Pursuit-Evasion Games

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Valerii Patsko, Sergey Kumkov, and Varvara Turova

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V. Patsko (✉) · S. Kumkov
N. N. Krasovskii Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russia
e-mail: patsko@imm.uran.ru; sskumk@gmail.com

V. Turova
Department of Mathematics, Technical University of Munich, Garching, Germany
e-mail: turova@ma.tum.de

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Abstract
Applied problems whose investigation involves methods of pursuit-evasion differential games are described. The main focus of this chapter is on time-optimal problems close to R. Isaacs' "homicidal chauffeur" game and to linear differential games of fixed terminal time with J. Shinar's space interception problem as the major example. These problems are taken because after a change of variables they can be reduced to models with two state variables. This allows us to provide adequate graphical representations of the level sets of the value functions being obtained numerically and emphasize important peculiarities of these sets. Also, other conflict control problems and control problems with uncertainties being extensively investigated nowadays are briefly outlined.

Keywords
Differential game · Homicidal chauffeur · Space interception · Semipermeable curves · Barriers · Singular surfaces · Maximal stable bridge

1 Introduction

1. Pioneering works on differential games were accomplished by R. Isaacs and published, starting from 1951, in his reports for the RAND Corporation. Already in the first of them (1951), R. Isaacs used the term "differential game" and formulated the "homicidal chauffeur" game, which later became one of the most famous problems. Note that in the beginning of the 1950s, modern mathematical optimal control theory relevant to accounting for "geometric constraints" on control had only just started to develop. The theorem on necessary conditions for optimal open-loop control, which received the name "Pontryagin maximum principle" was published in 1956–1957. A little earlier, the works of D. W. Bushaw and A. A. Feldbaum on construction of optimal feedback control in linear control problems in the plane had appeared.

In the homicidal chauffeur game a "car" pursues a "pedestrian." The car (a point in the plane) has constant velocity magnitude, but the direction of the velocity vector cannot change instantaneously because the angle velocity is bounded. In other words, there is a restriction on the turn radius of the car. The pedestrian (another point in the plane) is a non-inertia object whose velocity magnitude is bounded, but the direction can change instantaneously. Such non-inertia objects were called by R. Isaacs as objects with "simple motion." The pursuer minimizes the time of capture in a given neighborhood of the evader, while the pedestrian hinders this.

Of course, when considering this game, R. Isaacs kept in mind an applied problem, in which a torpedo pursues an evading small ship (Breitner 2005). It was the genius of R. Isaacs to give "catchy names" to applied problems in his reports and his book “Differential Games” (Isaacs 1965) and to leave only principal features in mathematical description, putting away a plenty of details that always exist when investigating applied problems. Studying differential games (in R. Isaacs’ understanding) is first to model the main important features of complex antagonist problems and then to solve them using analytical or numerical methods.

2. Isaacs’ method is based on the consideration of a first-order partial differential equation for the value function, which is analogous to the well-known in mechanics Hamilton-Jacobi equation. R. Isaacs derived the corresponding equation by letting feedback controls as admissible classes of controls in zero-sum games and formulated the “transition principle,” or the “principle of guaranteed nondeterioration of result in the process of motion.” Nowadays, this is a well-known principle of backward constructions (dynamic programming), when the value function is recomputed by going back from the terminal conditions. The operator of such recalculation implements the operations of minimum over controls of one player and maximum over controls of another one. Thereby, R. Isaacs clearly realized that the value function is, as a rule, non-smooth or even discontinuous. The latter is typical for time-optimal games, in which the payoff is the capture time.

R. Isaacs introduced the notion of “singular” surfaces in the game space and performed a classification of possible types of surfaces. He considered the construction of singular surfaces as a basis for solving differential games. Correctly constructed singular surfaces form a skeleton of solution by generating a peculiar separation of solution into cells. In the interior of each cell a single smooth optimal trajectory goes through every point. On singular surfaces, kinks of optimal trajectories, violation of uniqueness, etc. occur.

After the publication of R. Isaacs’ book, theoretical investigation of singular surfaces, their analysis for particular applied problems were performed by J. Breakwell and his postgraduate students A. Merz, P. Bernhard, J. Lewin, and G.-J. Olsder. P. Bernhard in the paper (1977) and A. A. Melikyan in the book (1998) obtained differential equations for typical singular surfaces. Consideration of differential games with the use of singular surfaces was done in the book by J. Lewin (1994). However, one should be clearly aware of the fact that solving differential games by construction of singular surfaces requires enormous effort even for problems in the plane. In the latter case, we construct not singular surfaces but singular lines. Most likely, in high-dimensional differential games, detecting and classification of singular surfaces are to be appreciated as very useful research that should, however, be performed after constructing level sets of the value function.

3. The theory of differential games has been extensively developed in the Soviet Union in the 1960s–1980s. There existed four centers where differential games were intensively investigated: The mathematical school of L. S. Pontryagin in Moscow, the school of N. N. Krasovskii in Sverdlovsk (now Ekaterinburg), the school of B. N. Pschenichnyi in Kiev, and the school headed by L. A. Petrosyan in Leningrad (now St.-Petersburg).

In his works on the pursuit problem, L. S. Pontryagin assumed that the first player (the pursuer) discriminates the second player (the evader). The discrimination is reduced to the requirement of informing on a small current time interval about the second player's control. In the problem from the side of the second player, on the contrary, the first player is discriminated.

A similar concept was followed by B. N. Pschenichnyi and partly by L. A. Petrosyan.

From the very beginning of his investigations, N. N. Krasovskii followed positional formalization, in which the control is constructed using only current position of the game but is being applied using a discrete control scheme. The latter means that the control chosen at some time instant of a given time grid remains constant until the next time instant of the grid. When solving differential games in positional formalization, the final result is the generation of (optimal) feedback strategy that guarantees the best outcome to the respective player, provided the step width of the discrete control scheme goes to zero.

4. In two papers Pontryagin (1967a,b) devoted to differential games with linear dynamics, L. S. Pontryagin showed how one can account for the advantage of the pursuer over the evader using the notion of geometric difference (Minkowski difference) and how, based on the feedback procedure, the solvability set in the problem of approaching a given target set by a conflict-controlled system can be constructed. Among other papers by L. S. Pontryagin, let us note the works (Pontryagin and Mischenko 1971; Pontryagin 1971) devoted to evasion problem on an infinite time interval. For objects with linear dynamics, very "fine" condition of dynamic advantage of the second player over the first player has been formulated. Once this condition is fulfilled, the evader performs an evading maneuver in dangerous situations of approach. Then he is waiting for the next dangerous situation and so on.

In the book (1970), N. N. Krasovskii proposed effective methods for solving linear differential games based on the notion of reachable sets, i.e., on solving the problem in the class of open-loop controls. Though these methods give an optimal result only in the case where some "regularity conditions" are fulfilled, from the practical point of view, they can be also applied in cases where regularity conditions are not satisfied, because very often the difference between the optimal result and the result obtained is unessential. Moreover, these methods are very clear and can be easily understood by engineers.

Somewhat later, N. N. Krasovskii and A. I. Subbotin introduced (1974, 1988) for a wide class of differential games with nonlinear dynamics the notions of stable bridge and maximal stable bridge. The latter is the maximal set in the space (time $\times$ state vector), from which the first player can solve the problem of approaching a given target set under the assumption of discrimination of his opponent (the second player). Thus, being absolutely unacceptable from the point of view of the engineering practice, the idealized assumption on the discrimination was included into the theoretical construction. It was shown that if the stable bridge (or the maximal stable bridge) is somehow constructed, then an extremal to the stable bridge positional strategy of the first player holds trajectories of the control system in a sufficiently small neighborhood of the stable bridge, provided that the discrete control scheme with sufficiently small time step is used. The concept of stable bridges allowed N. N. Krasovskii and A. I. Subbotin to prove the existence of the value function for different classes of differential games. Efficient numerical methods for the construction of maximal stable bridges were developed in Ekaterinburg (Grigor'eva et al. 2005; Subbotin and Patsko 1984; Taras'yev et al. 1988; Ushakov 1998).

The theory of positional control is developed for differential games with nonlinear dynamics and separable controls of the players. Thereby, usually local Lipschitz condition and sublinear growth in state variable, measurability in time, and continuity in controls are required for the function in the right-hand side. It is extremely important that the results achieved are generalized (Krasovskii and Subbotin 1974, 1988; Subbotin and Chentsov 1981) to the case of systems with inseparable controls, including the case where Isaacs' condition (equality of minmax and maxmin-Hamiltonians) is not fulfilled. The above conditions are assumed to be satisfied also in numerical constructions. In numerical procedures, matching of time and spatial discretization step widths is required additionally.

First works accomplished by L. A. Petrosyan are related to the "lifeline" game that was introduced in the book by R. Isaacs. In this game, the evader strives to reach a given terminal set, whereas the pursuer tries to catch the evader as soon as possible. In the papers Petrosjan (1965) and Petrosyan and Dutkevich (1972), this game is completely solved in the case of simple pursuit (i.e., when the objects have dynamics with simple motion). In addition, it is revealed that for the considered class of dynamics of the pursuer and evader, in the case of point capture, the optimal strategy of the pursuer is the well-known in the engineering practice parallel approach strategy. Among the works by L. A. Petrosyan from the late 1960s, let us mention the paper Petrosyan (1970) where the problem with the evader information time lag is considered. Here, it is proved that the optimal strategy of the evader is mixed. On this topic, see also Petrosyan (1977, 1993). From the middle of 1970s, the school of L. A. Petrosyan started to pay more attention, along with zero-sum differential games, to noncooperative and cooperative dynamic games with many players, which find use in applied economic theory.

5. In the theory of differential games, problems with complete information are distinguished from those ones with incomplete information. The problems with complete information assume precise knowledge of the current position of the game by all participants. This is not the case in the problems with incomplete information. For example, the pursuer forms his control based not on precise information on the state of the evader but on information obtained from inexact measurements only. Moreover, in practice, one should account for information and processing delays. This creates difficulties even on the problem statement stage. Problems with incomplete information accomplished at the beginning of 1970s are presented in the books Chernous'ko and Melikyan (1978), Krasovskii (1970), Krasovskii and Subbotin (1974), Petrosyan (1977), and Kurzhanski (1977). In the years ahead, numerous attempts were made by the scientific school of

N. N. Krasovskii to develop a theory of problems with incomplete information in such a way that it would contain similar elements (maximal stable bridges, extremal positional strategies) as in the theory of problems with complete information. These approaches are reflected in the works Kurzhanski (2004), Osipov (2006), and Kryazhimskiy and Osipov (2010).

As problems with incomplete information, also settings can be considered, in which one of the conflicting players knows the precise phase state of the game at some instants only. There exist statements that assume a bounded number of observation times and that the respective player chooses a future observation instant based on the information about the phase state being available at the previous observation. Using the description of the informal problem from Neveu et al. (1995), let us imagine a helicopter that detected a submarine and tries to approach it (as projected to the horizontal plane) close enough to deliver a weapon. The submarine maneuvers in order to escape to a secure zone. The helicopter is only equipped with a dipping active sonar. Thus, to get information concerning the submarine position, the helicopter has to choose instant for dip position to detect and localize the submarine. After this instant, it chooses the next dip instant and so on. No information is available for the helicopter between the two dips. Therefore, from the point of view of the first player (helicopter), a problem of the choice of observation instants and of the construction of an open-loop control between two subsequent observation points that ensures appropriate result at the end of pursuit occurs. Considering the problem from the point of view of the second player (submarine) also yields in a statement with bounded number of observations of the game phase states but for the submarine.

Similar problems were considered by A. A. Melikyan (1973, 1975) in frames of differential game theory in the early 1970s. He found model examples, in which the correct (optimal) choice of the observation instants provides the equality of the best guaranteed result of the observing player and the optimal result that this player could guarantee under continuous observation of the current position of the game. The results obtained are included in the book Chernous'ko and Melikyan (1978). It occurred that questions on optimal choice of observation instants are closely connected to important theoretical questions related to the coincidence of the value of differential game (under condition of continuous observation of the game state) with the iterations of programming max-min function. Such programmed iterations were proposed by A. G. Chentsov (1976, 1978a). Corresponding results are included in the book Subbotin and Chentsov (1981).

At the beginning of the 1990s, topics related to bounded number of observation instants were further developed in the works by P. Bernhard, O. Pourtallier and their colleagues (Bernhard and Pourtallier 1994; Neveu et al. 1995; Olsder and Pourtallier 1995). Having in mind some applied problems, they investigated statements, in which the observing player needs some specified time to determine the phase state, and this player is immovable during the measurement (helicopter during acting a dipping active sonar). For games with simple motion and games with linear dynamics, construction of sets of initial states from which the reach of a given target set can be guaranteed under a specified number of observation intervals, as well as problems of minimization of the whole observation time, and some others, were considered.

6. In the 1970s–1990s, in different countries, investigations of pursuit-evasion games with objects separated into two groups were performed. For example, several objects collected in one group should capture all evaders combined in another group in finite time. Of course, from the point of view of existence of the value function and optimal feedback strategies, problems with many objects, as a rule, are included into the general theory of differential games. However, what would be effectively verifiable conditions of successful capture? How can one construct optimal strategies of the players? The papers Pschenichnyi et al. (1981), Petrov (1988), and Grigorenko (1989) and the books Chikrii (1997), Grigorenko (1990), and Blagodatskih and Petrov (2009) are devoted to the investigation of such questions for different classes of linear differential games. A great stimulating role in the creation of methods of group pursuit game theory had the work (1976) by B. N. Pschenichnyi, which considered the problem of the successful capture of one evader by a group of several pursuers in the case where all objects are identical and their dynamics are that of simple motion. In the paper Mishchenko et al. (1977), the local evasion maneuver from the work (1971) by L. S. Pontryagin and E. F. Mishchenko is extended to the situation of many pursuers. In the work (1976), for problems with simple motion, F. L. Chernous'ko suggested his method of preventing the capture of the evader by a group of pursuers. Under assumption of advantage in velocity of the evader over each pursuer, the method provides a certain distance evasion from all pursuers with keeping the motion inside a prescribed neighborhood of a given basic trajectory. Particular problems with evident applied character were considered in Hagedorn and Breakwell (1976) and Levchenkov and Pashkov (1990). In Petrosyan (1966) and Petrosyan and Shiryaev (1980), for differential games with several pursuers and several evaders, the notion of Nash equilibrium is used. Close results are presented in the book Petrosyan (1993). A survey of publications on pursuit-evasion games with many players is given in the paper Kumkov et al. (2017).

7. In the 1980s and early 1990s, the attention of many researchers was drawn to problems of aircraft control in the presence of wind disturbances. A tremendous role in the development of this topic played the publications by A. Miele and his collaborators (Miele et al. 1986, 1987, 1988), in which aircraft take off, landing, and abort landing problems were formulated for nonlinear system of vertical channel. These papers were followed by the works of other authors Leitmann and Pandey (1991), Bulirsch et al. (1991a), Bulirsch et al. (1991b), Botkin et al. (1984), and Patsko et al. (1994), in which various methods of optimal control theory and differential games were applied to similar problems.

Of course, the works mentioned were of research nature. The design of autopilots for different stages of aircraft motion is traditionally based on methods of the automatic regulation theory being intensively developed in the 1930s–1950s and relied heavily on achievements of stability theory for linear systems. The outcome of these algorithms when applied in mathematical modeling in the presence of wind disturbances is essentially worse compared to algorithms based on comprehensive mathematical optimal control theory (including differential game theory) that directly accounts for geometric bounds on deviations of steering mechanisms. However, such novel algorithms require as input data almost all state variables, many of which are difficult or even impossible to measure. The interest to study and to apply pursuit-evasion games to aircraft problems could be increased nowadays based on new realistic settings and comprehensive efficient numerical methods of differential game theory.

Many applied-oriented works accomplished in the 1980s are presented in the collection (Yavin et al. 1987) edited by Y. Yavin, M. Pachter, and E. Y. Rodin.

8. The presentation in this chapter is as follows. We describe several mathematical problems that should be regarded as model problems assimilating principal aspects of very important practical problems.

In Sect. 2, we consider the time-optimal homicidal chauffeur game and its modifications. For each problem, we give the statement and the corresponding references to journal publications. Then our results on numerical construction of the level sets of the value function and, for one of the modifications, results of modeling optimal strategies are presented.

In Sect. 3, a space interception problem with linear dynamics is considered. Here again, the main attention is paid to the computation of the level sets of the value function (maximal stable bridges with a given value of miss). It is stressed that the level sets in the space time $\times$ state vector can have narrow throats with complex geometry. Investigation of such seemingly pure mathematical peculiarities is important for understanding the structure of the solvability sets of the interception problem. Significant attention is paid to adaptive control of the first (minimizing) player, being developed by us for the case where, according to the statement of the problem, no geometric constraint on the control of the second player is specified.

Thus, our objective is to give a vivid presentation of two canonical classes of differential games and applied problems, which can be solved using numerical methods developed for these classes. The presentation is accompanied by a large number of figures to demonstrate the structure and peculiarities of the value function.

In the last subsections of Sects. 2 and 3, we mention comprehensive complex applied problems related to time-optimal games and to games with linear dynamics and stress that their investigation requires development of new efficient numerical methods.

The investigation of applied model problems considered in this chapter was initiated and to a large extent explored by the outstanding mathematicians: R. Isaacs, J. Breakwell, A. Merz, and J. Shinar. We place their photographs here (Figs. 22.1 and 22.2).

The Introduction and Sect. 2 of this chapter are written by V. S. Patsko and V. L. Turova, Sect. 3 and Conclusion are prepared by S. S. Kumkov and V. S. Patsko.

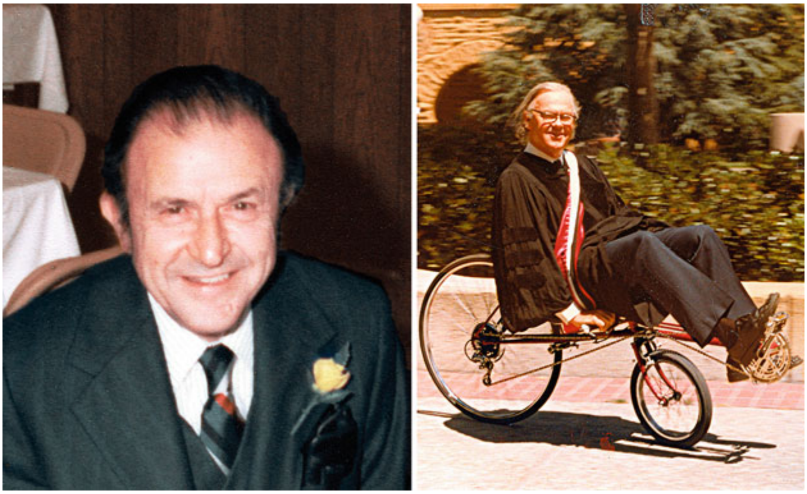


Fig. 22.1 Rufus Isaacs (1979) and John Valentine Breakwell ($\approx$ 1986)

Fig. 22.2 Antony Merz (2008) and Josef Shinar (2007)

2 Time-Optimal Problems: Homicidal Chauffeur Game and Its Modifications

Among pursuit-evasion games, the most popular ones are time-optimal problems, where one player wishes to minimize and another wishes to maximize the terminal time of the game. In turn, the most famous problem among time-optimal problems is the homicidal chauffeur game. It formed the basis of the book by R. Isaacs. After the publication of this book, a huge number of applied studies were performed on the homicidal chauffeur game and its modifications.

The significance of the problem is the following. On one side, many practical situations fall under this mathematical description, e.g., the abovementioned conflict situation between a controlled torpedo and an evading small motor boat or an aircraft pursuing a helicopter in a horizontal plane and so on. On the other hand, the problem is formulated mathematically in such a way that after passing to reduced coordinates we deal with two state variables. This was important in the middle of the last century as numerical investigation of the problem was not yet possible. Thus, intuition could help here. Also presently, when the application of numerical methods is not uncommon, systematic numerical analysis for various values of parameters with the aim to reveal regular and singular parts of solutions can really be performed only in the case where the problem is reduced to the one with two state variables. A third thing to mention is that the problem is very interesting as a test example when developing various numerical methods of differential game theory.

2.1 Dynamics of Conflicting Objects

Two moving objects, a "pedestrian" and a "car," present in the game.

The pedestrian is a non-inertia point object with coordinates x_e, y_e in the plane, which can change the direction of the motion instantaneously. The magnitude of the velocity v is bounded from above by a given number. Using differential equations, this can be expressed in the form

$$\begin{aligned} \dot{x}_e &= v_1, \\ \dot{y}_e &= v_2, \quad v = (v_1, v_2)', \ |v| \le \rho. \end{aligned} \tag{22.1}$$

Such an object was called by R. Isaacs as "object with simple motion." Here and below, the prime means transposition.

Dynamics of the car:

$$\begin{aligned} \dot{x}_p &= w \sin\theta, \\ \dot{y}_p &= w \cos\theta, \\ \dot{\theta} &= wu/R, \ |u| \le 1. \end{aligned} \tag{22.2}$$

Here x_p, y_p are the coordinates of the point object, θ is the angle specifying the direction of the velocity vector (measured clockwise from the $+y_p$-axis), $w = \text{const}$

is the given velocity magnitude, u is the scalar control (bounded in absolute value by 1), and R is the minimum turn radius.

By normalizing time and geometric coordinates, one can achieve $w = 1$, $R = 1$. In the new dimensionless variables, the dynamics of the objects is written as follows:

$$\begin{aligned} P: \dot{x}_p &= \sin\theta, & E: \dot{x}_e &= v_1, \\ \dot{y}_p &= \cos\theta, & \dot{y}_e &= v_2, \\ \dot{\theta} &= u, \quad |u| \le 1; & v &= (v_1, v_2)', \quad |v| \le \nu. \end{aligned} \tag{22.3}$$

The constraint on v in (22.3) has changed compared to (22.1) because of the joint normalization for (22.1) and (22.2). In the following, the notation $Q = \{v : |v| \le \nu\}$ will be often used.

It was R. Isaacs who introduced the name "car" for the object (22.2). After the normalization and assuming that the velocity $w = 1$, the path length run by such an object is $wT = T$, where T is the elapsed time. Therefore, minimization of time for the object (22.2) is equivalent to minimization of the path length.

A. A. Markov addressed in his paper (Markov 1889) four optimization problems for railway track laying. In the first two of them, he assumed that the movement along the railway track is performed with constant velocity, the curvature radius of the railway track is bounded, and the path length is used as an optimum criterion. This means that he studied practically time-optimal problems for the object with dynamics (22.2) but using other terms.

L. Dubins published the paper (Dubins 1957) on the line of shortest length, which connects two points in the plane. The lines whose curvature is bounded from below by the same number were admitted for comparison; herewith each line should have the same given outgoing direction at the initial point and the same given incoming direction at the terminal point. Obviously, movement along such lines is also described by system (22.2). It came about in works on theoretical robotics that objects with dynamics (22.2) are often called "Dubins' car."

The next in complexity model is the car model from the paper by J. Reeds and L. Shepp (1990):

$$\begin{aligned} \dot{x}_p &= w \sin\theta \\ \dot{y}_p &= w \cos\theta \\ \dot{\theta} &= u, \qquad |u| \le 1, \ |w| \le 1. \end{aligned} \tag{22.4}$$

The control u determines the angular velocity of motion. The control w is responsible for the instantaneous change of the linear velocity magnitude. In particular, the car can instantaneously change the direction of motion to the opposite one. A non-inertia change of the linear velocity magnitude is a mathematical idealization. But, citing (Reeds and Shepp 1990, p. 373), "for slowly moving vehicles, such as carts, this seems like a reasonable compromise to achieve tractability."

It is natural to consider problems where the range for changing the control w is $[a, 1]$. Here, $a \in [-1, 1]$ is the parameter of the problem. If $a = 1$, Dubins' car is obtained. For $a = -1$, one arrives at Reeds-Shepp's car.

2.2 Dynamics in Reduced Coordinates

Place the origin of the reduced coordinates x, y to the position of player P. Let $h(t)$ be a unit vector in the direction of motion of player P at time t. The orthogonal to $h(t)$ unit vector is denoted by $k(t)$ (see Fig. 22.3). We have

$$h(t) = \begin{pmatrix} \sin\theta(t) \\ \cos\theta(t) \end{pmatrix}, \quad k(t) = \begin{pmatrix} \cos\theta(t) \\ -\sin\theta(t) \end{pmatrix}.$$

Differentiating the relations

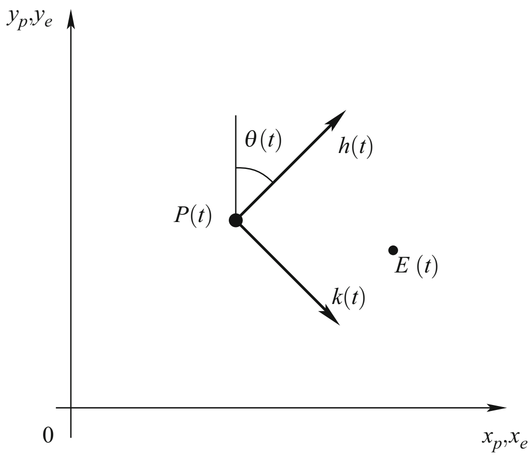
$$\begin{aligned} x(t) &= \cos\theta(t)(x_e(t) - x_p(t)) - \sin\theta(t)(y_e(t) - y_p(t)), \\ y(t) &= \sin\theta(t)(x_e(t) - x_p(t)) + \cos\theta(t)(y_e(t) - y_p(t)), \end{aligned}$$

we turn from system (22.3) to system

$$\begin{aligned} &\dot{x} = -yu + v_x, \\ &\dot{y} = xu - 1 + v_y, \\ &|u| \le 1, \ v = (v_x, v_y)', \ |v| \le \nu. \end{aligned} \tag{22.5}$$

Here $v_x = v_1 \cos\theta - v_2 \sin\theta$, $v_y = v_1 \sin\theta + v_2 \cos\theta$. Note that the form of the circular constraint on the control of player E remains the same in the reduced

Fig. 22.3 Movable reference system



coordinates. However, this might be not the case if the geometric constraint on the control of player E in (22.3) would be of another kind.

In the case where player P is described by (22.4), let the axis y of the relative coordinate system be directed toward the forward motion of the car to obtain

$$\begin{aligned} &\dot{x} = -yu + v_x, \\ &\dot{y} = xu - w + v_y, \\ &|u| \leq 1,\ w \in [a, 1],\ v = (v_x, v_y)',\ |v| \leq \nu. \end{aligned} \tag{22.6}$$

2.3 Isaacs' Method for Games of Kind and Games of Degree. Iterative Viability Methods

The problems considered in Sects. 2.5 and 2.6 were originally solved using Isaacs' method. The problem from Sect. 2.7 was initially investigated using an iterative method based on the concept of viability trajectories. Below, we give a schematic description of these methods.

1. In classical mathematics, smooth solutions to first-order partial differential equations are searched using Cauchy characteristics (see, e.g., Courant 1962; Evans 1998; Melikyan 1998).

Consider a partial differential equation

$$F\Big(x, J, \frac{dJ}{dx}\Big) = 0, \quad x \in R^n. \tag{22.7}$$

Here, F is a scalar function, J is the unknown function $x \to J(x)$, and $\frac{dJ}{dx}$ is its derivative. Depending on the context, the derivative of the scalar function of a vector argument will be considered either as a row matrix or a column matrix. Assume that $F \in C^2$. Also, let the function J satisfy a given boundary condition. Typically, values $J(x)$ are defined on a smooth manifold of dimension $n-1$. With some additional regularity condition, the theorem on local parametrization of the graph of the function $J \in C^2$ holds. The parametrization is performed using the system of ordinary differential equations:

$$\dot{x} = F_\psi(x, s, \psi),\ \ \dot{s} = \langle \psi,\ F_\psi(x, s, \psi)\rangle,\ \ \dot{\psi} = -F_x(x, s, \psi) - \psi F_s(x, s, \psi), \tag{22.8}$$

where the angular brackets denote the scalar product of two vectors.

The functions $t \to x(t)$ and $t \to s(t)$, being the solution (together with the function $t \to \psi(t)$) to system (22.8), define curves carpeting the graph of the function J. Initial values $x(t_*)$, $s(t_*)$, $\psi(t_*)$ correspond to the boundary condition on the function J and depend on some parameter ξ of dimension $n-1$. The scalar variable t taking values close to t_* is also a parameter. The functions $t \to x(t)$, $t \to s(t)$, and $t \to \psi(t)$ are called characteristics of equation (22.7).

Necessity. Given a solution J to equation (22.7) $\Longrightarrow$ system (22.8) defines a parametrization of the graph of the function J. Here, $s(t) = J(x(t))$, $\psi(t) = \dfrac{dJ}{dx}(x(t))$.

Sufficiency. Given functions $t \to x(t)$, $t \to s(t)$, and $t \to \psi(t)$, satisfying system (22.8) and relation $F(x(t), s(t), \psi(t)) = 0$. The initial values $x(t_*)$, $s(t_*)$, and $\psi(t_*)$ of these functions, being dependent on a parameter ξ, have the sense of a point on some sub-manifold of the initial manifold, the value $J(x(t_*))$ of the unknown function, and the value $\dfrac{dJ}{dx}(x(t_*))$ of its derivative, respectively. The regularity condition ensures that only one x-characteristics goes through each point $x = x(t)$ of this characteristics in some neighborhood of the distinguished sub-manifold. Then, the function J, being a solution to (22.7), is constructed in such neighborhood. The curves $t \to x(t)$ and $t \to s(t) = J(x(t))$ follow the graph of this function. Here, $\psi(t) = \dfrac{dJ}{dx}(x(t))$.

The proof of the sufficiency is constructive and gives a receipt for constructing the function J.

If the function F in (22.7) does not depend on J, then the system (22.8) is simplified as follows:

$$\dot{x} = F_\psi(x, \psi), \;\; \dot{s} = \langle \psi, \, F_\psi(x, \psi) \rangle, \;\; \dot{\psi} = -F_x(x, \psi). \tag{22.9}$$

The first and third equations in (22.9) are separated from the second equation and can be integrated independently. Once the functions $t \to x(t)$ and $t \to \psi(t)$ are found, the function $t \to s(t)$ can be then determined.

2. R. Isaacs applied the Cauchy characteristic method for solving differential games.

For clarity, let us consider time-optimal differential game in the plane with the dynamics:

$$\dot{x} = f_1(x, u) + f_2(x, v), \;\; x \in R^2, \; u \in P, \; v \in Q, \tag{22.10}$$

and a closed target set M. The first player has the control u at his disposal and minimizes the transfer time of system (22.10) to the set M. The second player being responsible for the control v has the opposite interest. The controls u and v are bounded by geometric constraints. The differential game is considered in the class of feedback controls.

In the theory of differential games, existence of the value function $x \to V(x)$ is established. Here, we do not go into details of a particular formalization. In typical examples, the value function is not differentiable or even continuous for $x \in R^2 \backslash M$. However, there exist regions (cells) in which $V \in C^2$. In each of such regions, the derivative of the function $t \to V(x(t))$ along an arbitrary trajectory $t \to x(t)$ can be computed:

$$\frac{dV}{dx}(x(t)) \cdot \dot{x}(t) = \frac{dV}{dx}(x(t)) \cdot (f_1(x, u(t)) + f_2(x, v(t))).$$

Accounting for the nondeterioration principle along optimal trajectories, we obtain

$$\min_{u \in P} \frac{dV}{dx}(x) f_1(x, u) + \max_{v \in Q} \frac{dV}{dx}(x) f_2(x, v) = -1. \tag{22.11}$$

Relation (22.11) is just R. Isaacs' transition rule (principle) written in terms of derivatives. Introducing the notation for the Hamiltonian

$$H(x, \psi, u, v) = \psi'(f_1(x, u) + f_2(x, v)),$$

we rewrite relation (22.11) as follows:

$$1 + \min_{u \in P} \max_{v \in Q} H(x, \frac{dV}{dx}(x), u, v) = 0. \tag{22.12}$$

Thus, the value function V satisfies the partial differential equation (22.12). Computing the extremal elements u^* and v^* in (22.12), assume additionally their smoothness in x. We obtain

$$1 + H(x, \frac{dV}{dx}(x), u^*(x), v^*(x)) = 0. \tag{22.13}$$

The equations of characteristics for (22.13) have the form

$$\dot{x} = H_\psi(x, \psi, u^*(x), v^*(x)), \quad \dot{\psi} = -H_x(x, \psi, u^*(x), v^*(x)). \tag{22.14}$$

Having under some boundary conditions a solution to (22.14), we obtain the functions $u^*(t) = u^*(x(t), \psi(t))$ and $v^*(t) = v^*(x(t), \psi(t))$. With that, relation (22.13) is fulfilled:

$$1 + H(x, \psi(t), u^*(t), v^*(t)) = 0. \tag{22.15}$$

Thereby,

$$\begin{aligned} &\min_{u \in P} \psi'(t) \cdot f_1(x(t), u) = \psi'(t) \cdot f_1(x(t), u^*(t)), \\ &\max_{v \in Q} \psi'(t) \cdot f_2(x(t), v) = \psi'(t) \cdot f_2(x(t), v^*(t)). \end{aligned} \tag{22.16}$$

Relationships (22.16) together with equations

$$\dot{x} = H_\psi(x, \psi, u^*(t), v^*(t)), \quad \dot{\psi} = -H_x(x, \psi, u^*(t), v^*(t)) \tag{22.17}$$

are similar to Pontryagin's maximum principle for optimal control problems. Here, they were derived in the form of necessary conditions based on the Cauchy characteristic method and on the assumption about smoothness of the function V and extremal elements u^* and v^* in (22.12).

We can also consider the question of constructing the unknown function V. Suppose that values of the function and its derivative are given on some curve. Consider the trajectories of system (22.10) that satisfy the minimax principle (22.16), (22.17), and, additionally, together with the function $t \to \psi(t)$, relation (22.15).

Assume that only a single trajectory satisfying the first equation of system (22.17) goes through each point x of some region that includes a curve on which values of the function V and its derivative are given. Then for every point x on the distinguished curve, we reconstruct the state $\xi(x)$ of the corresponding x-trajectory and the time $t(x)$ of passing through the point x. For every point x, let $u(x) = u^*(t)$, where $u^*(t)$ is defined by $\xi(x)$ and $t(x)$. Similarly, we introduce $v(x) = v^*(t)$. Assume that the functions $u(x)$ and $v(x)$ are continuously differentiable. Then the system

$$\begin{aligned}\dot{x} &= f_1(x, u(x)) + f_2(x, v(x)) = H_\psi(x, \psi, u(x), v(x)),\\ \dot{\psi} &= -H_x(x, \psi, u(x), v(x))\end{aligned}$$

can be put in correspondence with system (22.16), (22.17). With that,

$$1 + H(x(t), \psi(x(t)), u(x(t)), v(x(t))) = 0.$$

Further, the unknown function V is reconstructed in the region considered using the same technique as in the proof of sufficiency of the Cauchy characteristic method.

However, we cannot say that this is the value function of the differential game. This is merely a function that satisfies the partial differential equation, the given values, the values of derivatives, and boundary condition on the distinguished curve. To obtain the value function, a solution in the whole space should be found. If the reconstruction region spans the whole space, the problem is solved. However, this is a very rare situation. Typically, finding the value function (and associated with it optimal strategies) with Isaacs' method requires subsequent (backward in time) covering the whole game space with cells filled out with regular x-characteristics.

In more detail, the construction of the value function with the method of characteristics is described, apart from R. Isaacs' book, in the paper (Berkovitz 1994) and in the books (Başar and Olsder 1995; Lewin 1994).

3. In R. Isaacs' approach to differential games with complete information, two main ideas can be emphasized.

The first idea is related to believing that the solution region of the differential game typically is divided into cells in which interior the value function is smooth and can be found using the Cauchy characteristic method for a properly derived first-order partial differential equation. But how can one set boundary conditions for each cell? The only possible way is primary analysis of terminal conditions of the differential game. For example, for time-optimal differential game in the plane with the target set M, the parts of the boundary of M which can be penetrated by optimal trajectories should be determined. We accept that the value function on such parts is equal to zero and try to compute the derivative of the value function in x on these parts. Then Cauchy's characteristics are emitted *backward* in time for each of the parts. Analysis of x-characteristics enables to determine where such backward construction should be stopped, and examination of next cells should be started.

The second idea by R. Isaacs concerns singular lines (in general case, singular surfaces). Singular lines are the curves on which optimal trajectories lose usual regularity. For example, the dispersal line is a curve each point of which is approached by two backward x-characteristics with equal optimal result values. The universal line is a curve which, to the contrary, is leaved by two backward optimal trajectories. Hence, the direct time-optimal trajectories approach the universal line. Thereby, there exists an optimal motion that goes along the universal line, but the corresponding trajectory (being the universal line itself) is not a regular Cauchy's x-characteristic.

R. Isaacs discovered a new type of singular lines (surfaces), which he called equivocal. Optimal trajectories approach such lines in direct time; then each optimal trajectory splits into two branches: the first one comes with a kink to the other side of the singular line, and the second one goes along the singular line. The value function is continuous but not differentiable on equivocal lines. R. Isaacs mentioned that equivocal lines are inherent to differential games. In contrast to other singular lines, curves with such properties cannot exist in problems where only one player optimizing the dynamic system behavior is present.

One more types of singular lines called barriers are curves where the value function is discontinuous.

The book Isaacs (1965) contains many remarkable pictures explaining the sense of singular lines and singular surfaces. R. Isaacs described how singular lines can arise in the backward constructions.

Detection and construction of singular lines is a key to consideration of the next cell in Isaacs' method. Thus, Isaacs' approach is a backward construction of cells based on the analysis of arising singular lines (surfaces). The cells are filled with optimal trajectories. By constructing cells backward in time, we hope to cover with them the whole game space. The obtaining value function is, as a rule, not differentiable or even continuous.

4. The theory of differential games essentially influenced the development of the theory of partial differential equations. In the beginning of the 1980s, new notions of generalized solutions of the first-order partial differential equations have been introduced. The generalized solution suggested by M. G. Crandall and P.-J. Lions was called viscosity solution (Crandall et al. 1984; Crandall and Lions 1983; Lions 1982), whereas the concept proposed by A. I. Subbotin was specified as minimax solution (Subbotin 1980, 1984). The equivalence of these two notions was established, and new concepts of generalized solutions were developed to cover formulations of typical differential games with non-smooth or even discontinuous value function (Subbotin 1995). Many facts from the theory of singular lines and surfaces revealed in the theory of differential games earlier were reformulated for

generalized solutions of partial differential equations. On this way, A. A. Melikyan developed (Melikyan 1998) a theory of singular surfaces for the first-order partial differential equations. Numerical methods for solving Hamilton-Jacobi equations associated with differential games and based on the concept of viscosity solutions are being developed (see, e.g., Botkin et al. 2011; Chen et al. 2015; Falcone 2006; Grigor'eva et al. 2000).

5. Considering differential games, R. Isaacs distinguished between games of kind and games of degree. Let us explain this using differential game with dynamics (22.10) and a closed target set M. The terminal time of the control process is not fixed.

In the game of kind, we are interested in finding the set $\mathcal{A}$ of all initial states x_0 from which the first player guarantees approaching the set M within a finite time, using a feedback control $u(x)$ implemented in a discrete control scheme. For initial states in the set $R^2 \backslash \mathcal{A}$, such a guarantee is absent. Surely, in frames of accurate formalization, one should correctly specify what "approaching M" means. Namely, whether "approaching M" implies precise transition to M or transition to an arbitrarily small neighborhood of M. We will not do this here. In any case, there are only two possible (guaranteed) outcomes in the game of kind: yes (approaching is possible) or no (approaching is not possible).

In the game of degree, the first player minimizes the time of approach M. Here, compared to the game of kind, one should determine a minimum guaranteed time of transition to M for each initial state $x_0 \in \mathcal{A}$. It is established that this time coincides with the best guaranteed time of the second player (who maximizes the time of transition to M); therefore, one can speak about the value function $V(x_0)$. For initial states $x_0 \notin \mathcal{A}$, nothing new compared to the game of kind arises.

A common feature for the games of kind and degree is the construction of the boundary of the set $\mathcal{A}$. In the game of degree, we have $V(x_0) < \infty$ for $x_0 \in \mathcal{A}$, and $V(x_0) = \infty$ for $x_0 \in R^2 \backslash \mathcal{A}$. Therefore, the curves constituting the boundary of the set $\mathcal{A}$ are barrier lines.

R. Isaacs formulated the main property of smooth curves that comprise the boundary of $\mathcal{A}$. Namely, he considered the following relation:

$$\min_{u \in P} \max_{v \in Q} \ell'(x)(f_1(x,u) + f_2(x,v)) = 0. \tag{22.18}$$

Here $\ell(x)$ is the normal to the smooth curve at the point x. Let the side of the curve to which the normal is directed be referred as negative and the opposite side be indicated as positive. R. Isaacs called the smooth curves satisfying the relation (22.18) the semipermeable curves. Families of semipermeable curves are defined only by dynamics of the game (including constraints on the controls) and do not depend on the objectives of the players. Having the dynamics of the game, we can perform an analysis of families of such curves in the plane in advance to use them later for constructing barriers. Each semipermeable curve is often bounded. Considering one of the two directions of moving along the curve, one can specify its start point and end point. For each point x in the plane, one of the following possibilities is realized: there is no semipermeable curve, there is only one semipermeable curve, or there are several semipermeable curves going through x.

Relation (22.18) can be considered as a partial differential equation with respect to unknown scalar function $x \to J(x)$:

$$\min_{u \in P} \max_{v \in Q} \frac{dJ}{dx}(x)(f_1(x,u) + f_2(x,v)) = 0. \tag{22.19}$$

The family of smooth semipermeable curves is a family of x-characteristics of equation (22.19). On each x-characteristic, the value $J(x)$ is constant. The whole family can be found by specifying some (not arbitrary) curve in the plane and values of derivative $\frac{dJ}{dx}(x)$ in points of this curve, so that (22.19) holds.

It is useful to distinguish families of semipermeable curves of the first and second type. For families of the first type, when constructing semipermeable curves backward in time, the vector of moving direction along the curve is related to the vector ℓ by a clockwise rotation through the angle $\pi/2$. For families of semipermeable curves of the second type, the corresponding vectors are related by a counterclockwise rotation through the angle $\pi/2$.

Remark. The greater the number of families of semipermeable curves for a given dynamics is, the more complex the differential game with a particular payoff is.

6. Smooth semipermeable curves are the basis for solving games of kind in Isaacs' method. Suppose that the target set M is convex and has a smooth boundary. Visiting the boundary of M, find those parts of it through which the first player guarantees the transition of system (22.10) to the interior of the set M for any counteraction of the second player. R. Isaacs called such pieces "the usable part" (UP). Let for simplicity UP consists of a single arc. Taking an arbitrary internal point x of this arc and denoting by $\ell(x)$ the vector of outward normal to the set M at this point, write down the inequality:

$$\min_{u \in P} \ell'(x) f_1(x,u) + \max_{v \in Q} \ell'(x) f_2(x,v) < 0, \tag{22.20}$$

which provides a guaranteed approach of the interior of the set M by the trajectories of system (22.10) not only from the point x but also from the points outside M, being close to M. For two boundary points x_*, x^* of this arc (BUP), we obtain

$$\min_{u \in P} \ell'(x) f_1(x,u) + \max_{v \in Q} \ell'(x) f_2(x,v) = 0. \tag{22.21}$$

In the coarse case, for any other points $x \in \partial M$ outside the arc $[x_*, x^*]$, the inequality >0 holds for the left part of (22.21). Then the arc $[x_*, x^*]$ is a "gate," through which entering M is only possible. The next step is to construct smooth barrier lines, being "boards of the way" leading to the arc $[x_*, x^*]$, from the points x_*, x^*. Herewith, with respect to each of the two boards, the first player is able to prevent the transition of the system from the positive side of the board (faced to the way) to its negative side, no matter how the second player acts. Conversely, the second player is able to prevent the transition from the negative side to the positive side. Thus, one of the boards is smooth semipermeable curve of the first type and the other one is of the second type.

The most simple situation in the coarse case is realized, when the semipermeable curves under consideration intersect without tangency. In this case, it can be often proved that the set $\mathcal{A}$ of successful termination of the game of quality with respect to the first player is the union of the set M and the part of the plane bounded by the curve UP and by the pieces of two semipermeable curves between their start points x_*, x^* and the intersection point.

In the paper Patsko (1975), the game of kind for differential games in the plane with arbitrary linear dynamics, a point target set M, scalar control of the first player with a bounded absolute value, and an arbitrary convex polygonal constraint on the control of the second player is completely solved. An algorithm for constructing the set $\mathcal{A}$ for games of kind in the plane in the case of complex roots of the characteristic polynomial of linear system and arbitrary polygonal constraints on the controls of the first and second players is described in Turova (1984).

7. Isaacs' method for games of kind and games of degree uses constructions that provide a precise answer without any iterations. The error obtained is determined by only inaccuracy in the implementation of prescribed operations.

In the middle of the 1970s, A. G. Chentsov proposed the method of programmed iterations for various classes of differential games (Chentsov 1976, 1978a,b). In this method, the solution to the problem in the form of some set is obtained as a result of iterative descent to this set from above. The method is based on the concepts used in the scientific school of N. N. Krasovskii. As it was mentioned in the Introduction, the central notion in this school is the notion of maximal stable bridge. For example, for time-optimal games with stationary dynamics and a closed target set M, the maximal stable bridge terminating at the time T on the set M is the collection of all positions (t_*, x_*), from which the first player by discriminating the second one can bring the state vector x to the set M within the time not exceeding $T - t_*$. The corresponding "tube" W (maximal stable bridge) in the space t, x $(t \leq T)$ contains the cylinder $\{(t, x) : t \leq T,\ x \in M\}$. It is known that any t-section $W(t)$ is the level set (Lebesgue set) of the value function $x \to V(x)$ of the time-optimal game, i.e., $W(t) = \{x : V(x) \leq T - t\}$.

For the construction of the set W on some interval $[\bar{t}, T]$, $\bar{t} < T$, A. G. Chentsov proposed the following iterative procedure. The iterations start from the set $W^{(0)} = [\bar{t}, T] \times R^n$, where R^n is the phase space of the dynamic system. Then a closed subset $W^{(1)} \subset W^{(0)}$ is distinguished by the following property: for any position $(t_*, x_*) \in W^{(1)}$, any constant control $v \in Q$ of the second player on the interval $[t_*, T]$, the first player can choose his open-loop control $u(\cdot)$, so that the trajectory

$(t, x(t))$ generated by the controls $u(\cdot)$ and v comes to the cylindric set $\{(t, x) : t_* \le t \le T, x \in M\}$. Then the set $W^{(2)} \subset W^{(1)}$ is introduced, so that for any point $(t_*, x_*) \in W^{(2)}$ and any constant control $v \in Q$, the first player can choose his open-loop control such that the trajectory of the system generated by the controls $u(\cdot)$ and v satisfies the constraint $(t, x(t)) \in W^{(1)}$. Thus, the set $W^{(1)}$ plays the role of a closed state constraint. Further, the set $W^{(3)} \subset W^{(2)}$ is constructed, where the set $W^{(2)}$ being the state constraint, and so on. It is proved that $W^{(i)} \to W$ on $[\bar{t}, T]$ as $i \to \infty$.

A. G. Chentsov interpreted the method of programmed iterations as a method explaining the structure of the differential game and helping to establish particular theoretical facts. The method was formulated and proved for very wide variety of problems, but A. G. Chentsov did not attempt to develop efficient numerical procedures based on this approach.

P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre proposed a conceptually similar iterative method (Cardaliaguet et al. 1995, 1999), which, however, was directed to the numerical implementation. The method uses ideas of viability theory developed by J.-P. Aubin (1991). The most clear application of the method is its employment in the game of kind for the construction of maximal set $\mathcal{A}^*$, from every point of which the second player (by discriminating the first player) guarantees the evasion of the dynamic system from approaching the set M for infinite time. The iterations are computed in the set $K = R^n \setminus M$. It is required that this set be closed. Hence, the set M is supposed to be open.

Let us explain the iterations. Replace the original continuous-time dynamics by a discrete one. For all $x \in R^n$ and $u \in P$, introduce the set $G_\varepsilon(x, u)$, which is interpreted in the context of discrete dynamics as a reachable set with respect to $v(\cdot)$ for a fixed control u of the first player on some small time interval of the length ε. The discrete dynamics is chosen in a way that the set $G_\varepsilon(x, u)$ contains an analogous reachable set of the original system. The following sequence of sets is introduced:

$$\begin{cases} K_\varepsilon^0 = K, \\ K_\varepsilon^{i+1} = \{x \in K_\varepsilon^i : \forall u \in P,\ G_\varepsilon(x, u) \cap K_\varepsilon^i \neq \varnothing\}, \quad i = 0, 1, \dots \end{cases} \tag{22.22}$$

The set K_ε^{i+1} is the maximal subset of the set K_ε^i, from any point x of which the first player, by showing his constant control $u \in P$, cannot steer the system away from the set K_ε^i at the end of the interval ε. It is proved that the sequence K_ε^i converges from above to some set $\overrightarrow{Disc}_{G_\varepsilon}(K)$, which is called a *discrete discriminating kernel*.

This set possesses the following *viability* (stability) property: if the first player shows his constant control u for the interval ε in advance, then, in frames of discrete approximating dynamics, the second player holds the motion in the set $\overrightarrow{Disc}_{G_\varepsilon}(K)$ for infinite time. For points $x \in K \setminus \overrightarrow{Disc}_{G_\varepsilon}(K)$, on the contrary, there exists a positional control method for choosing control u with the step ε, such that the trajectory will approach the set M within a finite time for any actions of the second player.

We described solving the game of kind for discrete approximating dynamics for a fixed ε. Then grids K_h and P_h on the sets K and P with the step size h are introduced, and a grid approximation $\Gamma_{\varepsilon,h}(x_h, u_h)$ of the reachable set of the original system with respect to $v(\cdot)$ is considered. The approximation $\Gamma_{\varepsilon,h}$ depends on the parameters ε and h. For $i \to \infty$, the iterations $K^i_{\varepsilon,h}$ similar to (22.22) give a *fully discrete discriminating kernel* $\overrightarrow{Disc}_{\Gamma_{\varepsilon,h}}(K_h) \subset K_h$. This set converges (under certain relation between ε and h) to the ideal set $\mathcal{A}^*$ as $\varepsilon \to 0$, $h \to 0$. It is proved that $\mathcal{A} = R^n \setminus \mathcal{A}^*$.

In numerical implementations, essential difficulties arise, when the grid K_h cannot be chosen bounded. Therefore, revealing cases where the grid K_h (as well as P_h) can be taken finite is of great interest.

In time-optimal games (games of degree), the set M is assumed to be closed. The first player, using the control u, minimizes the time of approaching M, the second player has the opposite objective. The time t is considered as an additional state variable (even for a stationary system). A discrete approximating dynamics defined by the parameter ε and grids with the step width h in variables $t \geq 0$ and x in R^n as well as a grid Q_h in the set Q are introduced. Let Z_h be a grid obtained in $\{t \geq 0\} \times R^n$. A grid approximation $\Gamma_{\varepsilon,h}(t_h, x_h, v_h)$ of the reachable set with respect to $u(\cdot)$ for the original system is considered, depending on the parameters ε and h.

The following decreasing sequence of sets is introduced:

$$\begin{cases} K^0_{\varepsilon,h} = Z_h, \\ K^{i+1}_{\varepsilon,h} = \{(t,x) \in K^i_{\varepsilon,h} : \forall v_h \in Q_h,\ \Gamma_{\varepsilon,h}(t_h, x_h, v_h) \cap K^i_{\varepsilon,h} \neq \varnothing\}, \quad i = 0, 1, \ldots \end{cases}$$

It is proved that $\lim\limits_{i\to\infty} K^i_{\varepsilon,h}$ exists. This limit is denoted as $\overrightarrow{Disc}_{\Gamma_{\varepsilon,h}}(Z_h)$. It is shown that the set $\overrightarrow{Disc}_{\Gamma_{\varepsilon,h}}(Z_h)$ converges to the epigraph of the value function of time-optimal game as $\varepsilon \to 0$, $h \to 0$ (under consistent relation between ε and h). In so doing, the problem of constructing the value function of time-optimal game is solved.

An interesting observation is done in paper (Botkin 1993). Consider a differential game $\dot{x} = f(x, u, v)$, where f globally possesses standard properties formulated in Krasovskii and Subbotin (1988). Let a compact $K \subset R^n$ be the state constraint, and $N = (-\infty, T] \times K$. A subset $Z \subset N$ is called stable if the player "v" can ensure the inclusion $(t, x(t)) \in Z$, $t \in [t_0, t_0 + \varepsilon]$, whenever $\big(t_0, x(t_0)\big) \in Z$, and the player "$u$" shows his constant control on the time interval $[t_0, t_0 + \varepsilon]$ (cf. the case of stable subsets in R^n). If there exists at least one stable subset of N, then there exists a maximal stable subset, say W, of N. If all time sections, $W(t)$, $t \in (-\infty, T]$, are nonempty, then $W(t)$ converges to the discriminating kernel $\overrightarrow{Disc}(K)$ in the Hausdorff metric as $t \to -\infty$. This gives rise (see Botkin 1993) to a recurrent algorithm resembling (22.22).

2.4 Families of Semipermeable Curves

When solving differential games in the plane, it is useful (as it was mentioned in Sect. 2.3) to carry out a preliminary study of families of smooth semipermeable curves that are determined by the dynamics of the controlled system. The knowledge of these families for time-optimal problems allows to verify the correctness of the construction of barrier lines on which the value function is discontinuous.

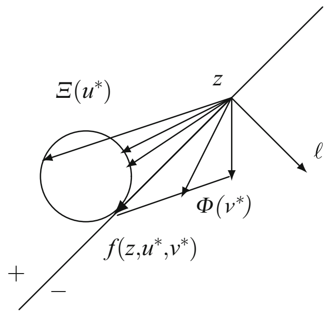
A smooth semipermeable curve is a line with the following preventing property: one of the players can prevent crossing the curve from the positive side to the negative one, the other player can prevent crossing the curve from the negative side to the positive one.

Let us explain the meaning of semipermeable curves. Introduce the minimax Hamiltonian of the game as follows:

$$H(\ell, z) = \min_u \max_v \ell' f(z, u, v) = \max_v \min_u \ell' f(z, u, v), \; z = (x, y)' \in R^2, \ell \in R^2.$$

Here $f(z, u, v) = p(z)u + v + g$, $p(z) = (-y, x)'$, $g = (0, -1)'$. Fix $z \in R^2$ and consider ℓ such that $H(\ell, z) = 0$. Denote $u^* = \arg\min_u \ell' f(z, u, v)$, $v^* = \arg\max_v \ell' f(z, u, v)$. It holds: $\ell' f(z, u^*, v) \leq 0$ for any $v \in Q$ and $\ell' f(z, u, v^*) \geq 0$ for any $u \in [-1, 1]$. This means that the direction $f(z, u^*, v^*)$ which is orthogonal to ℓ separates the vectograms $\Phi(v^*) = \bigcup_{u \in [-1,1]} f(z, u, v^*)$ and $\Xi(u^*) = \bigcup_{v \in Q} f(z, u^*, v)$ of the first and the second players (Fig. 22.4). Such a direction is called semipermeable. Thus, the semipermeable directions are defined by the roots of the equation $H(\ell, z) = 0$. We will distinguish the roots from "−" to "+" and the roots from "+" to "−". When defining these roots, we will suppose that $\ell \in E$, where E is a closed polygonal line around the origin. We say that ℓ_* is a root from "−" to "+" if $H(\ell_*, z) = 0$ and $H(\ell, z) < 0$ ($H(\ell, z) > 0$) for $\ell \prec \ell_*$ ($\ell \succ \ell_*$) that are sufficiently close to ℓ_*. The notation $\ell \prec \ell_*$ means that the direction of the vector ℓ can be obtained from the direction of the vector ℓ_* using the counterclockwise rotation by the angle not exceeding π. The roots from "−" to

Fig. 22.4 Semipermeable direction



"+" and the roots from "+" to "−" are called the roots of the first and of the second type, respectively.

One can prove that, in the game considered, the equation $H(\ell, z) = 0$ has at least one root of the first type and one root of the second type. Moreover, it has two roots of the first type and two roots of the second type at most. We denote the roots of the first type by $\ell^{(1),i}(z)$ and the roots of the second type by $\ell^{(2),i}(z)$. One can find the domains of the functions $\ell^{(j),i}(\cdot)$, $j = 1, 2$, $i = 1, 2$. The form of these domains is shown in Fig. 22.5.

It can be proved that the function $\ell^{(j),i}(\cdot)$ satisfies the Lipschitz condition in any closed subset of its domain. So, we can consider the following differential equation:

$$\frac{dz}{dt} = \Pi \ell^{(j),i}(z), \tag{22.23}$$

where Π is the matrix of rotation by the angle $\pi/2$ (the rotation's direction depends on j). Since the tangent at each point of phase trajectories of this equation is a semipermeable direction, the trajectories are semipermeable curves. It means that the first player can keep one side of the curve (say, positive side) and the second player can keep another side (negative side). So, the equation (22.23) specifies a family $\Lambda^{(j),i}$ of semipermeable curves.

In Fig. 22.6, the families of semipermeable curves for dynamics (22.5) are presented. There are families $\Lambda^{(1),1}$ and $\Lambda^{(1),2}$ of the first type and families $\Lambda^{(2),1}$ and $\Lambda^{(2),2}$ of the second type. The second upper index in the notation $\Lambda^{(j),i}$ indicates those of two extremal values of control u that corresponds to this family: $i = 1$ is related to curves which are trajectories for $u = 1$; $i = 2$ is related to curves which are trajectories for $u = -1$. The arrows show the direction of motion in reverse time. Due to symmetry properties of the dynamics, all families can be obtained from only one of them (e.g., $\Lambda^{(1),1}$) by means of reflections about the horizontal and vertical axes.

The construction of mentioned four families of smooth semipermeable curves can be explained as follows.

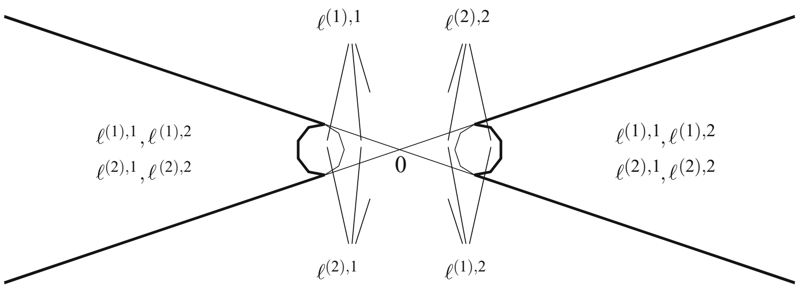


Fig. 22.5 Domains of $\ell^{(j),i}$

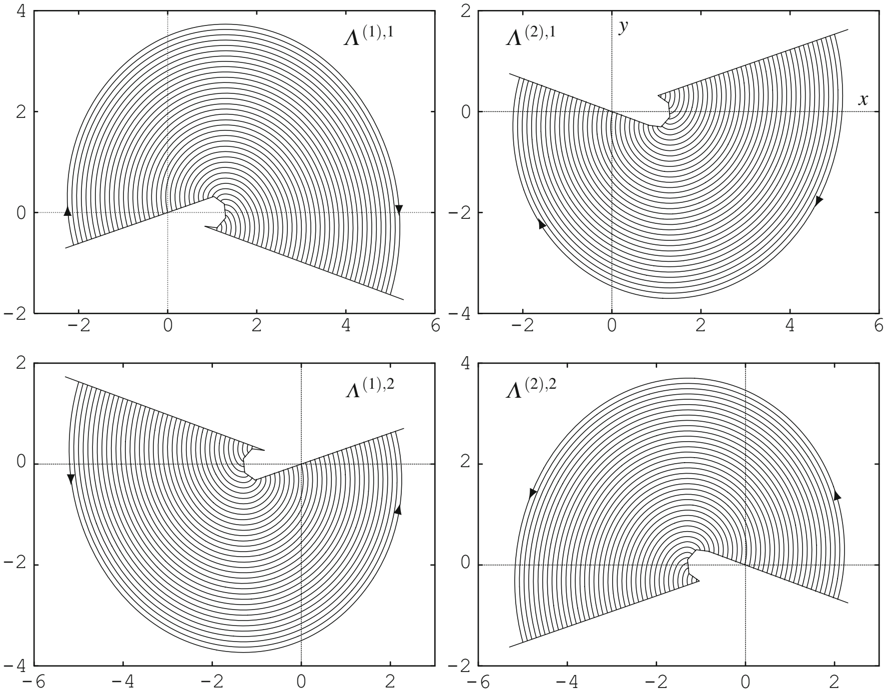


Fig. 22.6 Families of smooth semipermeable curves for the classical homicidal chauffeur dynamics

Assign the set

$$B_* = \{(x, y) : -y + v_x = 0,\ x - 1 + v_y = 0,\ v \in Q\}$$

to the control $u = 1$, and the set

$$A_* = \{(x, y) : y + v_x = 0,\ -x - 1 + v_y = 0,\ v \in Q\}$$

to the control $u = -1$. Hence, B_* is the set of all points in the plane x, y such that the velocity vector of system (22.5) vanishes at $u = 1$ and some $v \in Q$. We have $A_* = -B_*$.

Consider two tangents to the sets A_*, B_* passing through the origin (see Fig. 22.7), and mark arcs $a_1a_2a_3$ and $b_1b_2b_3$ on ∂A_* and ∂B_*, respectively.

Attach an inextensible string of a fixed length to the point b_1 and wind it up on the arc $b_1b_2b_3$. Then wind the string down keeping it taut in the clockwise direction. The end of the string traces an involute, which is a semipermeable curve of the family $\Lambda^{(1),1}$. The complete family $\Lambda^{(1),1}$ is obtained by changing the length of the string.

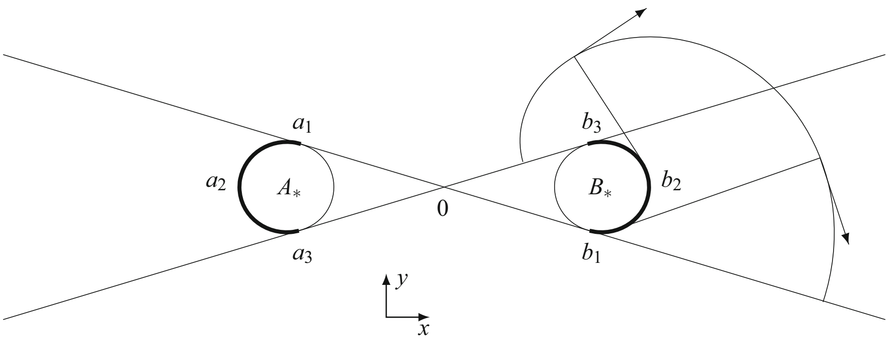


Fig. 22.7 Auxiliary arcs generating the families of smooth semipermeable curves for dynamics (22.5)

The family $\Lambda^{(2),2}$ is obtained as the collection of the counterclockwise involutes of the arc $a_1a_2a_3$ by attaching the string to the point a_3 .

The family $\Lambda^{(2),1}$ is generated by the clockwise involutes of the arc $b_1b_2b_3$ provided the string is attached to the point b_3.

The family $\Lambda^{(1),2}$ is composed of the counterclockwise involutes of the arc $a_1a_2a_3$ provided the string is attached to the point a_1.

The curves of different families belonging to the same type can be sewed in some cases so that the semipermeability property will be preserved. The procedure for computing the solvability set of the game of kind is based on the issuing two semipermeable curves (which are faced each to other with positive sides) from end points of M's usable part, on the analysis of their mutual disposition, and on the sewing semipermeable curves of different families belonging to the same type.

Families of semipermeable curves corresponding to dynamics (22.6) are arranged in a more complicated way (see Patsko and Turova 2009), and we do not present them here.

2.5 Classical Homicidal Chauffeur Problem

In the homicidal chauffeur problem described in the book by R. Isaacs, player P strives as soon as possible to bring the state vector of system (22.5) to a given closed bounded set M, whereas player E strives to prevent this.

2.5.1 Statement by R. Isaacs

Isaacs supposed that the terminal set M is a circle of radius r with the center at the origin. Thus, the description of the problem involves two independent parameters ν and r.

R. Isaacs investigated the problem for some parameter values using his method for solving differential games. The basis of the method is the backward computation

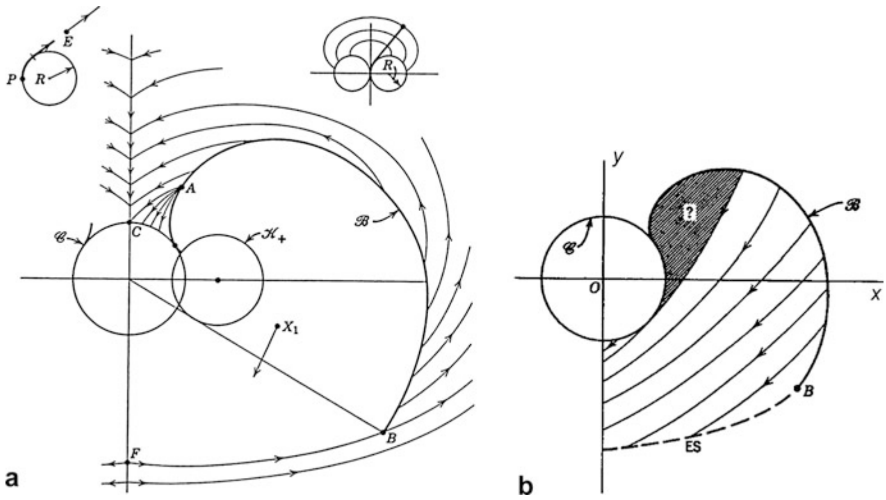


Fig. 22.8 Pictures by R. Isaacs from Isaacs (1965) explaining the solution to the homicidal chauffeur game

of characteristics for an appropriate partial differential equation. First, some primary region is filled out with regular characteristics, then secondary region is filled out, and so on. The final characteristics in the plane of state variables coincide with optimal trajectories.

Figure 22.8a shows a drawing from the book (Isaacs 1965) by R. Isaacs. The solution is symmetric with respect to the vertical axis. The upper part of the vertical axis is a singular line. Forward time-optimal trajectories meet this line at some angle and then go along it toward the target set M. According to the terminology by R. Isaacs, the line is called universal. The part of the vertical axis adjoining the target set from below is also a universal singular line. Optimal trajectories go down along it. The rest of the vertical axis below this universal part is dispersal: two optimal paths emanate from every point of it. On the barrier line $\mathcal{B}$, the value function is discontinuous. The side of the barrier line where the value of the game is smaller will be called positive. The opposite side is negative. One can see in Fig. 22.8a that the barrier line is a semipermeable curve of the first type. There is a similar line, but of the second type, in the left symmetric part.

The equivocal singular line ES emanates tangentially from the terminal point of the barrier (Fig. 22.8b). It separates two regular regions. Optimal trajectories that come to the equivocal curve split into two paths: the first one goes along the curve, and the second one leaves it and comes to the regular region on the right (optimal trajectories in this region are shown in Fig. 22.8a).

The equivocal curve is described through a differential equation which cannot be integrated explicitly. Therefore, any explicit description of the value function in the region between the equivocal and barrier lines is absent. The most difficult for the investigation is the “rear” part (Fig. 22.8b, shaded region) denoted by R. Isaacs with a question mark. He could not obtain a solution for this region.

The arising of singular lines (resp. singular surfaces in the case of higher dimension) that have no explicit description is a typical difficulty when investigating particular differential games. Having explicit formulas on initial stages of the backward construction of solution, we cannot come over a singular line and continue explicit description. Only qualitative investigation is possible. Because of that, development of numerical methods even for problems in the plane is necessary.

2.5.2 Brief Description of Numerical Algorithm for the Construction of Level Sets of the Value Function

In this subsection, we give a schematic description of the algorithm for the backward construction of level sets of the time-optimal value function. One can also say that, using the backward procedure, we construct the solvability sets of the game by a given time on some time grid. The equivalent term occurring in the literature is the backward (guaranteed) reachable set.

Primarily, the algorithm was developed (Patsko and Turova 1995, 1997) for linear time-optimal game problems in the plane. However, it turned out that the linearity of dynamics for time-optimal differential games does not give essential advantages in construction of level sets of the value function, since the level sets are often non-convex, which is typical for nonlinear dynamics. Therefore, for time-optimal games, nonlinear case is equivalent in difficulty to linear one. Hence, after some modernization, the algorithm have been used for nonlinear dynamics too (Patsko and Turova 2001).

The basic idea of the algorithm for approximate construction of the level sets $W_M(\tau) = \{(x, y) : V(x, y) \le \tau\}$ of the value function V is explained in the following.

We replace the set M by its polygonal approximation. Similarly, the geometric constraint Q on the control v is substituted by a polygon. The set $W_M(\tau)$ is formed via step-by-step backward procedure giving a sequence of embedded sets:

$$W_M(\Delta) \subset W_M(2\Delta) \subset W_M(3\Delta) \subset \ldots \subset W_M(i\Delta) \subset \ldots \subset W_M(\tau). \qquad (22.24)$$

Here Δ is the step of the backward procedure. Each set $W_M(i\Delta)$ consists of all initial points such that the first player brings system (22.5) into the set $W_M((i-1)\Delta)$ within the time duration Δ (we put $W_M(0) = M$).

The crucial point of the algorithm is the computation of “fronts.” The front F_i (Fig. 22.9) is the set of all points of $\partial W_M(i\Delta)$ for which the minimum guaranteeing time of reaching $W_M((i-1)\Delta)$ is equal to Δ. For other points of $\partial W_M(i\Delta)$ the optimal time is less than Δ. The line $\partial W_M(i\Delta) \setminus F_i$ possesses the properties of the barrier. The front F_i is constructed using the previous front F_{i-1}. For the first step of the backward procedure, F_0 coincides with the usable part Γ_0 of the boundary of M.

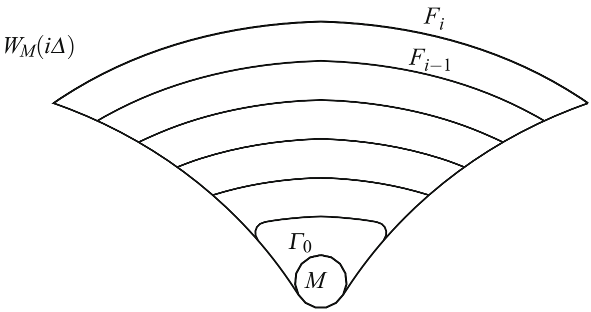


Fig. 22.9 Construction of the sets $W_M(i\,\Delta)$

Let us explain how the fronts can be constructed. Suppose the front F_{i-1} is a smooth curve. Let z_* be an arbitrary point of F_{i-1} and ℓ is the normal vector to the front at z_*. Let $u^\circ = \arg\min_{|u|\le 1} \ell' p(z_*)u, \quad v^\circ = \arg\max_{v\in Q} \ell' v.$

We call u°, v° the extremal controls. The controls u° and v° are chosen from the conditions of minimizing and, respectively, maximizing the projection of the velocity vector of (22.5) onto the direction ℓ. If the vector pointed to z_* is collinear to ℓ, then any control $u \in [-1, 1]$ is extremal. If Q is a polygon in the plane, and ℓ is collinear to some normal vector to an edge $[q_1, q_2]$ of Q, then any control $q \in [q_1, q_2]$ is extremal.

After computing the extremal controls, the extremal trajectories of system (22.5) issued from the front's points backward in time are considered: $z(\tau) = z_* - \tau(p(z_*)u^\circ + v^\circ + g)$. The ends of these trajectories at $\tau = \Delta$ form the next front F_i. If the extremal control u° is not unique at some point $z_* \in F_{i-1}$, then the segment $\Phi(z_*) = \{\bigcup_{u^\circ\in[-1,1]} (z_* - \Delta\,(p(z_*)u^\circ + v^\circ + g))\}$ is considered instead of the single point. Similarly, if the extremal control v° is not unique, the segment $\Xi(z_*) = \{\bigcup_{v^\circ\in[q_1,q_2]} (z_* - \Delta\,(p(z_*)u^\circ + v^\circ + g))\}$ is considered.

For each front, we distinguish points of the local convexity and points of the local concavity. In Fig. 22.10, d is a point of the local convexity, and e is a point of the local concavity. If z_* is a point of the local convexity and the extremal control u° is not unique, we obtain a local picture like the one shown in Fig. 22.11a after issuing the extremal trajectories from the point z_*. Here, the additional segment $\Phi(z_*)$ appears on the new front F_i. If the extremal control v° is not unique, we obtain a local picture similar to the one shown in Fig. 22.11b: the "swallow tail" $\beta_1\xi\beta_2$ does not belong to the new front F_i and it is taken away. For points of the local concavity, there is an inverse situation: if the extremal control u° is not unique, a swallow tail that should be removed appears; if the extremal control v° is not unique, an additional segment $\Xi(z_*)$ appears on the new front F_i. If both u° and v° are nonunique, the insertion or the swallow tail arises depending on which of segments $\Phi(z_*)$ or $\Xi(z_*)$ is greater.

In the course of numerical computations, we operate with polygonal lines instead of smooth curves. Two normal vectors to the links $[a, b]$, $[b, c]$ of the polygonal

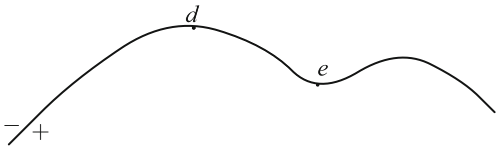


Fig. 22.10 Local convexity and concavity

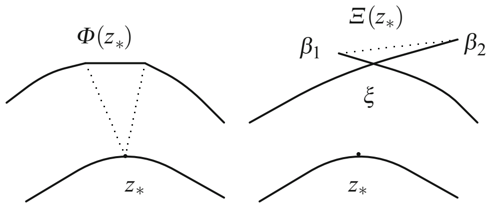


Fig. 22.11 Nonuniqueness of extremal controls in the case of local convexity

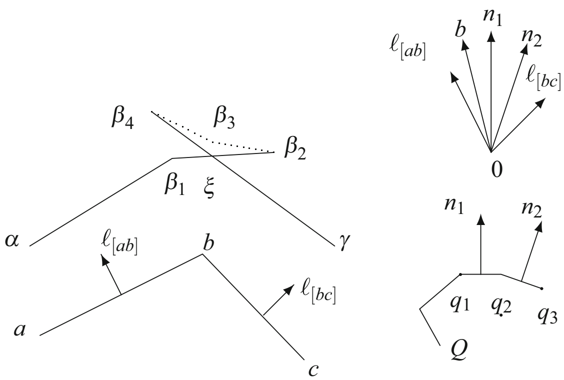


Fig. 22.12 Example of local constructions

line are considered at each vertex b (Fig. 22.12). The algorithm treats all possible variants of disposition of the vectors $\ell_{[ab]}, \ell_{[bc]}$, normals to the edges of Q, and the vector b. In Fig. 22.12, for instance, the case is shown where the vector b is between the vectors $\ell_{[ab]}, \ell_{[bc]}$, and the normals $n_1,\ n_2$ to the set Q are between the vectors b and $\ell_{[bc]}$. The extremal controls of the players are computed for each of these vectors, and the extremal trajectories are issued from the points a, b, c. The ends of these trajectories computed at $\tau = \Delta$ give a local picture shown in Fig. 22.12. In the case considered, four extremal trajectories were issued from the point b. Their ends are $\beta_1, \beta_2, \beta_3$, and β_4. The segment $[\beta_1, \beta_2]$ appears due to nonuniqueness of the extremal control u° for the vector b. The segments $[\beta_2, \beta_3]$ and $[\beta_3, \beta_4]$ arise due to nonuniqueness of the extremal control v° for the vectors n_1 and n_2. After removing the swallow tail $\xi\beta_4\beta_3\beta_2\xi$, the polygonal line $\alpha\beta_1\xi\gamma$ is obtained to be a fragment of the next front.

2.5.3 Two Examples of Numerical Solution of Classical Problem

Figure 22.13 shows level sets $W_M(\tau) = \{(x, y) : V(x, y) \leq \tau\}$ of the value function $V(x, y)$ for $\nu = 0.3, r = 0.3$. The numerical results presented in Fig. 22.13

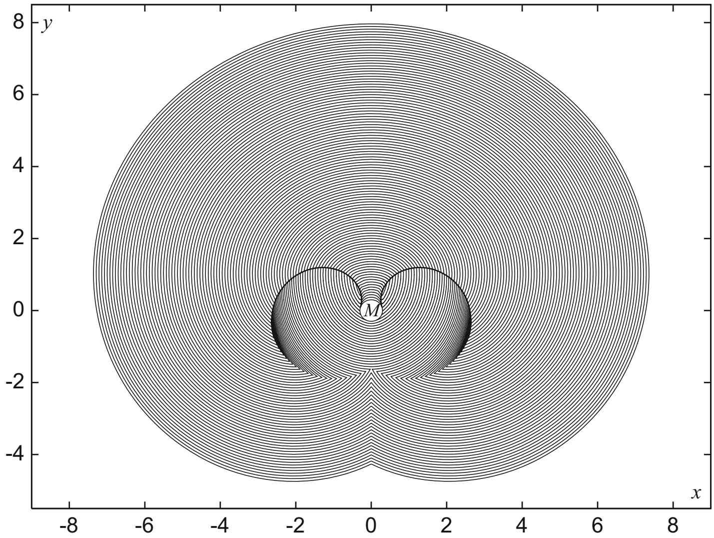


Fig. 22.13 Level sets of the value function for the classical problem; game parameters $\nu = 0.3$ and $r = 0.3$; backward computation is done till the time $\tau_f = 10.3$ with the time step $\Delta = 0.01$, output step for fronts $\delta = 0.1$

are obtained using the algorithm, which is described in the previous subsection. The lines on the boundary of the sets $W_M(\tau)$, $\tau > 0$, consisting of points (x, y) where the equality $V(x, y) = \tau$ holds, are fronts (isochrones). For the visualization of graphs of the value function in time-optimal differential games, a special computer program has been developed (Averbukh et al. 2000).

The computation for Fig. 22.13 is done with the time step $\Delta = 0.01$ till the time $\tau_f = 10.3$. The output step for fronts is $\delta = 0.1$. The set M is approximated by an inscribed regular 20-polygon and the set Q by a 24-polygon. Figure 22.14 presents the graph of the value function. The value function is discontinuous on the two barrier lines and on a part of the boundary of the target set. The barrier lines are arcs of semipermeable curves of the families $\Lambda^{(1),1}$ and $\Lambda^{(2),2}$. In the case considered, the value function is smooth in the abovementioned rear region.

If the center of the target circle is shifted from the y-axis, the symmetry of the solution with respect to y-axis is destroyed. The arising front structure to the negative side of barrier lines can be very complicated. One of such examples is presented in Fig. 22.15. The target circle of radius 0.075 is centered at the point with the coordinates $m_x = 1, m_y = 1.5$. The computation time step $\Delta = 0.01$. The maximal value of the game for computed fronts is 9.5, and it is attained at a point in the second quadrant. The fronts are depicted with the time step $\delta = 0.08$. Figure 22.16 presents the graph of the value function.

The real situation corresponding to a shifted target set M may be the following. Assume that the pursuer is able to create a small circular killing zone in some

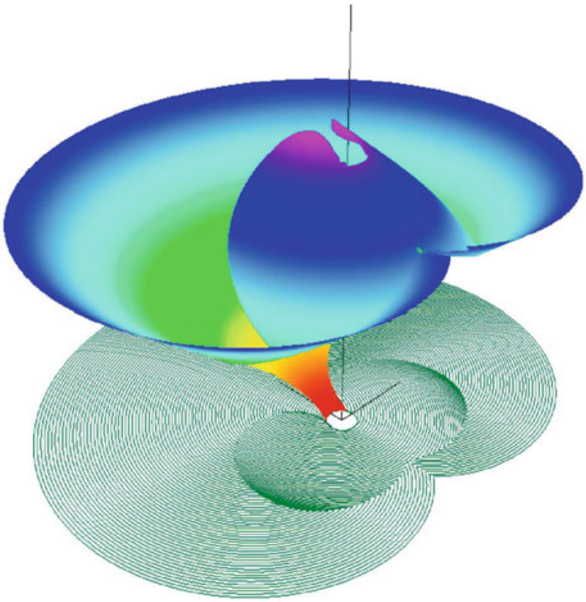


Fig. 22.14 Graph of the value function; $\nu = 0.3$, $r = 0.3$

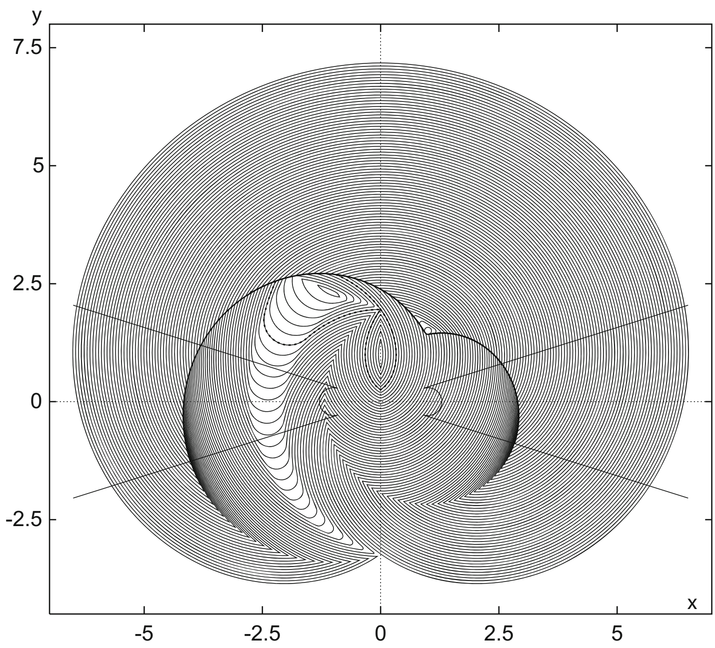


Fig. 22.15 Nontrivial structure of fronts for shifted target circle; $\nu = 0.3$, $\tau_f = 9.5$, $\Delta = 0.01$, $\delta = 0.08$. Target set is a circle of radius $r = 0.075$ centered at point $(1, 1.5)$

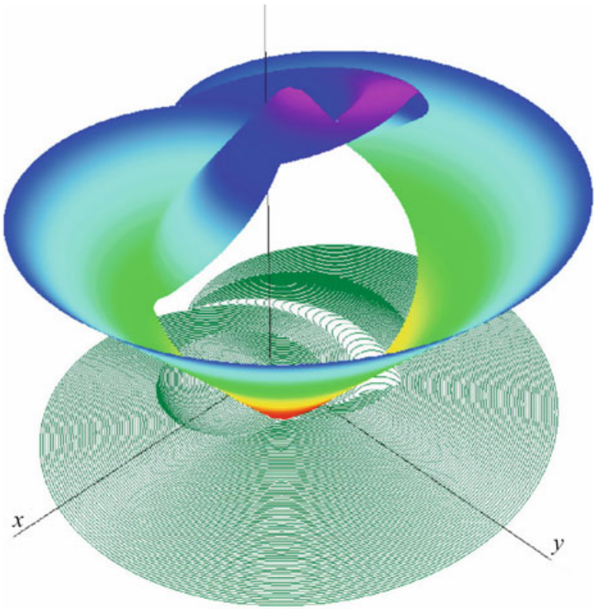


Fig. 22.16 Graph of the value function for shifted target circle; $\nu = 0.3$, $r = 0.075$

distance and at some angle to the velocity vector direction in front of himself. Then the pursuer minimizes the time of capture of the evader in such killing zone.

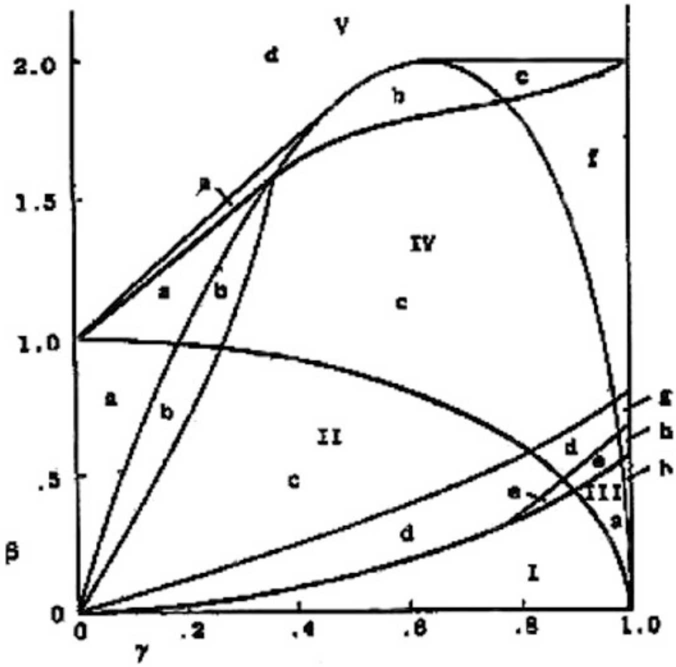
2.5.4 Investigations by J. Breakwell and A. Merz

J. Breakwell and A. Merz continued the investigation of the homicidal chauffeur game in the setting by R. Isaacs. Their results are partly and very briefly described in the papers (Breakwell and Merz 1969; Merz 1974). A complete solution is obtained by A. Merz in his PhD thesis (1971) at Stanford University.

A. Merz divided the two-dimensional parameter space into 20 subregions. He investigated the qualitative structure of the optimal paths and the type of singular lines for every subregion. All types of singular curves (dispersal, universal, equivocal, and switch lines) described by R. Isaacs for differential games in the plane appear in the homicidal chauffeur game for certain values of parameters. In his thesis, A. Merz suggested to distinguish some new types of singular lines and consider them separately. Namely, he introduced the notion of focal singular lines which are universal ones but with tangential approach of optimal paths. The value function is non-differentiable on the focal lines.

Figure 22.17 presents a picture and a table from the thesis by A. Merz that demonstrate the partition of two-dimensional parameter space ν, r into subregions with certain system of singular lines (A. Merz used symbols γ, β for the notation of parameters. He called singular lines as exceptional lines).

The thesis contains many pictures explaining the type of singular lines and the structure of optimal paths. By studying them, one can easily detect tendencies in the behavior of the solution depending on the change of the parameters.



DISTRIBUTION OF EXCEPTIONAL LINES

Region	Sub-region	Exceptional Lines Present in Each Subregion											
		S	UL	PDL_y	PDL	EDL_a	EDL_c	ID^+	ID^-	EL	SL	SE	FL
I		x	x										
II	a	x	x	x					x		x		
	b	x	x	x					x	x	x		
	c	x	x	x					x	x			
	d	x	x	x					x	x		x	
	e	x	x	x					x	x		x	x
III	a	x	x			x		x					
	b	x	x				x	x					
IV	a		x	x	x			x	x		x		
	b		x	x	x			x	x	x	x		
	c		x	x	x	x		x	x	x			
	d		x	x	x	x		x	x	x		x	
	e		x	x	x	x		x	x	x		x	x
	f		x	x	x		x	x	x	x			
	g		x	x	x		x	x	x	x		x	
	h		x	x	x		x	x	x	x		x	x
V	a		x	x				x					
	b		x	x		x		x					
	c		x	x			x	x					
	d		x	x									

Fig. 22.17 Decomposition of two-dimensional parameter space into subregions

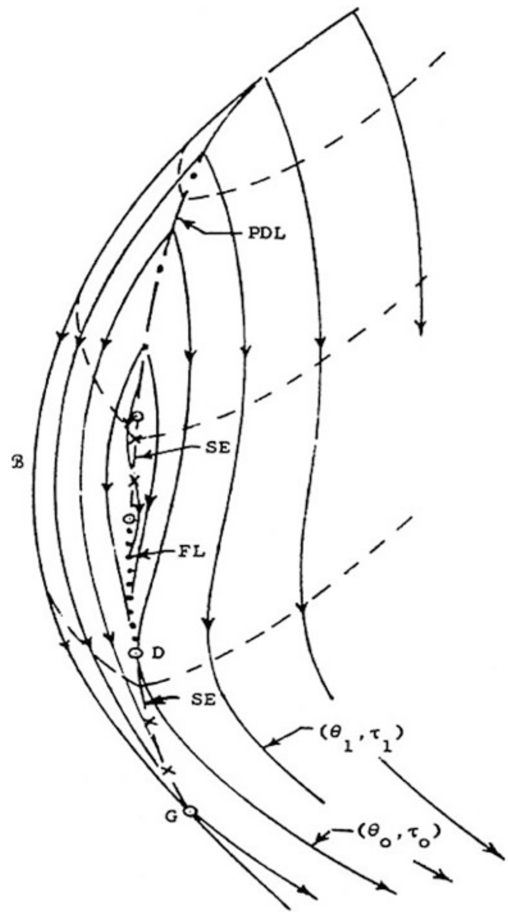


Fig. 22.18 Structure of optimal paths in the rear part for subregion IIe

In Fig. 22.18, the structure of optimal paths in that part of the plane that adjoins the negative side of the barrier is shown for the parameters corresponding to subregion IIe. This is the rear part denoted by R. Isaacs with a question mark. For subregion IIe, very complicated situation takes place.

PDL denotes the dispersal line controlled by player P. Two optimal trajectories emanate from every point of this line. Player P controls the choice of the side to which trajectories come down. Singular curve SE (the switch envelope) is specified as follows. Optimal trajectories approach it tangentially. Then one trajectory goes along this curve, and the other (equivalent) one leaves it at some angle. Therefore, line SE is similar to an equivocal singular line. The thesis contains arguments according to which the switch envelope should be better considered as an individual type of singular line.

FL denotes the focal line. The dotted curves mark boundaries of level sets (in other words, isochrones or fronts) of the value function.

The value function is not differentiable on the line composed of the curves PDL, SE, FL, and SE.

The authors of this chapter undertook many efforts to compute the value function for parameters from subregion IIe. But it was not successful, since we could not obtain corner points that must be present on fronts to the negative side of the barrier. One of the possible explanations to this failure can be the following: the effect is so subtle that it cannot be detected even for very fine discretizations. The computation of level sets of the value function for the subregions where the solution structure changes very rapidly, dependent on the parameters, can be considered as a challenge for differential game numerical methods being presently developed by different scientific teams.

Figure 22.19 demonstrates computation results for the case where fronts have corner points in the rear region. However, the values of parameters correspond not to subregion IIe but to subregion IId. For the latter case, singular curve SE remains, but focal line FL disappears.

For some subregions of parameters, barrier lines on which the value function is discontinuous disappear. A. Merz described a very interesting transformation of the barrier line into two close to each other dispersal curves of players P and E. In this case, there exist both optimal paths that go up and those that go down along

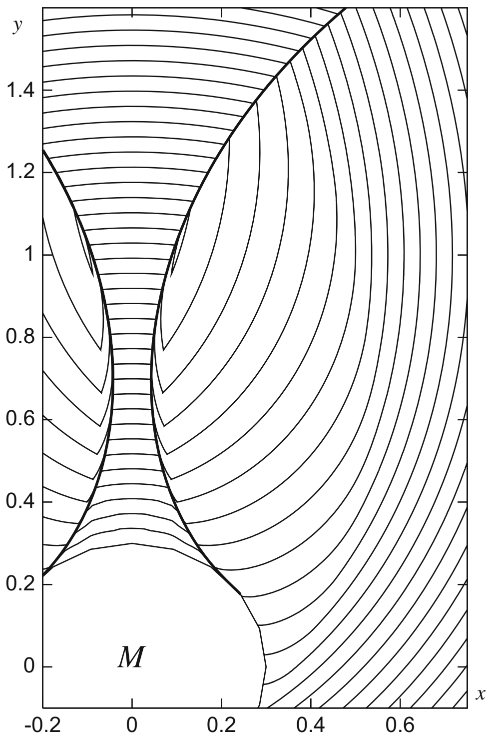


Fig. 22.19 Level sets of the value function for parameters from subregion IId; $\nu = 0.7$, $r = 0.3$; $\tau_f = 35.94$, $\Delta = 0.006$, $\delta = 0.12$

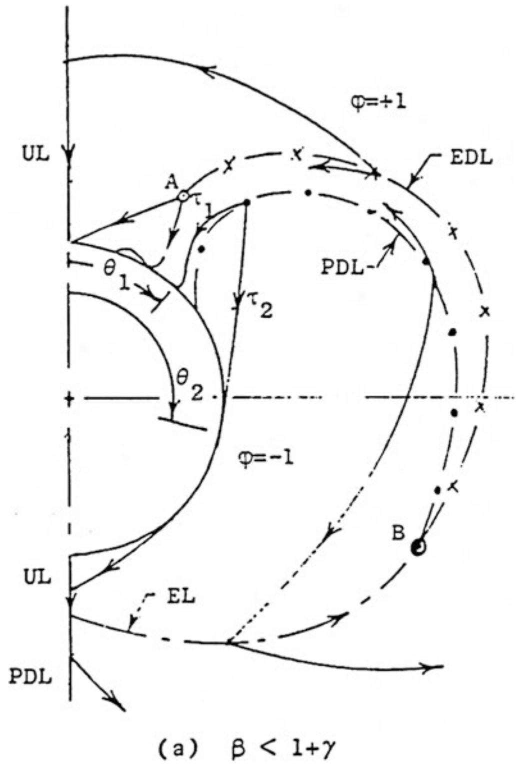


Fig. 22.20 Structure of optimal trajectories in subregion IVc

the boundary of the target set. The investigation of such a phenomenon is of great theoretical interest.

Figure 22.20 presents a picture from the thesis by A. Merz that corresponds to subregion IVc (A. Merz as well as R. Isaacs used the symbol φ for the notation of the control of player P. In this text, the corresponding notation is u). Numerically constructed level sets of the value function are shown in Fig. 22.21. When examining Fig. 22.21, it might seem that some barrier line exists. But this is not true. This again underlines the importance of theoretical investigation of particular differential games. Without the work completed by A. Merz, the presence of some barrier line in that place could erroneously be established based on the numerical outcome. Accounting for the results by A. Merz (obtained mainly analytically) enables refining our numerical constructions. Here, we have exactly the case like the one shown in Fig. 22.20. In Fig. 22.22, an enlarged fragment of Fig. 22.21 is given. The curve consisting of fronts' corner points above the accumulation region of fronts is the dispersal line of player E. The curve composed of corner points below the accumulation region is the dispersal line of player P. The value function is continuous in the accumulation region. To see where (in the considered part of the plane) the point of a maximal value of the game is located, additional fronts are

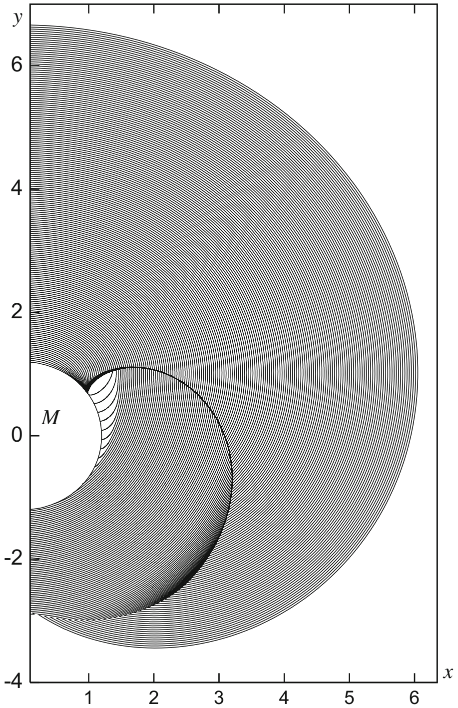


Fig. 22.21 Level sets of the value function; $\nu = 0.7$, $r = 1.2$; $\tau_f = 24.22$, $\Delta = 0.005$, $\delta = 0.1$

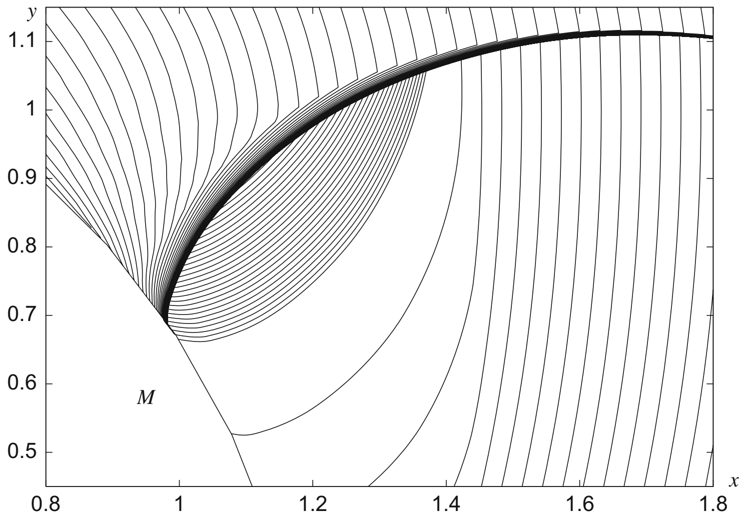


Fig. 22.22 Enlarged fragment of Fig. 22.21; $\tau_f = 24.22$. Output step for fronts close to the time τ_f is decreased up to $\delta = 0.005$

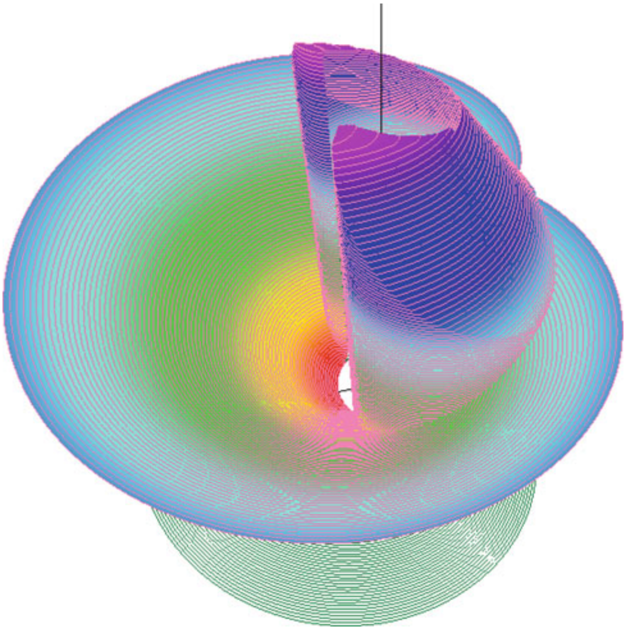


Fig. 22.23 Graph of the value function; $\nu = 0.7$, $r = 1.2$. Level lines are plotted. Salient curve corresponding to the line PDL from Fig. 22.20 is seen

shown. The point of the maximal value has coordinates $x = 1.1$, $y = 0.92$. The value function at this point is equal to 24.22.

The graph of the value function for the example considered is shown in Fig. 22.23. The level lines are plotted to make visible two curves consisting of salient points. Taking into account the symmetry with respect to the y-axis, the curves correspond to the dispersal singular lines EDL and PDL in the plane x, y (see Fig. 22.20).

2.6 Surveillance-Evasion Game

In the PhD thesis by J. Lewin (1973) (performed as well under the supervision of J. Breakwell), in the joint paper by J. Breakwell and J. Lewin (1975), and also in the paper by J. Lewin and G.-J. Olsder (1979), both dynamics and constraints on the controls of the players are the same as in Isaacs' setting but the objectives of the players differ from those in the classic statement. Namely, player E tries to decrease the time of reaching the target set M by the state vector, whereas player P strives to increase that time. In the first and second works, the target set is the complement (with respect to the plane) of an open circle centered at the origin. In the third publication, the target set is the complement of an open cone with the apex at the origin.

The meaning related to the original context concerning two moving objects is the following: player E tries, as soon as possible, to escape from some detection zone attached to the geometric position of player P, whereas player P strives to keep his opponent in the detection zone as long as possible. Such a problem was called the surveillance-evasion game. To solve it, J. Breakwell, J. Lewin, and G.-J. Olsder used Isaacs' method.

Below, level sets of the value function computed for the game with conic terminal set with the backward procedure from Sect. 2.5.2 are presented. Herewith, the extremal controls of the pursuer P and the evader E are determined via the relations $u^\circ = \operatorname{argmax}\{\ell' p(z_*)u : |u| \le 1\}$ and $v^\circ = \operatorname{argmin}\{\ell' v : v \in Q\}$ for every point z_* of local convexity and outer normal ℓ to the front at z_*. For the points of local concavity, the extremal controls of P and E are defined by the formulae $u^\circ = \operatorname{argmin}\{\ell' p(z_*)u : |u| \le 1\}$ and $v^\circ = \operatorname{argmax}\{\ell' v : v \in Q\}$, where ℓ is an inner normal to the front at z_*. So, the local constructions described earlier for the points of local convexity are now true for the points of local concavity and vice versa. In the results presented, the circle constraint of radius $\nu = 0.588$ on the control v of player E is substituted by an inscribed regular hexagon.

In the surveillance-evasion game with the conic target set M (the detection zone is the cone $R^2 \backslash M$ of semi-angle α), examples of transition from finite values of the game to infinite values are of interest and can be easily constructed.

Figure 22.24 shows level sets of the value function for five values of parameter $\alpha = 143°, 136.3°, 130°, 125.6°, 121°$. Since the solution to the problem is symmetric with respect to y-axis, only the right half-plane is shown for four of five figures. The pictures are ordered from greater to smaller α.

In the first picture, the value function is finite in the set that adjoins the target cone and is bounded by the curve $a'b'cba$. This set is filled out with the fronts (isochrones). The value function is zero within the target set. Outside the union of the target set and the set filled out with the fronts, the value function is infinite.

In the third picture, a situation of the accumulation of fronts is presented. Here, the value function is infinite on the line fe and finite on the arc ea. The value function has a finite discontinuity on the arc be.

The second picture demonstrates a transition case from the first to the third picture.

In the fifth picture, the fronts propagate slowly to the right and fill out (outside the target set) the right half-plane as the backward time τ goes to infinity. Figure 22.25 gives a graph of the value function for this case.

The fourth picture shows a transition case between the third and fifth pictures.

Note that all lines on which the value function is discontinuous (barrier lines) are arcs of families of semipermeable curves described in Sect. 2.4.

2.7 Acoustic Game

Let us return to problems where player P minimizes and player E maximizes the time of reaching the target set M. In papers (Cardaliaguet et al. 1995, 1999),

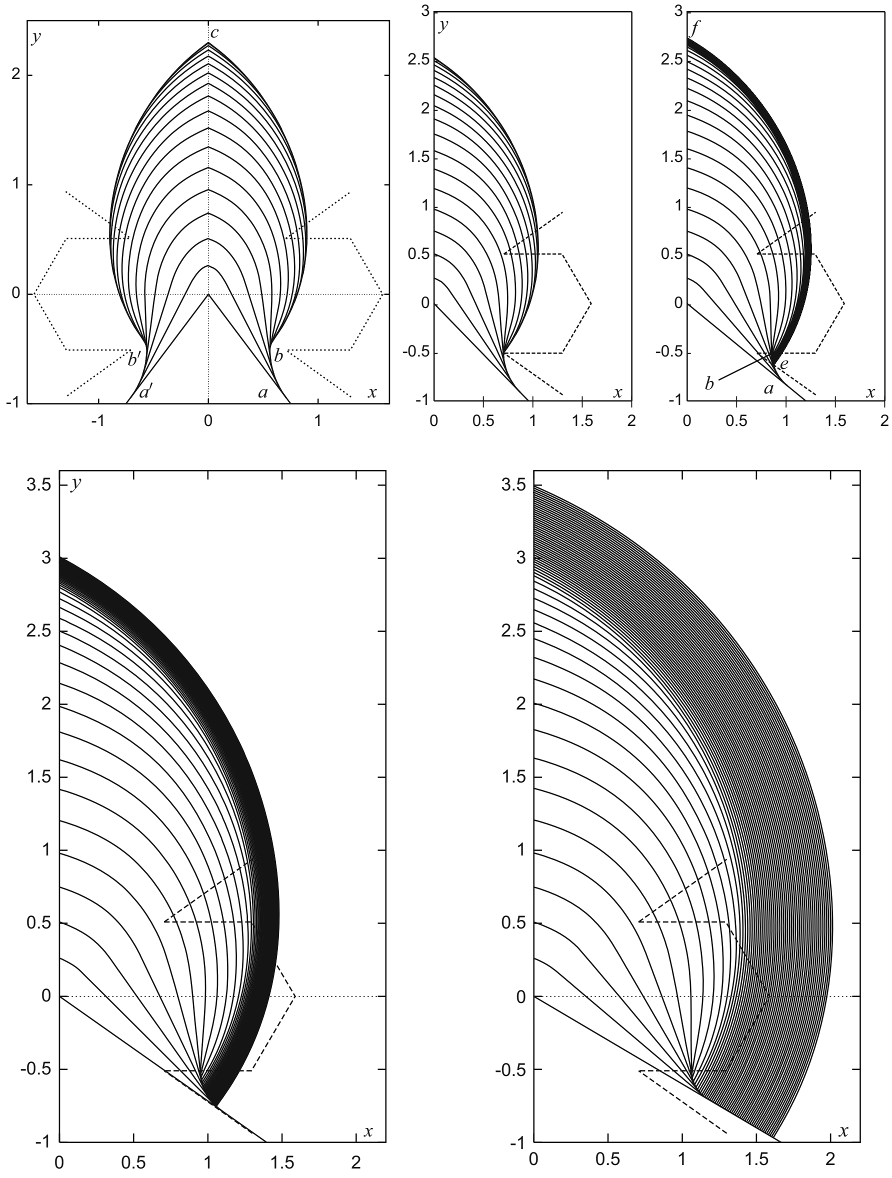


Fig. 22.24 Surveillance-evasion game. Change of the front structure depending on the semi-angle α of the nonconvex detection cone; $\nu = 0.588$, $\Delta = 0.017$, $\delta = 0.17$

P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre have considered an "acoustic" variant of the homicidal chauffeur problem. It is supposed that the constraint ν on the control of player E depends on the state (x, y). Namely,

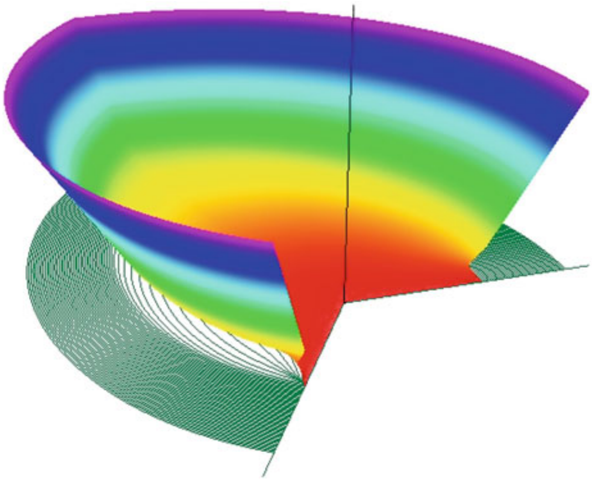


Fig. 22.25 Value function in the surveillance-evasion game: $\nu = 0.588$, $\alpha = 121°$

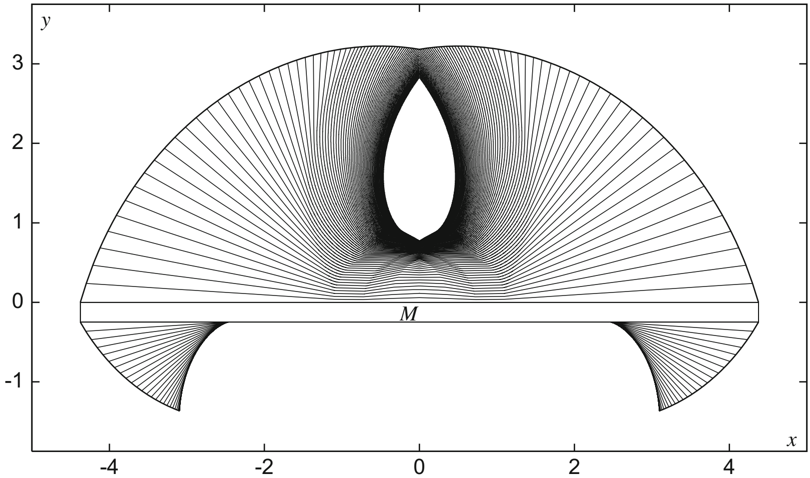


Fig. 22.26 Level sets of the value function in the acoustic problem; $\nu^* = 1.5$, $s = 0.9375$; $\Delta = 0.00625$, $\delta = 0.0625$

$$\nu(x, y) = \nu^* \min\left\{1, \sqrt{x^2 + y^2}/s\right\},\ s > 0.$$

Here, ν^* and s are the parameters of the problem.

The applied aspect of the acoustic game: object E should not be very loud if the distance between him and object P becomes less than a given value s. Such an applied aspect and its interpretation were suggested by P. Bernhard. Let us cite here work (Bernhard and Larrouturou 1989): "Emitted noise is a function of evader's speed, while perceived noise is also a function of the distance between the opponents."

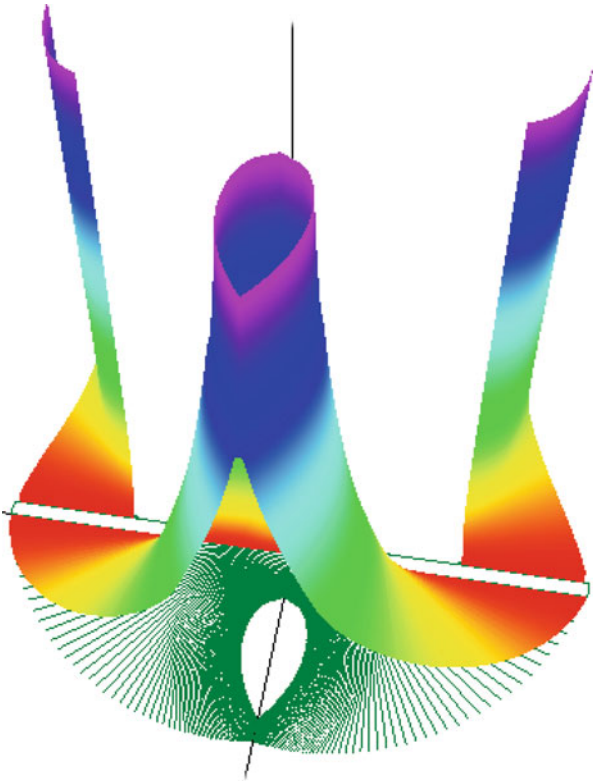


Fig. 22.27 Graph of the value function in the acoustic problem; $\nu^* = 1.5$, $s = 0.9375$

P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre investigated the acoustic problem using their method for numerical solving of differential games that was briefly described in Sect. 2.3 (item 7). It was revealed that one can choose the values of the parameters in such a way that the set of states where the value function is finite will contain a hole in which points the value function is infinite. Especially easy such a case can be obtained when the target set is a rectangle stretched along the horizontal axis.

Figures 22.26 and 22.27 demonstrate an example of the acoustic problem with the hole. The level sets of the value function and the graph of the value function are shown. The value of the game is infinite outside the set filled out with the fronts.

Let us underline that the abovementioned hole is separated from the target set. In Fig. 22.28, level sets for the parameters $\nu^* = 1.4$, $s = 2.5$ are presented. The graph of the value function is shown in Fig. 22.29. Also here, a hole with infinite magnitudes of the value function arises. But this hole touches the target set, which allows one to compute it easily through the barrier lines emanated from some points on the boundary of the target set.

The acoustic homicidal chauffeur problem is carefully investigated in Patsko and Turova (2001, 2004). These works also contain findings on families of

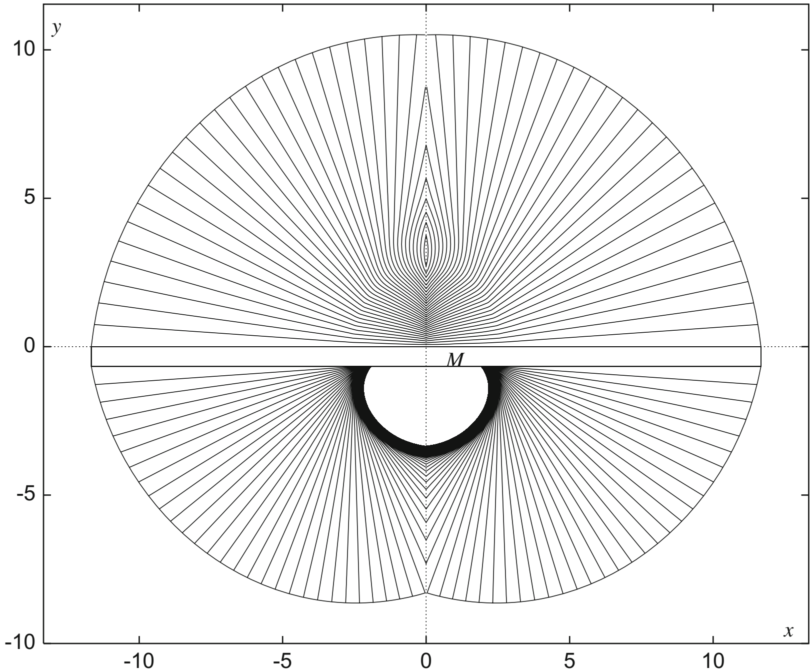
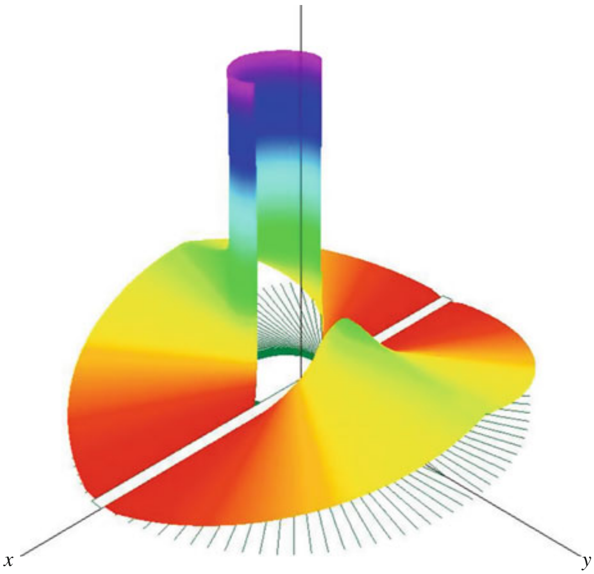


Fig. 22.28 Level sets of the value function in the acoustic problem; $\nu^* = 1.4$, $s = 2.5$; $\Delta = 1/30$, $\delta = 1/6$

Fig. 22.29 Graph of the value function in the acoustic problem; $\nu^* = 1.4$, $s = 2.5$



semipermeable curves of the first and second types arising in this problem under various values of parameters ν^*, s. With the help of semipermeable curves, the appearance of the hole with infinite value of the game (see Figs. 22.26 and 22.27) is explained.

2.8 Game with a More Agile Player *P*

Consider the homicidal chauffeur problem, in which player P controls the car that can change his linear velocity instantaneously. Here, we use dynamics equations (22.6). Accordingly, numerical procedures of Sect. 2.5.2 for the computation of level sets of the value function become more complicated. In more detail, the problem with a more agile player P is investigated in Patsko and Turova (2009).

2.8.1 Level Sets of the Value Function

In Fig. 22.30, the level sets of the value function which correspond to one and the same time $\tau = 3$ but to different values of the parameter a from -1 to 1 are presented. For all computations, the radius of the target set is $r = 0.3$ and the constraint on the control of player E is $\nu = 0.3$. In case $a = -1$, player P controls a Reeds-Shepp's car, and the obtained level set is symmetric with respect to both y-axis and x-axis. If $a = 1$, the level set for the classical homicidal chauffeur game is obtained.

Figure 22.31 shows the level sets of the value function for $a = -0.1$, $\nu = 0.3$, $r = 0.3$. The computation is done backward in time till $\tau_f = 4.89$. Precisely this value of the game corresponds to the last outer front and to the last inner front adjoining to the lower part of the boundary of the target circle M. The front structure is well seen in Fig. 22.32 showing an enlarged fragment of Fig. 22.31. One can see a nontrivial character of changing the fronts near the lower border of the accumulation region. The value function is discontinuous on the arc dhc. It is also discontinuous outside M on two short barrier lines emanating tangentially from the boundary of M. The right barrier is denoted by ce.

2.8.2 Optimal Strategies

When solving time-optimal differential games of the homicidal chauffeur type (with discontinuous value function), the most difficult task is the construction of the optimal (or ε-optimal) strategies of the players. Let us demonstrate such a construction using the last example.

We construct ε-optimal strategies using the extremal aiming procedure (Krasovskii 1985; Krasovskii and Subbotin 1988). The computed control remains unchanged during the next step of the discrete control scheme. The step of the control procedure is a modeling parameter. The strategy of player P (E) is defined using the extremal shift to the nearest point (extremal repulsion from the nearest point) of the corresponding front. If the trajectory comes to a prescribed layer

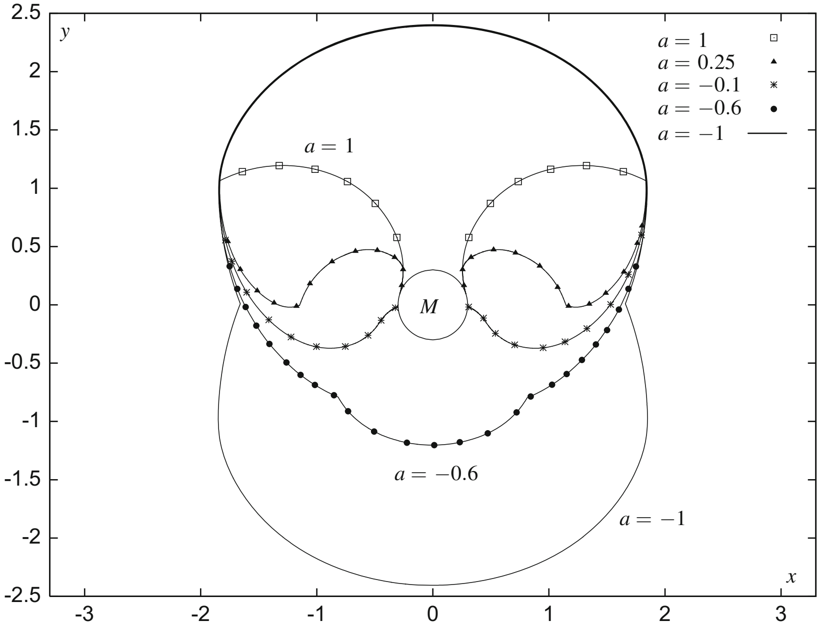


Fig. 22.30 Homicidal chauffeur game with more agile pursuer. Dependence of level sets of the value function on the parameter a for $\tau = 3$; $\nu = 0.3$, $r = 0.3$

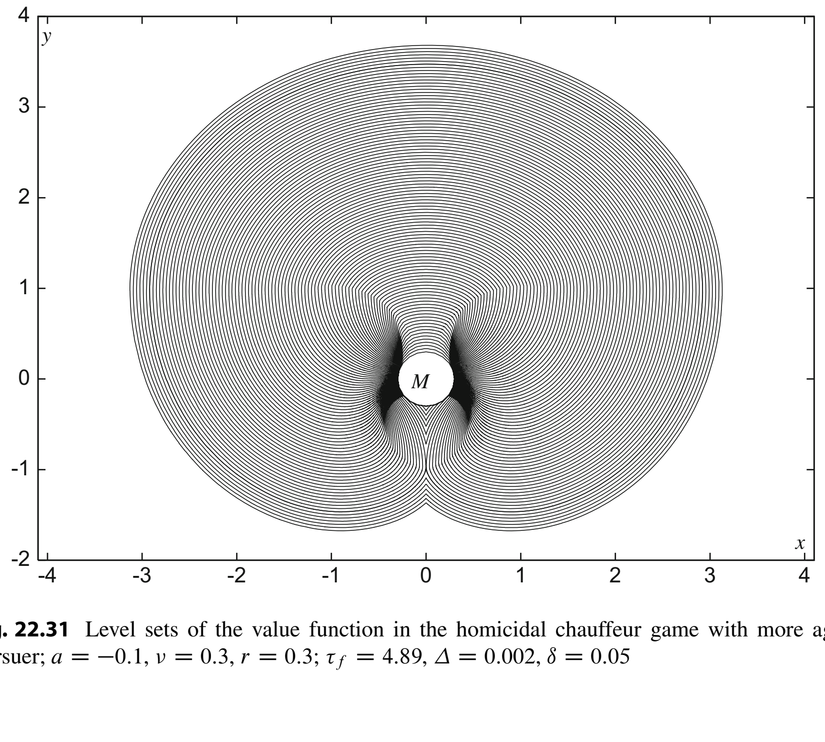


Fig. 22.31 Level sets of the value function in the homicidal chauffeur game with more agile pursuer; $a = -0.1$, $\nu = 0.3$, $r = 0.3$; $\tau_f = 4.89$, $\Delta = 0.002$, $\delta = 0.05$

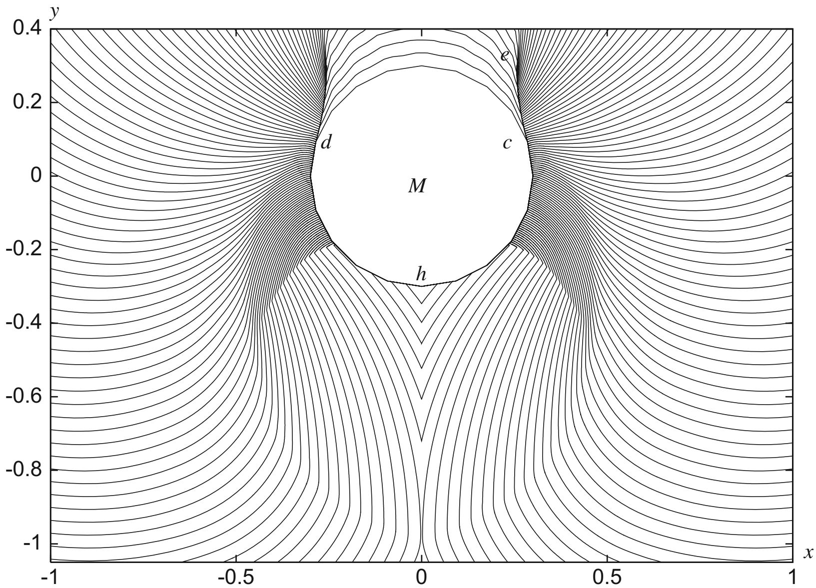


Fig. 22.32 Enlarged fragment of Fig. 22.31

attached to the positive (negative) side of the discontinuity line of the value function, then a control which pushes away from the discontinuity line is utilized.

Let us choose two initial points $a = (0.3, -0.4)$ and $b = (0.29, 0.1)$. The first point is located in the right half-plane below the front accumulation region, the second one is close to the barrier line on its negative side. The values of the game in the points a and b are $V(a) = 4.225$ and $V(b) = 1.918$, respectively.

In Fig. 22.33, the trajectories for ε-optimal strategies of the players are shown. The time step of the control procedure is 0.01. We obtain that the time of reaching the target set M is equal to 4.230 for the point a and 1.860 for the point b. Figure 22.33c demonstrates an enlarged fragment of the trajectory emanating from the initial point b. One can see a sliding mode along the negative side of the barrier.

Figure 22.34 presents trajectories for nonoptimal behavior of player E and optimal behavior of player P. The control of player E is computed using a random number generator (random choice of vertices of the polygon approximating the circle constraint of player E). The reaching time is 2.590 for the point a and 0.300 for the point b. One can see how the second trajectory penetrates the barrier line. In this case, the value of the game calculated along the trajectory drops jump-wise.

In Fig. 22.35, the trajectories for nonoptimal behavior of player P and optimal behavior of player E are shown. The control u of player P acts in optimal way, whereas the control w is nonoptimal. For Fig. 22.35a, $w \equiv 1$. The time of reaching the target set is 7.36. For Fig. 22.35b, c, $w \equiv -1$ until the trajectory comes to the vertical axis, after that $w \equiv 1$. Figure 22.35c demonstrates an enlarged fragment of

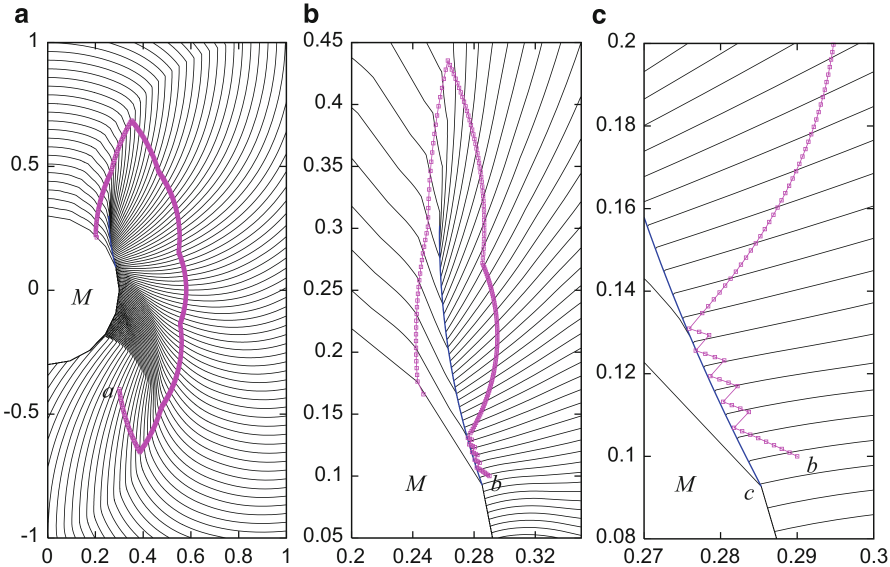


Fig. 22.33 Homicidal chauffeur game with more agile pursuer. Simulation results for optimal motions. (**a**) Initial point $a = (0.3, -0.4)$. (**b**) Initial point $b = (0.29, 0.1)$. (**c**) Enlarged fragment of the trajectory from the point b

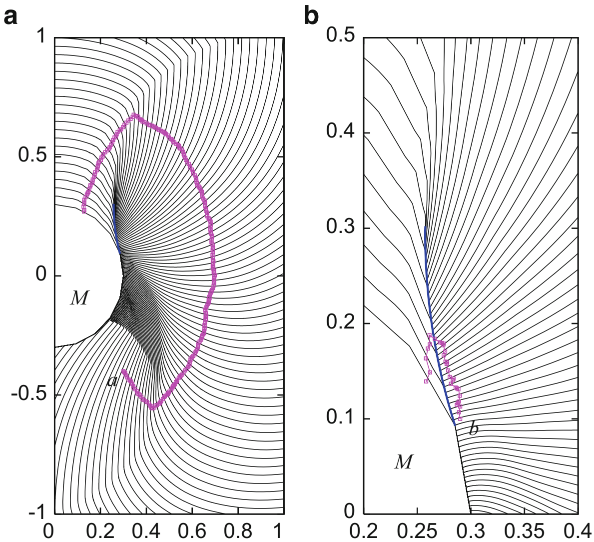


Fig. 22.34 Homicidal chauffeur game with more agile pursuer. Optimal behavior of player P and random action of player E. (**a**) Initial point $a = (0.3, -0.4)$. (**b**) Initial point $b = (0.29, 0.1)$

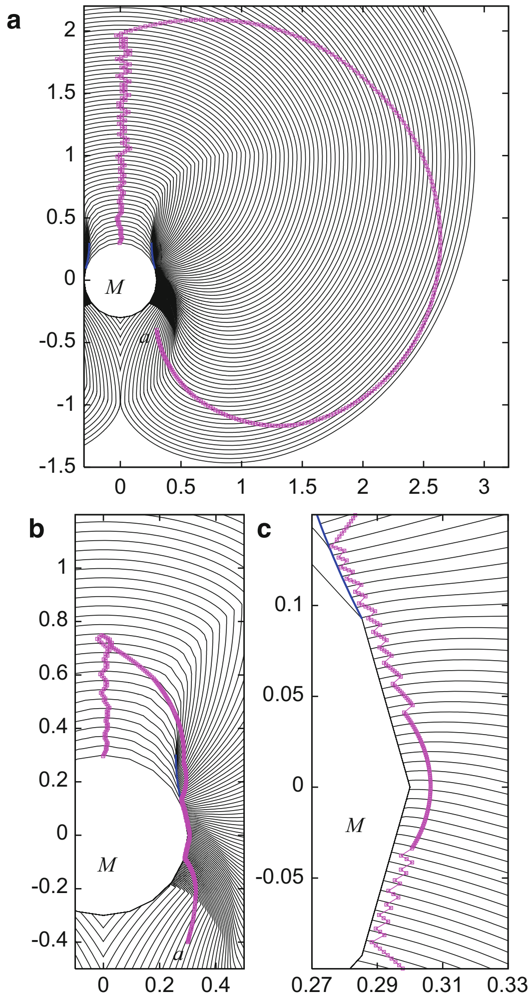


Fig. 22.35 Homicidal chauffeur game with more agile pursuer. Optimal behavior of player E and non-optimal control w of player P. Initial point $a = (0.3, -0.4)$. (**a**) $w \equiv 1$. (**b**) $w = -0.1$ until the trajectory comes to the vertical axis, after that $w = 1$. (**c**) Enlarged fragment of the trajectory on the left

the trajectory from Fig. 22.35b. The trajectory goes very close to the terminal set. The reaching time is 5.06.

2.9 Homicidal Chauffeur Game as a Test Example and a Stimulus for Investigation of New Comprehensive Problems

Presently, numerical methods and algorithms for solving zero-sum differential games are intensively developed. Often, the homicidal chauffeur game is used as a test or demonstration example (see, e.g., Botkin et al. 2011, 2013; Dvurechenskii 2013; Dvurechenskii and Ivanov 2012; Meyer et al. 2005; Mikhalev and Ushakov 2007; Mitchell 2002; Raivio and Ehtamo 2000).

In the reference coordinate frame, the game is of the second order in phase variables. Therefore, one can apply both general algorithms and algorithms taking into account the specifics of the plane. The nontriviality of the dynamics is in that the control u enters the right hand side of the two-dimensional control system as a factor by the state variables, and the constraint on the control v can depend on the phase state. Moreover, the control of player P can be two-dimensional, as it is in the modification discussed in Sect. 2.8.

Speaking about applied problems whose investigation was motivated by time-optimal differential games and, in particular, by the homicidal chauffeur problem, let us note the following:

1. There exist publications (see, e.g., Bakolas and Tsiotras 2012, 2013) related to the minimization of the traveling time of an object in the presence of drift field. In idealized settings, the spatial and velocity characteristics of such field are supposed to be known. A more realistic approach assumes the presence of uncertainties in the representation of drift field. The velocity field can be generated by the wind if the object is moving in the air, or by the undertow if movement of some deep-sea vehicle is considered.

2. Very important are problems in which a controlled object (an aircraft or a ship) should avoid collision with some other moving object. If our information on the movement of the second object is incomplete, we are again in the scope of differential game theory methods. For example, in the paper Exarchos et al. (2015), an object with car-like dynamics should avoid from a given circular neighborhood of an object with dynamics of simple motion. Here, at least locally, on some interval of possible collision, it is appropriate to consider the "homicidal pedestrian" problem.

3. Let a third object known as a "defender" join to the pursuer and the evader. Suppose that an aircraft performing its task is attacked by a missile. At the same time, the second missile begins to defend the aircraft. Hence, on a small time interval, an interaction between the evader (the aircraft), the pursuer (the first missile), and the defender (the second missile) takes place. The evader and the defender can share information on the current position of all objects, completely or in part. Conceptual formulation of such problems and some approaches to their analytic analysis are considered in Shaferman and Shima (2010), Shima (2011), and Pachter et al. (2014). It should be noted that in such problems, it is even not completely clear if the problem can be formulated as a zero-sum differential game and whether the value function of this game (being defined by equating guaranteed result of the evader and defender with the guaranteed result of the pursuer) exists. Moreover, it is desirable to account for the state constraints imposed on the movement of every of the three objects. Here, a new, very interesting direction of research arises. Of course, accurate solving of such problems is impossible without appropriate numerical methods. Issues related to the formulation of such problems and to numerical solution methods are discussed in Fisac and Sastry (2015).

4. In the book Blaquière et al. (1969), two-player differential games with two target sets are considered. Each of the players strives to steer the control system to his own target set prior to his opponent. The applied interpretation of such problems

can be a combat between two aircraft or between two ships (Getz and Pachter 1981). Who is the pursuer and who is the evader, if each of the combatants should hit his opponent? In the context of objects like those ones in the homicidal chauffeur game, the problem is considered in Davidovitz and Shinar (1989), Merz (1985), and Olsder and Breakwell (1974).

5. In addition to the homicidal chauffeur game, the book of R. Isaacs contains, among basic time-optimal problems, the game of "two cars" and the "isotropic rocket" game. The meaning of the first game is clear from its name. In the second game, the pursuer with the dynamics of a material point in the plane in the presence of a drag force (which depends on the magnitude of velocity) pursuers the evader with the dynamics of simple motion. Using some transformation of coordinates, both problems can be reduced to differential games with a three-dimensional state vector. Among publications on the game of two cars, let us mention the works Simakova (1967, 1968), Merz (1972), and Pachter and Getz (1980) and the works Bernhard (1970, 1979) on the isotropic rocket game. Time-optimal problem for a material point (isotropic rocket) in a resistant medium with a partially given drift field is investigated in Selvakumar and Bakolas (2015). However, these problems are not yet completely solved (including dependence of solution on parameters). In this connection, let us stress the necessity of invoking careful numerical modeling. For an example of application of numerical methods to the game of two cars, we refer to the paper Mitchell et al. (2005), in which, for some set of parameters, the game of two cars is used as a fragment of collision-avoidance problem of two aircraft.

6. It should be clearly understood that such problems as the "homicidal chauffeur game," the "game of two cars," and the "isotropic rockets" are problems which reflect, at the model level, typical features of applied problems. One should be careful when considering the opportunity to use solutions of model mathematical problems in real practical applications. We refer to Bolonkin and Murphey (2005) as an example of rational approach to solving the problem of avoiding encounter. In this work, the counteraction of two objects with car-like dynamics is considered. For given initial states, the question being asked is: Can the second object avoid an encounter with the first one? On some time grid, the reachable sets of each object are constructed. Using them, it is analyzed if there exists a motion of the second object, which, at every time grid point, is located outside the corresponding reachable set of the first object. By constructing such a motion, we wish to bring it to the position, starting from which the further avoidance of the second object from the first one becomes evident. If this is possible, the avoidance maneuver is found. Thus, the conclusion on the construction (in approximate, engineering sense) of an avoidance maneuver using an open-loop control is drawn merely on the base of computing reachable sets on a time grid. Fast construction of reachable sets for objects with car-like dynamics is well realizable on current onboard computers. If the problem with one pursuer and several evaders is considered and we are interested in the behavior of the pursuer, then, after the above described analysis, only those evaders for whom the avoidance maneuver is impossible will be left. The pursuer's control law that guarantees the capture of the evader is designed by choosing a single evader from the remaining ones. However, the realization of subtle mathematical results

of the differential games theory related to the construction of optimal feedback controls that guarantee the interception of the second object by the first one is hardly possible at this time. So, we should agree with simplifications dropping the attempt to compute optimal solutions.

7. Mathematical results on the problem, in which the moving car should, as soon as possible, intercept more than one moving object are published in Berdyshev (2002, 2008) and being included into the book Berdyshev (2015). Similar problems but in the game formulation are considered in Shevchenko (2012). As an application, the problem with false targets, in which an attacking vehicle that does not distinct between the true and false targets should visit several moving targets and hit each of them can be mentioned. Very important are also the problems, in which, on the contrary, several moving objects with car-like dynamics (e.g., unmanned vehicles) perform coordinated pursuit of a single moving target (see Shaferman and Shima 2008).

8. Consider one more variant of a pursuit-evasion problem with a false target. There are a pursuer and an evader. Dynamic abilities of the pursuer exceed the ones of the evader. At the initial instant, the pursuer knows only the area where the evader is located. Using this information, the pursuer begins the search. At some instant, the evader creates a false target, and the pursuer encounters with the problem of two targets: the ones is true and another is false. The pursuer's detection means work effectively only when the distance between the pursuer and the observed object is not larger than a given value. Moreover, it is possible to identify the target (true or false) only when the pursuer comes within a given small distance from the target. Generally, the pursuer seeks to minimize the guaranteed capture time of the true target. Some such problems for objects with the dynamics of simple motion in the plane and in three-dimensional space were formulated and investigated in Abramyants et al. (1981, 2007), Shevchenko (1997), Zheleznov et al. (1993). The books Kim (1993) and Petrosyan and Garnaev (1993) are devoted to search problems under uncertainty conditions.

9. Extremely hard are pursuit-evasion problems with two or several groups of interacting objects with car-like (aircraft) dynamics. Here, various problem settings are possible. For instance, which objects can share information on their states and so on. For an example of mathematical works related to this direction, see Stipanović et al. (2010, 2012). Numerical algorithms for such problems are considered in Chen et al. (2015).

10. One of the popular methods in the mathematical control theory is approximation of complex sets by ellipsoids (Chernous'ko 1993; Kurzhanski and Valyi 1997). In Kurzhanski (2015, 2016), ellipsoids whose orientation and size are changed with time are used to describe virtual containers enclosing a group of moving objects. These small objects (little ships) should not collide, whereas the container having them inside should solve his task of transfer from some initial position to a given position, passing through "straits" and rounding "islands." The presence of uncertainties transforms this problem to a very sophisticated pursuit-evasion game.

11. When designing wheel robots, one of the main criteria for the choice of robot dynamics is the following. Assume that the robot should track some

prescribed trajectory in the plane in the presence of disturbances. It turned out that dynamics of simplest car (Dubins' car) are poorly suited for such a problem. The reason is that, for such dynamics, for small time intervals, the reachable set (in geometric coordinates) at fixed time does not contain the initial point. In other words, the reachable set of Dubins' car at fixed final time does not coincide with his time-limited reachable set. For Reeds-Shepp's vehicle, this property is fulfilled. Therefore, it is better to have the dynamics of the robot to be similar to the dynamics of Reeds-Shepp's car. This example shows why comprehensive mathematical control theory and, in particular, its branch related to time-optimal games finds application in robotics (Laumond 1998; Laumond et al. 2014).

12. Among many contemporary applied problems, in which researchers are trying to apply methods of time-optimal games, it is worthy to mention those ones related to a jamming attack on the communication network of a team of moving objects. These problems are investigated by T. Başar and his collaborators (Bhattacharya and Başar 2012; Han et al. 2011). The simplest task is formulated in the following way. Two aircrafts (maybe, unmanned air vehicles) move in the horizontal plane. Communication requirements demand that the distance between them had to be not larger than a given value. At the initial instant, this condition is fulfilled. After that, the third object, a jammer aircraft, appears. This one tries to jam the connection between the mentioned pair of aircraft. To do this, the jammer uses its special equipment. The time-optimal zero-sum game is formulated as follows: the aerial jammer moves in a way to maximize the connection interruption time, whereas the team of the first and second aircraft tries to minimize this time by changing configuration of their motions.

3 Linear Differential Games with Fixed Termination Instant

Let the dynamics of a controlled object be described by a vector differential equation

$$\dot{z} = A(t)z + B(t)u + C(t)v, \quad u \in P \subset R^p,\ v \in Q \subset R^q. \tag{22.25}$$

Here, $z \in R^n$ is the phase vector, $A(t)$ is a square matrix $n \times n$, u is the vector control of the first player constrained by a compact set P, and v is the vector control of the second player constrained by a compact set Q. The matrices $B(t)$, $C(t)$ have appropriate sizes. All matrix functions are assumed to be continuous in time.

Let the termination instant t_f of the process be fixed and given. A scalar continuous function φ of the terminal payoff is defined. It can depend not on the entire phase vector z, but only on some m its components. The vector consisting of these components is denoted by z_m. The magnitude $\varphi\big(z_m(t_f)\big)$ at the terminal point of a system trajectory is minimized by the first player and maximized by the second one.

Thus, we have a differential game with linear dynamics, fixed termination instant and terminal payoff function. Such game has a continuous value function $(t,z) \mapsto \mathbf{V}(t,z)$.

3.1 Passage to Dynamics Without Phase Variable in the Right-Hand Side

Let $Z(t_f, t)$ be the fundamental Cauchy matrix corresponding to the matrix $A(t)$ participating in system (22.25). The symbol $Z_m(t_f, t)$ denotes a submatrix of the matrix $Z(t_f, t)$ composed of the m rows that correspond to the components of the phase vector z, which define the payoff function φ. The position of system (22.25) computed to the terminal instant t_f from the current instant t and current position $z(t)$ under zero players' controls $u \equiv 0$, $v \equiv 0$ is defined by the formula $Z(t_f, t)z(t)$. Respectively, the forecast value of the chosen m components is computed as $Z_m(t_f, t)z(t)$.

Introduce a new variable $y(t) = Z_m(t_f, t)z(t)$. Then

$$\dot{y}(t) = \frac{d}{dt}\big(Z_m(t_f, t)z(t)\big) = Z_m(t_f, t)B(t)u + Z_m(t_f, t)C(t)v.$$

Denote

$$D(t) = Z_m(t_f, t)B(t), \quad E(t) = Z_m(t_f, t)C(t)$$

and consider a game with the following dynamics

$$\dot{y}(t) = D(t)u + E(t)v, \quad u \in P, \quad v \in Q, \tag{22.26}$$

fixed termination instant t_f and the terminal payoff function φ. If in some interval $[t_*, t_f]$, the players' controls $t \mapsto u(t)$ and $t \mapsto v(t)$ in systems (22.25) and (22.26) are the same and $y(t_*) = Z_m(t_f, t_*)z(t_*)$, then $y(t_f) = z_m(t_f)$. This fact is the basis for the proof of equivalence of games (22.25) and (22.26): the magnitudes of the value function $\mathbf{V}(t, z)$ in game (22.25) and the value function $V(t, y)$ in game (22.26) are connected by relation

$$\mathbf{V}(t, z) = V\big(t, Z_m(t_f, t)z\big).$$

The benefits of dynamics (22.26) are the following:

(1) In the right-hand side of dynamics (22.26), there is no phase variable; this permits to solve problem (22.26) by simpler numerical methods;
(2) The dimension of the phase vector of system (22.26) equals m; if $m < n$, then it can simplify constructions. Often, in applied problems, one has $m = 2$, therefore, $y \in R^2$, and one should make constructions in the plane.

Replacement of dynamics (22.25) by dynamics (22.26) in linear differential games with fixed termination instant is used actively in mathematical and engineering literature since the end of the 1960s (Bryson and Ho 1975; Krasovskii 1970, 1971).

3.2 Lebesgue Sets of Value Function

Let $M_c = \{y : \varphi(y) \leqslant c\}$ be a Lebesgue set (a level set) of the payoff function that corresponds to the number c. Denote $W_c = \{(t, y) : t \leq t_f,\ V(t, y) \leq c\}$. The set W_c is the Lebesgue set of the value function that corresponds to the number c. Let $W_c(t) = \{y : V(t, y) \leq c\}$, $t \leq t_f$, be a time section (a t-section) of the tube W_c at the instant t. A remarkable property of the tube is that W_c is the set maximal by inclusion in the space $t \times y$, which possesses the stability property (weak-invariant set): for any initial position $(t_*, y_*) \in W_c$ for any constant control $t \mapsto v(t) \equiv v^* \in Q$ of the second player, there is a measurable control $t \mapsto u(t)$ of the first player that has its values in P and such that the trajectory $t \mapsto y\big(t; t_*, y_*, u(\cdot), v^*\big)$ of system (22.26) at any instant $t \in [t_*, t_f]$ stays in the set $W_c(t)$. A set that possesses such a property is called also u-stable bridge. Thus, W_c is the maximal stable bridge that stops at the set M_c.

Here, we must involve measurable controls of the first player (not piecewise-continuous) because only measurable controls provide closure of the trajectory bundle. From ideological point of view, one can imagine piecewise-continuous controls that are easier to understand.

Discriminating the second player (since he shows his further control in some future time interval), the first player can guide system (22.26) through the section $W_c(t)$, and at the instant t_f, the system is delivered to the set M_c. If to reject the discrimination of the second player, then the positional strategy of the first player extremal to the set W_c (Krasovskii 1985; Krasovskii and Subbotin 1974, 1988) and applied in a discrete control scheme with some sufficiently small time step keeps any trajectory of system (22.26) near the tube W_c up to the instant t_f.

The maximal stable bridge is (Krasovskii and Subbotin 1974, 1988) a closed set. If the set M_c is convex, then any t-section $W_c(t)$ is convex too (see Krasovskii and Subbotin 1988, p. 87) due to linearity of system (22.26).

The closure of the set $\{(t, y) : t \leq t_f\} \setminus W_c$ has analogous property but under discrimination of the first player and is called the maximal v-stable bridge.

Applying analytic or numerical procedures for constructing sets W_c in some grid of values of the parameter c, one can obtain a collection of exact or approximate Lebesgue sets of the value function of game (22.26). On the basis of such a collection, optimal positional controls of the players can be constructed approximately: being at some position (t, y), we choose the first player's control $u(t, y)$ that maximally pulls the system to the closest smaller set $W_c(t)$ and the second player's control $v(t, y)$ that maximally pulls the system from this set.

3.3 Backward Procedure for Constructing Sets $W_c(t)$ in the Convex Case

Now, let us describe the method for constructing level sets of the value function (maximal stable bridges) of games of the type (22.26) in the case of continuous quasiconvex payoff φ. (A function is called quasiconvex if all its Lebesgue sets are convex.) The procedure is of the backward type and can be treated as the dynamic programming principle applied to differential games.

To do the numerical construction, let us take a sequence of instants $t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = t_f$ in the time interval $[t_0, t_f]$ of the game. Uniformity of the grid is unessential. For a given constant c, the result of the procedure is a collection of sets, each corresponding to a chosen time instant t_i and approximating the time section $W_c(t_i)$ of the level set $W_c = \{(t, y) : V(t, y) \leq c\}$ of the value function V of the game (22.26) at this instant. The symbol $\widetilde{W}_c(t_i)$ will denote the set approximating the original time section $W_c(t_i)$.

Change the dynamics of the game (22.26) by a piecewise-constant dynamics:

$$\dot{y} = \widetilde{D}(t)u + \widetilde{E}(t)v, \quad \widetilde{D}(t) = D(t_i), \ \widetilde{E}(t) = E(t_i), \ t \in [t_i, t_{i+1}). \tag{22.27}$$

Instead of the original constraints P and Q for the controls of the players, let us consider their polyhedral approximations $\widetilde{P}$ and $\widetilde{Q}$. Let $\widetilde{\varphi}$ be the approximating payoff function. It is defined so that for any number c, its level set $\widetilde{M}_c = \{y : \widetilde{\varphi}(y) \leq c\}$ is a convex polyhedron close in Hausdorff metrics to the level set M_c of the original payoff function.

The approximating game (22.27) is chosen such that in each step $[t_i, t_{i+1}]$ of the backward procedure we deal with a game with simple motion and polyhedral convex constraints for the players' controls. At the first step $[t_{N-1}, t_N] = [t_{N-1}, t_f]$, a solvability set $\widetilde{W}_c(t_{N-1})$ for a game of homing with target set $\widetilde{W}_c(t_N) = \widetilde{M}_c$ is constructed. Here, the first player tries to guide the system to the set $\widetilde{W}_c(t_N)$ at the instant t_N, and the second one opposes this. Continuing in the same way, a set $\widetilde{W}_c(t_{N-2})$ is constructed on the base of $\widetilde{W}_c(t_{N-1})$, and so on. As a result, we obtain a collection of convex polyhedra $\widetilde{W}_c(t_i)$ approximating (Botkin 1982; Polovinkin et al. 2001; Ponomarev and Rozov 1978) the sections $W_c(t_i)$ of the original level set W_c of the value function of the game (22.26) in Hausdorff metrics.

The procedure of moving from the section $\widetilde{W}_c(t_{i+1})$ to the next one $\widetilde{W}_c(t_i)$ is described in terms of support functions of the sets under consideration. Recall that the value $\rho(l, A)$ of the support function of a bounded closed set A on the vector l is calculated by formula

$$\rho(l, A) = \max_{a \in A} \langle l, a \rangle.$$

Here, the symbol $\langle \cdot, \cdot \rangle$ denotes the dot product of two vectors.

Introduce denotations $\mathscr{P}(t_i) = D(t_i)\widetilde{P}$, $\mathscr{Q}(t_i) = E(t_i)\widetilde{Q}$ for vectograms of the players (i.e., for the sets of velocities, which the players can give to the system). One knows (Pontryagin 1967b; Pschenichnyi and Sagaidak 1970) that in the convex case the set $\widetilde{W}_c(t_i)$ is represented as

$$\widetilde{W}_c(t_i) = \Big(\widetilde{W}_c(t_{i+1}) + (t_{i+1} - t_i) \cdot \big(-\mathscr{P}(t_i)\big)\Big) \overset{*}{-} (t_{i+1} - t_i) \cdot \mathscr{Q}(t_i).$$

Here, the symbol "+" denotes the Minkowski sum (the algebraic sum), and the symbol "$\stackrel{*}{-}$" denotes the Minkowski difference (the geometric difference). These operations are defined as

$$\mathscr{A} + \mathscr{B} = \{a + b : a \in \mathscr{A},\, b \in \mathscr{B}\}, \quad \mathscr{A} \stackrel{*}{-} \mathscr{B} = \bigcap_{b \in \mathscr{B}} (\mathscr{A} - b).$$

The support function of the Minkowski sum of two sets equals the sum of the support functions of these sets. For two convex compact sets, the support function of their Minkowski difference coincides with the convex hull of the difference of the support functions of these sets. Therefore, $\rho\big(\cdot, \widetilde{W}_c(t_i)\big) = \operatorname{co}\gamma(\cdot, t_i)$, where the value of the function $\gamma(\cdot, t_i)$ on a vector l is defined by the formula

$$\gamma(l, t_i) = \rho\big(l, \widetilde{W}_c(t_{i+1})\big) + (t_{i+1} - t_i) \cdot \rho\big(l, -\mathscr{P}(t_i)\big) - (t_{i+1} - t_i) \cdot \rho\big(l, \mathscr{Q}(t_i)\big). \tag{22.28}$$

The symbol "co" denotes the operation of taking the convex hull of a function.

The function $\gamma(\cdot, t_i)$ is positively homogeneous and piecewise-linear (because the support functions of the polyhedra $\widetilde{W}_c(t_{i+1})$, $-\mathscr{P}(t_i)$ and $\mathscr{Q}(t_i)$ are of this type). The property of local convexity can be violated only on the boundary of the linearity cones of the function $\rho\big(\cdot, \mathscr{Q}(t_i)\big)$, that is, on the boundary of the cones generated by normals to the neighboring faces of the polyhedron $\mathscr{Q}(t_i)$. This can be taken into account during construction of the convex hull of the function $\gamma(\cdot, t_i)$.

As a result of the backward procedure in the interval $[t_0, t_f]$, one obtains a collection of the sets $\widetilde{W}_c(t_i)$ for a value of the parameter c.

3.4 Constructing Sets $\widetilde{W}_c(t)$ in the Two-Dimensional Case

When the phase vector is two-dimensional, constructions described in the previous subsection can be realized very effectively. The procedure of convex hull construction is very fast because we have information about places of possible violation of local convexity. In Fig. 22.36, the structure of the function γ graph is shown schematically. Dash lines point out "corrections" of the function during convexification process. Herewith, this process stops after a few steps.

Below, the argument t_i of the function γ is omitted to simplify denotations.

Cones of linearity of the function γ are defined by the outer normals to edges of the convex polygons $\widetilde{W}_c(t_{i+1})$, $-\mathscr{P}(t_i)$, $\mathscr{Q}(t_i)$. Gathering these normals and ordering them clockwise, we obtain a collection L of vectors. The collection of magnitudes $\gamma(l)$ of the function γ (22.28) on the vectors $l \in L$ is denoted as Φ. The collections L, Φ define completely the function γ (with taking into account its positive homogeneity).

The set of outer normals to edges of the polygon $\mathscr{Q}(t_i)$ ordered clockwise is denoted S. The collection S is called set of "suspicious" vectors. This name is due to the fact that the function γ is certainly convex in any cone whose interior does

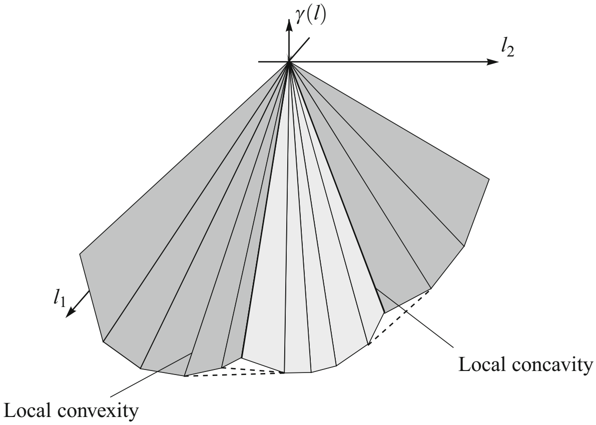


Fig. 22.36 A scheme of constructing convex hull of a positively homogeneous piecewise-linear function

not include any vector from the set S. And, respectively, violation of convexity can happen only in cones that do include at least one of such vectors.

Assume $L^{(1)} = L, \Phi^{(1)} = \Phi, S^{(1)} = S$. The $(k+1)$th step of the convexification process is made by replacing sets $L^{(k)}$, $\Phi^{(k)}$ by some other collections $L^{(k+1)} \subset L^{(k)}$, $\Phi^{(k+1)} \subset \Phi^{(k)}$. With that, $S^{(k)}$ is also changed by a new collection $S^{(k+1)}$.

Now, let us describe one step of the convexification procedure. Suppose that the angle between any two neighbor vectors from the collection $L^{(k)}$ counted clockwise is less then π. Let $l \mapsto \gamma^{(k)}(l)$ be a piecewise-linear positively homogeneous function described by the collections $L^{(k)}$, $\Phi^{(k)}$. Since

$$L^{(k)} \subset L^{(k-1)} \subset \cdots \subset L^{(1)}, \quad \Phi^{(k)} \subset \Phi^{(k-1)} \subset \cdots \subset \Phi^{(1)},$$

then for any vector $l \in L^{(k)}$ the magnitude $\gamma^{(k)}(l)$ equals $\gamma(l)$.

Take any vector $l_* \in S^{(k)}$ and two its neighboring in $L^{(k)}$ vectors l_- (taken counterclockwise from l_*) and l_+ (taken clockwise from l_*). Check whether the inequality $\langle l_*, y\rangle \leqslant \gamma^{(k)}(l_*)$ is essential in the triple of inequalities $\langle l_-, y\rangle \leqslant \gamma^{(k)}(l_-)$, $\langle l_*, y\rangle \leqslant \gamma^{(k)}(l_*)$, $\langle l_+, y\rangle \leqslant \gamma^{(k)}(l_+)$. The inequality $\langle l_*, y\rangle \leqslant \gamma^{(k)}(l_*)$ is essential, if for the set

$$A = \{y : \langle l_-, y\rangle \leqslant \gamma^{(k)}(l_-),\ \langle l_+, y\rangle \leqslant \gamma^{(k)}(l_+)\}$$

the following relation holds:

$$A \neq A \bigcap \{y : \langle l_*, y\rangle \leqslant \gamma^{(k)}(l_*)\}.$$

From essentiality of the inequality $\langle l_*, y \rangle \leqslant \gamma^{(k)}(l_*)$, it follows that the positively homogeneous function $\gamma^{(k)}$ is locally convex in the cone produced by the vectors l_-, l_*, l_+.

The algorithm for checking essentiality: find the point y_* of intersection of lines $\langle l_-, y \rangle = \gamma^{(k)}(l_-)$ and $\langle l_*, y \rangle = \gamma^{(k)}(l_*)$. Further, check the inequality $\langle l_+, y_* \rangle < \gamma^{(k)}(l_+)$. If it holds, then the middle inequality is essential. Therefore, the local convexity takes place.

If the local convexity holds, the vector l_* is excluded from the collection $S^{(k)}$. The new collection is denoted $S^{(k+1)}$. With that, $L^{(k+1)} = L^{(k)}$, $\Phi^{(k+1)} = \Phi^{(k)}$.

If the local convexity is violated, there are two situations. Let α be the angle between vectors l_- and l_+ counted clockwise. If

- $\alpha < \pi$, then the vector l_* is excluded from the collection $S^{(k)}$ and simultaneously the vectors l_- and l_+ are added there. (One or both can be in the collection $S^{(k)}$.) The collection obtained after these operations is the new set of "suspicious" vectors and is denoted by $S^{(k+1)}$. The new collection $L^{(k+1)}$ is obtained from $L^{(k)}$ by excluding the vector l_*. In the same, to get $\Phi^{(k+1)}$ from $\Phi^{(k)}$ we exclude the value $\gamma^{(k)}(l_*) = \gamma(l_*)$;
- $\alpha \geqslant \pi$, then either the considered triple of inequalities is inconsistent (the convex hull of the initial function does not exist) and, therefore, $\widetilde{W}_c(t_i) = \varnothing$. Or $\widetilde{W}_c(t_i)$ is a degenerated polygon, that is $\widetilde{W}_c(t_i)$ is a point or a segment. In both these cases, further constructions are ceased.

These are the actions made during one step of convex hull construction. The process is stopped at some step j, when $S^{(j)} = \varnothing$, that is, when there is no "suspicious" vectors. This means that the function $\gamma^{(j)}$ defined by the collections $L^{(j)}$ and $\Phi^{(j)}$ is locally convex on all vectors, that is, it is convex. Therefore, it is the convex hull of the initial function γ. Also stop can be caused by ceasing constructions; in this case, either $\widetilde{W}_c(t_i) = \varnothing$ or the polygon $\widetilde{W}_c(t_i)$ is degenerated.

The collection of sets $\widetilde{W}_c(t_i)$ is used by a visualization software to construct a solid tube to be drawn.

The algorithm for constructing t-sections $\widetilde{W}_c(t_i)$ is described in more details in Isakova et al. (1984) and Patsko (1996). The algorithm is very effective because before constructing the convex hull of a function γ we know places where its local convexity can be violated. Proof of convergence of the algorithm and some approximational schemes close to it are given in Botkin (1982) and Ponomarev and Rozov (1978). Convergence of analogous schemes is justified in Ponomarev and Rozov (1978). An algorithm for a posteriori estimating the numerical construction error is developed in Botkin and Zarkh (1984).

Let us give an example of numerical construction of maximal stable bridges for the following game:

$$\begin{aligned} &\dot{z}_1 = z_2 + v, \quad \dot{z}_2 = -z_1 + u, \qquad t \in [0, 8], \\ &|u| \leqslant 1, \quad |v| \leqslant 0.9, \quad \varphi(z_1, z_2) = z_1^2 + z_2^2. \end{aligned} \tag{22.29}$$

In this case, $n = m = 2$. So, when the value function V is constructed in the coordinates y, we can pass back to the original coordinates z taking into account the relation $z = Z^{-1}(t_f, t)y$.

In Fig. 22.37, three Lebesgue sets of the value function (maximal stable bridges) are shown in the coordinates τ, z_1, z_2. They are computed in the interval [0, 8] of the backward time $\tau = t_f - t$ and correspond to the values $c = 1.05, 1.4, 2.7$ of the payoff function. In Fig. 22.37b, two external bridges are cut off by a plane parallel to the axes z_1, τ. In Fig. 22.37a, all three are cut off.

The break of the internal bridge is sharp. But for other examples, there are magnitudes of the payoff such that the corresponding Lebesgue set of the value function has a degenerated t-section $W_c(\bar{\tau})$ at some instant $\bar{\tau}$ (e.g., the section has no interior). Further, for $\tau > \bar{\tau}$, the sections grow and have interior. Below, tubes of this type are called *critical*. At the beginning of numerical studies of linear differential games, it seemed that critical tubes can appear in rare model examples only, but not in practical problems. But further, it have turned out that such a point of view is incorrect. J. Shinar found that critical tubes and connected to them narrow throats are quite typical for problems of space interception.

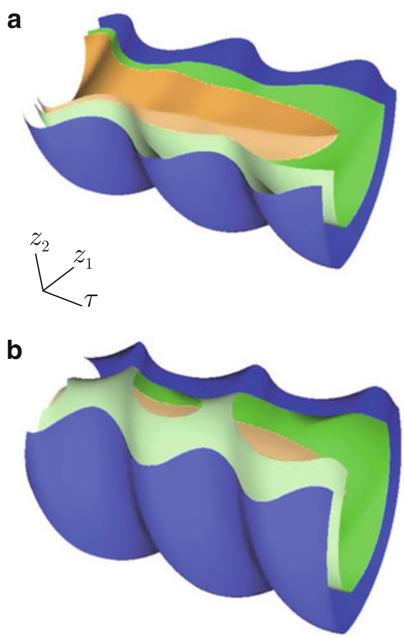


Fig. 22.37 Three maximal stable bridges for game (22.29)

3.5 J. Shinar's Problem of Space Interception

3.5.1 Problem Formulation

In the works Shinar et al. (1984), Shinar and Zarkh (1996), and Melikyan and Shinar (2000), a three-dimensional air-to-air interception problems has been formulated as a pursuit-evasion game by J. Shinar. The study by J. Shinar was based on earlier works Gutman and Leitmann (1975, 1976), Gutman (1979), and Shinar and Gutman (1980).

The pursuer is the interceptor missile, while the evader is a maneuverable aerial target (an aircraft or another missile). The natural payoff function of the game is the distance of closest approach, the miss distance, to be minimized by the pursuer and maximized by the evader. For the sake of simplicity, point mass models with velocities of constant magnitudes V_P, V_E were selected. The lateral accelerations of both players, normal to the respective velocity vectors, have constant bounds a_P and a_E. The evader controls its maneuver with ideal dynamics, while the pursuer's maneuver dynamics is represented by a first-order transfer function with the time constant τ_P.

In Fig. 22.38, the origin of the Cartesian coordinate system is collocated with the pursuer. The direction of the X-axis is along the initial line of sight. The XY plane is the nominal "collision plane" determined by the initial velocity vector of the evader $(V_E)_0$ and the initial line of sight. The Z-axis completes a right-handed coordinate system.

It is assumed the initial conditions are near to a "collision course," defined by

$$V_P \sin(\chi_P)_{\text{col}} = V_E \sin(\chi_E)_0, \tag{22.30}$$

and the actual velocity vector $V_P(t)$ of the pursuer remains close to the collision requirement $(V_P)_{\text{col}}$, satisfying

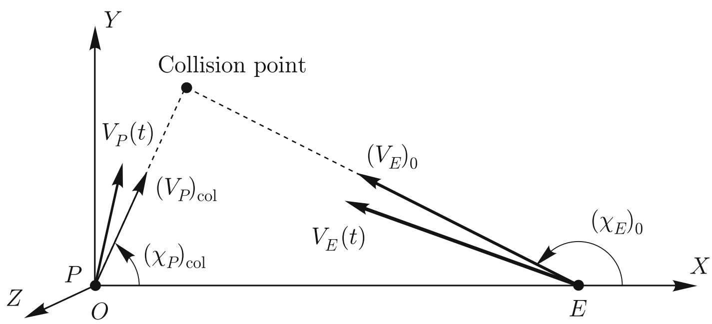


Fig. 22.38 The system of coordinates in the problem of three-dimensional pursuit. The actual vectors of $V_P(t)$ and $V_E(t)$ differ only slightly during the engagement from the nominal vectors $(V_P)_{\text{col}}$ and $(V_E)_0$, respectively

$$\sin\Big(\chi_P(t)-(\chi_P)_{\text{col}}\Big) \approx \chi_P(t)-(\chi_P)_{\text{col}}, \quad \cos\Big(\chi_P(t)-(\chi_P)_{\text{col}}\Big) \approx 1. \tag{22.31}$$

It is also assumed that the actual velocity vector $V_E(t)$ of the evader will remain close enough to its initial direction satisfying

$$\sin\Big(\chi_E(t) - (\chi_E)_0\Big) \approx \chi_E(t) - (\chi_E)_0, \quad \cos\Big(\chi_E(t) - (\chi_E)_0\Big) \approx 1. \tag{22.32}$$

Based on the small angle assumptions (22.31) and (22.32), the relative trajectories can be linearized with respect to the initial line of sight. Moreover, the relative motion in the X direction can be considered as uniform. Thus, this coordinate can be replaced by the time, transforming the original three-dimensional motion to a two-dimensional motion in the YZ plane. For a given initial range, the uniform closing velocity determines the final time t_f of the engagement. Therefore, the problem of minimizing (maximizing) the three-dimensional miss distance at a free terminal time can be changed by the minimization (maximization) of the distance in the YZ plane at the fixed terminal time of the nominal collision (two-dimensional miss).

Since in general the velocity vectors $(V_P)_{\text{col}}$ and $(V_E)_0$ of the players are not aligned with the initial line of sight, the projections of the originally circular control constraints, normal to the respective velocity vectors, become elliptical.

The equations of motion of the linearized pursuit-evasion game are

$$\begin{array}{ll} \ddot{\xi}_P = F, & \\ \dot{F} = (u - F)/\tau_P, & t \in [0, t_f],\ \xi_P,\ \xi_E \in R^2,\ \ u \in P,\ \ v \in Q, \\ & \varphi\big(\xi_P(t_f), \xi_E(t_f)\big) = \big|\xi_P(t_f) - \xi_E(t_f)\big|, \\ \ddot{\xi}_E = v, & \end{array} \tag{22.33}$$

where ξ_P and ξ_E are the positions of the players in the plane normal to the initial line of sight, and u and v are their respective acceleration command signals.

To reduce dynamics (22.33) to form (22.25), a variable change

$$\begin{array}{lll} z_1 = \xi_{P,1} - \xi_{E,1}, & z_2 = \dot{\xi}_{P,1} - \dot{\xi}_{E,1}, & z_3 = \ddot{\xi}_{P,1}, \\ z_4 = \xi_{P,2} - \xi_{E,2}, & z_5 = \dot{\xi}_{P,2} - \dot{\xi}_{E,2}, & z_6 = \ddot{\xi}_{P,2} \end{array}$$

can be applied, leading to the following standard form of the game

$$\begin{gathered} \dot{z} = Az + Bu + Cv, \\ A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_1 \end{array}\right], \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/\tau_P \end{bmatrix}, \\ B' = (1/\tau_P)\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad C' = \left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right] \end{gathered} \tag{22.34}$$

with constraints for the players' controls taken as ellipses

$$u \in P = \left\{ u \, : \, u' \begin{bmatrix} 1/\cos^2(\chi_P)_{\text{col}} & 0 \\ 0 & 1 \end{bmatrix} u \leq a_P^2 \right\},$$
$$v \in Q = \left\{ v \, : \, v' \begin{bmatrix} 1/\cos^2(\chi_E)_0 & 0 \\ 0 & 1 \end{bmatrix} v \leq a_E^2 \right\},$$

and the payoff function $\varphi\big(z_1(t_f), z_4(t_f)\big) = \sqrt{z_1^2(t_f) + z_4^2(t_f)}$.

The passage to the equivalent game yields

$$\dot{y} = D(t)u + E(t)v, \ \ t \in [0, t_f], \ \ y \in R^2, \ \ u \in P, \ \ v \in Q, \ \ \varphi\big(y(t_f)\big) = \big|y(t_f)\big|,$$

where

$$D(t) = \zeta(t) \cdot I_2, \quad \zeta(t) = (t_f - t) + \tau_P e^{-(t_f - t)/\tau_P} - \tau_P, \tag{22.35}$$
$$E(t) = \eta(t) \cdot I_2, \quad \eta(t) = -(t_f - t) \tag{22.36}$$

and I_2 is the 2×2 unit matrix.

In order that achieving a small miss distance be feasible, in all realistic pursuit-evasion examples, the pursuer must have some advantage in maximum lateral acceleration in every direction. This means that the control constraint set P of the pursuer has to cover completely the control constraint set Q of the evader. In other words, the inequalities

$$a_P/a_E > 1, \qquad a_P|\cos(\chi_P)_{\text{col}}| > a_E|\cos(\chi_E)_0|, \tag{22.37}$$

describing the relations of the semiaxes of the ellipses P and Q have to be valid. Such an advantage allows reducing an initial launching error and overcoming long duration constant evader maneuvers. However, due to the first-order dynamics of the pursuer's control function and the ideal dynamics of the evader, zero miss distance cannot be achieved against an optimally maneuvering evader.

In Shinar et al. (1984) as well as in Melikyan and Shinar (2000), the parameters of the problem were of an interception of a manned aircraft, assuming that $V_P > V_E$. Thus using (22.30), one obtains $\big|\cos(\chi_P)_{\text{col}}\big| > \big|\cos(\chi_E)_0\big|$.

In Shinar and Zarkh (1996), an interception of a tactical ballistic missile was considered with $V_P < V_E$, leading to $\big|\cos(\chi_P)_{\text{col}}\big| < \big|\cos(\chi_E)_0\big|$.

This difference influences considerably the form of the critical tube in the equivalent game and the associated singular surfaces.

3.5.2 Maximal Stable Bridges: Case of Fast Pursuer

Let us start with the results of numerical investigations for the case when the pursuer's velocity V_P is greater than the velocity V_E of the evader.

Relation (22.30) of the nominal collision and the relation $V_P > V_E$ of the players' velocities yield that the eccentricity of the ellipse P is smaller than the eccentricity of the ellipse Q (see Fig. 22.39).

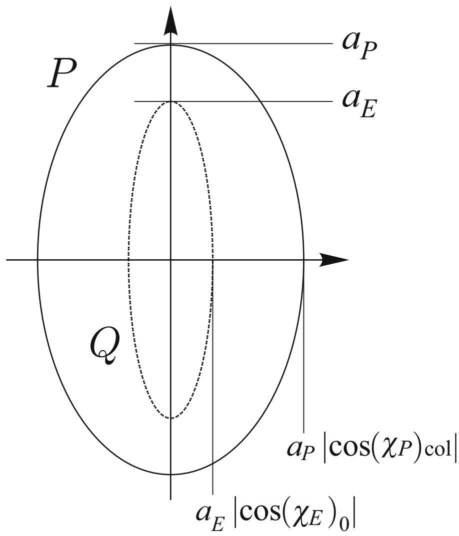


Fig. 22.39 Elliptical constraints of the players' controls in the case of a faster pursuer. The ellipse P is drawn by a *solid line*, and Q is drawn by a *dashed one*. The eccentricity of P is smaller than the eccentricity of Q

In the numerical investigation, the following data were chosen:

$$\frac{V_E}{V_P} = 0.666, \quad \frac{a_P}{a_E} = 5.0, \quad |\cos(\chi_P)_{\rm col}| = 0.87, \quad |\cos(\chi_E)_0| = 0.66, \quad \tau_P = 1\,\text{s}.$$

Consequently, the elliptical control constraints are

$$P = \left\{ u \in R^2 : \frac{u_1^2}{0.87^2} + \frac{u_2^2}{1.00^2} \le 5.0^2 \right\}, \quad Q = \left\{ v \in R^2 : \frac{v_1^2}{0.66^2} + \frac{v_2^2}{1.00^2} \le 1 \right\}.$$

This example has been computed in the interval $\tau \in [0, 2]$ s of backward time. The time step Δ was taken equal to 0.025 s. The circles of the level sets of the payoff function and the ellipses P and Q of constraints for the players' controls were approximated by 100-gons (polygons with 100 vertices).

In Fig. 22.40, the vectogram tubes for this example are shown. The first player's tube is light gray, the second one's is dark gray.

An enlarged part of the previous picture from another point of view can be seen in Fig. 22.41. On the vectogram tube of the second player, the contours of some time sections are shown.

Since $\mathscr{Q}(t) = E(t)Q = \eta(t)I_2Q = \eta(t)Q$, where $\eta(\cdot)$ is described by (22.36), the dark gray tube grows linearly with τ. For the tube $\mathscr{P}$, we have $\mathscr{P}(t) = \zeta(t)P$, where $\zeta(\cdot)$ is taken from (22.35). So, initially (for small values of τ), the light gray tube grows slower than linearly, but for large values of τ, it becomes almost linear and starts to grow faster than the tube $\mathscr{Q}$ does. This faster growth is provided by inequalities (22.37).

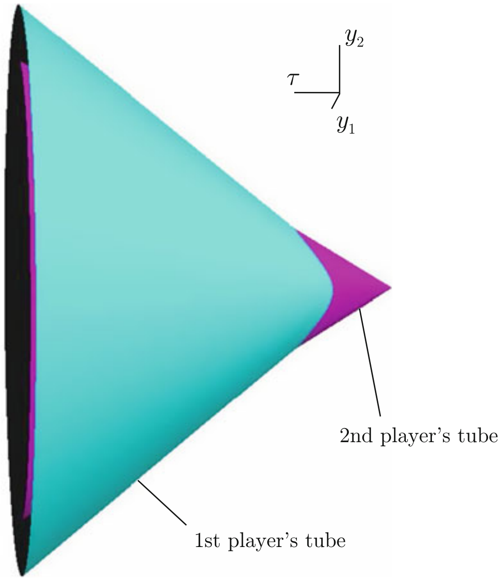


Fig. 22.40 Vectogram tubes for the case of a faster pursuer

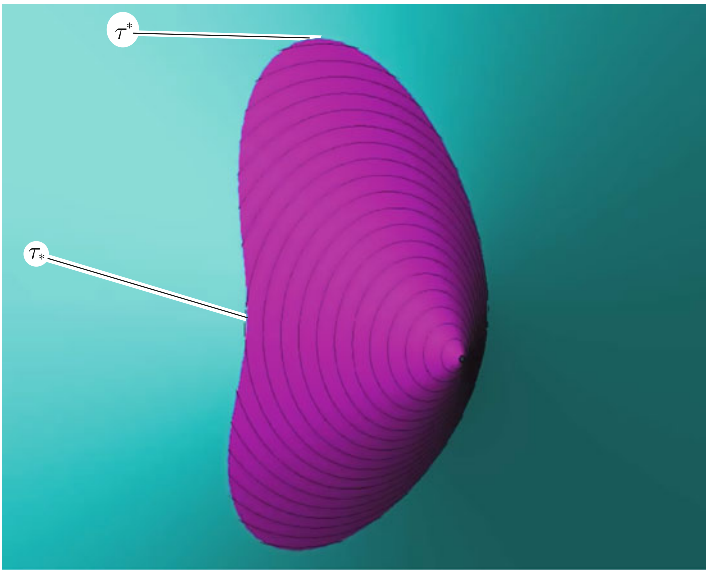


Fig. 22.41 An enlarged fragment of the vectogram tubes. The first player gains advantage in horizontal direction at the instant τ_* and a complete advantage at the instant τ^*

So, for $\tau < \tau_*$, the second player (the maximizer) has a complete advantage, that is, the vectogram $\mathscr{Q}(\tau)$ of the second player completely covers the vectogram $\mathscr{P}(\tau)$ of the first player (Fig. 22.42a). The instant τ_* is characterized by the fact that the horizontal size of the ellipses $\mathscr{P}(\tau_*)$ and $\mathscr{Q}(\tau_*)$ are equal (Fig. 22.42b). In the

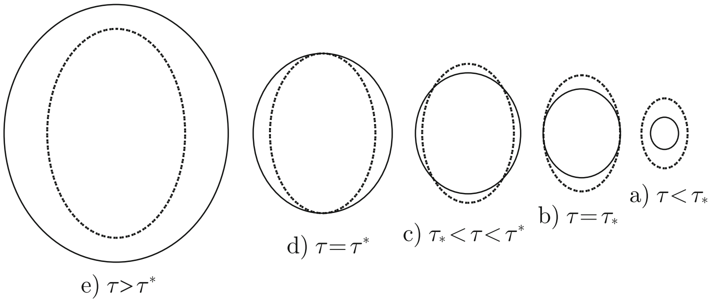


Fig. 22.42 Sections of the vectogram tubes at some time instants. The vectograms of the first player are shown by the *solid lines*, the *dashed lines* denote the vectograms of the second player

interval $\tau_* < \tau < \tau^*$, none of the players has a complete advantage: the first player is stronger in horizontal direction, the second player is stronger in directions near to the vertical (Fig. 22.42c). When $\tau = \tau^*$, the vertical sizes of the ellipses become equal (Fig. 22.42d) and for $\tau > \tau^*$ the first player has complete advantage (Fig. 22.42e). This change in the relationship of the vectograms $\mathscr{P}(\tau)$ and $\mathscr{Q}(\tau)$ can be explained by the difference between the eccentricities and the sizes of the ellipses P and Q and the form of the functions $\zeta(t)$ and $\eta(t)$.

Such a shift of the advantage from the maximizing player to the minimizer leads to creating a narrow throat. Figure 22.43 shows a level set close to the critical one. This level set is computed for $c = 0.141$ m. The narrow throat is located at $\tau^* = 0.925$ s. Contours of some time sections of the level set are shown. One can see that the t-sections $\widetilde{W}_c(t)$ of the level set near the narrow throat are elongated horizontally. This is due to the relation of the players' capabilities. For $\tau < \tau_*$, the second player is stronger in the vertical direction than horizontally. According to this, the sections $\widetilde{W}_c(t)$ are compressed more in the vertical direction. When τ is slightly greater than τ^*, the first player's advantage is stronger in the horizontal direction (Fig. 22.42e), which leads to a horizontal expansion of the sections. For sufficiently large values of τ the first player's advantage in vertical direction becomes greater than in the horizontal one, so, the t-sections start to grow vertically faster than in the horizontal direction, and at some instant, the elongation becomes vertical. For the presented example, this happens outside the time interval of Fig. 22.43.

In Fig. 22.44, scene is given that contains a level set close to the critical one. The τ-axis goes from the right to left, and the axis y_2 is directed upward. The axis y_1 is orthogonal to the sheet. Both vectogram tubes are transparent now. Such an overlap demonstrates clearly the influence of the players' vectograms on the geometry of the level set surface. For example, one can easily see that, when the first player gains complete advantage (at $\tau^* = 0.925$ s), the narrow throat ends (the tube of the level set starts to enlarge). In addition, it is seen that before that instant, the tube contracts due to the advantage of the second player.

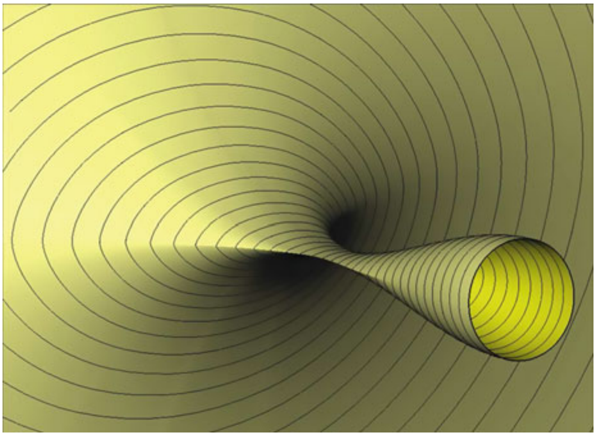


Fig. 22.43 A level set close to the critical one, $c = 0.141$ m. The instant of the most narrow place $\tau^* = 0.925$ s

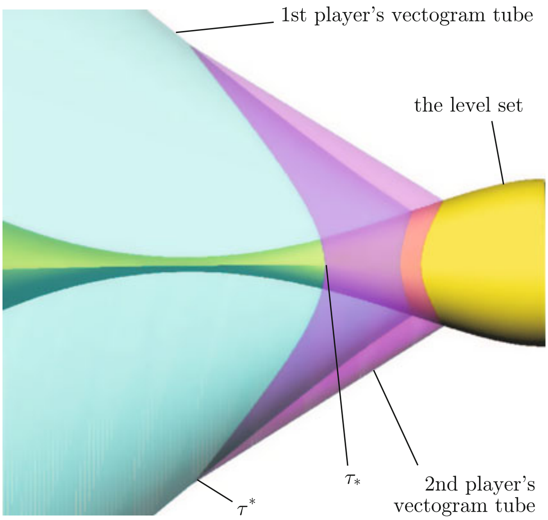


Fig. 22.44 Superposition of the vectogram tubes and the level set close to the critical one. The players' vectogram tubes are transparent

The results shown here agree qualitatively with the ones obtained in an analytical investigation of the problem with a faster pursuer made in Shinar et al. (1984) and Melikyan and Shinar (2000). In these papers, it is shown that in the case of a faster pursuer the geometry of the critical level set is the same for any combination of parameters of the problem.

3.5.3 Maximal Stable Bridges: Case of Slow Pursuer

In this subsection, the results with a slower pursuer $V_P < V_E$ are presented. The eccentricity of the ellipse P is greater than the eccentricity of the ellipse Q (see Fig. 22.45).

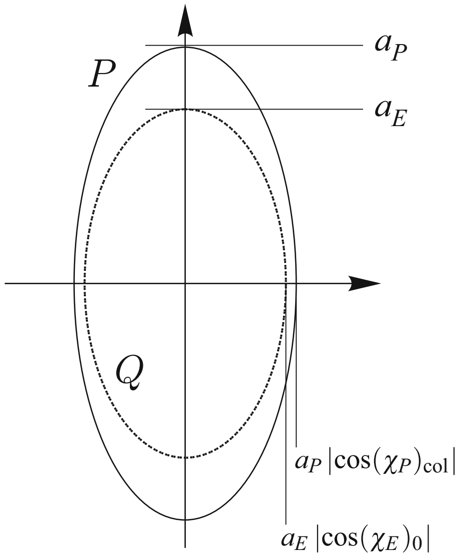


Fig. 22.45 Elliptical control constraints of the players in the case of a slower pursuer. The ellipse P is drawn by a *solid line*, Q is drawn by a *dashed one*. The eccentricity of P is greater than the eccentricity of Q

Based on the data of the original problem

$$\frac{V_E}{V_P} = 1.054, \quad \frac{a_P}{a_E} = 1.3, \quad \left|\cos(\chi_P)_{\rm col}\right| = 0.67, \quad \left|\cos(\chi_E)_0\right| = 0.71, \quad \tau_P = 1\,\text{s}$$

in the construction, the following data were used:

$$P = \left\{u \in R^2 : \frac{u_1^2}{0.67^2} + \frac{u_2^2}{1.00^2} \le 1.30^2\right\}, \quad Q = \left\{v \in R^2 : \frac{v_1^2}{0.71^2} + \frac{v_2^2}{1.00^2} \le 1\right\}.$$

This example has been computed in the interval $\tau \in [0, 7]$ s with the time step $\Delta = 0.01$ s. The circles of the level sets of the payoff function and the ellipses of the players' control constraints, P and Q, were approximated by 100-gons.

Like in the example of the previous subsection, there is a narrow throat also here. Figure 22.46 shows a general view of the level set $\widetilde{W}_c$ computed for the parameter $c = 1.546$ m, which is slightly greater than the critical one. But unlike the example described above, here the narrow throat has a much more complex structure: the orientation of the t-sections' elongation changes very tricky near the throat. An enlarged view of the throat is shown in Fig. 22.47.

Let us use the players' vectogram tubes for this problem to explain the shape of the level set. The vectogram tubes are shown in Fig. 22.48. The tube of vectograms of the first player ($\mathscr{P}$) is drawn in red, and the tube of the second player ($\mathscr{Q}$) is in green. Here also, the tube $\mathscr{Q}$ grows linearly with τ, whereas the tube $\mathscr{P}$ grows slower than linearly at small values of τ and becomes almost linear later. Eventually, for large values of τ, it will grow faster than the tube $\mathscr{Q}$, because (22.37).

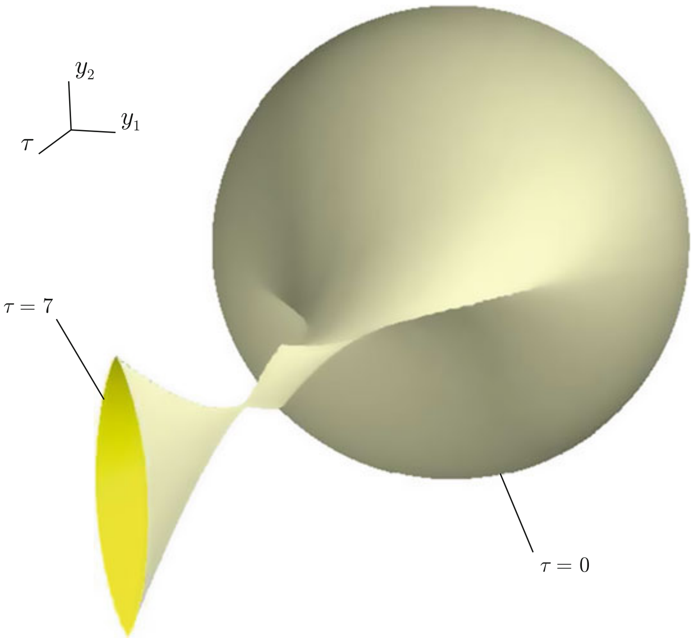


Fig. 22.46 General view of the level set of the value function with a narrow throat

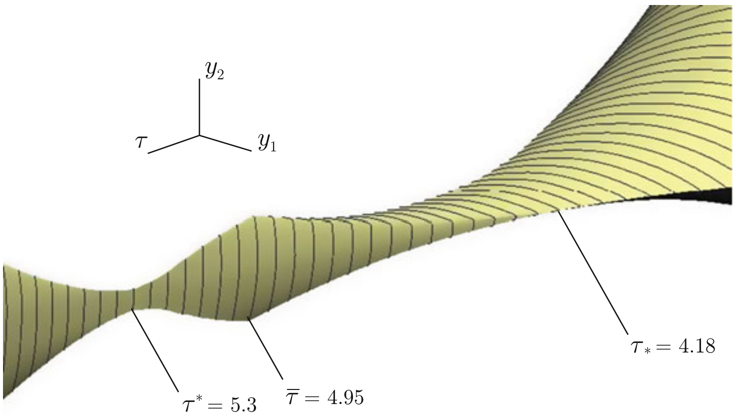


Fig. 22.47 Enlarged view of the narrow throat

Since the ellipses P and Q have different eccentricities, the first player's ellipse $\mathscr{P}(\tau)$ starts to cover the ellipse $\mathscr{Q}(\tau)$ of the second player in different directions at different instants. So, for $\tau < \tau_* = 4.18$ s, the ellipse $\mathscr{Q}(\tau)$ includes the ellipse $\mathscr{P}(\tau)$ completely (see Fig. 22.49a). At $\tau = \tau_*$, the first player's ellipse reaches the ellipse of the second player in the vertical direction (see Fig. 22.49b). In

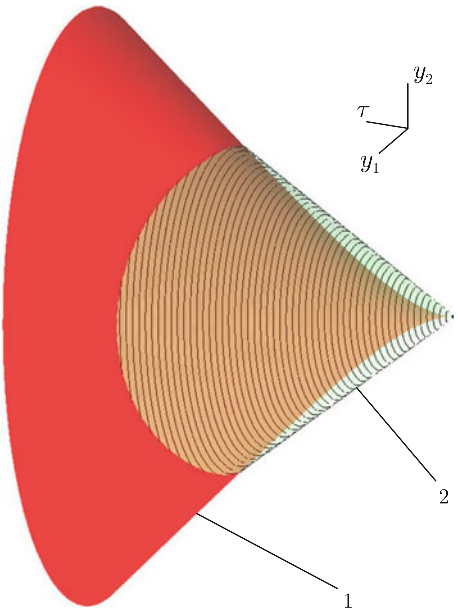


Fig. 22.48 General view of the vectogram tubes of the first (*1*) and second (*2*) players. The vectogram tube of the second player is transparent, showing contours of some sections

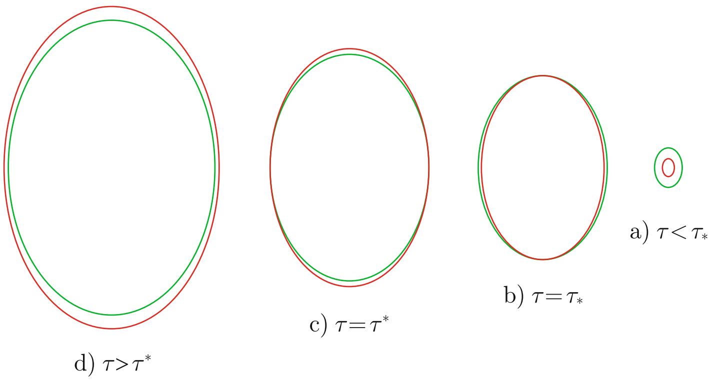


Fig. 22.49 Sections of the vectogram tubes at some time instants. The vectograms of the first player are shown by the *red lines*; the *green lines* denote the vectograms of the second player

the interval $\tau_* < \tau < \tau^*$, the ellipse $\mathscr{P}(\tau)$ covers more and more the ellipse $\mathscr{Q}(\tau)$. Finally, at $\tau = \tau^* = 5.3$ s the set $\mathscr{P}(\tau)$ covers the set $\mathscr{Q}(\tau)$ even in the horizontal direction (see Fig. 22.49c). And for $\tau > \tau_*$, $\mathscr{P}(\tau)$ covers $\mathscr{Q}(\tau)$ completely (see Fig. 22.49d).

The relationship between the players' vectograms leads to an intricate changing of the level set's t-sections near the narrow throat, as it can be seen in Figs. 22.46, 22.47, and 22.50. The latter shows groups of sections in different intervals of τ to demonstrate the different phases of the sections' changing.

For $\tau < \tau_*$, the second player has complete advantage over the first one. Since in backward time the second player tries to contract the sections of the level sets

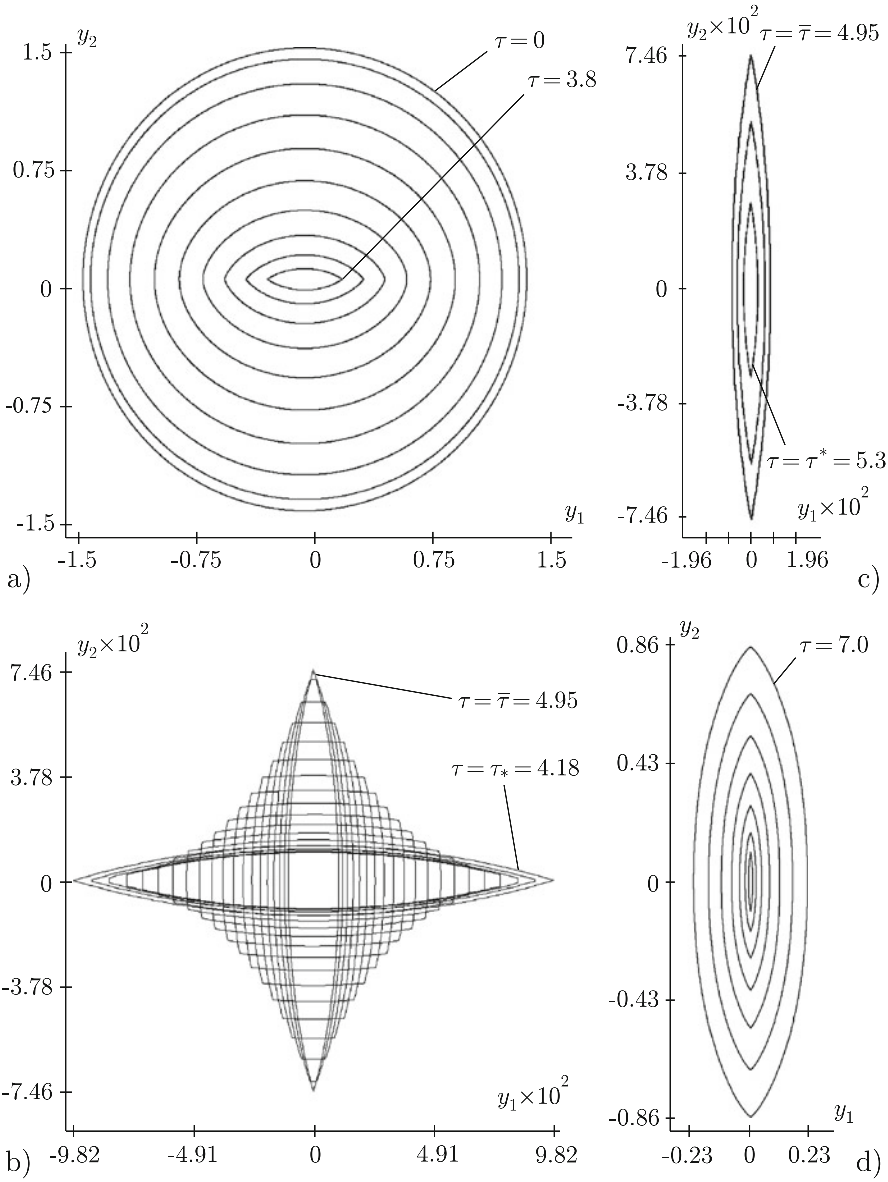


Fig. 22.50 Groups of time sections of a level set close to the critical tube in some intervals of the backward time: (**a**) $\tau \in [0, 3.8]$ s; (**b**) $\tau \in [\tau_*, \bar{\tau}]$; (**c**) $\tau \in [\bar{\tau}, \tau^*]$; (**d**) $\tau \in [5.41, 7.0]$ s

as much as possible, the t-sections of the tube $\widetilde{W}_c$ are reduced in the interval $0 < \tau < \tau_*$. In Fig. 22.50a, the sections are shown in the interval $\tau \in [0, 3.8]$ s. Since the second player's advantage is greater in the vertical direction, the tube starts to contract more in the vertical direction than in the horizontal one. Therefore, at $\tau = \tau_*$ the t-section of the level set is elongated horizontally.

In the interval $\tau_* < \tau < \tau^*$, the first player gains advantage gradually, starting in the vertical direction, while the second player keeps its horizontal advantage. For this reason the t-sections of the level set start expanding vertically while being reduced in the horizontal direction. This interval can be subdivided into two parts.

Between τ_* and $\bar{\tau} = 4.95$ s the time sections have the shape of "curvilinear rectangles" as it can be seen in Fig. 22.50b. Their form is gradually changing from a horizontal elongation to a vertical one.

At $\bar{\tau}$ the horizontal arcs disappear, and the t-sections start having a vertical lens shape. Simultaneously, the vertical expansion becomes a contraction despite of the vertical advantage of the first player (Fig. 22.50c), because the horizontal contraction enforces a contraction due to the lens shape.

Finally, at $\tau = \tau^*$, when the first player gains a complete advantage, one obtains the narrowest section of the throat (Figs. 22.47 and 22.50c) with vertical elongation. For $\tau > \tau^*$ the first player keeps the complete advantage and the t-sections start to expand in all directions monotonically. The rate of expansion is, however, nonuniform, but the direction of elongation remains vertical (see Fig. 22.50d).

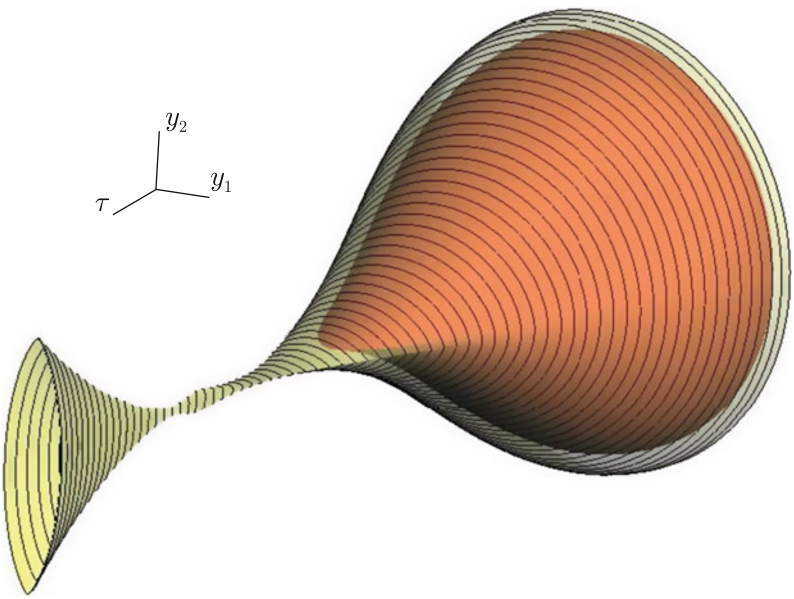


Fig. 22.51 The level set with narrow throat for the parameter $c = 1.546$ m (*yellow transparent*) and the level set for a less value of the parameter $c = 1.48$ m (*red*)

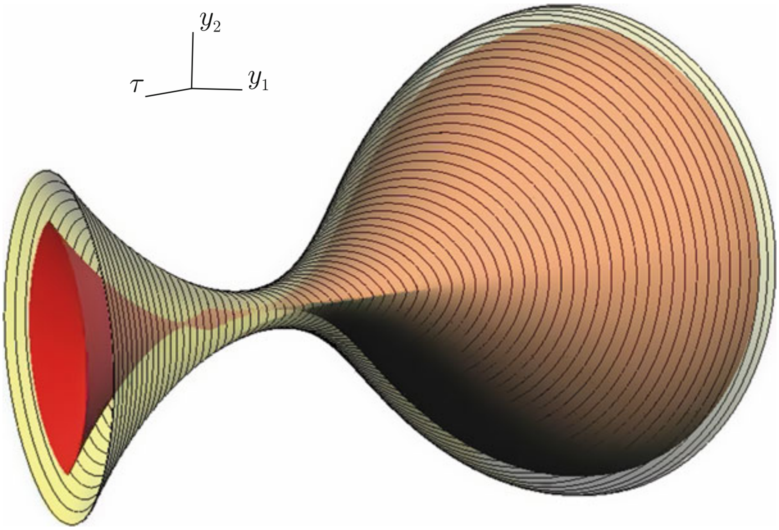


Fig. 22.52 The level set with narrow throat for the parameter $c = 1.546$ m (*red*) and the level set for a greater value of the parameter $c = 1.67$ m (*yellow transparent*)

The following two figures show the critical level set in comparison with level sets close to it. Figure 22.51 shows the critical tube (drawn in transparent yellow) and the tube computed for the value of $c = 1.48$ m of the payoff function, which is less than the critical one. This tube is finite in time and drawn in red. In Fig. 22.52, the critical level set (in red) and the one computed for $c = 1.67$ m (in transparent yellow) are presented. One can see that the latter has smooth boundary. These figures demonstrate that the majority of peculiarities of the value function are found near the narrow throat, emphasizing the necessity of extremely accurate computations near the throat.

The analytical results of the paper Shinar and Zarkh (1996) shows that in the case of a slower pursuer the geometry of the critical tube differs qualitatively for different combinations of the parameters of the problem. The dependence of the critical tube geometry on the parameters of the problem (and how it affects the singular surfaces) is investigated in that paper. The example computed numerically in this subsection corresponds to the case of the most complicated structure of the narrow throat.

3.6 Adaptive Control on the Basis of Differential Game Methods

In the framework of linear differential games, let us describe a method for constructing the first player's control, which can be reasonably called *adaptive*.

Consider a system with linear dynamics

$$\dot{z} = A(t)z + B(t)u + C(t)v,\\ z \in R^n,\ t \in T,\ u \in P \subset R^p,\ v \in R^q, \tag{22.38}$$

analogous to system (22.25) except that here there is no any compact constraint for the second player's control v. Here, $T = [t_0, t_f]$ is the time interval of process. Assume that the set P contains the origin of the space R^p. Also, assume that the matrix function B is Lipschitzian in the interval T.

The first player tries to guide m chosen components of the phase vector of system (22.38) at the instant t_f to the given terminal set M. The set M is assumed to be a convex compactum in the set of these m coordinates of the phase vector z. Suppose also that the interior of M is not empty and contains the origin, which will be considered as a "center" of M. An additional objective of the first player is to guide these m components of the vector z as closer to the center of M as possible.

As it is described in Sect. 3.1, let us pass to a system without the phase variable in the right-hand side of the dynamics:

$$\dot{y} = D(t)u + E(t)v,\\ y \in R^m,\ t \in T,\ u \in P \subset R^p,\ v \in R^q. \tag{22.39}$$

The first player tries to guide the phase vector of system (22.39) at the instant t_f to the set M as closer to its center as possible.

All constructions below are for system (22.39). The obtained adaptive control $U(t, y)$ can be applied to system (22.38) too as $U\big(t, Z_m(t_f, t)z\big)$.

3.6.1 System of Stable Bridges

Let the symbol $S(t) = \big\{y \in R^m : (t, y) \in S\big\}$ denote the time section of the set $S \subset T \times R^m$ at the instant $t \in T$. Denote by $O(\varepsilon) = \{y \in R^m : |y| \leqslant \varepsilon\}$ the ball with the radius ε and center at the origin of the space R^m.

Stable bridges. Consider in the interval $[t_0, t_f]$ a zero-sum differential game with a terminal set $\mathbb{M}$ and geometric constraints $\mathbb{P}$, $\mathbb{Q}$ for the players' controls:

$$\dot{y} = D(t)u + E(t)v,\\ y \in R^m,\ t \in T,\ \mathbb{M},\ u \in \mathbb{P},\ v \in \mathbb{Q}. \tag{22.40}$$

Here, the matrices $D(t)$, $E(t)$ are the same as in system (22.39). The sets $\mathbb{M}$, $\mathbb{P}$, $\mathbb{Q}$ are assumed to be convex compacta. They are regarded as parameters of the game.

Let $u(\cdot)$ and $v(\cdot)$ be measurable functions with their values in the compact sets $\mathbb{P}$ and $\mathbb{Q}$, respectively. A motion of system (22.40) (and, therefore, of system (22.39)) emanated from the point y_* at the instant t_* under controls $u(\cdot)$ and $v(\cdot)$ is denoted by $y\big(\cdot; t_*, y_*, u(\cdot), v(\cdot)\big)$.

Below, the symbol $W_{\mathbb{M}}$ denotes the maximal stable bridge in game (22.40) that stops at the set $\mathbb{M}$ at the instant t_f.

3.6.2 Constructing System of Stable Bridges

1°. Take a set $Q_{\max} \subset R^q$, which is regarded as the maximal constraint for the second player's control can be treated as "reasonable" by the first player when guiding system (22.39) to the terminal set M. Assume that the set $Q_{\max}$ contains the origin of its space. Denote by W_{main} the maximal stable bridge for system (22.40) that corresponds to the parameters $\mathbb{P} = P$, $\mathbb{Q} = Q_{\max}$, $\mathbb{M} = M$. Below, it is called the *main bridge* for brevity.

Suppose additionally that the set $Q_{\max}$ is chosen in such a way that for some $\varepsilon > 0$ for every $t \in T$, the following inclusion holds:

$$O(\varepsilon) \subset W_{\text{main}}(t). \tag{22.41}$$

The value of ε is fixed for further reasonings.

Thus, W_{main} is a closed tube in the space $T \times R^n$ that stops at the set M at the instant t_f. Any t-section $W_{\text{main}}(t)$ is convex and contains the origin of the space R^n with some neighborhood.

2°. Introduce some *additional* closed tube $W_{\text{add}} \subset T \times R^n$ such that any t-section $W_{\text{add}}(t)$ is the reachable set of system (22.40) at the instant t with the initial set $O(\varepsilon)$ taken at the instant t_0. That is, constructing the tube W_{add}, we assume that in dynamics (22.40) the first player is absent ($u \equiv 0$) and the control of the second player is constrained by $Q_{\max}$. One can easily see that W_{add} is the maximal stable bridge for system (22.40) with

$$\mathbb{P} = \{0\}, \ \ \mathbb{Q} = Q_{\max}, \ \ \mathbb{M} = W_{\text{add}}(t_f).$$

For any $t \in T$, the t-section $W_{\text{add}}(t)$ is convex, and the following inclusion holds:

$$O(\varepsilon) \subset W_{\text{add}}(t). \tag{22.42}$$

3°. Consider a collection of tubes $W_k \subset T \times R^m$, $k \geqslant 0$, whose t-section $W_k(t)$ are defined as

$$W_k(t) = \begin{cases} kW_{\text{main}}(t), & 0 \leqslant k \leqslant 1, \\ W_{\text{main}}(t) + (k-1)W_{\text{add}}(t), & k > 1. \end{cases}$$

The sets $W_k(t)$ are compact and convex. For any numbers $0 \leqslant k_1 < k_2 \leqslant 1 < k_3 < k_4$ due to relations (22.41), (22.42) strict inclusions hold

$$W_{k_1}(t) \subset W_{k_2}(t) \subset W_{k_3}(t) \subset W_{k_4}(t).$$

In works Ganebny et al. (2006, 2007), the following important properties have been justified. A tube W_k for $0 \leqslant k \leqslant 1$ is the maximal stable bridge for system (22.40) that corresponds to a constraint kP for the first player's control, a constraint $kQ_{\max}$ for the second player's control, and a terminal set kM. For $k > 1$, a set W_k is a stable bridge (but, generally speaking, not the maximal one) for the parameters

$$\mathbb{P} = P, \quad \mathbb{Q} = kQ_{\max}, \quad \mathbb{M} = M + (k-1)W_{\text{add}}(t_f).$$

Thus, one has a growing system of stable bridges, where each greater bridge corresponds to a greater constraints for the second player's control. This system is generated only by two tubes W_{main} and W_{add} by means of operation of Minkowski sum and multiplication by a nonnegative number parameter.

Feedback control. The adaptive control $(t, y) \mapsto U(t, y)$ itself is constructed in the following way.

Fix a number $r > 0$. Consider a position (t, y). If $|y| \le r$, assume $U(t, y) = 0$. If $|y| > r$, find the minimal number k^* such that the distance between the point y and the t-section $W_{k^*}(t)$ of the bridge W_{k^*} equals r. On the boundary of the set $W_{k^*}(t)$ find the point y^* closest to y. One has $|y^* - y| = r$. Define a vector $u^* \in P_{k^*}$ from the extremum condition

$$(y^* - y)' D(t) u^* = \max \left\{ (y^* - y)' D(t) u : u \in P_{k^*} \right\}.$$

Assume $U(t, y) = u^*$.

Thus, the control U is generated on the basis of the extremal shift rule (i.e., well known in the theory of differential games) and is applied in the *discrete scheme* (Krasovskii 1985; Krasovskii and Subbotin 1974, 1988) with the time step Δ_U. The control is chosen at the beginning of each time step of length Δ_U and kept during the step. In Ganebny et al. (2009), a theorem about the result guaranteed by the control U is formulated and proved.

3.7 Adaptive Control in J. Shinar's Problem

To apply the adaptive method of control, one should introduce an auxiliary constraint $Q_{\max}$. To do this, let us take a reasonable value $a_{E\,\max}$ bounding the lateral acceleration of the evader. This value defines the constraint $Q_{\max}$ as an ellipse

$$Q_{\max} = \left\{ v \in R^2 : \frac{v_1^2}{A_E^2} + \frac{v_2^2}{B_E^2} \le 1 \right\}$$

where the semiaxes A_E, B_E are parallel to the coordinate axes and are computed on the basis of the value $a_{E\,\max}$ and cosine of the angle $(\chi_E)_{\text{nom}}$.

Let us show the simulation results for the case

$$\tau_P = 1.0\,\text{s}, t_f = 10.0\,\text{s}, a_P = 1.3\,\text{m/s}^2, (\chi_P)_{\text{nom}} = 47.94°, (\chi_E)_{\text{nom}} = 45°.$$

The ellipse P, therefore, has the semiaxes equal to $A_P = 1.3$, $B_P = 0.87$. The radius of the terminal circle is taken to be equal to 2.

Let us choose the value $a_{E\ \max} = 1.0\,\mathrm{m/s}^2$. Then, the ellipse $Q_{\max}$ has semiaxes $A_E = 1.0$, $B_E = 0.71$. To construct the adaptive control, one should introduce also the parameter r. Let us take $r = 0.01$. The adaptive control U is applied in the discrete scheme with the time step $\Delta_U = 0.01$ s.

The initial phase vector in the difference coordinates is taken as $(z_1^0, z_4^0)' = \xi_P^0 - \xi_E^0 = (-3\text{ m}, 0\text{ m})$, $(z_2^0, z_5^0)' = \dot{\xi}_P^0 - \dot{\xi}_E^0 = (0\text{ m/s}, 2\text{ m/s})$, $(z_3^0, z_6^0)' = F^0 = 0$. The disturbance control is generated as a piecewise-constant function, which values are in the ellipse $1.5Q_{\max}$ and which stays constant for a random time periods not longer than 3 s. The random procedure for choosing the next value from the ellipse is the following: at first, uniformly we choose an angle from the interval $[0, 2\pi)$, then also uniformly in the radius-vector a point is chosen between the origin and the boundary of ellipse.

In Fig. 22.53a, the phase trajectory of system (22.34) is shown in difference coordinates z_1, z_4. The initial point is denoted by an asterisk and the final one by a black circle. The circle of the terminal set is shown.

In Fig. 22.53b, c, one can see hodographs of the realizations of the controls $u(t)$ and $v(t)$. The hodograph of the control $u(t)$ is inside the ellipse P, the initial and final points are also marked by an asterisk and a black circle. The hodograph of the control $v(t)$ in some time intervals goes outside the ellipse $Q_{\max}$.

Figure 22.54a, b shows graphs of levels of the vector control $u(t)$ with respect to the ellipse P and of the vector disturbance $v(t)$ with respect to the ellipse $Q_{\max}$. There are two intervals of maximality of the useful control: at the beginning of the

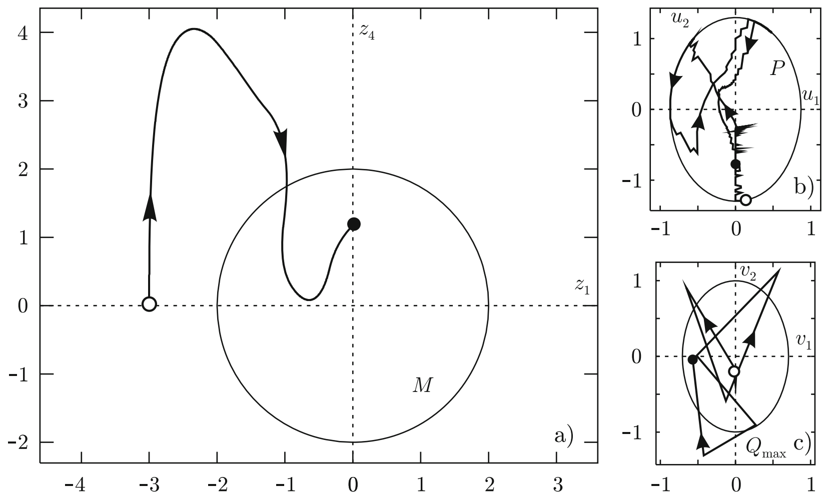


Fig. 22.53 Simulation results: (**a**) the phase trajectory of the system in difference geometric coordinates; (**b**) the hodograph of the useful control u; (**c**) the hodograph of the disturbance v, the ellipse $Q_{\max}$ is shown

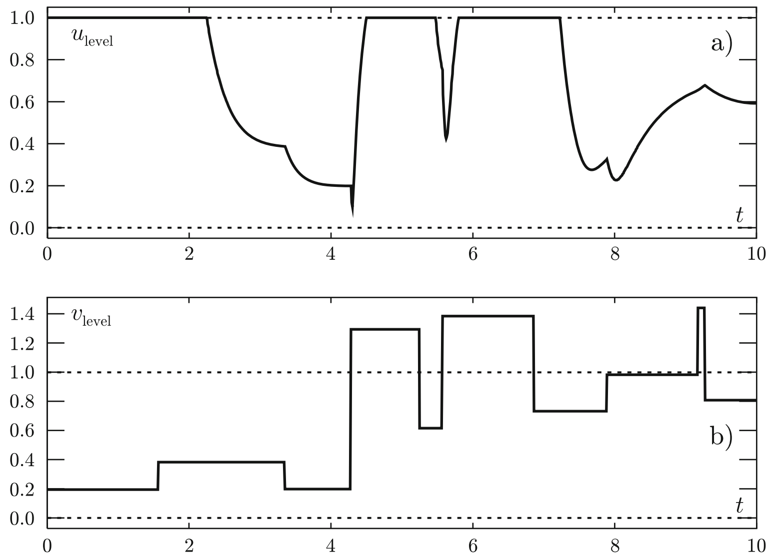


Fig. 22.54 Realizations of controls: (**a**) the graph of the level of the useful control u; (**b**) the graph of the level of the disturbance v with respect to the chosen constraint $Q_{\max}$ (the level 1.0)

process, when the initial deviation is diminished, and in the middle of the process, when the disturbance is sufficiently outside the forecasted ellipse $Q_{\max}$. In other intervals, the useful control level is less then maximally possible. One can see how the useful control reaches a level, which corresponds to the level of the disturbance in the next time interval.

Looking at Fig. 22.53a, one can also see that despite the disturbance realization is greater than the chosen level $Q_{\max}$, the process termination is successful: the system is guided inside the terminal set.

3.8 One-Dimensional Differential Games. Linear Problems with Positional Functional. Linear-Quadratic Problems

Differential games with linear dynamics and fixed terminal time permit to reduce the dimension of the phase vector when constructing u-stable bridges; the new phase vector has a dimension equal to the dimension of the target set. Moreover, the t-sections of the bridges in the new coordinates keep the convexity property if the target set is convex. In J. Shinar's problem (Sect. 3.5), a two-dimensional geometric miss is measured at the terminal instant; therefore, the new phase variable is two-dimensional.

1. In problems dealing with space pursuit of one weak-maneuvering object by another one, it is quite reasonable to disjoin the two-dimensional miss in the plane orthogonal to the nominal line-of-sight to two orthogonal components and

to consider two problems each having a one-dimensional miss. In this case, both new problems have one-dimensional phase variable, and maximal stable bridges are constructed in the space *time* $\times$ *phase variable* of dimension 1×1. The construction actually is reduced to one-dimensional integrating to obtain two lines that are upper and lower boundaries of the bridge. Moreover, all upper boundaries as well as lower ones differ from each other by a vertical shift.

J. Shinar and his group used these facts effectively to study dependence of the problem solution on the parameters of the game in the case of one-dimensional miss. Since the maximal u-stable bridge can be regarded as the solvability set of the problem with a given level of the miss (or as a Lebesgue set of the value function), the analysis of its peculiarities is significant both from theoretical and practical points of view.

In particular, in works by J. Shinar and his collaborators, the following questions have been studied. (1) If the object has both air rudders and jet engine, what period of the final stage of the pursuit is more suitable for spending the jet impulse (at the beginning, in the middle, or at the end)? How does it influence to enlargement of the solvability set (Shinar et al. 2012)? (2) What part of the object is more suitable for locating the air rudders (head, middle or rear part) (Shima 2005; Shima and Golan 2006)?

The shape of the solvability set depends significantly on the character of transient processes in the servomechanisms. A study of this question is made in Shinar et al. (2013).

2. In applied problems, the miss distance is often measured not at the terminal instant only but also at some prescribed intermediate ones. With that, the first player minimizes and the second one maximizes some functional that depends on all these misses. The differential games that involve such nonterminal functionals are studied in book A.N.Krasovskii and N.N.Krasovskii (1995). In this book, the concept of positional functional is introduced for which the optimal strategies of the first and second players do not depend on the history of the process as it is for the case of the terminal payoff. For systems with linear dynamics with convex positional functional, the book suggests a constructive method for computing the value function based on a convex hull construction operation. In works Lukoyanov (1998), Kornev (2012), and Gomoyunov et al. (2015), the main stages of this method, proofs, and corresponding numerical procedures are considered in details.

3. In Sect. 3, we consider differential games with linear dynamics, fixed terminal instant, and geometric constraints on the players' controls. Taking into account such "hard-bounded control limits" complicates sufficiently optimal feedback control laws in comparison with linear-quadratic (LQ) formulations, where constraints on the players' controls are introduced "softly" by means of integral quadratic functionals. Linear feedback control laws obtained in the framework of LQ formulations are very popular in engineering practice. Among a large number of works, we would like to mention book (1998) by J.Z. Ben-Asher and I. Yaesh. This book is oriented to students and engineers specializing in missile guidance. It is written in informal mathematical style. There are a lot of topics and examples from the simplest LQ classical optimal control problems to game LQ methods under inexact parameters

or inexact time-to-go of the control process. The book includes listings of MAPLE and MATLAB subroutines for certain problems of missile guidance.

4 Conclusion

The theory of differential games is very lucky because its first author and founder R. Isaacs being an excellent mathematician (the number theory was his first area of interests in young age) turned to the new work at the beginning of the 1950s having before his eyes a bundle of practical problems. His book "Differential Games" is an enthusiastic anthem to problems of this type. With that, as well as L.S. Pontryagin in the theory of optimal control, he started to investigate one of the most difficult classes, the time-optimal problems. The object that R. Isaacs called "car" is now extremely widespread in works on aviation, robotics, sea navigation, etc.

Just a time-optimal game including exactly this object (the "homicidal chauffeur" game) is chosen by us as the central one for the first part of this chapter (Sect. 2). We have collected some applied problems (first of all, those have been considered in works by J.V. Breakwell and A.W. Merz) to show how deeply they can be studied and how various they can be.

In the second part of this chapter (Sect. 3), we have concentrated on differential games with linear dynamics and fixed terminal instant. J. Shinar showed plenty of problems from the aerospace navigation, in which linearization of the dynamics and fixation of the terminal instant are relevant. The presence of both these factors allows one to pass to an equivalent differential game with the phase variable having the dimension, possibly, sufficiently lower than in the original game. Moreover, if the terminal payoff is convex, then the level sets of the value function (the solvability sets, the maximal stable bridges) of such a problem have convex t-sections. This simplifies greatly the solution of the game.

Choosing such topics for our chapter, we have tried to solve numerically several problems taken from these areas. The numerical method used for solving time-optimal problems in the first part is heuristic in many respects. Computation of each problem needs an "individual supervision" of the program execution. That is why the method is set forth very schematically, only its main idea is described. Our major objective is to give quite exact pictures of the value function, which is often discontinuous and, generally speaking, has non-convex level sets. Also, there can be places of fast change of the value function that are expressed in condensation of contour curves, which is a difficult situation from the computational point of view. Our results have been checked on the basis of many examples computed by other methods and by other authors.

Numerical constructions in the second part are mainly connected with convexification operation (construction of the convex hull). If the problems are of optimal control type, that is, if they do not include the second player, then in the situation of convex payoff, the t-sections of the level sets would be convex automatically without any additional convexifications. In the game problems, even if the payoff is convex, at each step we should involve the convex hull construction procedure that complicates the algorithm. At the same time, our procedure is much faster in comparison with general convex hull construction methods because in the framework of our algorithm we have information about places of possible local convexity violation of the processed function. If the phase variable of the game is two-dimensional, this information can be effectively used. The convexification algorithm is given in sufficient details, but descriptively, not in a strict programmatic manner. As in the first part, we use our own visualization software when demonstrating the results of numerical constructions.

In the last subsections of both parts, we touch upon some publications on applied problems close to those have been considered but much more difficult. The corresponding text includes short descriptions of these problems and some remarks on them.

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References

Abramyants TG, Maslov EP, Yakhno VP (2007) Evasion from detection in the three-dimensional space. J Comput Syst Sci Int 46(5):675–680

Abramyants TG, Maslov EP, Rubinovich EYa (1981) A simplest differential game of alternate pursuit. Autom Remote Control 41(8):1043–1052

Aubin JP (1991) Viability theory. Birkhäuser, Basel

Averbukh VL, Ismagilov TR, Patsko VS, Pykhteev OA, Turova VL (2000) Visualization of value function in time-optimal differential games. In: Handlovičová A, Komorníková M, Mikula K, Ševčovič D (eds) Algoritmy 2000, 15th Conference on Scientific Computing, Vysoke Tatry – Podbanske, 10–15 Sept 2000, pp 207–216

Başar T, Olsder GJ (1995) Dynamic noncooperative game theory. Academic Press, London

Bakolas E, Tsiotras P (2012) Feedback navigation in an uncertain flow-field and connections with pursuit strategies. J Guid Control Dyn 35(4):1268–1279

Bakolas E, Tsiotras P (2013) Optimal synthesis of the Zermelo-Markov-Dubins problem in a constant drift field. JOTA 156:469–492

Ben-Asher JZ, Yaesh I (1998) Advances in missile guidance theory. Progress in astronautics and aeronautics. AIAA, New York

Berdyshev YI (2002) A problem of the sequential approach to a group of moving points by a third-order non-linear control system. J Appl Math Mech 66(5):709–718

Berdyshev YI (2008) On a nonlinear problem of a sequential control with a parameter. J Comput Syst Sci Int 47(3):380–385

Berdyshev YI (2015) Nonlinear problems of sequential control and their application. UrO RAN, Ekaterinburg (in Russian)

Berkovitz LD (1994) A theory of differential games. In: Başar T, Haurie A (eds) Annals of the international society of dynamic games. Advances in dynamic games and applications, vol 1. Birkhäuser, Boston, pp 3–22
Bernhard P (1970) Linear differential games and the isotropic rocket. Ph.D. thesis, Stanford University
Bernhard P (1977) Singular surfaces in differential games, an introduction. In: Haggedorn P, Knobloch HV, Olsder GJ (eds) Differential games and applications. Lecture notes in control and information sciences. Springer, Berlin, pp 1–33
Bernhard P (1979) Contribution à l'étude des jeux différentiels à deux joueurs, somme nulle et information parfaite. thèse de doctorat d'État
Bernhard P, Larrouturou B (1989) Etude de la barriere pour un probleme de fuite optimale dans le plan. rapport de recherche
Bernhard P, Pourtallier O (1994) Pursuit evasion game with costly information. Dyn Control 4(4):365–382
Bhattacharya S, Başar T (2012) Differential game-theoretic approach to a spatial jamming problem. In: Cardaliaguet P, Cressman R (eds) Advances in dynamic game: theory, applications and numerical methods for differential and stochastic games. Annals of the international society of dynamic games, vol 12. Birkhäuser, Boston, pp 245–268
Blagodatskih AI, Petrov NN (2009) Conflict interaction controlled objects groups. Udmurt State University, Izhevsk (in Russian)
Blaquière A, Gérard F, Leitmann G (1969) Quantitative and qualitative differential games. Academic Press, New York
Bolonkin AA, Murphey RA (2005) Geometry-based parametric modeling for single-pursuer multiple evader problems. J Guid Control Dyn 28(1):145–149
Botkin ND (1982) Evaluation of numerical construction error in differential game with fixed terminal time. Probl Control Inf Theor 11:283–295
Botkin ND (1993) Asymptotic behavior of solution in differential games. viability domains of differential inclusions. Russian Acad Sci Dokl Math 46(1):8–11
Botkin ND, Hoffmann KH, Mayer N, Turova VL (2013) Computation of value functions in non-linear differential games with state constraints. In: Hömberg D, Tröltzsch F (eds) Proceedings of the 25th IFIP TC7 Conference on System Modeling and Optimization, pp 235–244
Botkin ND, Hoffmann KH, Turova VL (2011) Stable numerical schemes for solving Hamilton-Jacobi-Bellmann-Isaacs equations. SIAM J Sci Comput 33(2):992–1007
Botkin ND, Kein VM, Patsko VS (1984) The model problem of controlling the lateral motion of an aircraft during landing. J Appl Math Mech 48(4):395–400
Botkin ND, Zarkh MA. Estimation of error for numerical constructing absorption set in linear differential game. In: Subbotin and Patsko (1984), pp 39–80 (in Russian)
Breakwell JV, Merz AW (1969) Toward a complete solution of the homicidal chauffeur game. In: Proceedings of the 1st International Conference on the Theory and Application of Differential Games, Amherst, pp III-1–III-5
Breitner MH (2005) The genesis of differential games in light of Isaacs' contributions. JOTA 124(3):523–559
Bryson AE, Ho YC (1975) Applied optimal control. Optimization, estimation and control. Hemisphere Publishing Corporation/John Wiley and Sons, New York
Bulirsch R, Montrone F, Pesch HJ (1991) Abort landing in the presence of windshear as a minimax optimal control problem, Part 1: necessary conditions. JOTA 70(1):1–23
Bulirsch R, Montrone F, Pesch HJ (1991) Abort landing in the presence of windshear as a minimax optimal control problem, Part 2: multiple shooting and homotopy. JOTA 70(2):223–254
Cardaliaguet P, Quincampoix M, Saint-Pierre P (1995) Numerical methods for optimal control and differential games. Technical report, Ceremade CNRS URA 749, University of Paris – Dauphine
Cardaliaguet P, Quincampoix M, Saint-Pierre P (1999) Set-valued numerical analysis for optimal control and differential games. In: Bardi M, Raghavan TES, Parthasarathy T (eds) Stochastic and differential games: theory and numerical methods. Annals of the international society of dynamic games, vol 4. Birkhäuser, Boston, pp 177–247

Chen M, Fisac J, Sastry S, Tomlin C (2015) Safe sequential path planning of multi-vehicle systems via double-obstacle Hamilton-Jacobi-Isaacs variational inequality. In: Proceedings of the 14th European Control Conference, pp 3304–3309
Chentsov AG (1976) On a game problem of converging at a given instant of time. Math USSR-Sb 99(141)(3):394–420
Chentsov AG (1978) An iterative program construction for a differential game with fixed termination time. Soviet Math Dokl 19(3):559–562
Chentsov AG (1978) On a game problem of converging at a given instant of time. Math USSR-Sb 19(3):559–562
Chernous'ko FL (1976) A problem of evasion from many pursuers. J Appl Math Mech 4(1):11–20
Chernous'ko FL (1993) State estimation for dynamic systems. CRC Press, Boca Raton
Chernous'ko FL, Melikyan AA (1978) Game problems of control and search. Nauka, Moscow (in Russian)
Chikrii AA (1997) Conflict-controlled processes. Mathematics and its applications, vol 405. Kluwer Academic Publishers Group, Dordrecht
Courant R (1962) Partial differential equations. InterScience, New York
Crandall MG, Evans LC, Lions PL (1984) Some properties of viscosity solutions of Hamilton-Jacobi equations. Trans Am Math Soc 282(1):487–502
Crandall MG, Lions PL (1983) Viscosity solutions of Hamilton-Jacobi equations. Trans Am Math Soc 277(1):1–42
Davidovitz A, Shinar J (1989) Two-target game model of an air combat with fire-and-forget all-aspect missiles. JOTA 63(2):133–165
Dubins LE (1957) On curves of minimal length with a constraint on average curvature and with prescribed initial and terminal positions and tangents. Am J Math 79:497–516
Dvurechenskii PE (2013) Algorithms for construction of ε-optimal strategies in nonlinear differential games in the plane. Ph.D. thesis, Moscow Institute of Physics and Technology, Moscow (in Russian)
Dvurechenskii PE, Ivanov GE (2012) An algorithm for constructing optimal strategy in nonlinear differential game with nonfixed time of termination. Proc Moscow Inst Phys Tech 4(4):51–61 (in Russian)
Evans LC (1998) Partial differential equations. AMS, New York
Exarchos I, Tsiotras P, Pachter M (2015) On the suicidal pedestrian differential game. Dyn Games Appl 5(3):297–317
Falcone M (2006) Numerical methods for differential games based on partial differential equations. Int Game Theory Rev 8(2):231–272
Fisac JF, Sastry S (2015) The pursuit-evasion-defense differential game in dynamic constrained environments. In: Proceedings of the IEEE 54th Annual Conference on Decision and Control (CDC), pp 4549–4556
Ganebny SA, Kumkov SS, Patsko VS (2006) Control design in problems with an unknown level of dynamic disturbance. J Appl Math Mech 70(5):680–695
Ganebny SA, Kumkov SS, Patsko VS (2009) Extremal aiming in problems with an unknown level of dynamic disturbance. J Appl Math Mech 73:411–420
Ganebny SA, Kumkov SS, Patsko VS, Pyatko SG (2007) Constructing robust control in differential games: application to aircraft control during landing. In: Jørgensen S, Quincampoix M, Vincent T (eds) Annals of the international society of dynamic games, vol 9. Birkhäuser, Boston, pp 69–92
Getz WM, Pachter M (1981) Two-target pursuit-evasion differential games in the plane. JOTA 34(3):383–403
Gomoyunov MI, Kornev DV, Lukoyanov NY (2015) On the numerical solution of a minimax control problem with a positional functional. Proc Steklov Inst Math 291(1):S77–S95
Grigorenko NL (1989) The pursuit problem in n-person differential games. Math USSR-Sb 63(1):35–45
Grigorenko NL (1990) Mathematical methods of control of multiple dynamic processes. Moscow State University, Moscow (in Russian)

Grigor'eva SV, Pakhotinskikh VYu, Uspenskii AA, Ushakov VN (2005) Construction of solutions in some differential games with phase constraints. Sb Math 196(4):513–539
Grigor'eva SV, Tarasyev AM, Uspenskii AA, Ushakov VN (2000) Constructions of the differential game theory for solving the Hamilton-Jacobi equations. Proc Steklov Inst Math (Suppl Issues) Suppl 2:S38–S53
Gutman S (1979) On optimal guidance for homing missile. J Guid Control 3(4):296–300
Gutman S, Leitmann G (1975) On a class of linear differential games. JOTA 17(5–6):511–522
Gutman S, Leitmann G (1976) Optimal strategies in the neighborhood of a collision course. AIAA J 14(9):1210–1212
Hagedorn P, Breakwell JV (1976) A differential game with two pursuers and one evader. JOTA 18(2):15–29
Han Z, Niyato D, Saad W, Başar T, Hjorungnes A (2011) Game theory in wireless and communication networks: theory, models, and applications. Cambridge University Press, Cambridge/ New York
Isaacs R (1951) Games of pursuit. Scientific Report of the RAND Corporation. Technical report, RAND Corporation, Santa Monica
Isaacs R (1965) Differential games. John Wiley and Sons, New York
Isakova EA, Logunova GV, Patsko VS Computation of stable bridges for linear differential games with fixed time of termination. In: Subbotin and Patsko (1984), pp 127–158 (in Russian)
Kim DP (1993) Methods of search and pursuit of mobile objects. Nauka, Moscow (in Russian)
Kornev DV (2012) On numerical solution of positional differential game with nonterminal payoff. Autom Remote Control 73:1808–1821
Krasovskii AN, Krasovskii NN (1995) Control under lack of information. Birkhäuser, Berlin
Krasovskii NN (1970) Game-theoretic problems on the encounter of motions. Nauka, Moscow (in Russian)
Krasovskii NN (1971) Rendezvous game problems. National Technical Information Service, Springfield
Krasovskii NN (1985) Control of a dynamic system. The minimum problem of a guaranteed result. Nauka, Moscow (in Russian)
Krasovskii NN, Subbotin AI (1974) Positional differential games. Nauka, Moscow (in Russian)
Krasovskii NN, Subbotin AI (1988) Game-theoretical control problems. Springer, New York
Kryazhimskiy AV, Osipov YS (2010) Idealized program packages and problems of positional control with incomplete information. Proc Steklov Inst Math 268:155–174
Kumkov SS, Le Ménec S, Patsko VS (2017) Zero-sum pursuit-evasion differential games with many objects: survey of publications. Dyn Games Appl 7(4):609–633
Kurzhanski AB (2015) On a team control problem under obstacles. Proc Steklov Inst Math 291:128–142
Kurzhanski AB (2016) Problem of collision avoidance for a team motion with obstacles. Proc Steklov Inst Math 293:120–136
Kurzhanski AB, Valyi I (1997) Ellipsoidal calculus for estimation and control. Birkhäuser, Boston
Kurzhanski AB (1977) Control and observation under uncertainty. Nauka, Moscow (in Russian)
Kurzhanski AB (2004) The problem of measurement feedback control. J Appl Math Mech 68(4):487–501
Laumond JP (ed) (1998) Robot motion planning and control. Lecture notes in control and information science, vol 229. Springer, New York
Laumond JP, Mansard N, Lasserre JB (2014) Optimality in robot motion: optimal versus optimized motion. Commun ACM 57(9):82–89
Leitmann G, Pandey S (1991) Aircraft control for flight in an uncertain environment: take-off in windshear. JOTA 70(1):25–55
Levchenkov AY, Pashkov AG (1990) Differential game of optimal approach of two inertial pursuers to a noninertial evader. JOTA 65(3):501–518
Lewin J (1973) Decoy in pursuit-evasion games. Ph.D. thesis, Stanford University
Lewin J (1994) Differential games. Springer, London
Lewin J, Breakwell JV (1975) The surveillance-evasion game of degree. JOTA 16(3–4):339–353

Lewin J, Olsder GJ (1979) Conic surveillance evasion. JOTA 27(1):107–125
Lions PL (1982) Generalized solutions of Hamilton-Jacobi equations. Research notes in mathematics, vol 69. Pitman, Boston
Lukoyanov NY (1998) The problem of computing the value of a differential game for a positional functional. J Appl Math Mech 62(2):177–186
Markov AA (1889) Some examples of the solution of a special kind of problem on greatest and least quantities. Soobscenija Charkovskogo matematiceskogo obscestva 2, 1(5, 6):250–276 (in Russian)
Melikian AA (1975) On optimum selection of noise intervals in differential games of encounter. J Appl Math Mech 39(2):195–203
Melikyan AA (1973) On minimal observations in a game of encounter. J Appl Math Mech 37(3):407–414
Melikyan AA (1998) Generalized characteristics of first order PDEs. Birkhäuser, Boston
Melikyan AA, Shinar J (2000) Identification and construction of singular surface in pursuit-evasion games. In: Filar JA, Gaitsgory V, Mizukami K (eds) Annals of the international society of dynamic games, vol 5. Birkhäuser, Boston, pp 151–176
Merz AW (1971) The homicidal chauffeur – a differential game. Ph.D. thesis, Stanford University
Merz AW (1972) The game of two identical cars. JOTA 9(5):324–343
Merz AW (1974) The homicidal chauffeur. AIAA J 12(3):259–260
Merz AW (1985) To pursue or to evade – that is the question. J Guid Control Dyn 8(2): 161–166
Meyer A, Breitner MH, Kriesell M (2005) A pictured memorandum on synthesis phenomena occurring in the homicidal chauffeur game. In: Martin-Herran G, Zaccour G (eds) Proceedings of the Fifth International ISDG Workshop, Segovia, 21–24 Sept 2005, pp 17–32
Miele A, Wang T, Melvin WW (1986) Optimal take-off trajectories in the presence of windshear. JOTA 49(1):1–45
Miele A, Wang T, Tzeng CY, Melvin WW (1987) Optimal abort landing trajectories in the presence of windshear. JOTA 55(2):165–202
Miele A, Wang T, Wang H, Melvin WW (1988) Optimal penetration landing trajectories in the presence of windshear. JOTA 57(1):1–40
Mikhalev DK, Ushakov VN (2007) Two algorithms for approximate construction of the set of positional absorption in the game problem of pursuit. Autom Remote Control 68(11): 2056–2070
Mishchenko EF, Nikol'skii MS, Satimov NYu (1977) The problem of avoiding encounter in n-person differential games. Trudy Mat Inst Steklov 143:105–128
Mitchell I (2002) Application of level set methods to control and reachability problems in continuous and hybrid systems. Ph.D. thesis, Stanford University
Mitchell I, Bayen A, Tomlin C (2005) A time-dependent Hamilton-Jacobi formulation of reachable sets for continuous dynamic games. IEEE Trans Autom Control 50(7):947–957
Neveu D, Pignon JP, Raimondo A, Nicolas JM, Pourtallier O (1995) Pursuit games with costly information: application to the ASW helicopter versus submarine game. In: Olsder GJ (ed) New trends in dynamic games and applications. Annals of the international society of dynamic games, vol 3. Birkhäuser, Boston, pp 247–257
Olsder GJ, Breakwell JV (1974) Role determination in aerial dogfight. Int J Game Theory 3:47–66
Olsder GJ, Pourtallier O (1995) Optimal selection of observation times in a costly information game. In: Olsder GJ (ed) New trends in dynamic games and applications. Annals of the interantional society of dynamic games, vol 3. Birkhäuser, Boston, pp 227–246
Osipov YS (2006) Program packages: an approach to the solution of positional control problems with incomplete information. Russ Math Surv 61(4):611–661
Pachter M, Garcia E, Casbeer D (2014) Active target defense differential game. In: Proceedings of the 52nd Allerton Conference on Communication, Control and Computing, Monticello, pp 46–53
Pachter M, Getz W (1980) The geometry of the barrier in the game of two cars. Optim Control Appl Met 1(2):103–118

Patsko VS (1975) Problem of quality in linear differential games of the second order. In: Kurzhanski AB (ed) Differential games and control problems. UrSC AS USSR, Sverdlovsk, pp 167–227 (in Russian)

Patsko VS (1996) Special aspects of convex hull constructing in linear differential games of small dimension. In: Control applications of optimization 1995. A Postprint Volume from the IFAC Workshop, Haifa, 19–21 Dec 1995. Pergamon, New York, pp 19–24

Patsko VS, Botkin ND, Kein VM, Turova VL, Zarkh MA (1994) Control of an aircraft landing in windshear. JOTA 83(2):237–267

Patsko VS, Turova VL (1995) Numerical solution of two-dimensional differential games. Technical report, UrO RAN. Institute of Mathematics and Mechanics, Ekaterinburg, Ekaterinburg

Patsko VS, Turova VL (1997) Numerical solutions to the minimum-time problem for linear second-order conflict-controlled systems. In: Bainov D (ed) Proceedings of the Seventh International Colloquium on Differential Equations, Plovdiv, 18–23 Aug 1996, pp 329–338

Patsko VS, Turova VL (2001) Level sets of the value function in differential games with the homicidal chauffeur dynamics. IGTR 3(1):67–112

Patsko VS, Turova VL (2004) Families of semipermeable curves in differential games with the homicidal chauffeur dynamics. Automatica 40(12):2059–2068

Patsko VS, Turova VL (2009) Numerical investigation of the value function for the homicidal chauffeur problem with a more agile pursuer. In: Bernhard P, Gaitsgory V, Pourtallier O (eds) Advances in dynamic games and their applications: analytical and numerical developments. Annals of the international society of dynamic games, vol 10. Birkhäuser, Boston, pp 231–258

Petrosjan LA (1965) A family of differential survival games in the space R^n. Soviet Math Dokl 6(2):377–380

Petrosyan LA (1966) Pursuit lifeline games with many participants. Proc Armen Acad Sci Ser Math 1:331–340

Petrosyan LA (1970) Differential games with incomplete information. Soviet Math Dokl 11(6):1524–1528

Petrosyan LA (1977) Differential games of pursuit. Leningrad State University, Leningrad, USSR (in Russian)

Petrosyan LA (1993) Differential games of pursuit. World Scientific Publisher, London

Petrosyan LA, Dutkevich YG (1972) Games with a "lifeline". The case of l-capture. SIAM J Control 10(1):40–47

Petrosyan LA, Garnaev AY (1993) Search games. St. Petersburg Gos. University, St.-Petersburg (in Russian)

Petrosyan LA, Shiryaev VD (1980) Group pursuit by one pursuer of many evaders. Viestn Leningr Univ 13(3):50–59

Petrov NN (1988) A group pursuit problem with phase constraints. J Appl Math Mech 52(6): 803–806

Polovinkin ES, Ivanov GE, Balashov MV, Konstantiov RV, Khorev AV (2001) An algorithm for the numerical solution of linear differential games. Sb Math 192(10):1515–1542

Ponomarev AP, Rozov NH (1978) Stability and convergence of alternating Pontrjagin sums. Univ, Ser. 15: Vycisl. Mat. i Kibernet 1:82–90 (in Russian)

Pontryagin LS, Mischenko EF (1971) The problem of evasion in linear differential games. Diff Urav 7(2):436–445 (in Russian)

Pontryagin LS (1967) Linear differential games, 1. Soviet Math Dokl 8:769–771

Pontryagin LS (1967) Linear differential games, 2. Soviet Math Dokl 8:910–912

Pontryagin LS (1971) A linear differential escape game. Proc Steklov Inst Math 112:27–60

Pschenichnyi BN (1976) Simple pursuit by several objects. Cybern Syst Anal 12(3):484–485

Pschenichnyi BN, Chikrii AA, Rappoport IS (1981) An efficient method of solving differential games with many pursuers. Soviet Math Dokl 23(1):104–109

Pschenichnyi BN, Sagaidak MI (1970) Differential games of prescribed duration. Cybernetics 6(2):72–83

Raivio T, Ehtamo H (2000) On the numerical solution of a class of pursuit-evasion games. In: Filar JA, Gaitsgory V, Mizukami K (eds) Advances in dynamic games and applications. Annals of the international society of dynamic games, vol 5. Birkhäuser, Boston, pp 177–192

Reeds JA, Shepp LA (1990) Optimal paths for a car that goes both forwards and backwards. Pac J Math 145(2):367–393

Selvakumar J, Bakolas E (2015) Optimal guidance of the isotropic rocket in a partially uncertain flow. In: Proceedings of European Control Conference, Linz, pp 3328–3333

Shaferman V, Shima T (2008) Unmanned aerial vehicles cooperative tracking of moving ground target in urban environments. J Guid Control Dyn 31(5):1360–1371

Shaferman V, Shima T (2010) Cooperative multiple model adaptive guidance for an aircraft defending missile. J Guid Control Dyn 33(6):1801–1813

Shevchenko I (1997) Successive pursuit with a bounded detection domain. JOTA 95(1):25–48

Shevchenko I (2012) Locally optimizing strategies for approaching the furthest evader. In: Petrosyan LA, Zenkevich NA (eds) Contributions to game theory and management, vol 5. Graduate School of Management, St. Petersburg State University, St.Petersburg, pp 293–303

Shima T (2005) Capture conditions in a pursuit-evasion game between players with biproper dynamics. JOTA 126(3):503–528. https://doi.org/10.1007/s10957-005-5495-3

Shima T (2011) Optimal cooperative pursuit and evasion strategies against a homing missile. J Guid Control Dyn 34(2):414–425

Shima T, Golan OM (2006) Bounded differential games guidance law for dual-controlled missiles. IEEE Trans Control Syst Tech 14(4):719–724

Shinar J, Glizer VY, Turetsky V (2012) Complete solution of a pursuit-evasion differential game with hybrid evader dynamics. IGTR 14(3):1–31

Shinar J, Glizer VY, Turetsky V (2013) The effect of pursuer dynamics on the value of linear pursuit-evasion games with bounded controls. In: Krivan V, Zaccour G (eds) Annals of the international society of dynamic games, vol 13. Dynamic games – theory, applications, and numerical methods. Birkhäuser, Basel, pp 313–350. https://doi.org/10.1007/978-3-319-02690-9_15

Shinar J, Gutman S (1980) Three-dimensional optimal pursuit and evasion with bounded controls. IEEE Trans Autom Control AC-25(3):492–496

Shinar J, Medinah M, Biton M (1984) Singular surfaces in a linear pursuit-evasion game with elliptical vectograms. J Optim Theory Appl 43(3):431–456

Shinar J, Zarkh M (1996) Pursuit of a faster evader – a linear game with elliptical vectograms. In: Proceedings of the Seventh International Symposium on Dynamic Games, Yokosuka, pp 855–868

Simakova EN (1967) On one differential pursuit game. Autom Remote Control 2:173–181

Simakova EN (1968) Concerning certain problem of persuit on the plane. Autom Remote Control 29(7):1031–1034

Stipanović DM, Melikyan AA, Hovakimyan N (2010) Guaranteed strategies for nonlinear multi-player pursuit-evasion games. IGTR 12(1):1–17

Stipanović DM, Tomlin C, Leitmann G (2012) Monotone approximations of minimum and maximum functions and multi-objective problems. Appl Math Opt 66(3):455–473

Subbotin AI (1980) A generalization of the basic equation of the theory of differential games. Soviet Math Dokl 22:358–362

Subbotin AI (1984) Generalization of the main equation of differential game theory. JOTA 43(1):103–133

Subbotin AI (1995) Generalized solutions of first-order PDEs. The dynamical optimization perspective. Birkhäuser, Boston

Subbotin AI, Chentsov AG (1981) Optimization of the guarantee in control problems. Nauka, Moscow (in Russian)

Subbotin AI, Patsko VS (eds) (1984) Algorithms and programs for solving linear differential games. Ural Scientific Center, Institute of Mathematics and Mechanics, Academy of Sciences of USSR, Sverdlovsk (in Russian)

Taras'yev AM, Ushakov VN, Khripunov AP (1988) On a computational algorithm for solving game control problems. J Appl Math Mech 51(2):167–172

Turova VL Linear differential game of quality. In: Subbotin and Patsko (1984), pp 191–248 (in Russian)

Ushakov VN (1998) Construction of solutions in differential games of pursuit-evasion. In: Differential inclusions and optimal control. Lecture notes in nonlinear analysis, vol 2. Nicholas Copernicus University, Torun, pp 269–281
Yavin Y, Pachter M, Rodin EY (eds) (1987) Pursuit-Evasion differential games. International series in modern applied mathematics and computer science, vol 14. Pergamon Press, Oxford
Zheleznov VS, Ivanov MN, Kurskii EA, Maslov EP (1993) Avoidance of detection in 3D. Comput Math Appl 26(6):55–66