\varphi & 0 &\le \theta_e < 2\pi \\
|\varphi| &\le 1
\end{aligned}$$

Here, (X_p, Y_p) is the position of the pursuer (car) in the plane, V_p is his velocity, θ_p is the heading of the pursuer, R is the minimum turning radius, and ϕ is the control of the pursuer; (X_e, Y_e), V_e, θ_e are the position, the velocity, and the control of the evader (pedestrian).

The pursuer strives, as soon as possible, to capture the non-inertial evader by changing the rate of turn. The objective of the evader is the opposite. The capture occurs if the distance between the players becomes less than a given number ℓ. By normalizing the time and the distance, one can achieve

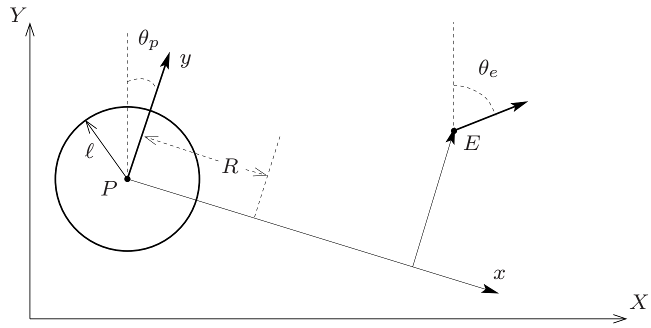


Fig. 3: Coordinate system.

that the minimum turning radius R of the pursuer and his velocity become equal to 1. Therefore, we can use the dimensionless capture radius $\beta = \ell/R$ and the evader's dimensionless velocity $\gamma = V_e/V_p$ as the parameters of the game. Rewriting the dynamics in the relative coordinate system, with (x, y) being the relative position of the evader with respect to the pursuer, yields the two-dimensional in the state variables differential game

$$\begin{aligned} \dot{x} &= -\varphi\, y + \gamma \sin\psi \\ \dot{y} &= \quad \varphi\, x + \gamma \cos\psi - 1 \\ |\varphi| &\le 1, \quad \psi \in [0, 2\pi). \end{aligned} \tag{1}$$

Here, ψ is the clockwise angle from the positive y-axis to the direction of E's velocity vector. So, φ is the control of the pursuer, and ψ is the control of the evader.

3.2 Isaacs' solution

In the book by R. Isaacs, the solution to the homicidal chauffeur game was given only for a narrow range of parameters. The left panel of Fig. 4 shows a drawing from his book. The horizontal axis measures x in (1), and the vertical axis measures y. Isaacs used the symbol $\mathscr{C}$ of the usable part to denote the points on the boundary of the terminal circle through which the player P can guarantee leading the trajectory to the inside of the terminal set. He found that if parameters β, γ satisfy the inequality $\beta^2 + \gamma^2 \le 1$, then two barrier lines emanate tangentially to the boundary of the terminal set from the endpoints of the usable part in reverse time.

The solution is symmetric with respect to the vertical axis. The upper part of the vertical axis is a singular line. Forward time optimal trajectories meet this line at a certain angle and then go along it towards the target set. According to the terminology of R. Isaacs, this line is called universal. The part of the vertical axis adjoining the target set from below is also a universal singular line. Optimal trajectories go down along it. The rest of the vertical axis below this universal part is dispersal: two optimal paths emanate from every point of it. On the barrier line $\mathscr{B}$, the value function is discontinuous. The equivocal singular line emanates tangentially from the terminal point B of the barrier (right panel of Fig. 4). It separates two regions. Optimal trajectories that approach the equivocal curve split into two paths: the first one goes along the curve, and the second one leaves it and comes to the regular region on the right (optimal trajectories in this region are shown in the left panel of Fig. 4).

The equivocal curve is described through a differential equation which can not be integrated explicitly. Therefore, an explicit description of the Value function in the region between the equivocal and barrier lines is troublesome. The most difficult, for the investigation, is the part (denoted by R. Isaacs with a question mark) adjoining to the boundary of the terminal circle and the downside of the barrier (Fig. 4, right panel). He could not obtain a solution for this part.

Isaacs also analyzed the case where the right and left barriers intersect on the vertical axis.

Let us quote the words of Merz from his dissertation, page 3: "The pioneering work by Isaacs describes a treatment of the problem which is complete only for certain rather narrow ranges of the two parameters... Specifically, a barrier and an equivocal line were found to enclose 'turn-away' zones in the relative space, and the y-axis was found to be either a universal line or a pursuer's dispersal line."

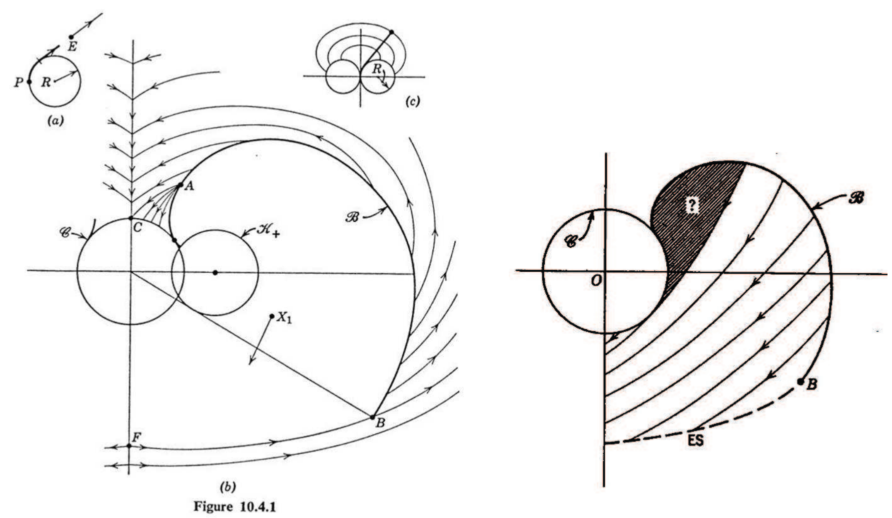


Fig. 4: Figures from Isaacs' book. Left panel: Solution in primary region. Right panel: Turn-away zone.

3.3 Partition of parameter space into subregions with different types of solutions

Merz obtained a complete solution to the problem for all possible values of the two parameters β, γ. The curve c_1 shown in Fig. 5 (in part in the left panel and in whole in the right panel) corresponds to the equation $\beta^2 + \gamma^2 = 1$. The inequality $\beta^2 + \gamma^2 \le 1$ holds below this curve. The curve c_2 is described by the relation

$$\beta = \gamma \sin^{-1} \gamma + \sqrt{1 - \gamma^2} - 1.$$

For the parameters from region II above this curve, the right and, symmetrical to this, the left barrier do not intersect. Below this line they do intersect. Accordingly, in the first case, the capture set for the player P is the whole plane, whereas in the second case it is a part of the plane bounded by the pieces of the barrier lines up to the point of their intersection. The curve c_3 satisfies the relation

$$\cos(\frac{\beta}{2\gamma}) + \gamma = 0.$$

For the parameters in region II to the right of the curve c_3, there is a point A (Fig. 4, left panel) on the barrier which is a dispersal point for optimal trajectories above the barrier. It is the player E's dispersal point, from which a bundle of optimal motions emanates for $\varphi = 1$. This implies that the barrier is continued by an equivocal singular line (Fig. 4, right panel). The curve c_4 is calculated numerically by using relations that guarantee that below the barrier line the optimal control of the player P is $\varphi = -1$. The case corresponding to region I and subregion IIc, was called the classical one by Merz. The solution for it was described and discussed in the second chapter of his thesis.

In parameter region I, the solution coincides with that described in Isaacs' book. For parameters from subregion IIc belonging to the primary part (outside the turn-away zone), the solution also coincides with that given by Isaacs. In the interior of each of the two turn-away zones, the Value function is smooth and the corresponding field of optimal motions is regular. The optimal control of the player P in the right (left) turn-away zone is $\varphi = -1$ ($\varphi = 1$). In fact, Merz used this condition to distinguish the parameter subregion IIc. Of particular interest are safe-contact optimal motions going along the right (left) part of the boundary of the terminal circle. For the right part, such "limiting" motions are generated by optimal trajectories coming from the shaded region in Fig. 4, right panel.

In the third chapter, the solution to the classical case was extended for the subregions IIa, IIb, IId, and IIe, to cover the range of parameters satisfying the inequality $\beta^2 + \gamma^2 \le 1$. In the fourth chapter, a solution was given for the values of parameters satisfying the inequality $\beta^2 + \gamma^2 > 1$.

Merz divided the parameter space into 20 subregions (Fig. 5, right panel). The thesis contains the description (explicit or implicit) of all curves separating the subregions. For every subregion, the structure of the solution and all possible singular lines were established (Fig. 6).

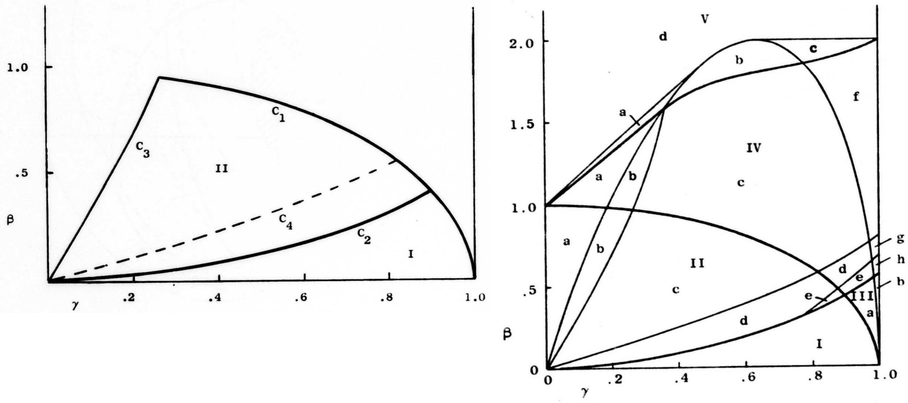


Fig. 5: Figures from Merz' thesis. The horizontal axis measures γ, the vertical axis measures β. Left panel: Separation of parameter area in subregions for classical case. Right panel: Separation of parameter space in subregions for general case.

DISTRIBUTION OF EXCEPTIONAL LINES

Region	Sub-region	Exceptional Lines Present in Each Subregion											
		$\mathscr{B}$	UL	PDL_y	PDL	EDL_a	EDL_c	$1D^+$	$1D^-$	EL	SL	SE	FL
I		x	x										
II	a	x	x	x					x		x		
	b	x	x	x					x	x	x		
	c	x	x	x					x	x			
	d	x	x	x					x	x		x	
	e	x	x	x					x	x		x	x
III	a	x	x			x		x					
	b	x	x				x	x					
IV	a		x	x	x			x	x		x		
	b		x	x	x			x	x	x	x		
	c		x	x	x	x		x	x	x			
	d		x	x	x	x		x	x	x		x	
	e		x	x	x	x		x	x	x		x	x
	f		x	x	x		x	x	x	x			
	g		x	x	x		x	x	x	x		x	
	h		x	x	x		x	x	x	x		x	x
V	a		x	x				x					
	b		x	x		x		x					
	c		x	x			x	x					
	d		x	x									

Fig. 6: Table from Merz' thesis showing the type of solution for each subregion of parameter partition. Notation: $\mathscr{B}$ = barrier, UL = universal line, PDL_y = pursuer's dispersal line on the y-axis, PDL = pursuer's dispersal line outside the y-axis, EDL_a = evader's dispersal line emanating from a point A, EDL_c = evader's dispersal line emanating from a point C, $1\mathrm{D}^+$ and $1\mathrm{D}^-$ = two variants of safe contact motions, EL = equivocal line, SE = switch envelope line, SL = switch line, FL = focal line.

3.4 Singular lines

The detection of singular lines and singular surfaces was thought by Isaacs to be the basis for the study of differential games. For problems in the plane, those lines are singular, on which the regularity of the field of optimal trajectories is lost. In simplest cases this results in kinks in the field, whereas in more complicated situations it implies merging or divergence of optimal trajectories and, associated with that, the loss of the uniqueness of optimal motions. Since the notion of optimal motion is connected with the notion of the Value function, the continuity and differentiability of the Value function are analyzed on singular lines. It is supposed that the Value function is locally, continuously differentiable on either side of a singular line.

In the Merz' dissertation, the term "exceptional lines" is used instead of "singular lines". Let us quote from the thesis, page 3: "Loci ... which separate two qualitatively distinct families of relative trajectories and which may or may not be trajectories themselves, will be termed 'exceptional lines.' As will be made clear by later results, Isaacs' terminology 'singular line' has an undesirable connotation to most control theoreticians, and is not sufficiently broad in meaning. The present choice is made to accommodate as large class of such arcs as possible..."

However, the term "exceptional lines" never caught on, while the term "singular lines" is still used in differential games.

Let us briefly describe, at a heuristic level, the types of singular lines in the plane which occur in the homicidal chauffeur problem.

In Figure 7a and b, barriers and universal lines are schematically shown. Optimal trajectories cannot penetrate the barriers and should go around their ends. The Value function is discontinuous across barrier lines. Optimal trajectories meet the universal line at some angle and then go along it. The Value function is smooth across the universal line.

At first sight, the focal line (Fig. 7c) is similar to the universal line (Fig. 7b). However, optimal trajectories meet this line tangentially and then go along it. The Value function is nonsmooth across the focal line.

The switch line (Fig. 7d) is characterized by switching the optimal control of the pursuer. One optimal motion comes to this singular line at some non-zero angle, and another leaves it also at a non-zero angle. Such a type of singular line is one of the simplest types. The Value function is smooth across this line.

Optimal trajectories meet the equivocal line at some angle (Fig. 8a). After arrival, they split into two equivalent optimal trajectories: one goes along the equivocal line, whereas another optimal trajectory leaves it at some angle. One says that the equivocal line is "controlled" by the player E, if it is his actions on the singular line that determine the subsequent optimal motion. In the case when the player E chooses a control he used before coming to the equivocal line, the optimal motion goes along the line with some intermediate control of the player P. If, however, the player E switches to the control which is prescribed for him on the other side of singular line, the motion is "dropped" to this side. Such a type of equivocal line is related to the local convexity of the Value function in the neighborhood of the point considered. In the case of local concavity of the Value function, the equivocal line is "controlled" by the player P. Isaacs showed that equivocal singular lines can exist in differ-

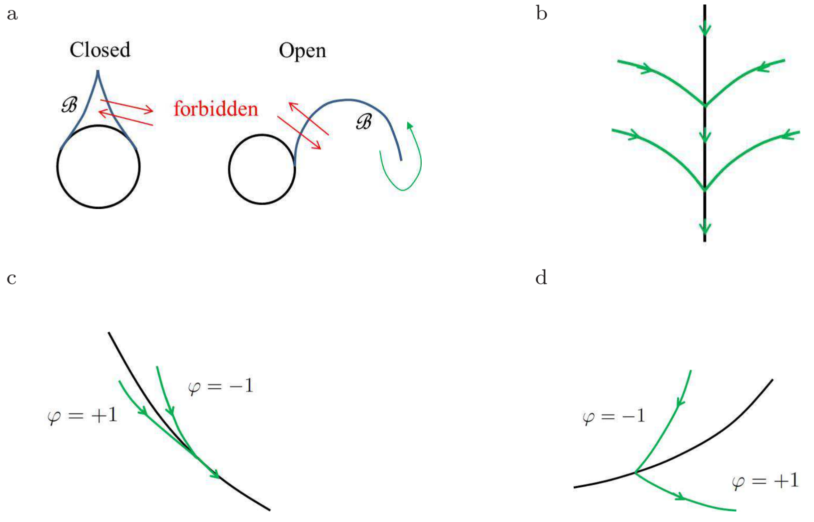


Fig. 7: a: Barrier; b: Universal line; c: Focal line; d: Switch line.

ential games only; they are absent in control problems. The Value function is nonsmooth across the equivocal line.

The switch envelope (Fig. 8b) is similar to the equivocal line, but optimal trajectories come tangentially to it and then go along it or leave, depending on the control choice of the pursuer. The Value function is nonsmooth across the switch envelope. It may well be that the switch envelope is an equivocal line controlled by the player P. The tangential approach of the singular line is associated with the "circle" constraint on the control of player E , having a smooth boundary.

Optimal trajectories leave the dispersal line (Fig. 8c and d) of the pursuer to the left or the right at some angle, depending on the pursuer's control choice. In all cases, except for region III, the lower part of the negative y-axis is the pursuer's dispersal line. The Value function is locally concave in this case. For the dispersal line of the evader, optimal trajectories depart to the left or to the right (Fig. 8e), depending on the evader's control choice. Here, the Value function is locally convex. The Value function is continuous but nonsmooth across the dispersal lines.

The safe contact trajectories (Fig. 9a and b) are parts of the boundary of the capture set, which can be tracked by optimal motions. The existence of such motions is one of the particular features of time-optimal differential games.

We have described the types of singular lines presenting in the homicidal chauffeur game. There are also some single points from which a bundle of optimal motions emanate. These are the point A in Fig. 4, left panel, as well as points A depicted in the thesis in Figs. 4.6 and B-1.

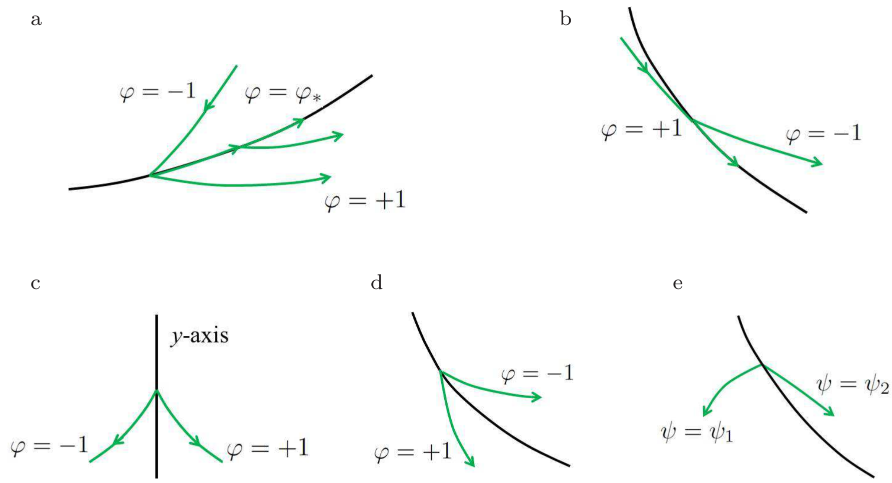


Fig. 8: a: Equivocal line; b: Switch envelope; c, d: Dispersal line of the pursuer; e: Dispersal line of the evader.

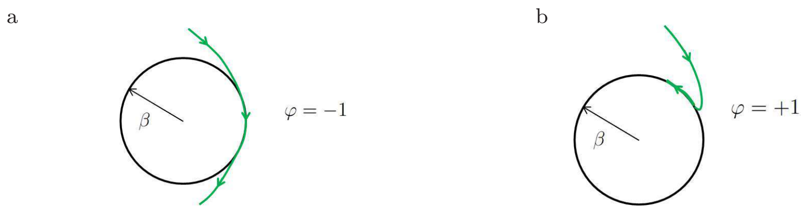


Fig. 9: a, b: Safe contact.

The discovery of equivocal singular lines is one of the most important achievements of Isaacs. Focal lines were discovered by Breakwell and Merz.

3.5 On some very sharp results

It is worth mentioning that the thesis is written very briefly. Some authors have tried to verify the solution in more detail or by using other methods (see, e.g., [Bardi et al., 1999, Botkin et al., 2013, Meyer et al., 2005, Mikhalev and Ushakov, 2007, Mitchell, 2002, Raivio and Ehtamo, 2000]). Let us especially note the paper [Pachter and Coates, 2018] where detailed analytic calculations related to the solution of the homicidal chauffeur problem for the parameters β, γ corresponding to the region IIc of the classical case, are performed. In these works [Coates et al., 2017a,b], it is shown how the scheme of the solution for the homicidal chauffeur problem can be applied in investigation of time-optimal control problems for the Dubins car with different terminal sets.

The authors of this paper have also applied their numerical algorithm [Patsko and Turova, 2001] to computing solutions to the homicidal chauffeur problem. In Figure 10, results of such computations for the classical game are presented: level sets of the Value function (left panel) and a 3-dimensional graph of the Value function (right panel). It is clearly seen that the Value function is discontinuous along the barrier lines.

The basis of our numerical algorithm, which is directed to time-optimal differential games in the plane, is the construction of level sets of the Value function. Moving backward in time with a sufficiently small step size from the terminal set, we build up the next level set. At the same time, the boundary of the level set constructed on the previous step is analyzed and the parts with barrier properties are distinguished. These parts are taken into account when designing the next level set.

The algorithm cannot replace an analytic investigation of the problem. We used it to obtain a visual geometric demonstration of level sets and a graph of the Value function, as well as, in some cases, to verify the solution obtained by A. Merz.

We tried to compute solutions for some complicated cases, particularly for parameter values from the subregion IIe where a focal line arises. In this case, there is a piece in the turn-away zone on the right of the axis y, where the optimal control of the player P is $\varphi = 1$ (recall that the solution is symmetric with respect to exis y). This part adjoins the barrier line and is separated from the part where $\varphi = -1$ by a complex line composed (see Fig. 11, left panel) of the dispersal line PDL (erroneously not mentioned in the table in Fig. 6), two parts of the switch envelope line SE, and the focal line FL located between them.

However, our efforts were not successful. Even very fine discretization in the numerical algorithm (replacement of the circled terminal set by a polygon,

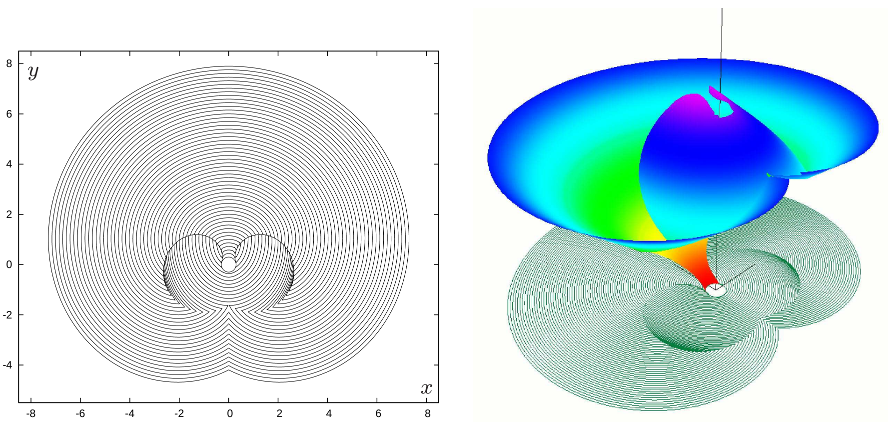


Fig. 10: Classical problem for parameter values $\beta = 0.3$, $\gamma = 0.3$. Left panel: Level sets of the Value function. Right panel: Graph of the Value function.

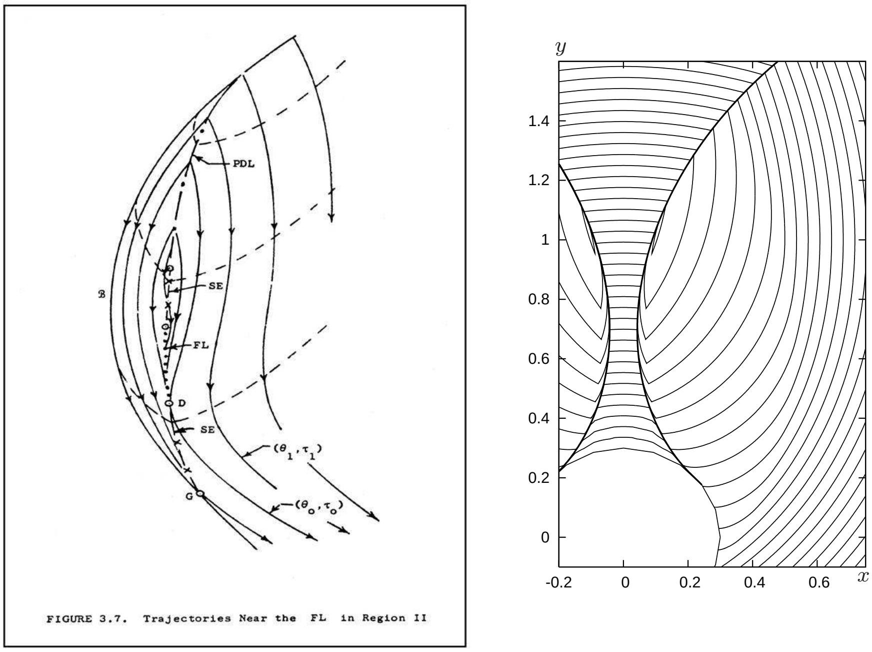


Fig. 11: Left panel: Structure of optimal paths in the turn-away zone for subregion IIe (Figure 3.7 from the thesis). The Value function is not differentiable on the line composed of the pursuer's dispersal line PDL, the switch envelope SE, and the focal line FL. Right panel: Level sets of the Value function for parameters from subregion IId computed using the algorithm [Patsko and Turova, 2001] for $\beta = 0.3$, $\gamma = 0.7$.

finite step size of the backward construction of the Value function's isochrones, etc.) for parameters β, γ from the narrow subregion IIe, did not help reveal corner points which must be present on level sets in the turn-away zone by virtue of the fact that the Value function is not differentiable across the separation line mentioned above.

An example of a numerical solution with nonsmooth isochrones in the turn-away zone is shown in the right panel of Fig. 11, which, however, corresponds to the values of parameters from the subregion IId. In this case, there is also a part of the turn-away zone where the optimal control of the player P is $\varphi = 1$. But the focal line disappears.

In Figure 12, a very interesting example for the values of parameters from the subregion IVc is demonstrated. To the left, the picture from the thesis is presented, where the barrier is replaced by the dispersal lines of the pursuer and evader. To the right, a result of the computation of the level sets of the Value function obtained by our algorithm is given. It seems that the thick line bounding the turn-away zone is a barrier line, but this is not the case. In fact,

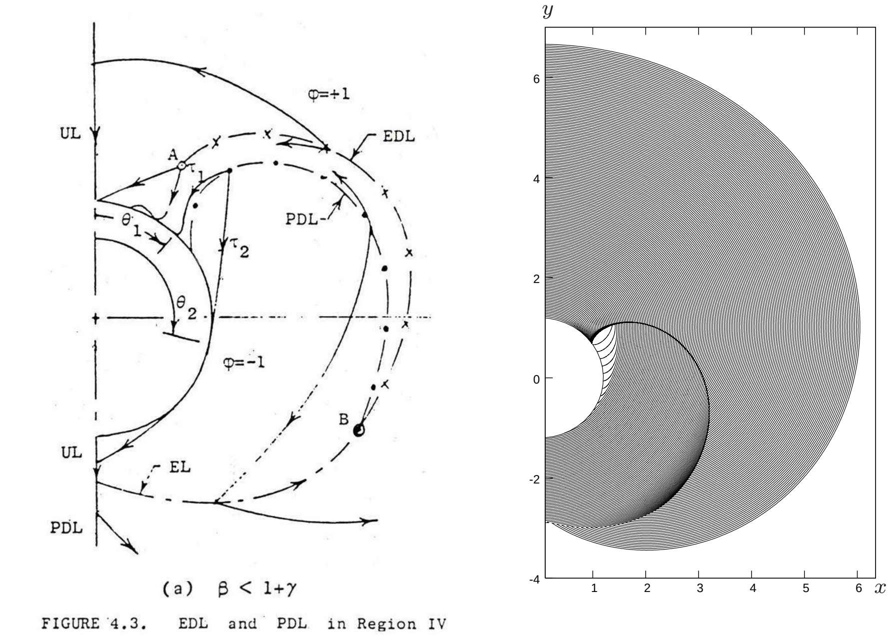


Fig. 12: Left panel: Structure of optimal trajectories in subregion IVc; Right panel: Level sets of the Value function for $\beta = 1.2$, $\gamma = 0.7$.

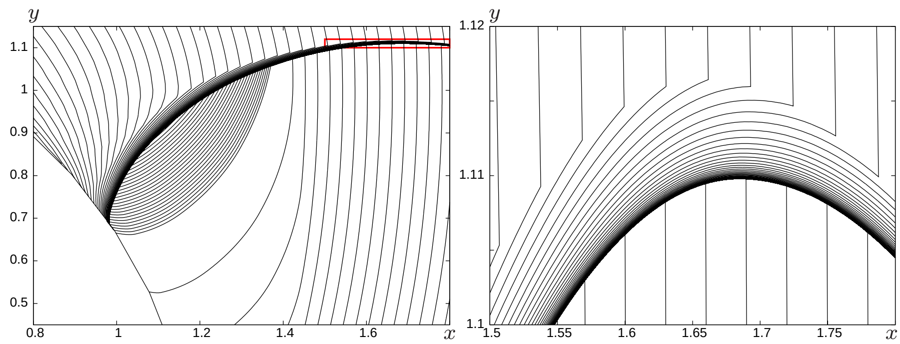


Fig. 13: Left panel: Enlarged fragment of the level sets depicted in the right panel of Fig. 12. Right panel: Zoomed fragment of level sets inside the small rectangle shown in red in the left panel.

the structure of level sets in the accumulation region corresponds exactly to the solution structure shown in the left panel.

Fragments of the accumulation region from Fig. 12, right panel, are shown in Fig. 13. The curve composed of the level sets' corner points above the accumulation region is the dispersal line of the evader. The curve composed of the corner points below the accumulation region is the dispersal line of the pursuer. The Value function is continuous in the accumulation region.

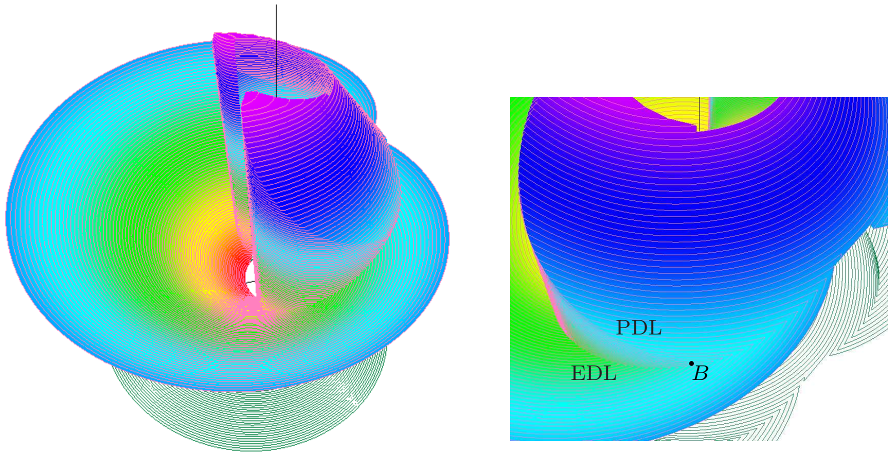


Fig. 14: Left panel: Graph of the Value function for $\beta = 1.2$, $\gamma = 0.7$. Right panel: Fragment of the graph of the Value function shown to the left. The salient curve, corresponding to the line PDL from Fig. 11, starts at the point B.

The graph of the Value function for the example from Figs. 12 and 13 is presented in Fig. 14 to the left. It confirms the continuity of the Value function in the accumulation region. In the right panel of Fig. 14, a fragment of the graph is shown where the dispersal lines of the pursuer and evader, as well as the equivocal line emanating from the point B, are clearly seen.

We finish the story about Merz' dissertation with a quote from his memoirs: "I was glad that I had not embarrassed Prof. John Breakwell, for whom I had written my thesis, titled 'The Homicidal Chauffeur: a Differential Game.'"

4 Works completed after defense of the thesis

4.1 The game of two cars

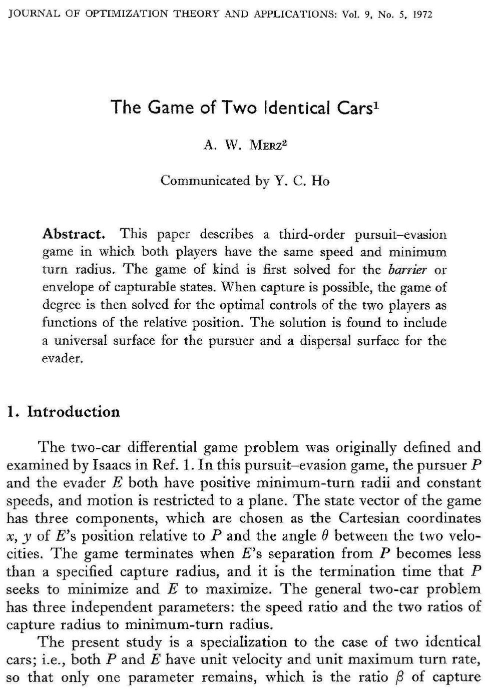
It is not natural for applications that one of the two objects in a pursuit problem is non-inertial. If, in the homicidal chauffeur game, the non-inertial evader is replaced by an object with the dynamics of the car, then we obtain the game of two cars ("the most natural generalization", see quote from the thesis at the end of Subsection 3.1), which is described in Isaacs' book. The dynamics of this game in relative coordinates reads:

$$\begin{aligned}
&\dot{x} = -\omega_1 u_1 y + v_2 \sin\theta,\\
&\dot{y} = \omega_1 u_1 x - v_1 + v_2 \cos\theta,\\
&\dot{\theta} = -\omega_1 u_1 + \omega_2 u_2,\\
&|u_1| \le 1,\ \ |u_2| \le 1.
\end{aligned}$$

Here, ω_1, ω_2 are the maximum magnitudes of the angular velocities of the first and second objects; v_1, v_2 are the constant magnitudes of linear velocities; θ is the relative heading angle of the second object; u_1, u_2 are controls restricted by the geometric constraints. The first player minimizes the time needed for the phase point to reach a given terminal set. The second player has the opposite interest. Usually, the terminal set is taken as a circle of radius β in the plane x, y with the center at the origin. Very often Merz used the symbol A for the first object and the symbol B for the second one. The dimension of the state vector is equal to three. This is the main reason why the game of two cars has not been solved completely up to now.

A. Merz investigated the game of two cars in the case of two identical cars. The solution in this case depends on a single parameter, which, after normalization, can be chosen as the radius β of the terminal circle. Here, the values of $\omega_1 = \omega_2$, $v_1 = v_2$ are set equal to 1. In the left panel of Fig. 15, the first page of the paper [5] published in JOTA in 1972 is shown.

In the case of two identical cars, the three-dimensional set, from where the first player guarantees that the phase point will reach the terminal set within a finite time, is bounded. This set is called "capture set." In the right panel of



JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 9, No. 5, 1972

The Game of Two Identical Cars[1]

A. W. Merz[2]

Communicated by Y. C. Ho

Abstract. This paper describes a third-order pursuit–evasion game in which both players have the same speed and minimum turn radius. The game of kind is first solved for the *barrier* or envelope of capturable states. When capture is possible, the game of degree is then solved for the optimal controls of the two players as functions of the relative position. The solution is found to include a universal surface for the pursuer and a dispersal surface for the evader.

1. Introduction

The two-car differential game problem was originally defined and examined by Isaacs in Ref. 1. In this pursuit–evasion game, the pursuer P and the evader E both have positive minimum-turn radii and constant speeds, and motion is restricted to a plane. The state vector of the game has three components, which are chosen as the Cartesian coordinates x, y of E's position relative to P and the angle θ between the two velocities. The game terminates when E's separation from P becomes less than a specified capture radius, and it is the termination time that P seeks to minimize and E to maximize. The general two-car problem has three independent parameters: the speed ratio and the two ratios of capture radius to minimum-turn radius.

The present study is a specialization to the case of two identical cars; i.e., both P and E have unit velocity and unit maximum turn rate, so that only one parameter remains, which is the ratio β of capture

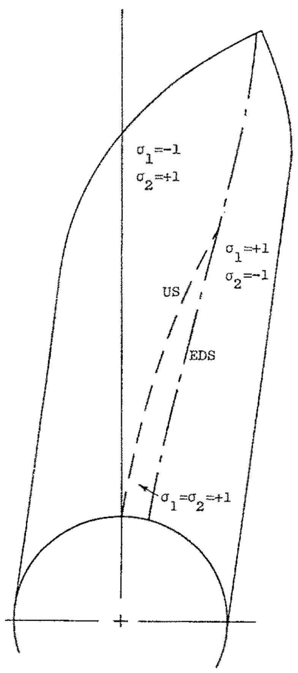


Fig. 15: Left panel: First page of the paper by Merz about the game of two cars. Right panel: Figure 8a from the paper about the game of two cars showing optimal strategies in the capture set for $\beta = 0.5$ and $\theta = -165$ degrees.

Fig. 15, the section of the capture set for the angle coordinate $\theta = -165$ degrees is shown. In Merz' paper such sections are presented for some collection of angles θ. Computations are performed for the terminal circle of radius $\beta = 0.5$. In the time-optimal game, Merz described two singular surfaces. These are the dispersal surface for the second player and the universal surface for the first player. Unfortunately, in the paper, the type of line (respectively, two-dimensional surface) in the upper part of the section is not described exactly.

We think that the paper by Merz can presently be used for testing three-dimensional computer algorithms aimed at solving time-optimal differential games. In particular, it is very interesting to compute the three-dimensional level sets of the Value function for the game of two identical cars.

4.2 One-on-one combat problem: Role determination

Among aviation tasks, aerial combat problems are very important. The basic one is the one-on-one combat problem. In this problem, we want to know from where the aircraft A wins and, vice versa, from where the aircraft B wins. If this problem is investigated in the horizontal plane, then the dynamics of two cars is appropriate. From the mathematical point of view, two terminal sets are considered: Target A and Target B (Fig. 16). In the relative coordinates, we try to find a capture region for the aircraft A (from where it wins) and a capture region for the aircraft B.

The paper by Olsder and Breakwell [1974] was the first of its kind in differential game literature. The Target A in their paper corresponds to situations for which the geometric position of the aircraft B is in the direction of aircraft A's velocity vector, and the distance between A and B is not greater than the given value l_A (Fig. 16). Here, it does not matter what value takes the angle θ.

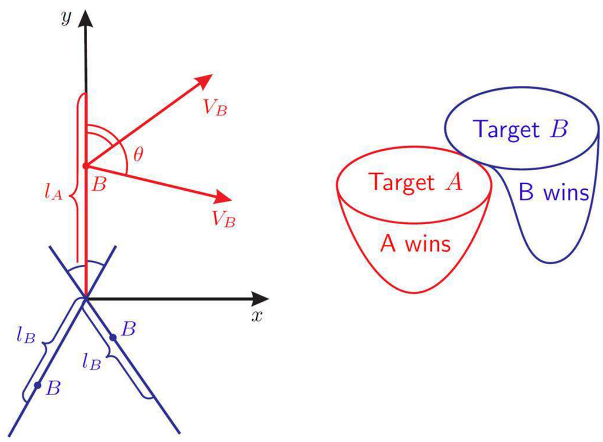


Fig. 16: One-on-one aerial combat problem.

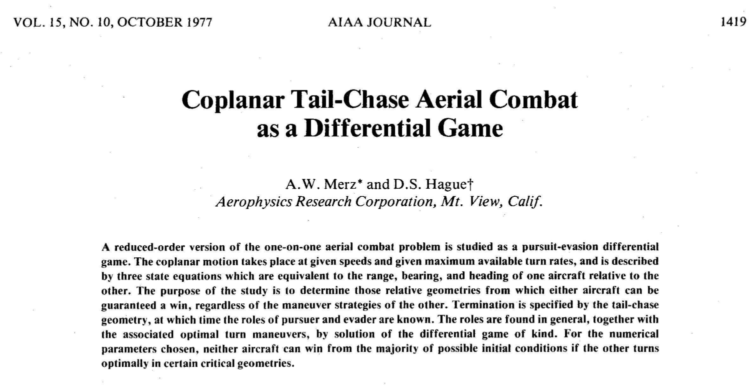
The same is true for the Target B; the distance between A and B is not greater than the given value l_B. The authors stress that the definitions of the target sets depend on aircraft weapon systems. Olsder and Breakwell point out that “the first attempt of the division of three-dimensional state space on two parts was made in an unpublished paper by A.W. Merz” for the case of identical combatants.

In Figure 17, the introductory part of the paper [25] on aerial combat by Merz and Hague is shown. They consider a case where the linear velocity of the aircraft A is greater than that of the aircraft B, but the aircraft A is less maneuverable.

Figure 18 explains the terminal conditions characterizing the Targets A and B. Instead of constraints l_A and l_B, we have constraints H_A and H_B on heading angles. In this way, the authors describe the tail-chase situation.

The paper was based on NASA reports [16, 21], in which the problem was described in more detail.

In Figure 19, two θ-sections of the three-dimensional capture regions for the aircraft A and B are shown. In the paper, the θ-sections are computed for some collection of values of the angle θ. A question arises. Would it be possible, using the numerical methods of differential game theory, to repeat or maybe correct the results obtained by Merz? A numerical algorithm must be oriented towards three-dimensional dynamics and must take into account state constraints. In the pictures a) and b) of Fig. 19, lines which separate the capture regions, as well as the capture regions and the domain “draw” where neither aircraft can win, are sections of barrier surfaces. The notation $B_{RL}A_L$, for example, indicates that the corresponding curve in the picture is composed using direct time motions with a right turn at some time interval, followed by a left turn for the aircraft B, and a left turn for the aircraft A.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Coplanar Tail-Chase Aerial Combat as a Differential Game

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A reduced-order version of the one-on-one aerial combat problem is studied as a pursuit-evasion differential game. The coplanar motion takes place at given speeds and given maximum available turn rates, and is described by three state equations which are equivalent to the range, bearing, and heading of one aircraft relative to the other. The purpose of the study is to determine those relative geometries from which either aircraft can be guaranteed a win, regardless of the maneuver strategies of the other. Termination is specified by the tail-chase geometry, at which time the roles of pursuer and evader are known. The roles are found in general, together with the associated optimal turn maneuvers, by solution of the differential game of kind. For the numerical parameters chosen, neither aircraft can win from the majority of possible initial conditions if the other turns optimally in certain critical geometries.

Fig. 17: Introductory part of the paper [25] on the aerial combat problem.

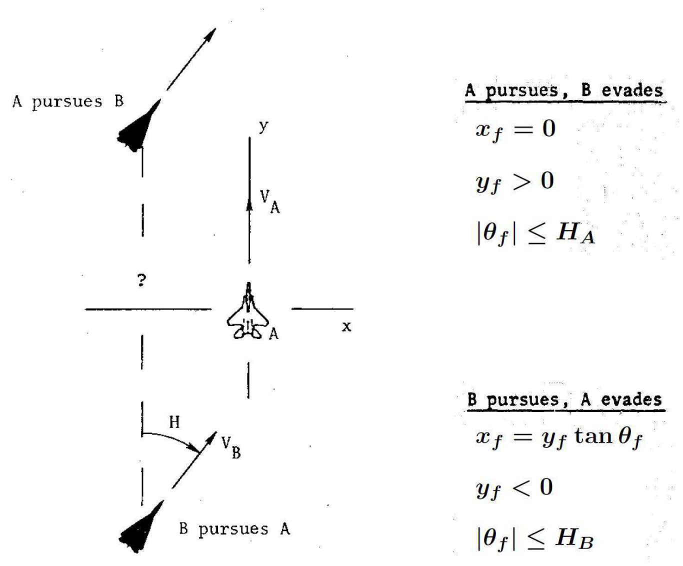


Fig. 18: Figure 1 from the paper [25] showing terminal conditions in the aerial combat problem.

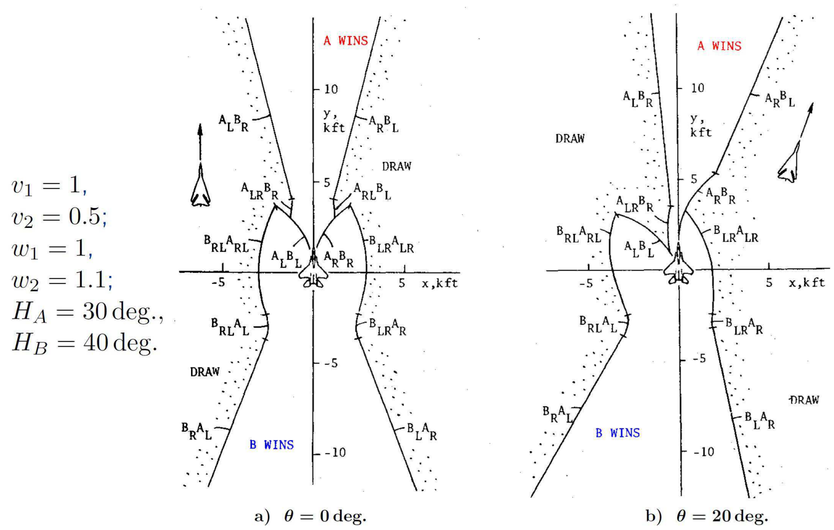


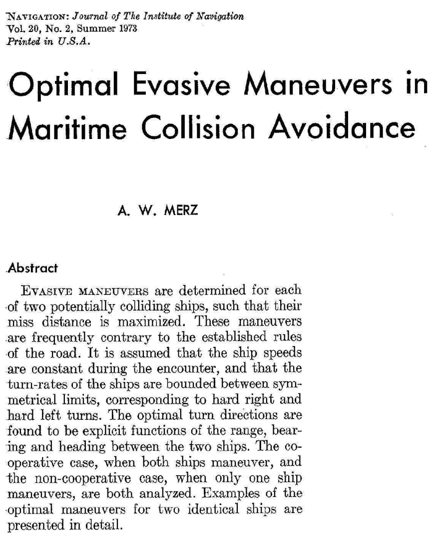
Fig. 19: Figures 4a and b from the paper [25] showing barriers and capture regions in the aerial combat problem.

4.3 Problem of maritime collision avoidance

An initial part of the first paper [7] by Merz on the problem of maritime collision avoidance is shown in the left panel of Fig. 20. The problem of collision avoidance is not related to differential games, but to optimal control theory. The payoff is the closest distance reached between two ships as they approach each other. The short-term motion of each ship is represented by a constant speed model with lateral accelerations being the only means of control. Each ship can maneuver by changing its heading. The turn rates of both ships are assumed to be constrained by symmetric bounds, corresponding to hard left or hard right turns. So, this is also the dynamics of two cars. It is supposed that the controls of the cars are coordinated.

In the introduction, Merz indicates that officially recommended maneuvers sometimes lead to a collision, and shows a corresponding example. In the right panel of Fig. 20 taken from the paper, one can see what happens if the official instructions are applied. The picture c) to the right corresponds to the solution obtained with optimal control theory.

The final result of this paper by Merz is the pictures showing the synthesis of optimal coordinated controls of the ships. Using the (x, y) relative position and the known relative heading angle θ, the optimal controls of the ships A and B are obtained. For example, in Fig. 21 the right turn of the ship A and the left turn of the ship B are the optimal controls. Then the miss-distance r_f is 1.0 in some normalized units. There are singular dispersal lines (θ-sections of three-dimensional singular surface) in this problem where the optimal control is nonunique. Merz points out that the non-uniqueness of optimal controls is the hidden reason of many of the catastrophes recorded in maritime history.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Printed in U.S.A.

Optimal Evasive Maneuvers in Maritime Collision Avoidance

A. W. MERZ

Abstract

EVASIVE MANEUVERS are determined for each of two potentially colliding ships, such that their miss distance is maximized. These maneuvers are frequently contrary to the established rules of the road. It is assumed that the ship speeds are constant during the encounter, and that the turn-rates of the ships are bounded between symmetrical limits, corresponding to hard right and hard left turns. The optimal turn directions are found to be explicit functions of the range, bearing and heading between the two ships. The cooperative case, when both ships maneuver, and the non-cooperative case, when only one ship maneuvers, are both analyzed. Examples of the optimal maneuvers for two identical ships are presented in detail.

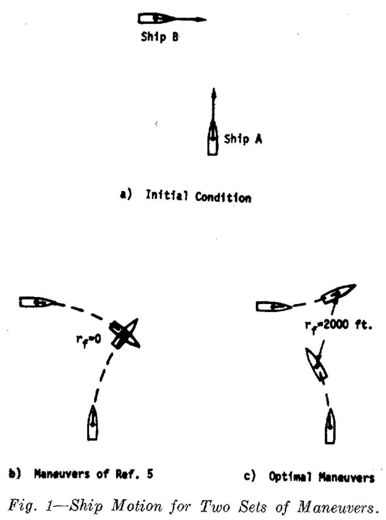


Fig. 20: Left panel: Introductory part of the paper [7] on the maritime collision avoidance problem. Right panel: Figure 1 from the paper [7] showing a) initial condition, b) avoidance maneuver according to existing instructions, and c) non-evident avoidance maneuver according to optimal control theory.

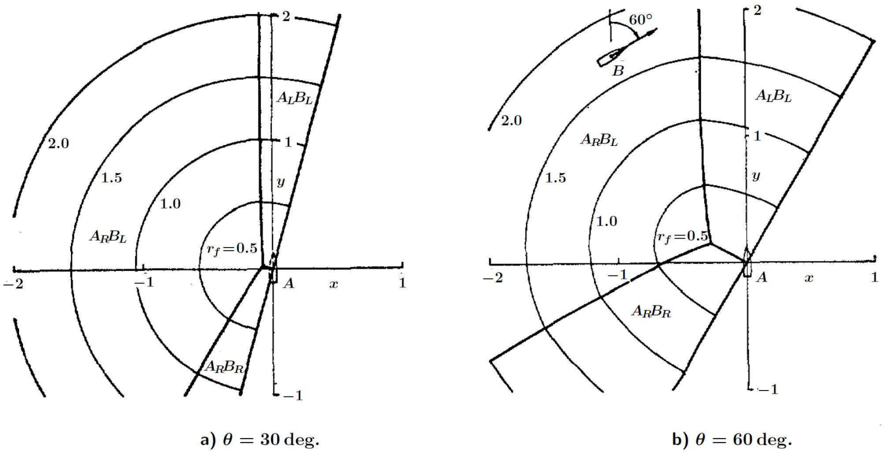


Fig. 21: Optimal control synthesis for identical ships, cooperative case, in the maritime collision avoidance problem. Figures 3a and b from the paper [7].

In the paper, there are also pictures of synthesis for cases of the non-cooperative control of two ships when one of them goes by its previous course and the other executes an optimal maneuver.

This paper was followed by other papers by Merz and coauthors [18, 22, 24, 31], in which the influence of technical details (for example, the accuracy of radar systems, sizes of the ships) was also analyzed.

Problems of collision avoidance for two aircraft were considered by A. Merz and coauthors in [6, 8, 9, 10, 12, 37].

The problem of collision avoidance is a very popular topic in modern optimal control literature, and all the authors cite the works by Merz.

4.4 American football as a differential game

Breakwell's son, John Alexander Breakwell, wrote to the authors: "...There is a little story about the differential games seminars at Stanford you might find interesting. Later games involved pursuit and evasion along a boundary (a straight line often). These seminars were attended by the young Stanford (American) football coach, who picked up the idea that the optimal evasion was often to aim for the boundary. This coach, whose name was Bill Walsh, went on to be the most successful professional football coach in American history."

Under the influence of such contacts, Breakwell and Merz wrote the paper [39], in which they considered three typical situations encountered while playing American football: the one-on-one equal-speed problem; the three-on-three equal-speed problem; the one-on-one problem with the evader E moving faster than the pursuer P. The motions of the players are assumed to be non-inertial.

The pursuer captures the evader if the distance between them becomes less than a given magnitude L.

Briefly consider the first two problems:

One-on-one problem (Fig. 22). The evader E tries to maximize his distance covered in the direction of the end zone before being intercepted by the pursuer. If the pursuer is located at the point P_2, then the best equilibrium interception point is the point in the center with the coordinate $y \approx 30$. But if the pursuer is at the point P_1, then the best equilibrium point lies on the left border of the field. In this case, we have some analogy with the lifeline differential games.

Three-on-three problem (Fig. 23). The initial positions of the players are fixed. The evader E_1 has the ball, and the other two evaders must be optimally defended against the two pursuers P_2 and P_3. Under optimal behavior, the evader E_1 runs to the point A. If the pursuer P_1 acts non-optimally, then the evader E_1 will further run along the border of the field. In the case of the optimal behavior of the pursuer P_1, the interception will be at the point A. Here, the evader E_1 passes the ball through the air to the evader E_2 or E_3. In anticipation, the evaders E_2 and E_3 move so that, having received a ball, each of them can run with it as far as possible towards the end of the field until his interception. Some explanations related to the formulation and solution of an arising differential game are given in the paper. In particular, the behavior of the pursuer P_3 is non-trivial. He watches both the behavior of E_2 and that of E_3.

American football is a very complicated game. During game preparations, the coach and all the team players analyze and learn by heart optimal combinations. It may well be that the application of differential game theory can help in the analysis of certain situations. Let's quote the words of Breakwell's

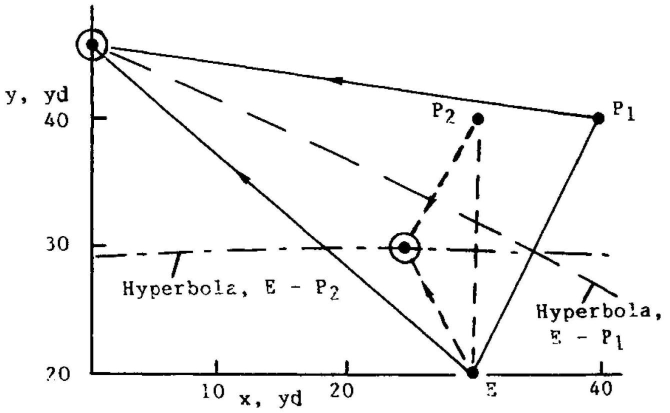


Fig. 2 Hyperbolic loci in the equal-speed case.

Fig. 22: Figure 2 from the paper [39] on football as a differential game showing breakthrough to the field side in the one-on-one equal-speed problem.

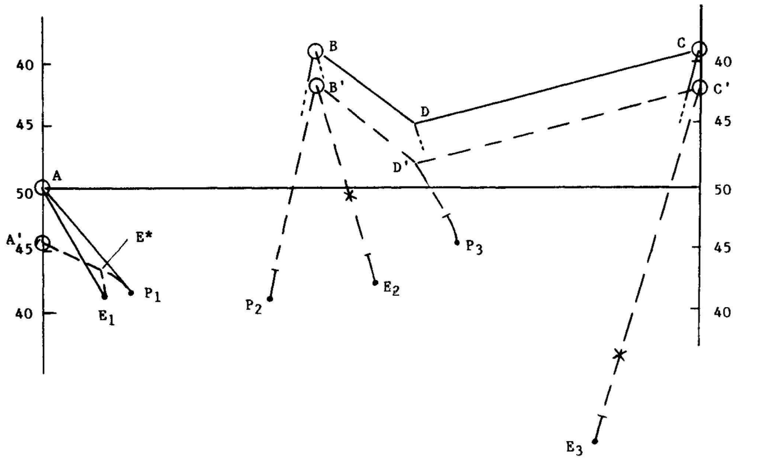


Fig. 23: Figure 3 from the paper [39] on football as a differential game showing optimal and suboptimal play in the three-on-three equal-speed problem.

son: “That the differential games theory had some impact on football is a conjecture, but a pretty good one. Evaluation requires expertise in both football and differential games. This is rare. But the essence is that the optimal strategy for the evader was to head for the boundary. This both as practiced in football today, and as shown by the theory.”

5 Bibliography of Merz’ works

Merz published many (of course, by the standards of those years) works on the problems of applied character. We give a list of these works. They can be conditionally assigned to the topics: homicidal chauffeur game [3, 4, 11], game of two cars with application mainly to aeronautic problems [5, 23, 29], maritime collision avoidance [7, 18, 22, 24, 31, 35], aircraft collision avoidance [6, 8, 9, 10, 12, 37], to pursue or to evade [16, 21, 25, 28, 32, 33, 35], optimal team tactics [39, 40, 42], satellite pursuit-evasion [34, 36], missile attitude stabilization [1, 2], optimization of airfoil [13, 14, 17], state, velocity, and orbit estimation [26, 27, 30, 31, 38, 41, 42, 43], and trajectory optimization for upper atmosphere sampling flights [15, 19, 20].

6 Afterword

The papers by A. Merz inspired many researchers around the world to work on control and differential game problems. It seems that nowadays, the development of numerical methods has increased the attention given to Merz’ works.

First of all, this is related to the impact of his signature work, the homicidal chauffeur game.

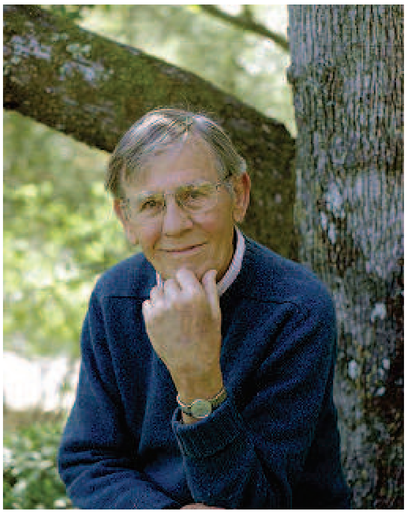
Currently, the theory of differential games is an advanced mathematical discipline. However, the number of model problems with good applied focus which have been solved is very small. Besides, almost every article on this subject suffers from the absence of a rigorous problem statement. The fact is that strict definitions, which are used in theoretical works, are good and convenient only for theoretical investigations related to e.g. the existence of the Value function, extremal properties of feedback optimal strategies, etc. They can hardly be utilized when solving concrete problems. What is, for example, "optimal motion" in differential games? Each researcher involved in applied topics understands this in his own, often nonstrict, sense.

So there may be many questions about Merz' thesis and his subsequent works in the field of differential games. Nevertheless, 50 years ago he took up work that is hardly possible for anyone even now. His dissertation is a challenge to modern mathematicians working in the field of differential games and numerical methods. Can anyone verify his results, strictly substantiate them, find possible mistakes?

We end the article with quotes from two letters of A. Merz to the authors.

August 1, 2008: "...the complexity of even low-order differential games can be very great, and it appears at this time that more realistic dynamic models are virtually out of reach from a computational point of view. For this reason it appears that using optimal controls derived from low-order dynamic models in actual physical models of higher-order may be the best way of testing the theory of differential games. These methods may show that the results found are 'better than other known methods,' and are therefore of practical use in applications."

March 22, 2010: "I have had frequent thoughts of the Game recently(despite the 39-year time interval since its publication), and only wish that I had done some work for the Three-Dimensional problem. This is much closer to the realistic case, with roll-angle as a second control for P and with E still capable of acceleration in any direction normal to E's velocity. I don't think it would be simple, and might not be solvable, but I still feel some sorrow at not having advanced our work in this way."



Antony Willits Merz, 2008.

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